The Eightfold Way Model and the Cartan Subalgebra Revisited and its Implications for Nuclear Physics

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Abstract

It was shown recently by the author [1], that a proper study of the Eightfold Way model vis-a-vis the SU(3) model shows, that the adjoint representation has certain unique features which provides it with a basic fundamentality which was missed out in the earlier interpretations. That paper [1] also showed that the Lie Algebra gives a more basic and complete description of the particle physics reality than the corresponding group does. In this paper we revisit the Eightfold Way Model and provide further support to the conclusions arrived in Ref. [1]. This demands that a proper Cartan Subalgebra be used for the description of the adjoint representation. This in turn allows us to make non-trivial statements about as to how nucleus may be understood as made up, not only of protons and neutrons treated as indistinguishable particles as in the SU(2)-isospin group, but also as another independent structure where the nucleus behaves as if it is made up of protons and neutrons wherein they are treated as distinguishable fermions.

Keywords: Lie Algegra, Adjoint Representation, Cartan Sublgebra, Eightfold Way Model, SU(2)-isospin, Nuclear Models

In a recent paper [1] the author had studied the Eightfold Way Model and the SU(3) flavour model and as to how they are related to each other. It is known that there is no baryon number associated with the spin 1/2 baryon octet in the Eightfold Way Model. However in the SU(3) model baryon number arises internally as Y = B + S. Hence the spin 1/2 baryon octet arising from the product 3 X 3 X 3 does have a baryon number. It was shown by the author [1], that the current understanding on this issue is fundamentally wrong. A correct understanding of this requires that the adjoint representation has certain unique features which provides it a basic fundamentality which was not realized in the earlier interpretations [1]. In this paper we revisit the Eightfold Way Model and provide further support to the conclusions arrived in Ref. [1]. We show that a proper Cartan Subalgebra be used to describe this adjoint representation. This in turn allows us to make non-trivial statements about how nucleus may be treated as made up, not only of protons and neutrons treated as indistinguishable particles as in the SU(2)-isospin group, but also in another independent structure where nucleus is shown to behave as if it is made up of protons and neutrons wherein they are treated as distinguishable entities.

It was also shown [1] that the SU(3) group structure itself was not rich enough to accommodate the difference in the structure between the spin 1/2 baryon octet in the Eightfold Way and the SU(3) models. It was shown that intrinsically the SU(3) Lie Algebra is richer than the Group itself to accommodate the difference between the spin 1/2 baryons in the Eightfold Way and the SU(3) models. Thus that paper [1] showed that the Lie Algebra gives a more basic and complete description of the particle physics reality than the corresponding group does.

Note that there has been a long-standing issue as to whether it is the Lie Group or the Lie Algebra which provides a more complete and fundamental description of the particle structure. So to say, is the Lie Algebra tied to its Lie group and/or it can go beyond Lie group in describing nature [2]. Our work [1] favours the latter opinion.

Given the two diagonal genertors in SU(3) in standard notation:

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{1}$$

and with $T_a = \frac{\lambda_a}{2}$. If these are taken as representation of the Cartan subalgebra, then in standard notation isospin and hypercharge are respectively T_3 and $Y = \frac{2}{\sqrt{3}} T_8$. Then the fundamental representation as eigenstates of these diagonal generators is given in terms of three quarks (u,d,s). In this the spin 1/2 octet arises in the product 3 X 3 X 3 = 1 + 8 + 8 + 10.

Though the quantum numbers of the octet members is the same as of the octet in the Eightfold way model, but these are not identically the same as was shown recently by the author [1].This is due the fact that in the above SU(3) model starting with the fundamental representation, there is an intrinsic baryon number Y=B+S; while there exists no baryon number for these same baryons in the Eightfold Way model.

To understand this let us study the proper Cartan Subalgebraic structure for the adjoint representation as given in the Eightfold Way model. For this we refer to Cahn (Ref. [3], p. 10-14).

For the Lie algebra [x,y]=z in the adjoint representation necessarily [ad(x), ad(y)] = ad(z). In SU(3) ad(T_3) and ad(Y) are diagonal and as 8X8 matrices are represented as:

$$ad(T_3) = \begin{pmatrix} \frac{1}{2} &&&&&&& \\ & \frac{-1}{2} &&&&&& \\ && 1 &&&&& \\ &&& 0 &&&& \\ &&&& -1 &&& \\ &&&&& 0 && \\ &&&&&& \frac{1}{2} & \\ &&&&&&& \frac{-1}{2} \end{pmatrix} \tag{2}$$

$$ad(Y) = \begin{pmatrix} 1 &&&&&&& \\ & -1 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& -1 & \\ &&&&&&& -1 \end{pmatrix} \tag{3}$$

Note that our 8-dimensional vector is ordered differently from Cahn's and is as per Dean's notation [4, p. 61].

Thus if we take X = a T_3 + b Y, then ad(X) is diagonal:

$$ad(X) = \begin{pmatrix} \frac{1}{2}a+b & & & & & & & \\ & \frac{-1}{2}a+b & & & & & & \\ & & a & & & & & \\ & & & 0 & & & & \\ & & & & -a & & & \\ & & & & & 0 & & \\ & & & & & & \frac{1}{2}a-b & \\ & & & & & & & \frac{-1}{2}a-b \end{pmatrix} \quad (4)$$

So we have now a two-dimensional subalgebra [3] (all linear combinations of T_3 and Y) which is abelian (that is, if X and Y are in the subalgebra [X,Y]=0). The basis we have chosen is such that for the rest of the Lie algebra, each element of the basis is an eigenvector of ad(X).

Now note that for a=1 and b=1/2 , X is thus the electric charge Q = $T_3 + \frac{1}{2}Y$ with these diagonal matrices defined in eqn. (1) above. ad(X) acting on the 8-dimensional basis vector given as (in this order):($p, n, \Sigma^+, \Sigma^0, \Sigma^-, \Lambda, \Xi^0, \Xi^-$). These are specified by the electric charge and supercharge Y in the Eightfold way model as [Q,Y]=0. So we see that actually the spin 1/2 baryon octet in the Eightfold model is an eigenstate of the charge Q and the hypercharge Y only. This is how the structure of the baryon in the Eightfold Way model is to be understood mathematically.

Note the difference. With the octet baryon in the SU(3) model is given as eigenstate of the two diagonal generators λ_3 and λ_8 (eqn. (1)). The two models differ as to the octet baryon and yet, as describing the same entities, these bring about a fundamental duality [1].

How is the Eightfold Way model description of the octet baryon as eigenstate of electric charge Q and hypercharge Y to be understood? It seems that the Casimir operator does not specify the adjoint representation here. Now in SU(3) starting with the smallest fundamental representation and building other 8-, 10-dimensional representations, the Casimir operators is essential to distinguish between various representations. But no Casimir operator is needed to label the adjoint representation. This conclusion is matched by referring to the Appendix where discussion after eqn. (6) indicates that for the adjoint representation one does not need the Casimir operator. This is amazing but actually straightforward to see. After all, the adjoint representation is fixed (besides being unique [1]) and gets specified by being eigenstates of the charge and the hypercharge operators only.

Let us refer to the Appendix. It shows (generalizing the arguments for SU(3)) that the Casimir operator is essential to distinguish between the various representation in the SU(3) model where all representation are built from the fundamental representation. But for the adjoint representation, the eight-dimensional vector is specified fully just be the abelian generators for SU(3). However, we see above that in SU(3), to get these charged states these abelian generators in SU(3) are not the simple diagonal ones λ_3 or λ_8 (eqn. (1)). But the abelian generators giving the proper Cartan subalgebra for the adjoint representation in SU(3) as [Q,Y]=0. Thus the electric charge Q and the hypercharge Y specify this 8-dimensional vector state completely. Note that in the notation of the Appendix, the proper Cartan subalgebra for the Eightfold Way arises as $[\ H_1,\ S_1\ ] = 0$, with H_1 = Y and S_1 = Q, and which label the 8-dimensional eigenvector as given above.

It is important to see how our understanding of hadronic physics would be affected by the fact that we do not need Casimir operator in the adjoint representation. However to start with, as to this reality, we find interesting support from nuclear physics.

Let us look at the (p.n) subset of the spin 1/2 octet baryon in the Eightfold Way and the SU(3) models. In the SU(3) model we have the isospin SU(2) degree of freedom essential to describe a system of protons and neutrons. Hence in a nucleus SU(2) isospin manifests itself by demanding that protons and neutrons be treated as identical particles which have to be antisymmetrized on exchange of proton and neutron in an n-p pair. This is manifested as the Generalized Pauli Exclusion Principle . And today's successful picture of the nucleus is based on Generalized Pauli Exclusion Principle which also justifies the Shell Model structure of the nucleus.

But we find that for the neutron-proton pair in the Eightfold Way Model there is no Casimir operator, i.e. no I^2 term. And hence there is no isospin terms to specify this state . And hence, clearly there cannot be any Generalized Pauli Exclusion Principle for the description of the proton-neutron pair in the Eightofld way model structure. Therefore what the Eightfold Model is showing unambiguously, is that in it the proton-neutron subset should not be treated as indistinguishable. So as per this model proton and neutron should be treated as distinguishable particles. This is an amazing prediction of the Eightfold Way model. However both of these structures exist simultaneously in hadron physics [1]. So as per this prediction, nuclear physics should be consistently described not only as when proton-neutron pair is treated as being indistinguishable, but in addition it should be consistent to study the nucleus as being made up of proton and neutron treated as distinguishable fermions. Any support for this unique prediction? Yes, indeed, strong support of this picture exists in nuclear physics.

Today, the dominant Independent Particle Model of nuclear physics is modeled after the SU(2) isopsin group with Charge Independence and Generalized Pauli Principle as its base. But the reality is that the nuclear physics phenomenon can be equally well described by treating the nucleus as made up of two independent Fermi seas of protons and neutron treated as separate. In that picture the neutron and protons are treated as distinguishable. In fact it has been shown convincingly at several places that these two pictures of the nucleus, as consisting of independent and distinguishable proton and neutron seas, and the other one where the Generalized Pauli Exclusion Principle treating p-n as identical, both give equivalent descriptions of the nucleus. This is well recorded, for example in Blatt and Weisskopf [5, p. 153-156], Brink [6. p. 16-18] , Lawson [7, p. 107-122].

Thus we find that the Eightfold way model gives independent eigenstates for the spin 1/2 baryon octet which is quite distinct from the same octet described in the SU(3) model. This gives unequivocal support of the conclusion of authors recent work [1]. The new predictions, arising from the Eightfold Way model, find strong support in nuclear physics.

Most interesting is the fact that firstly, there is a fundamental duality in describing spin 1/2 octet baryon by two independent structures as arising from the Eightfold Way model and the other one from the fundamental representation in the SU(3) model; and secondly, that the same is matched by two simultaneously coexisting and independent structures (one by treating n-p as indistinguishable and the other one where these are treated as distinguishable) in nuclear physics. Analogy with the wave-particle duality in quantum mechanics is very striking.

Appendix

We follow Dean's terminology (Ref. [4] p. 29-31) in this section. For a Lie group let the corresponding Lie algebra be given by $[X_\nu, X_\mu] = i f_{\nu\mu\lambda} X_\lambda$. Let $H_i, i = (1, 2...)$ be the maximal set of mutually commuting generators (Cartan's subalgebra) (e.g. in SO(3) there is only one such generator: $H_1 = J_3$). Let the Casimir operators of the group be given as C_μ (e.g. in SO(3) only one such generator J^2). In addition there may be other functions of the generators S_k which commute with all of the H_i but not with the full set of X_ν.

As H_i, C_μ and S_k are mutually commuting, these can be diagonalized simultaneously. Let us choose basis vector which is simultaneously eigenstate of all these specified by quantum numbers c=$c_1, c_2, ..$, h=$h_1, h_2, ..$ and s=$s_1, s_2, ..$ respectively. Let us write the basis vector as $\mid chs >$. For example in SO(3) the basis is thus fully specified as $\mid jm >$. In general we may distinguish between a particular irreducible representation with respect to the other irreducible representation by the set 'c' , the eigenvalue of the Casimir operators. Thus a particular representation matrix may be given for different eigenstates as

$$[x_\nu]_{hs,h^1s^1} = < chs \mid X_\nu \mid ch^1 s^1 > \tag{5}$$

However for the adjoint representation

$$[F_\nu]_{\mu\lambda} = i f_{\nu\mu\lambda} \tag{6}$$

These matrices are antisymmetric and hence a unitary transformation is made to obtain an equivalent representation for which H_i and S_k are diagonal. So only these two eigenvalues are needed to specify the vector in the adjoint representation.

Note that as this is for a specific representation, i.e. the adjoint representation, the Casimir operator does not figure in the matrix representation of the same. This is in contrast to the representations built from the fundamental representation, as above, where it is essential to distinguish between the various representations.

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