Chapter 4A from book **"Non-Rocket Space Launch and Flight",** by Alexander Bolonkin, Elsevier, 2007. 488 pgs**.**
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Attachment 4 pp.383-422

# Optimal Trajectories of Air and Space Vehicles

## Summary

The author has developed a theory on optimal trajectories for air vehicles with variable wing areas and with conventional wings. He applied a new theory of singular optimal solutions and obtained in many cases the optimal flight. The wing drag of a variable area wing does not depend on air speed and air density. At first glance the results may seem strange, however, this is the case and this chapter will show how the new theory may be used. The equations that follow enable computations of the optimal control and optimal trajectories of subsonic aircraft with pistons, jets, and rocket engines, supersonic aircraft, winged bombs with and without engines, hypersonic warheads, and missiles with wings.

The main idea of the research is to use the vehicle's kinetic energy to increase the range of missiles and projectiles.

The author shows that the range of a ballistic warhead can be increased 3–4 times if an optimal wing is added to it, especially a wing with variable area. If we do not need increased range, the warhead mass can be increased. The range of large gun shells can also be increased 3–9 times. The range of an aircraft may be improved by 3–15% or more.

The results can be used for the design of aircraft, missiles, flying bombs and shells for large guns.

## **Nomenclature** (in metric system)

$a$ – the speed of sound, m/s,
$a_1, b_1, a_2, b_2$ – coefficients of exponential atmosphere,
$C_L$ – lift coefficient,
$C_D$ – drag coefficient,
$C_{Do}$ – drag coefficient for $C_L = 0$,
$C_{DW}$ – wave wing drag coefficient when $\alpha = 0$,
$C_{Db}$ – body drag coefficient,
$c$ – relative thickness of a wing,
$c_b$ – relative thickness of a body,
$c_1$ – relative thickness of a vehicle body,
$\boldsymbol{c_s}$ – fuel consumption, kg/s/ kg thrust,

$\overline{D}$ – drag of vehicle, N,
$D$ – drag of vehicle without $\alpha$, N,
$D_{0W}$ – wave wing drag when $\alpha = 0$, N,
$D_{0b}$ – drag of a vehicle body, N,
$H$ – Hamiltonian,
$h$ – altitude, m,
$K = C_L/C_D$ – the wing efficiency coefficient,
$k_1, k_2, k_3$ – vehicle average aerodynamic efficiencies for sub-distances 1, 2, 3 respectively,

$L$ – range,
$M = V/a$ – Mach number,
$m$ – mass of vehicle, kg,

$p = m/S$ – load on a square meter of wing,
$q=\rho V^2/2$ – a dynamic air pressure,

$R$ – aircraft range or $R$ = distance from flight vehicle to Earth center;
$R = R_o + h$, where $R_o$ = 6378 km is Earth radius,

$t$– time,
$T = V_e\beta$ – thrust, N,
$V$ – vehicle speed, m/s,
$V_e$ – speed of throw back mass (air for propeller engine, jet for jet and rocket engine), m/s,
$S$ – wing area, $m^2$,
$s$ – length of trajectory,
$T$ – engine thrust, N,

$Y$ – lift force, N,
$\alpha$ – wing attack angle,
$\beta$ – fuel consumption,
$\theta$ – angle between the vehicle velocity and the horizon,
$\omega$ – thrust angle between thrust and velocity,
$\omega_E$ – Earth angle speed,
$\varphi_E$ – lesser angle between the Earth's Polar axis and a perpendicular to a flight plate,
$\rho$ – air density. $kg/m^3$.

## Introduction

The topic of the optimal flight of air vehicles is very important. There are numerous articles and books about the optimal trajectories of rockets, missiles, and aircraft. The classical research of this topic is by Miele[1]. Unfortunately, the optimal theory of this problem is very complex. In most cases, the researchers obtained complex equations, that allow one to compute a single optimal trajectory for a given aircraft and for given conditions, but the structure of optimal flight is not clear and simple formulas of optimal control (which depend only on flight conditions) are absent.

The author's new theory of singular optimal solutions, developed earlier[2–14], does not contain unknown coefficients or variables as previous theories have. He found that the optimal flight path depends only on the flight conditions and the addition of certain variable wing structures.

In conclusion, the author applies his solution to ballistic missiles, warheads, flying bombs, large gun shells, and subsonic, supersonic, and hypersonic aircraft with rocket, turbo-jet, and propeller engines. He shows that the range of these air vehicles can be increased 3–9 times.

### 1. General equations

Let us consider the movement of an air vehicle given the following conditions: (1) The vehicle moves in a plane containing the Earth's center. (2) The vehicle design allows the wing area to be changed (this will prove important in the remainder of this chapter). (3) We ignore the centrifugal force from the Earth's rotation (it is less then 1%). (4) Earth has a curvature.

Then the equations for flying vehicle (in a system of coordinates where the center of the system is located at the center of gravity of the flying vehicle, the $x$-axis is in the direction of flight, the $y$-axis is perpendicular to the $x$- axis, Fig. A4.1) are

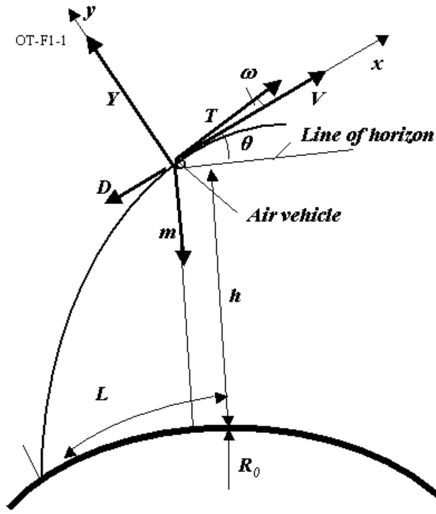


**Fig. A4.1** Vehicle forces and coordinate system.

$$\frac{dL}{dt} = V\cos\theta \quad ,$$

$$\frac{dh}{dt} = V\sin\theta \quad , \qquad \text{(A4.1)–(A4.2)}$$

$$\frac{dV}{dt} = \frac{T(h,V,\beta)\cos\omega - \overline{D}(\alpha,V,h)}{m} - g\sin\theta ,$$

$$\frac{d\theta}{dt} = \frac{T(h,V,\beta)\sin\omega + Y(\alpha,V,h)}{mV} - \frac{g}{V}\cos\theta + \frac{V\cos\theta}{R} + 2\omega_E\cos\varphi_E \ , \qquad \text{(A4.3) – (A4.5)}$$

$$\frac{dm}{dt} = -\beta \quad .$$

All values are in the metric system and all angles are taken to be in radians.

## Flight with a small change of vehicle mass and flight path angle

Most air vehicles fly at an angle $\theta$ in the range $\pm 15^{\circ}$ ($\theta = \pm 0.2618$ rad), with the engine located along the velocity vector. This means

$$\sin\theta = \theta, \qquad \cos\theta = 1 , \qquad \omega = 0 , \qquad \text{(A4.6) – (A4.8)}$$

because $\sin 15^{\circ} = 0.25882$, $\cos 15^{\circ} = 0.9659$.

Let as substitute (A4.6) – (A4.8) into (A4.1) – (A4.5)

$$\frac{dL}{dt} = V \quad ,$$

$$\frac{dh}{dt} = V\theta \quad , \qquad \text{(A4.9) – (A4.10)}$$

$$\frac{dV}{dt} = \frac{T(h,V) - \overline{D}(\alpha,V,h)}{m} - g\theta ,$$

$$\frac{d\theta}{dt} = \frac{Y(\alpha,V,h)}{mV} - \frac{g}{V} + \frac{V}{R} + 2\omega_E\cos\varphi_E \ , \qquad \text{(A4.11) – (A4.12)}$$

$$\frac{dm}{dt} = -\beta \quad , \qquad \text{(A4.13)}$$

where

$$|\theta| \le \theta_{\max} . \qquad \text{(A4.14)}$$

Many air vehicles fly with a low angular speed of $d\theta/dt$. The change of mass is also low in flight. This means $m$ = const, $dm/dt \cong 0$.

$$d\theta/dt \approx 0, \quad dm/dt = 0 . \qquad \text{(A4.15) – (A4.16)}$$

Let us take a new independent variable $s$ = length of trajectory

$$dt = ds/V, \qquad \text{(A4.17)}$$

and substitute (A4.14)-(A4.17) in (A4.9)-(A4.13). Then system (A4.9)-(A4.13) takes the form

$$\frac{dL}{ds} = 1 ,$$
$$\frac{dh}{ds} = \theta ,$$
$$\frac{dV}{ds} = \frac{T(h,V) - \overline{D}(\alpha,V,h)}{mV} - \frac{g}{V}\theta , \qquad \text{(A4.18) – (A4.21)}$$
$$0 = \frac{Y(\alpha,V,h)}{mV} - \frac{g}{V} + \frac{V}{R} + 2\omega_E \cos\varphi_E .$$

Let us re-write equation (A4.21) in the form

$$Y(\alpha,V,h) - mg + \frac{mV^2}{R} + 2mV\omega_E \cos\varphi_E = 0 . \qquad \text{(A4.22)}$$

If we ignore the last element, equation (A4.22) takes the form

$$Y(\alpha,V,h) - mg + \frac{mV^2}{R} = 0 . \qquad \text{(A4.22)'}$$

If $V$ is not very large ($V <$ 3 km/s), the two last elements in equation (A4.21) are small and they may be ignored. Equations (A4.22) and (A4.22)' can be used for deleting $\alpha$ from $\overline{D}$.

Note the new drag without $\alpha$ is

$$D = D(h,V). \qquad \text{(A4.23)}$$

If we substitute $\alpha$ from (A4.22) into equation (A4.20) the equation system take the form

$$\frac{dL}{ds} = 1 ,$$
$$\frac{dh}{ds} = \theta , \qquad \text{(A4.24) – (A4.26)}$$
$$\frac{dV}{ds} = \frac{T(h,V) - D(V,h)}{mV} - \frac{g}{V}\theta ,$$

Here the variable $\theta$ is new control limited by

$$|\theta| \le \theta_{\max} . \qquad \text{(A4.27)}$$

## Statement of the problem

Consider the problem: finding the maximum range of an air vehicle described by equations (A4.24) – (A4.26) for the limitation (A4.27). This problem may be solved using conventional methods. However, it is a non-linear problem but contains the linear control, which means the problem has a singular solution. To find this singular solution, we will use methods developed previously[2, 4].

Write the Hamiltonian (for purpose – minimum of time):

$$H = 1 + \lambda_1\theta + \lambda_2 \frac{1}{V}\left(\frac{T-D}{m} - g\theta\right) , \qquad \text{(A4.28)}$$

where $\lambda_1(s), \lambda_2(s)$ are unknown multipliers. Application of the conventional method gives

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial h} = -\lambda_2 \frac{1}{V}\left(\frac{T_h' - D_h'}{m}\right) \quad ,$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial V} = -\lambda_2\left[-\frac{1}{V^2}\left(\frac{T-D}{m} - g\theta\right) + \frac{1}{V}\left(\frac{T_V' - D_V'}{m}\right)\right] \quad , \qquad \text{(A24.29) – (A4.31)}$$

$$\theta = \max_{\theta} H = \theta_{\max} sign\left[\lambda_1 - \lambda_2 \frac{g}{V}\right] \quad .$$

Where $D_h^{'}, D_v^{'}, T_h^{'}, T_v^{'}$ denote the first partial derivatives of *D*, *T* by *h*, *V* respectively.

The last equation shows that the control $\theta$ can have only two values $\pm\theta_{\max}$. We consider the singular case when

$$A = \lambda_1 - \lambda_2 \frac{g}{V} \equiv 0 \ . \qquad \text{(A4.32)}$$

This equation has two unknown variables $\lambda_1$ and $\lambda_2$ and does not contain information about the control $\theta$. Let us to differentiate equation (A4.32) for the independent variable *s*. After substitution the equations (A4.26), (A4.29), (A4.30), and (A4.32) into the result of differentiation , we obtain the relation for $\lambda_1 \neq 0$, $\lambda_2 \neq 0$

$$V\left(T_h' - D_h'\right) = g\left(T_V' - D_V'\right) \qquad \text{(A4.33)}$$

This equation does not contain $\theta$ either, but it contains the important relation between the variables *V* and *h* on the optimal trajectory.

If we have the formulas (or graphs)

$$D = D(h,V), \qquad \text{(A5.34)}$$

$$T = T(h,V), \qquad \text{(A4.35)}$$

we could find the relation

$$h = h(V) \qquad \text{(A4.36)}$$

and the optimal trajectory for a given air vehicle.

This also gives important information about the structure of the optimal solution. Investigation of equation (A4.33) shows that the equation has one solution in each of the subsonic, supersonic, and hypersonic fields. The equation can have two solutions for a transonic field.

This means the optimal trajectory in most cases has three parts (see Fig. A4.2):

a) When climbing and in flight a vehicle moves from the initial point *A* with the angle $\pm\theta_{\max}$ up to the optimal curve (A4.36)**,** then continues along the optimal curve (A4.36) and moves with at an angle $\pm\theta_{\max}$ to point *B*.
b) When descending and in flight (Fig. A4.3) a vehicle moves from the initial point *A* with the angle $\pm\theta_{\max}$ (up or down) to the optimal curve (A4.36)**,** then continues down the optimal curve (A4.36), and moves at an angle $\pm\theta_{\max}$ (up or down) to the point *B*.

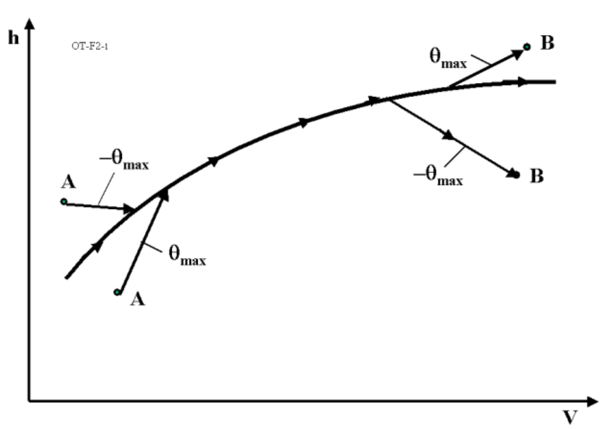

**Fig. A4.2.**. Optimal trajectory for air vehicle climb and flight.

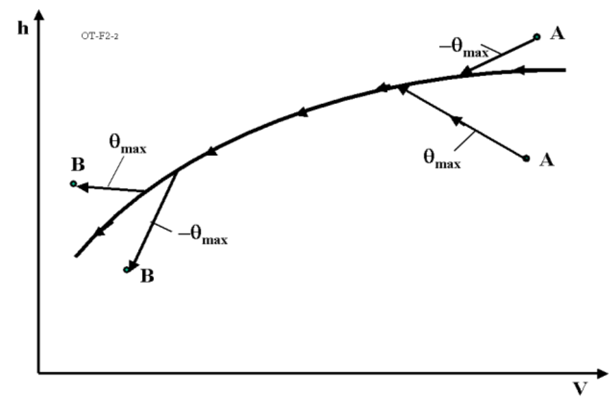


**Fig. A4.3**. Optimal trajectory for air vehicle descent and flight.

The selection of direction (up or down, with $\theta_{\max}$ or $-\theta_{\max}$ respectively) depends only on the position of the initial and end points *A* and *B*.

For air vehicles with rocket engines *T* = const, equation (A4.33) has a very simple form

$$VD'_h = gD'_V \quad . \qquad \text{(A4.37)}$$

The same form (same curve) also applies for a ballistic warhead, which does not have engine thrust (after its short initial burn) ($T = 0$).

If we want to find an equation for the control $\theta$, we continue to differentiate equation (A4.33) with the independent variable *s*, and substitute into the equations (A4.25), (A4.26), (A4.29), (A4.30), (A4.32), and (A4.33). We obtain the relation for $\theta$ if $\lambda_1 \neq 0$, $\lambda_2 \neq 0$

$$\theta = \frac{B_1(T-D)}{mV\left(B_1\dfrac{g}{V} - B_2\right)} \quad , \qquad \text{(A4.38)}$$

where

$$B_1 = (T'_h - D'_h) + V(T''_{hV} - D''_{hV}) - g(T''_{VV} - D''_{VV}) \quad ,$$
$$B_2 = V(T''_{hh} - D''_{hh}) - g(T''_{hV} - D''_{hV}) \quad . \qquad \text{(A4.39)–(A4.40)}$$

Here signs in form $D''_{hV}$ are the second partial derivates *D* for *h*, *V*.

$$D''_{hV} = \frac{\partial^2 D}{\partial h \partial V} \quad . \qquad \text{(A4.41)}$$

If the thrust does not depend on *h*, *V* (*T* = const) or no engine (*T* = 0), the equation for $\theta$ becames simpler

$$\theta = \frac{[(gD''_{VV} - D'_h) - VD''_{hV}](T-D)}{m[g(gD''_{VV} - D'_h) + V^2 D''_{hh}} . \qquad \text{(A4.42)}$$

In accordance with other publications[2–8] (e,g, equation (4.2)[4]) the necessary condition for optimal trajectory is

$$-(-1)^k \frac{\partial}{\partial \theta}\left[\frac{d^{2k}}{ds^{2k}}\left(\frac{\partial H}{\partial \theta}\right)\right] \geq 0 . \qquad \text{(A4.43)}$$

where *k* = 1.

To obtain results for different forms of the drags and thrusts, we must take formulas (or graphs) for subsonic, transonic, supersonic, or hypersonic speed, and specific formulas for the thrust

and substitute them in the equation (A4.33) and (A4.38). Consider two cases: subsonic and hypersonic speeds.

**Subsonic speed ($V$ < 270 m/s) and different engines.**

Lift, drag, and derivative equations for subsonic speed are

$$L = mg = \varsigma\alpha \frac{\rho V^2}{2} S, \quad \overline{D} = C_D \frac{\rho V^2}{2} S, \quad C_D = C_{Do} + \varepsilon\alpha^2, \quad D = \left[ C_{Do} + \varepsilon \left( \frac{2mg}{\varsigma \rho V^2 S} \right)^2 \right] \frac{\rho V^2}{2} S,$$

$$\rho = a_1 e^{-h/b_1}, \quad \frac{\partial D}{\partial V} = \left[ C_{Do} - \varepsilon \left( \frac{2mg}{\varsigma \rho V^2 S} \right)^2 \right] \rho V S, \quad \frac{\partial D}{\partial h} = -\frac{1}{b_1} \left[ C_{Do} - \varepsilon \left( \frac{2mg}{\varsigma \rho V^2 S} \right)^2 \right] \frac{\rho V^2}{2} S, \qquad \text{(A4.44)}$$

where $\varsigma = \dfrac{6.24\lambda}{\lambda + 2}$, $\quad \varepsilon = \dfrac{\varsigma^2}{\pi\lambda}$, magnitude $\varepsilon \approx \zeta^2/\pi\lambda$ is an induced drag coefficient, $\lambda = l^2/S$, $l$ is a wing span.

It is known in conventional aerodynamics that the coefficient of flight efficiency $k$ is

$$k = \frac{C_L}{C_D} = \frac{\varsigma\alpha}{C_{Do} + \varepsilon\alpha^2}, \quad \textit{from} \quad \max_{\alpha} k \quad \textit{we obtain} \quad \alpha_{opt} = \sqrt{\frac{C_{Do}}{\varepsilon}}, \quad k_{\max} = \frac{\varsigma}{2\sqrt{\varepsilon C_{Do}}}. \qquad \text{(A4.45)}$$

**a) Aircraft with rocket engine.** For this aircraft the thrust $T$ is constant or *0*. Equation (A4.33) has form (A4.37). Find the partial derivatives

$$T'_V = 0, \quad T'_h = 0. \qquad \text{(A4.46)}$$

Substituting (A4.44) to (A4.46) in (A4.37) we obtain the relation between air density $\rho$, altitude $h$, and aircraft speed $V$:

$$\rho = \frac{2gp}{\varsigma V^2} \sqrt{\frac{\varepsilon}{C_{Do}}}, \quad p = \frac{m}{S}, \quad h = b_1 \ln \frac{a_1}{\rho} \quad , \qquad \text{(A4.47)}$$

where $p = m/S$ is the load on a square meter of wing. For a diapason of $h$ = 0–11 km the coefficients $a_1$ = 1.225, $b_1$ = 9086.

Results of this computation are presented in Fig. A4.4.

b) **Aircraft with turbo-jet engine**. The thrust for this engine is

$$T = T_0 \frac{\rho}{\rho_0}, \quad T'_h = -\frac{T}{b_1}, \quad T'_V = 0. \qquad \text{(A4.48)}$$

Substitute (A4.48) in (A4.33). We obtain

$$V\left( -\frac{T}{b_1} - D'_h \right) = -gD'_V \quad \textit{or} \quad T = \frac{b_1}{V}\left( gD'_V - VD'_h \right), \qquad \text{(A4.48)'}$$

and substituting (A4.44) and (A4.48) in (A4.33), we obtain

$$\frac{1}{p}\left( \frac{V^2}{2b_1} + g \right)\left[ C_{Do} - \varepsilon \left( \frac{2pg}{\varsigma \rho V^2} \right)^2 \right] = \frac{\overline{T}_0}{b_1 \rho_0}, \quad \textit{where} \quad \overline{T}_0 = \frac{T_0}{m}. \qquad \text{(A4.49)}$$

We can then find $\rho$, $h$ from (A4.49)

$$\rho = \frac{2pg\sqrt{\varepsilon}}{\varsigma V^2 \sqrt{A_2}}, \quad \textit{where} \quad A_2 = C_{Do} - \frac{2p\overline{T}_0}{\rho_0 (V^2 + 2b_1 g)} \quad \overline{T}_0 = \frac{T_0}{m}, \quad h = b_1 \ln \frac{a_1}{\rho}. \qquad \text{(A4.50)}$$

Results of computation for the different $p$, $T$ = 0.8 *N/kg*, $a_1$ = 1.225, $b_1$ = 9086 are presented in Fig. A4.5.

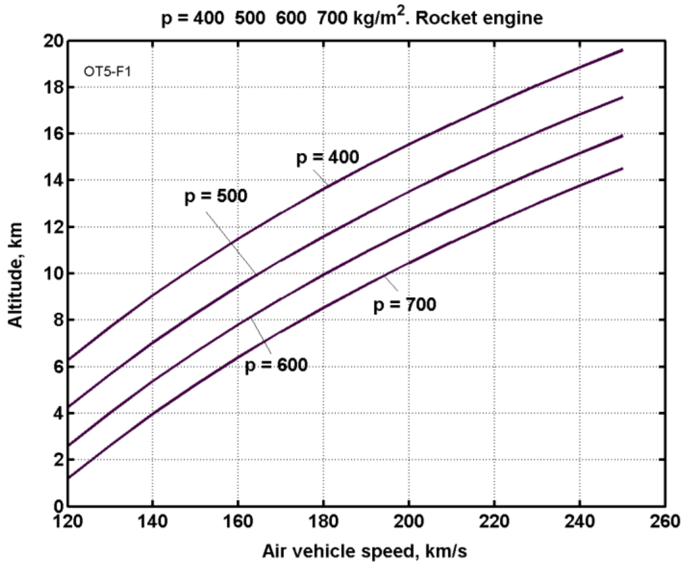


**Fig. A4.4**. Air vehicle altitude versus speed for wing load $p$ = 400, 500, 600, 700 kg/m$^2$ and a rocket engine.

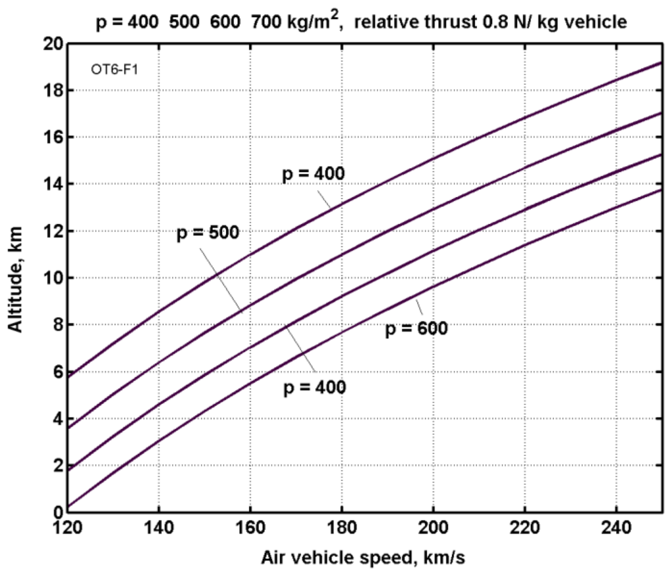


**Fig. A4.5**. Air vehicle altitude versus speed for wing load $p$ = 400, 500, 600, 700 kg/m$^2$, turbo-jet engine, and relative thrust 0.8 N/kg vehicle.

c) **Piston and turbo engines with propeller.** All current propeller engines have propellers with variable pitch. The propeller coefficient efficiency, $\eta$, approximately is constant. The thrust of this engine is

$$T = \frac{N_0}{V}\frac{\rho}{\rho_0}, \quad T'_V = -\frac{T}{V}, \quad T'_h = -\frac{T}{b_1}, \tag{A4.51}$$

where $N_0 = N_e\eta$, $N_e$ is engine power at $h = 0$.

Substituting (A4.44) in (A4.33). We obtain the equation for thrust

$$V\left(\frac{T}{b_1} + D'_h\right) = g\left(\frac{T}{V} + D'_V\right) \quad or \quad T = \frac{b_1 V(gD'_V - VD'_h)}{V^2 - gb_1} \ . \qquad \text{(A4.51)'}$$

Substitute (A4.44) and (A4.51) in (A4.33). We obtain

$$\frac{V}{p}\left(\frac{V^2}{b_1} - g\right)\left[C_{Do} - \varepsilon\left(\frac{2pg}{\rho V^2}\right)^2\right] = \frac{\overline{N}_0}{\rho_0}\left(\frac{g}{V^2} - \frac{1}{b_1}\right), \quad where \quad \overline{N}_0 = \frac{N_0}{m}, \quad p = \frac{m}{S}. \tag{A4.52}$$

We can then find $\rho$, $h$ from (A4.52)

$$\rho = \frac{2pg\sqrt{\varepsilon}}{\varsigma V^2\sqrt{A_3}}, \quad where \quad A_3 = C_{Do} + \frac{p\overline{N}_0}{\rho_0 V^3}, \quad h = b_1 \ln\frac{a_1}{\rho}. \qquad \text{(A4.53)}$$

Results of computation for $C_{Do} = 0.025$, $\lambda = 10$, for different values of $p$, $N$ are presented in Fig. A4.6.

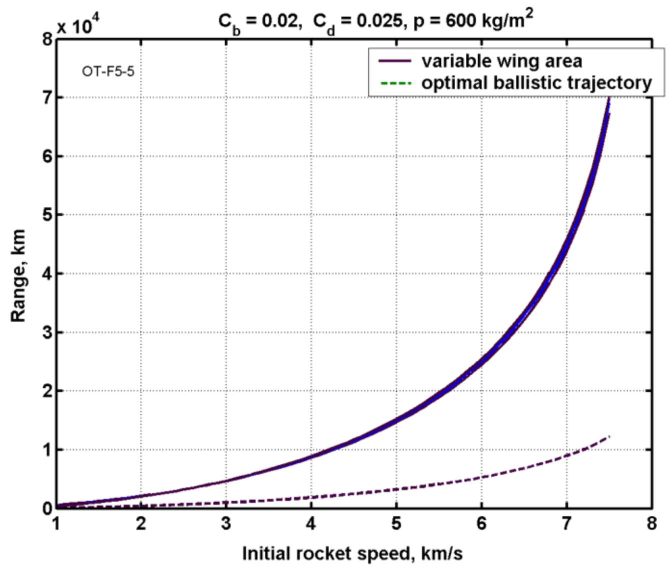


**Fig. A4.6.** Air vehicle range versus speed for wing load $p$ = 250, 300, 350, 400 kg/m$^2$, piston (propeller) engine, and relative engine power 100 W/kg vehicle.

**Hypersonic speed** (1 km/s < $V$ < 7 km/s).

The lift and drag forces in hypersonic flight are approximately (see (A4.22)')

$$L(\alpha, V, h) = mg - \frac{mV^2}{R} = \varsigma\alpha\frac{a\rho V}{2}S, \quad \overline{D} = (C_{DW} + \varepsilon\alpha^2)\frac{a\rho V}{2}S + C_{Db}\frac{a\rho V}{2}S_b,$$

$$\alpha = \frac{2p(g - V^2/R)}{\varsigma\rho a V}, \quad D = \left[C_{DW}\frac{a\rho V}{2} + \frac{2\varepsilon}{\rho a V}\left(\frac{m(g - V^2/R)}{\varsigma S}\right)^2\right]S + C_{Db}\frac{a\rho V}{2}S_{b,} \qquad \text{(A4.54)}$$

$$or \quad \frac{D}{m} = \left[C_{DW}\frac{q}{p} + \frac{\varepsilon p}{q}\left(\frac{g - V^2/R}{\varsigma}\right)^2\right] + C_{Db}\frac{q}{p_b}, \quad q = \frac{\rho a V}{2}.$$

Note

$$D_{0W} = C_{DW}\frac{\rho a V}{2}S, \quad D_{0b} = C_{Db}\frac{\rho a V}{2}S_b, \quad C_{DW} = 4c, \quad C_{Db} = 2c_b, \quad \rho = a_2 e^{-\frac{h-11000}{b_2}}, \qquad \text{(A4.55)}$$

The derivatives of $D$ by $V$, $h$ are

$$D'_V = \frac{D_{0W}}{V} + \frac{D_{0b}}{V} - \frac{2\varepsilon mp}{\varsigma^2\rho a}\left(g - \frac{V^2}{R}\right)\left(\frac{3}{R} + \frac{g}{V^2}\right),$$

$$D'_h = D'_\rho \rho'_h = -\frac{1}{b_2}\left(D_{0W} + D_{0b} - \frac{2\varepsilon mp(g - V^2/R)^2}{\varsigma^2\rho a V}\right). \qquad \text{(A4.56)}$$

a) **Rocket engine or hypersonic glider.** The derivatives from $T = const$ and $T = 0$ are

$$T'_V = 0, \quad T'_h = 0. \qquad \text{(A4.57)}$$

Substituting (A4.55) in (A4.56), and expressions (A4.56) and (A4.57) in (A4.37) to find $\rho$, $h$, we obtain for $h$ > 11,000 m

$$\rho = \frac{2p\sqrt{\varepsilon}}{\varsigma a}\sqrt{A_4}\,, \quad A_4 = \frac{\left(g - \frac{V^2}{R}\right)\left[g\left(\frac{3}{R} + \frac{g}{V^2}\right) + \frac{1}{b_2}\left(g - \frac{V^2}{R}\right)\right]}{\left(\frac{V^2}{b_2} + g\right)\left[C_{DW} + C_{Db}\left(\frac{S_b}{S}\right)\right]}, \quad h = 11000 + b_2 \ln\frac{a_2}{\rho}\,, \qquad \text{(A4.58)}$$

where $a_2 = 0.365$, $b_2 = 6997$ are coefficients of the exponent atmosphere for the stratosphere at 11 to 60 km.

If we ignore the small term $g\left(\frac{3}{R} + \frac{g}{V^2}\right)$ for $M > 3$ in (A4.58), the equations take the form

$$\rho = \frac{2p\left(g - V^2/R\right)\sqrt{\varepsilon}}{\varsigma a}\sqrt{A_5}\,, \quad A_5 = \frac{1}{C_{Do}(V^2 + gb_2)}, \quad where \quad C_{Do} = C_{0W} + C_{Db}\left(\frac{S_b}{S}\right),$$

where $C_{DW} \approx 4c$. If we ignore the term $gb_2$ (for $M > 3$), then

$$\rho = \frac{2p(g - V^2/R)}{\varsigma a V}\sqrt{\frac{\varepsilon}{C_{Do}}}\,. \qquad \text{(A4.59)}$$

In the limit as $R \to \infty$ in (2-54), we find

$$\rho = \frac{2pg}{\varsigma a V}\sqrt{\frac{\varepsilon}{C_{Do}}}\,. \qquad \text{(A4.59)'}$$

Here $\sqrt{C_{Do}/\varepsilon} = \alpha_{opt}$ is an optimal (maximum $C_L/C_D$) wing attack angle of the horizontal flight. Results of the computation in (A4.58) are presented in Fig. A4.7.

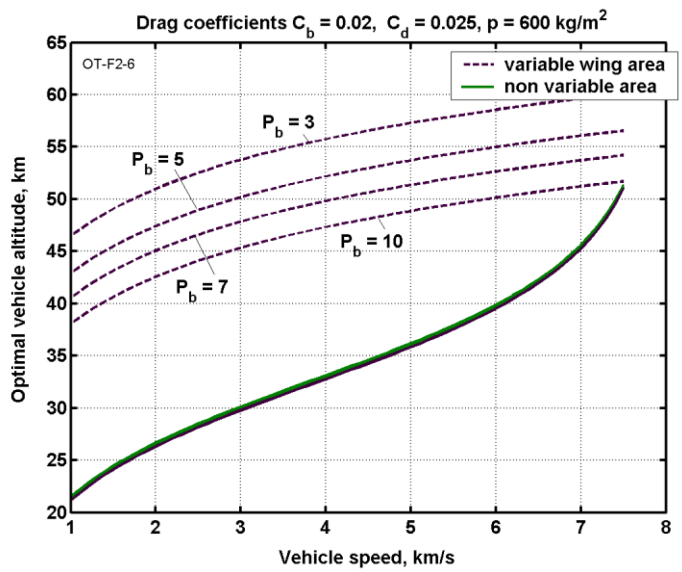


**Fig. A4.7**. Optimal vehicle altitude versus speed for specific body load $P_b = 3, 5, 7, 10$ ton/m$^2$, body drag coefficient $C_b = 0.02$, wing drag coefficient $C_d = 0.025$, wing load $p = 600$ kg/m$^2$.

**b) Ramjet engine.** The thrust of the jet engine is approximately ($M < 4$)

$$T = \xi\frac{\rho}{\rho_2}V^2, \quad T'_V = \frac{2T}{V}, \quad T'_h = -\frac{T}{b_2}\,, \qquad \text{(A4.60)}$$

where ξ is a numerical coefficient, $\rho_2$ is the air density at the lower end of the selected atmospheric diapason (in our case 11 km).

Substituting (A4.60) and (A4.56) in our main equation (A4.33), by repeat reasoning we can obtain the equation for the given engine

$$\rho = \frac{2p\sqrt{\varepsilon}}{\varsigma a}\sqrt{A_6}\ , \quad A_6 = \frac{\left(g - V^2/g\right)\left[g\left(\frac{3}{R} + \frac{g}{V^2}\right) + \frac{1}{b_2}\left(g - V^2/R\right)\right]}{\left[C_{DW} + C_{Db}\left(\frac{S_b}{S}\right)\right]\left[\left(\frac{V^2}{b_2} + g\right) - \frac{2\overline{T}_0 p}{a\rho_0}\left(\frac{V}{b_2} - \frac{2g}{V}\right)\right]}, \quad \overline{T} = \frac{T_0}{m}, \quad \text{(A4.61)}$$

where $T_0$ is taken at the lower end of the exponent atmospheric diapason (in our case 11 km). The curve of air density versus altitude $h$ is computed similarly to (A4.58).

## Optimal wing area

The lift force and drag of any wing may be written as

$$Y = mg = Y(\alpha, q, S), \quad D = D(\alpha^2, q, S). \qquad \text{(A4.62)}$$

Substituting (A4.62) in (A4.28) and finding the minimum $H$ versus $S$, we obtain the equation

$$D + \overline{D}'_\alpha \alpha'_S S = 0, \quad or \quad D + D'_S S = 0, \qquad \text{(A4.63)}$$

where $\alpha$ is the value found from the first equation (A4.62). Equation (A4.63) is the general equation for the optimal wing area and optimal specific load $p = m/S$ on a wing area.

**a) Subsonic speed**. Lift force and drag of the subsonic wing are

$$Y = mg = \varsigma\alpha q S \quad or \quad \alpha = \frac{mg}{\varsigma q S}, \quad \overline{D} = \left(C_{Do} + \varepsilon\alpha^2\right)qS \quad D = C_{DW} qS + \varepsilon\left(\frac{mg}{\varsigma}\right)^2 \frac{1}{qS}\ , \qquad \text{(A4.62)'}$$

where $q = \rho V^2/2$ is a dynamic air pressure for subsonic speed.

Substituting the last equation in (A4.62) into the first equation in (A4.63), we obtain the optimal specific load on the wing area

$$p_{opt} = \frac{\varsigma q}{g}\sqrt{\frac{C_{DW}}{\varepsilon}}\ . \qquad \text{(A4.63)'}$$

Substituting $\alpha$ from (A4.62)' into the last equation in (A4.62)' and dividing both sides by vehicle mass $m$, we obtain

$$\frac{D}{m} = \left[C_{DW}\frac{1}{p} + \varepsilon\left(\frac{g}{\varsigma q}\right)^2 p\right]q. \qquad \text{(A4.64)}$$

Here $D/m$ is specific drag (drag per unit weight for the vehicle). Substituting (A4.63)' into (A4.64). We abtain the minimum drag for a variable wing

$$\min\left(\frac{D}{m}\right) = 2\frac{g}{\varsigma}\sqrt{\varepsilon C_{DW}}\ , \qquad \text{(A4.64)'}$$

where the term on the right is wing drag for the lift of one unit of weight for the vehicle. We discover the important fact than the **optimal** wing drag of a variable wing **does not depend** on air speed, it depends **only** on the geometry of the wing. This may look wrong, but consider the following example. Wing drag is $D = mg/K$, where $K = C_L/C_D$ is the wing efficiency coefficient. The value $D/m$ does not depend on speed.

If the air vehicle has a body, the minimum drag is

$$\min\left(\frac{D}{m}\right) = 2\frac{g}{\varsigma}\sqrt{\varepsilon C_{DW}} + C_{Db}\frac{q}{p_b}, \quad q = \frac{\rho V^2}{2}. \qquad \text{(A4.65)}$$

Full vehicle drag depends on speed because the body drag depends on $V$.

Substituting the (A4.63)' term for α into (A4.62)', we obtain the optimal attack angle

$$\alpha_{opt} = \sqrt{\frac{C_{DW}}{\varepsilon}}\ . \qquad \text{(A4.66)}$$

This is the angle of optimal efficiency, but $C_{DW}$ is the wing drag coefficient **only** when $\alpha = 0$ (not the full vehicle as in conventional aerodynamics). The coefficient of flight efficiency

$$k = \frac{g}{D/m} \quad or \quad k_{\max} = \frac{g}{\min(D/m)}\ . \qquad \text{(A4.67)}$$

**b) Hypersonic speed.** The equations of wing lift force and wing air drag for hypersonic speed are as follows:

$$Y = \varsigma \alpha q S = m\left(g - \frac{V^2}{R}\right), \quad or \quad \alpha = \frac{p\left(g - V^2/R\right)}{\varsigma q}, \quad \overline{D} = \left(C_{DW} + \varepsilon\alpha^2\right)qS, \quad q = \frac{\rho a V}{2}. \qquad \text{(A4.68)}$$

Substituting α from (A4.68) into $\overline{D}$, we obtain

$$D = \left[C_{DW} + \varepsilon\left(\frac{m\left(g - V^2/R\right)}{\varsigma q S}\right)^2\right] qS. \qquad \text{(A4.68)'}$$

Substituting the wing load $p = m/S$ into (A4.68)', we obtain

$$\frac{D}{m} = \left[C_{DW}\frac{1}{p} + \varepsilon\left(\frac{g - V^2/R}{\xi q}\right)^2 p\right] q. \qquad \text{(A4.69)}$$

To find the minimum the air drag $D$ for $p$, we take the derivatives and set them equal to zero, then we obtain

$$p_{opt} = \frac{\varsigma q}{(g - V^2/R)}\sqrt{\frac{C_{DW}}{\varepsilon}}. \qquad \text{(A4.70)}$$

Substituting (A4.70) into (A4.69), we find the minimum wing drag

$$\min\left(\frac{D}{m}\right)_w = \frac{2}{\varsigma}\left(g - \frac{V^2}{R}\right)\sqrt{\varepsilon C_{DW}}.$$

The sum of the minimum vehicle drag plus body drag is

$$\min\left(\frac{D}{m}\right) = \frac{2}{\varsigma}\left(g - \frac{V^2}{R}\right)\sqrt{\varepsilon C_{DW}} + C_{Db}\frac{q}{p_b}, \quad q = \frac{\rho a V}{2}, \quad p_b = \frac{m}{S_b}. \qquad \text{(A4.71)}$$

Substituting (A4.70) into the term for $\alpha$ in (A4.65), we find the optimal attack angle of a vehicle without a body

$$\alpha_{opt} = \sqrt{C_{DW}/\varepsilon}. \qquad \text{(A4.72)}$$

The coefficient of flight efficiency $k = Y/D$ is

$$k = \frac{g - V^2/R}{D/m}, \quad k_{\max} = \frac{g - V^2/R}{\min(D/m)}.$$

For hypersonic speed the coefficients are approximately

$$\varsigma = 4, \quad \varepsilon = 2, \quad C_{DW} = 4c^2, \quad C_{Db} = 2c_1^2, \quad C_L = \varsigma\alpha, \quad C_{Do} = C_{DW} + C_{Db}. \qquad \text{(A4.73)}$$

In numerical computation the angle $\theta$ can be found from (A4.25) as $\theta = \Delta h/\Delta R_g$.
For the rocket engine or gliding flight we find the following relation: when $S$ is optimum (variable), the partial derivatives from (A4.71) are

$$D'_V = -\frac{4V}{\varsigma R}\sqrt{\varepsilon C_{DW}} + C_{Db}\frac{\rho a}{2p_b}, \quad D'_h = -\frac{C_{Db}\rho a V}{2b_2 p_b}.$$

Substituting these into (A4.37), we find the relationship between speed, altitude, and optimal wing load for a hypersonic vehicle with a rocket engine and **variable** optimal wing:

$$\rho = \frac{8 g p_b V\sqrt{\varepsilon C_{DW}}}{\varsigma a C_{Db} R(g + V^2/b_2)}, \quad h = 11000 + b_2 \ln\frac{a_2}{\rho}. \qquad \text{(A4.74)}$$

For ζ = 4, ε = 2 equation (A4.73)' has the form

$$\rho = \frac{2 g p_b V\sqrt{2C_{DW}}}{C_{Db} a R(g + V^2/b_2)}, \quad h = 11000 + b_2 \ln\frac{a_2}{\rho}, \qquad \text{(A4.74)'}$$

Results of computation using (A4.74)' for ζ = 4, ε = 2, $a_2 = 0.365$, $b_2 = 6997$ and different $p_b$ are presented in Fig. A4.7 (dashed lines). As you see, the variable area wing saves kinetic energy, because its curve is located over an invariable (fixed) wing. This is advantageous only at orbital speed (7.9 km/s) because no lift force is necessary.

# Estimation of flight range

## Air and space vehicles without thrust

The aircraft range can be found from equation (A4.26)

$$R_a = \int_{V_1}^{V_2} \frac{mVdV}{T - D - mg\theta}, \quad V_1 > V_2 \quad or \quad R_a = \int_{V_2}^{V_1} \frac{VdV}{D/m + g\theta}, \quad if \quad T = 0. \qquad \text{(A4.75)}$$

Consider a missile with the ***optimal variable wing*** in a descent trajectory with thrust $T = 0$.

a) Make the simplest estimation using equations for kinetic energy from classical mechanics. Separate the flight into two stages: hypersonic and subsonic. If we have the ratio of vehicle efficiency $k_1 = C_L / C_D$, $k_2 = C_L / C_D$, where $k_1$, $k_2$ are the ratios of flight efficiency for the hypersonic and subsonic stages respectively, we find the following equations for a range in each region:

$$\frac{m}{2}\left(V_1^2 - V_2^2\right) = \frac{m(g - V^2/R)}{k_1} R_1, \quad R_1 = \frac{k_1\left(V_1^2 - V_2^2\right)}{2(g - V^2/R)}, \quad R_2 = k_2 h, \quad R_a = R_1 + R_2,$$

Or more exactly

$$d\left(\frac{mV^2}{2}\right) = \frac{m(g - V^2/R)}{k_1} dR_1, \quad R_1 = -\frac{k_1 R}{2} \ln\left(\frac{g - V_2^2/R}{g - V_1^2/R}\right), \qquad \text{(A4.76)}$$

where $R_1$ is the hypersonic part of the range, $R_2$ is the subsonic part of the range, $V_1$ is the initial (maximum) vehicle hypersonic speed, $V_2$ is a final hypersonic speed, and $h$ is the altitude at the initial stage of the subsonic part of the trajectory.

b) To be more precise. Assume in (A4.75) $\rho = const$ (taking average air density).

1. For the **hypersonic** part of the trajectory: substitute (A4.71) into (A4.76). We then have

$$R_{1H} = \int_{V_1}^{V_2} \frac{VdV}{aV^2 + bV + c}, \quad or \quad R_{1H} = \int_{V_1}^{V_2} \frac{VdV}{X}, \quad where \quad X = aV^2 + bV + c,$$

$$a = \frac{2\sqrt{\varepsilon C_{DW}}}{\xi R}, \quad b = -C_{Db}\frac{\rho a}{2p_b}, \quad c = \frac{T}{m} - \frac{2g}{\varsigma} - g\theta, \quad \Delta = 4ac - b^2, \qquad \text{(A4.77)}$$

$$R_{1H} = \left[\frac{1}{2a}\ln X - \frac{b}{2a}\int\frac{dV}{X}\right]_{V_1}^{V_2}, \quad \int\frac{dV}{X} = \frac{2}{\sqrt{\Delta}}\arg\tan\frac{2aV + b}{\sqrt{\Delta}} \quad for \quad \Delta \ge 0,$$

$$\int\frac{dV}{X} = -\frac{2}{\sqrt{-\Delta}}\arg\tanh\frac{2aV + b}{\sqrt{-\Delta}} = \frac{1}{\sqrt{-\Delta}}\ln\frac{2aV + b - \sqrt{-\Delta}}{2aV + b + \sqrt{-\Delta}} \quad for \quad \Delta \le 0.$$

2. For the s**ubsonic** part of the trajectory: substitute (A4.65) into (A4.75). We then have

$$R_{1S} = -\frac{1}{2C_2}\ln\left|\frac{C_1 - C_2V_2^2}{C_1 - C_2V_1^2}\right|, \qquad \text{(A4.78)}$$

where the values for $C_1$, $C_2$ are

$$C_1 = \frac{T}{m} - g\left(\frac{2\sqrt{\varepsilon C_{DW}}}{a\varsigma} + \theta\right), \quad C_2 = C_{Db}\frac{\rho}{2p_b}. \qquad \text{(A4.79)}$$

The trajectory (without the rocket part of the trajectory) is

$$R_1 = R_{1H} + R_{1S} \quad or \quad R_g = R_{1H} + R_{1S} + R_2. \qquad \text{(A4.80)}$$

where $R_2 = k_2 h$ computed for altitude $h$ at the end of the kinetic part of the subsonic trajectory.

3. The **ballistic** trajectory of a wingless missile without atmosphere drag is

$$h = \frac{gt^2}{2}, \quad t = \sqrt{\frac{2h}{g}}, \quad R_b = V_1 t = V_1\sqrt{\frac{2h}{g}}, \quad V_i = \sqrt{V_1^2 + V_y^2}, \quad V_y^2 = 2h(g - V^2/R), \qquad \text{(A4.81)}$$

where $h$ is the initial altitude, $V_1$ is the initial horizontal speed of the wingless missile at altitude $h$, $V_y$ is initial (shot) vertical speed at $h$ = 0, $V_i$ is the full initial (shot) speed at $h$ = 0 .

For the hypersonic interval 5 < V < 7.5 km/s, we can use the more exact equation

$$R_b = V_1\sqrt{\frac{2h}{(g - V_1^2 / R)}} \quad , \qquad \text{(A4.82)}$$

where $R$ = 6378 km is the radius of Earth. The full range of a ballistic rocket plus the range of a winged missile is

$$R_f = R_b + R_a + R_g, \qquad \text{(A4.83)}$$

where $R_g = kh$ is the vehicles gliding range from the final altitude $h_2$ (see Fig. A4.11) with aerodynamic efficiency $k$.

The classical method finding of the optimal shot ballistic range for spherical Earth without atmosphere is

$$R_b = 2R\beta_{opt} , \quad \tan\beta_{opt} = \frac{\nu_A}{2\sqrt{1-\nu_A}}, \quad \nu_A = \frac{V_A^2}{V_c^2} \ , \qquad \text{(A4.84)}$$

where $\beta_{opt}$ is the optimal shot angle, $V_A$ is the shot projectile speed, and $V_c$ is an orbital speed for a circular orbit at a given altitude.

4. **Cannon projectile**. We divide the distance into three sub-distances: 1) 1.2$M$ < $M$, 2) 0.9$M$ < $M$ < 1.2$M$, 3) 0 < $M$ < 0.9$M$. The range of the wing cannon projectile may be estimated using the equation

$$R = \frac{k_1}{2g}(V_1^2 - V_2^2) + \frac{k_2}{2g}(V_2^2 - V_3^2) + \frac{k_3}{2g}(V_3^2 - V_0^2), \quad where \quad 0 < V_0 < V_3 < V_2 < V_1 \ , \qquad \text{(A4.85)}$$

where $k_1$, $k_2$, $k_3$ are the average aerodynamic efficiencies for sub-distances 1, 2, 3 respectively. Conventionally, these coefficients have the following values: subsonic $k_3$ = 8–15, near sonic $k_2$ = 2–3, supersonic and hypersonic $k_1$ = 4–9. If $V >$ 600 m/s, the first term in (A4.85) has the greatest value and we can use the more simple equation for range estimation:

$$R = \frac{k_1}{2g}V_1^2 . \qquad \text{(A4.84)'}$$

At the top of its trajectory, a modern projectile can have an additional impulse from small rocket engines. Their weight is 10–15% of the full mass of the projectile and increases the maximum range by 7–14 km. In this case we must substitute $V = V_1 + dV$ into (A4.84)', where $dV$ is the additional impulse (150–270 m/s).

## Subsonic aircraft with thrust. Horizontal flight

The optimal climb and descent of a subsonic aircraft with a constant mass and fixed wing is described by equations (A4.50) and (A4.47). Any given point in a climb curve may be used for horizontal flight (with different efficiency). We consider in more detail the horizontal flight when the aircraft mass decreases because the fuel is spent. This consumption may reach 40% of the initial aircraft mass. The optimal horizontal flight range may be computed in the following way:

$$dR = Vdt, \quad dt = \frac{dm}{c_s T} = \frac{gdm}{c_s D}, \quad dR = \frac{gV}{c_s D(m)}dm, \quad R = \frac{gV}{c_s}\int_{m_k}^{m}\frac{dm}{D(m)}, \qquad \text{(A4.86)}$$

where $m$ is fuel mass, $c_s$ is fuel consumption, kg/s/ kg thrust.

a) For a **fixed wing**, we have (from (A4.44))

$$D = C_{Do} qS + \frac{\varepsilon}{qS}\left(\frac{g}{\varsigma}\right)^2 m^2, \quad where \quad C_{Do} = C_{Dw} + C_{Db}\left(\frac{S_b}{S}\right), \quad q = \frac{\rho V^2}{2} \quad . \qquad \text{(A.87)}$$

Substituting (A4.87) into (A4.86), we obtain

$$R = \frac{gV}{c_s\sqrt{C_1 C_2}} \arg\tan \frac{\sqrt{C_1/C_2}\,(m - m_k)}{1 + (C_1/C_2)mm_k}, \quad where \quad C_1 = \frac{\varepsilon}{qS}\left(\frac{g}{\varsigma}\right)^2, \quad C_2 = C_{Do} qS. \quad \text{(A4.88)}$$

b) For a **variable wing** we have (from (A4.65)

$$R = \frac{gV}{c_s C_1} \ln \frac{C_1 m - C_2}{C_1 m_k - C_2}, \quad where \quad C_1 = 2\frac{g}{\varsigma}\sqrt{\varepsilon C_{DW}}, \quad C_2 = C_{Db} qS_b, \quad \rho = \rho_0 e^{-h/b_1}. \quad \text{(A4.89)}$$

Results of the computation are presented in Fig. A4.8. The aircraft have the following parameters: $C_{DW}$ = 0.02; $C_{Db}$ = 0.08; $b_1$ = 9086; $S$ = 120 m$^2$; $m$ = 100 tons, $m_k$ = 80 tons, $c_s$ = 0.00019 kg/s/kg thrust; wing ratio $\lambda$ = 10.

As you see, the specific fuel consumption does not depend on speed and altitude, a good aircraft design reaches the maximum range only at one point, in one flight regime: when the aircraft flies at the maximum speed possible for the critical Mach number, at the maximum altitude possible for that engine. The deviation from this point decreases in the range in 5–10–15 percent or more. The variable wing increases efficiency of the other regime, which that approximately reduces the losses by a half.

The coefficient of flight efficiency may be computed using equation $k = g/(D/m)$, where the values

$$\frac{D}{m} = C_{DW}\frac{q}{p} + \frac{\varepsilon p}{q}\left(\frac{g}{\varsigma}\right)^2 + C_{Db}\frac{q}{p_b}, \quad \left(\frac{D}{m}\right)_1 = 2\frac{g}{\varsigma}\sqrt{\varepsilon C_{DW}} + C_{Db}\frac{q}{p_b}, \qquad \text{(A4.90)}$$

apply for fixed and variable wings respectively. Results of computation are presented in Fig. A4.9. The curve of the variable wing is the round curve of the fixed wing.

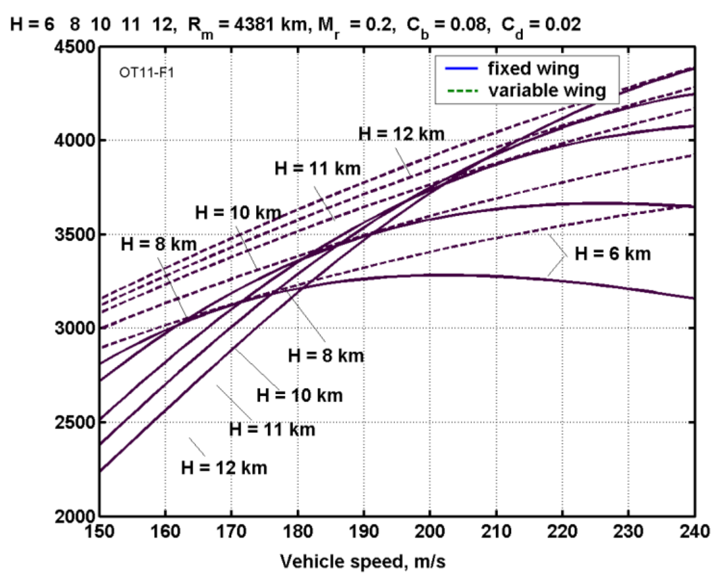


**Fig. A4.8**. Aircraft range for altitude $H$ = 6, 8, 10, 11, 12 km; maximum range $R_m$ = 4361 km; relative fuel mass $M_r$ = 0.2; body drag coefficient $C_b$ = 0.08; wing drag coefficient $C_d$ = 0.02.

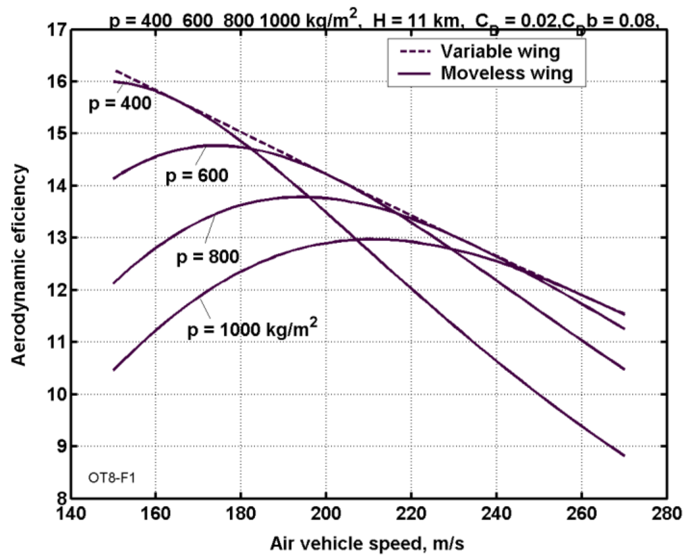


**Fig. A4.9.** Aerodynamic efficiency of non-variable and variable wings for wing load $p$ = 400, 600, 800, 1000 kg/m$^2$, wing drag $C_D$ = 0.02, body drag $C_{Db}$ = 0.08, wing ratio 10.

## Optimal engine control for constant flight pass angle

Let us to consider equations (A4.1) – (A4.5) for a constant angle of trajectory, $\theta$ = const. Substituting $\theta$ = constant, thrust $T = V_e\beta$, and a new independent variable $s = Vt$ (where $s$ is the length of the trajectory) into the equation system (A4.1) – (A4.5). We obtain the following equations

$$\frac{dL}{ds} = \cos\theta \quad ,$$

$$\frac{dh}{ds} = \sin\theta \quad ,$$

$$\frac{dV}{ds} = \frac{V_e(h,V)\beta - \overline{D}(\alpha,V,h)}{mV} - \frac{g}{V}\sin\theta\,, \qquad \text{(A4.91) – (A4.96)}$$

$$\frac{dm}{ds} = -\frac{1}{V}\beta \quad ,$$

$$Y(\alpha,V,h) - gm\cos\theta + \frac{mV^2}{R} + 2mV\omega_E\cos\varphi_E = 0\,,$$

$$0 \le \beta \le \beta_{\max} \quad .$$

Equation (A4.95) is used to substitute for $\alpha$ in equation (A4.93) and for a change of air drag

$$\overline{D}(\alpha,V,h) = D(V,h). \qquad \text{(A4.97)}$$

We find a non-linear system with a linear fuel control $\beta$. This means the system can have a singular solution.

**Solution**

Consider the maximum range for vehicles described by equation (A4.91) – (A4.96).
Let us write the Hamiltonian $H$

$$H = \cos\theta + \lambda_1\sin\theta + \lambda_2\left[\frac{V_e(h,V)\beta - D(V,h)}{mV} - \frac{g}{V}\sin\theta\right] - \lambda_3\frac{1}{V}\beta \quad , \qquad \text{(A4.98)}$$

where $\lambda_1(s), \lambda_2(s), \lambda_3(s)$ are unknown multipliers. Application of conventional methods gives

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial V} = -\lambda_2\left[\left(-\frac{1}{V^2}\right)\left(\frac{V_e\beta - D(V,h)}{m} - g\sin\theta\right) - \frac{D'_V}{mV}\right] - \lambda_3\frac{1}{V^2}\beta \quad ,$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial m} = \lambda_2\frac{V_e\beta - D}{m^2V}, \qquad \text{(A4.99) – (A4.101)}$$

$$\beta = \max_{\beta} H = \beta_{\max} sign\left[\lambda_2 V_e - \lambda_3 m\right] \quad .$$

Where $D'_V$ is the first partial derivate of $D$ by $V$.

The last equation shows that the fuel control $\beta$ can have only two values, $\pm\beta_{\max}$. We consider the singular case when

$$A = \lambda_2 V_e - \lambda_3 m \equiv 0. \qquad \text{(A4.102)}$$

This equation has two unknown variables, $\lambda_2$ and $\lambda_3$, and does not contain information about fuel control $\beta$.

The first two equations (A4.91) – (A4.92) do not depend on variabls and can be integrated

$$L = s\cos\theta, \qquad \text{(A4.103)}$$

$$H = s\sin\theta. \qquad \text{(A4.104)}$$

In accordance with the References[2] let us differentiate equation (A4.102) by the independent variable $s$. After substitution into equations (A4.93) – (A4.95), (A4.97), (A4.99), (A4.100), (A4.102), and (A4.104) we obtain the relation for $\lambda_2 \neq 0$, $\lambda_3 \neq 0$:

$$\dot{A} = VD - mVD'_m + V_e(-D - mg\sin\theta + VD'_V) - VV'_{e,V}(D - mg\sin\theta) + mV^2V'_{e,s} = 0\,. \qquad \text{(A4.105)}$$

This equation also does not contain $\beta$, however it does contain an important relation between variables $m$, $h$ and $V$, on an optimal trajectory. This is a 3-dimentional surface. If we know

$$D = D(h,V)\,, \qquad \text{(A4.106)}$$

$$V_e = V_e(h,V)\,, \qquad \text{(A4.107)}$$

The mass of our apparatus $m$, and its altitude $h$, we can find the optimal flight speed. This means we can calculate the necessary thrust and the fuel consumption for every point $m$, $h$, $V$ (Fig. A4.10).

If we want to find an equation for the fuel control $\beta$, we continue to differentiate equation (A4.105) to find the independent variable $s$ and substitute in equations (A4.91) – (A4.104). If we calculate the relation for $\beta$, if $\lambda_2 \neq 0$, $\lambda_3 \neq 0$, $V_e$ = const, then

$$\beta = \frac{\dot{A}'_V(D + mg\sin\theta) - mV\dot{A}'_s}{V_e\dot{A}'_V - m\dot{A}'_m}, \qquad \text{(A4.108)}$$

where

$$\dot{A}'_V = \frac{\partial}{\partial V}\left(\frac{dA}{ds}\right), \quad \dot{A}'_s = \frac{\partial}{\partial s}\left(\frac{dA}{ds}\right). \qquad \text{(A4.109)}$$

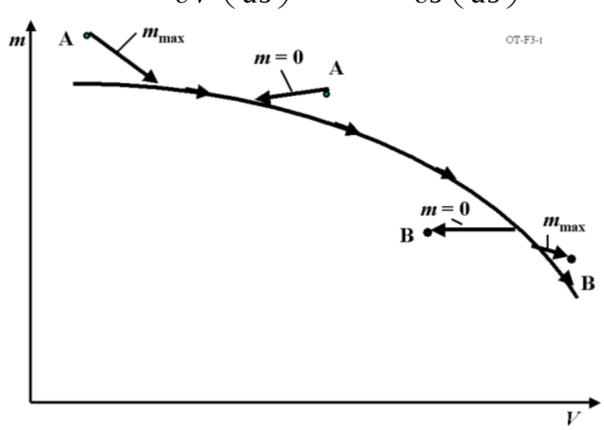


**Fig. A4.10**. Optimal fuel consumption of flight vehicles.

The necessary condition of the optimal trajectory as it is shown in the References[2–8] (see for example, equation (4.2)[4]) is

$$-(-1)^k \frac{\partial}{\partial\theta}\left[\frac{d^{2k}}{ds^{2k}}\left(\frac{\partial H}{\partial s}\right)\right] \geq 0 . \qquad \text{(A4.110)}$$

where $k = 1$.

If the flight is horizontal ($\theta = 0$), the expression (A4.108) is very simply

$$\beta = \frac{D}{V_e} . \qquad \text{(A4.111)}$$

This means the thrust equals the drag, a fact that is well known in aerodynamic science.

To obtain the specific equations for different forms of drag and thrust, we must take formulas (or graphs) for subsonic, transonic, supersonic and hypersonic speed for thrust and substitute them into the equations (A4.105) and (A4.108).

## Simultaneous optimization of the path angle and fuel consumption

Consider the case where the path angle and the fuel consumption are simultaneously optimized.
In this case the general equations (A4.1) – (A4.5) have the form:

$$\frac{dL}{ds} = 1 ,$$

$$\frac{dh}{ds} = \theta ,$$

$$\frac{dV}{ds} = \frac{V_e(h,V)\beta - D(m,V,h)}{mV} - \frac{g}{V}\theta , \qquad \text{(A4.112) – (A4.116)}$$

$$\frac{dm}{ds} = -\frac{1}{V}\beta ,$$

$$Y(\alpha, V.h) = mg + \frac{mV^2}{R} + 2mV\omega_E \cos\varphi_E .$$

Let us write the Hamiltonian

$$H = 1 + \lambda_1\theta + \lambda_2\left(\frac{V_e(h,V)\beta - D(m,V,h)}{mV} - \frac{g}{V}\theta\right) - \lambda_3\frac{1}{V}\beta . \qquad \text{(A4.117)}$$

The necessary conditions of optima give

$$A = \frac{\partial H}{\partial\theta} = V\lambda_1 - g\lambda_2 = 0 ,$$

$$B = \frac{\partial H}{\partial\beta} = V_e\lambda_2 - m\lambda_3 = 0 , \qquad \text{(A4.118) – (A4.119)}$$

The lambda equations are

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial h} = -\lambda_2 \frac{V'_{e,h}\ \beta - D'_h}{mV} ,$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial V} = -\lambda_2\left[\frac{(V'_{e,V}\beta - D'_V)V - (V_e\beta - D)}{mV^2} + \frac{g}{V^2}\theta\right] - \lambda_3\frac{1}{V^2}\beta , \qquad \text{(A4.120) – (A4.122)}$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial m} = \lambda_2 \frac{V_e\beta - D + mD'_m}{m^2 V} .$$

If we differentiate $A$ (A4.118), from $dA/ds = 0$, we find the optimal fuel consumption

$$\beta = \frac{gVD'_V - V^2 D'_h}{g(V_e + V'_{e,V}V) - V'_{e,h}V^2} \ . \qquad \text{(A4.123)}$$

Then we differentiate $B$ (A4.119), from $dB/ds = 0$ we find the optimal path angle

$$\theta = \frac{V'_{e,V}D - V_e D'_V - V_e D/V - D + mD'_m}{m(g + V'_{e,h}V - V'_{e,V}g)} \ . \qquad \text{(A4.124)}$$

We have used the conventional forms for the partial derivatives in (A4.120)–(A4.124) as in the earlier sections of the chapter (see for example (A4.51)).

If we know from analytical formulas or graphical functions *Ve*, *D*, *Y* we can find the optimal trajectory of the air vehicle.

In the general case, this trajectory includes four parts:

1. Moving between limitations $\theta$ and $\beta$.
2. Moving between one limitation $\theta$ or $\beta$ and one optimal control $\beta$ or $\theta$.
3. Moving simultaneously with both optimal controls $\theta$ and $\beta$.
4. Moving at a given point along one limitation and/or both limitations

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## Application to aircraft, rocket missiles, and cannon projectiles

**A) Application to rocket vehicles and missiles**.

Let us apply the previous results to typical current middle- and long-distance rockets with warheads. We will show: if the warhead has wings and uses the optimal trajectory, the range of the warhead (or its useful load) is increased dramatically in most cases. We will compute the optimal trajectories for a rocket-launched warhead at a particular altitude (20–60 km) and speed (1–7.5 km/s). Point *B* is located on the curve (A4.58) for a fixed wing and on curve (A4.73)' for a variable wing (Fig. A4.11). Further, the winged warhead flies (descends) along the optimal trajectory *BD* (Fig. A4.58) according to equations (A4.58) (fixed wing) or equations (A4.73)' (variable wing) respectively. When the speed is reduced by a small amount (for example, 1 km/s) (point *D* in Fig. A4.11), the winged warhead glides (distance *DE* in Fig. A4.11).

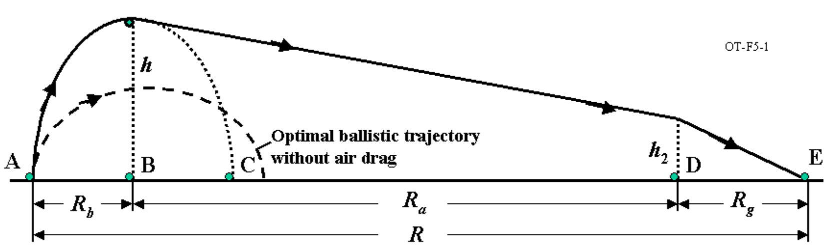


**Fig. A4.11.** Trajectory of flying vehicles.

The following equations are used for computation:

1. ***The optimal trajectory for a fixed wing space vehicle.***

a) Equation (A4.58) is used to calculate $h = h(V)$ to find the optimal trajectory of a warhead with a non-variable fixed wing in the speed interval $1 < V < 7.5$ km/s. The result is presented in Fig. A4.7.

b) Equation (A4.54) gives the magnitude ($D/m$).

c) The equation (A4.75) in the form

$$\Delta R_a = \frac{V\Delta V}{(D/m) + g\theta}, \quad R_a = \Sigma \Delta R_a, \quad \theta = -\frac{\Delta h}{\Delta R_a}, \quad k = \frac{g - V_0^2/R}{D/m}, \quad R_g = h_0 k \ , \qquad \text{(A4.125)}$$

is used for computation in the intervals $R_a$, $R_g$ (Fig. A4.11). Here $R_g$ is the range of a gliding vehicle.

d) Equation (A4.75) is used to calculate $R_b$ in the launch interval *AB* (Fig. A4.11).

e) The full range, $R$, of a warhead with a fixed wing and the full ballistic warhead range, $R_w$, are

$$R = R_b + R_a + R_g\,, \quad R_w = 2R_b\,. \qquad (A4.126)$$

f) Equation (A4.84) is used to calculate the optimal **ballistic** trajectory of a shot without air drag (a vehicle **without** wings). The range of this trajectory, as it is known, may be significantly more than the range in the atmosphere.

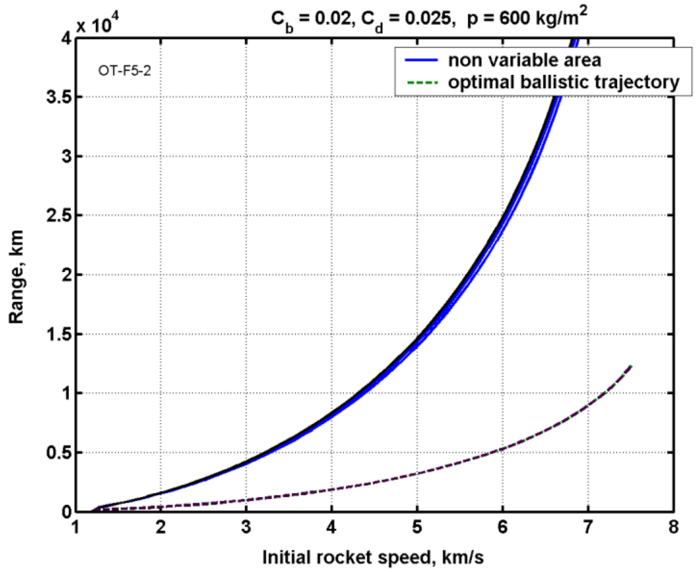


**Fig. A4.12**. Range of NON-VARIABLE wing vehicle for body drag coefficient $C_b = 0.02$, wing drag coefficient $C_d = 0.025$, wing load $p = 600$ kg/m$^2$.

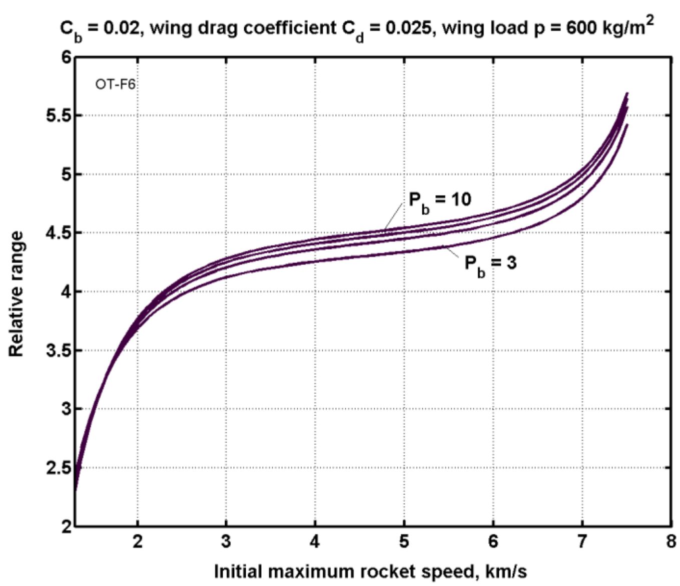


**Fig. A4.12.** The relative range of a **non**-variable wing vehicle for the body drag coefficient $C_b = 0.02$, wing drag coefficient $C_d = 0.025$, wing load $p = 600$ kg/m$^2$, body load $P_b = 3$–$10$ ton/m$^2$.

The results are presented in Fig. A4.12. Computation of the relative range (for different $p_b$) using the formula

$$R_r = \frac{R_f}{R_b} \qquad (A4.127)$$

is presented in Fig. A4.12. The optimal range of the winged vehicle is approximately 4.5 times that of the ideal ballistic rocket computed without air drag. In the atmosphere this difference will be significantly more.

## 2. *Rockets, missiles and space vehicles with* variable *wings*

The computation is the same. For computing $\rho$, $h$, $D/m$ we can use equations (A4.73)' and (A4.71) respectively. The results for different body loads are presented in Fig. A4.7. The optimal trajectories of vehicles with variable wing areas have less slope. This means the vehicle loses less energy when it moves. It travels above the optimal trajectory of a vehicle with fixed wings, which means it needs a lot more time (10–20) and more wing area than a fixed wing space vehicle (Fig. A4.14). The computation of the optimal variable wing area is presented in Fig. A4.15. The relative range (equation (A4.127)) is presented in Fig. A4.16.

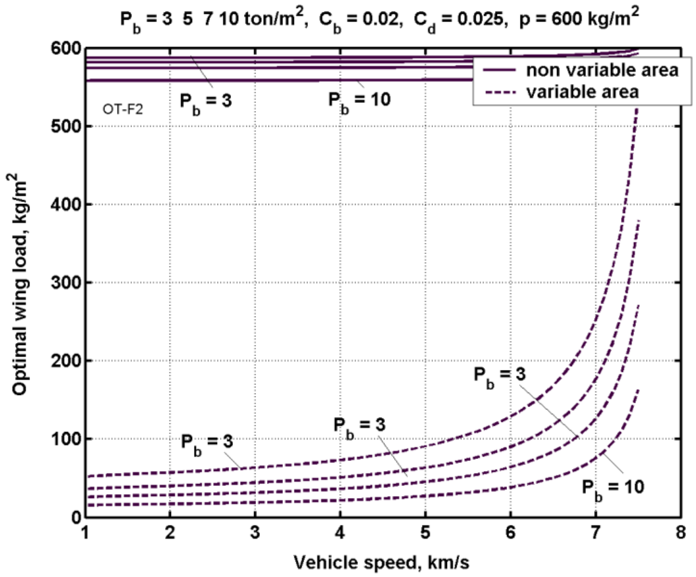


**Fig. A4.14**. Optimal wing load versus speed for specific body load $P_b = 3, 5, 7, 10$ ton/m$^2$, body drag coefficient $C_b = 0.02$, wing drag coefficient $C_d = 0.025$, wing load $p = 600$ kg/m$^2$.

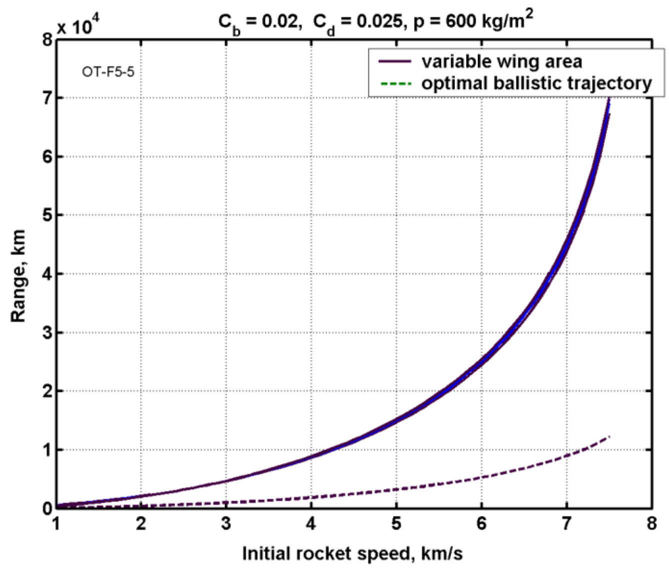


**Fig. A4.15**. Range of a variable wing vehicle for the body drag coefficient $C_b = 0.02$, the wing drag coefficient $C_d = 0.025$, the wing load $p = 600$ kg/m$^2$.

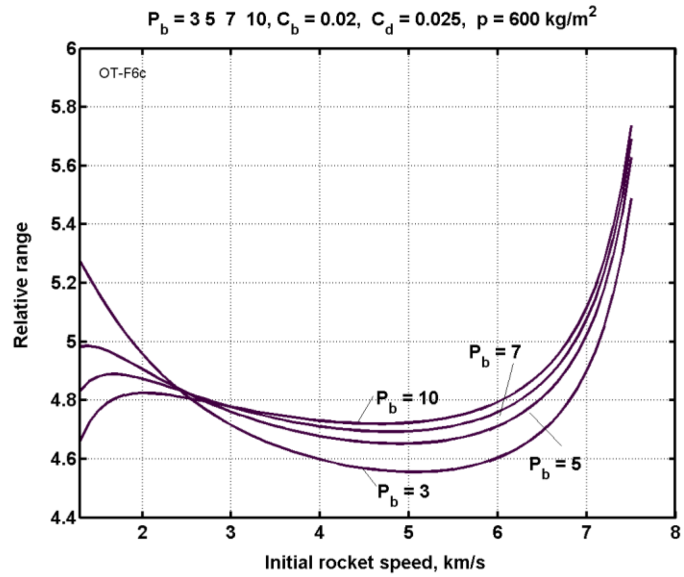


**Fig. A4.16**. Relative range of variable wing vehicle for the body drag coefficient $C_b = 0.02$, the wing drag coefficient $C_d = 0.025$, the wing load $p$ = 600 kg/m$^2$, the body load $P_b$ = 3–10 ton/m$^2$.

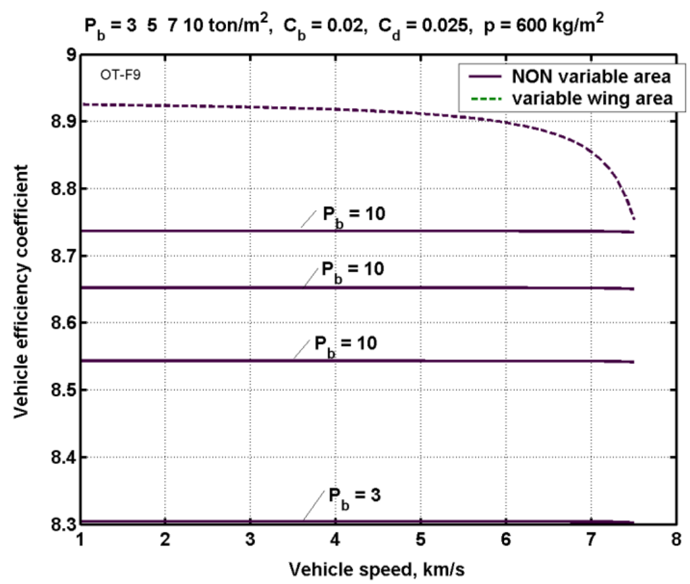


**Fig. A4.17**. Vehicle efficiency coefficient versus speed for specific body load $P_b$ = 3, 5, 7, 10 ton/m$^2$, body drag coefficient $C_b = 0.02$, wing drag coefficient $C_d = 0.025$, wing load $p$ = 600 kg/m$^2$.

The aerodynamic efficiency of vehicles with fixed (for different $p_b$ bodies) and optimal variable wings computed using equations (A4.125) and (A4.67) respectively is presented in Fig. A4.12. The difference between vehicles with fixed and variable wings reaches 0.2–0.6 . The slope of the trajectory to horizontal is small (Fig. A4.18).

The range of the **fixed** wing vehicle computed using equation (A4.125) is presented in Fig. A4.12. The range of the **variable** wing vehicle computed using equation (A4.126) is presented in Fig. A4.15. The curve is practically the same (see Figs. A4.12 and A4.15).

3. ***Increasing the rocket payload for the same range.*** If we do not need to increase the range, the winged vehicle can be used to increase the payload, or to save rocket fuel. We can change the mass of the fuel or the payload. The additional payload may be estimated by the following equation

$$\mu = 1 - e^{-\frac{\Delta V}{V_e}}, \qquad \text{(A4.128)}$$

where $\mu = m/m_b$ is relative mass (the ratio of rocket mass of the winged vehicle to the ballistic rocket), $\Delta V = V_b - V$ is the difference between the optimal ballistic rocket speed (equation (A4.84)) and the rocket with a winged vehicle (equation (A4.126)) for given range (see Fig. A4.12). Results of computation are presented in Fig. A4.19. The mass of the rocket with a winged vehicle may be only 20–35% of the optimal ballistic rocket flown without air drag.

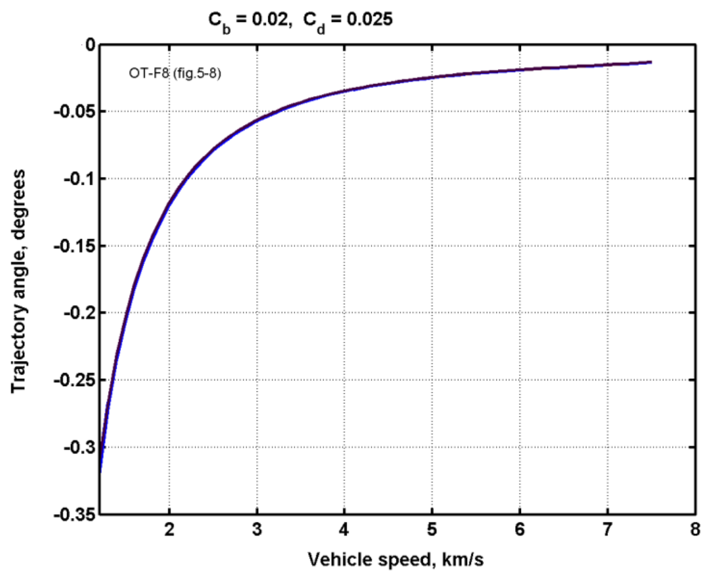


**Fig. A4.18.** Trajectory angle versus speed for body drag coefficient $C_b = 0.02$, wing drag coefficient $C_d = 0.025$.

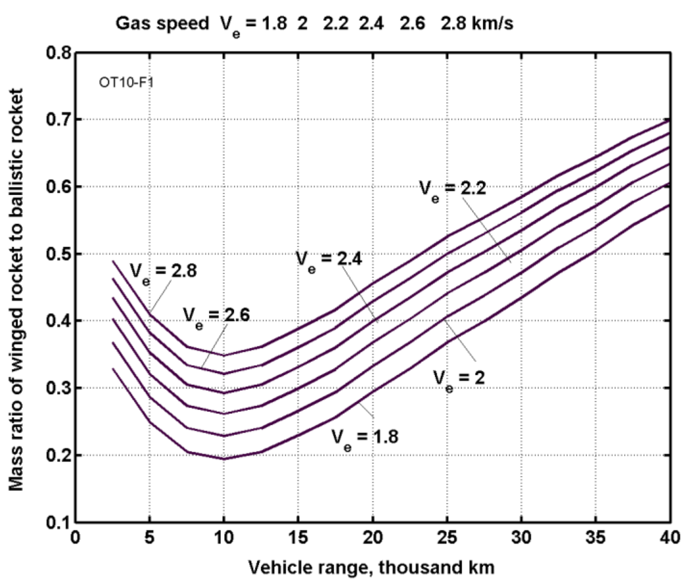


**Fig. A4.19.** Ratio of mass of winged rocket to ballistic rocket for specific engine run-out gas speed $V_e$ = 1.8, 2, 2.2, 2.4, 2.6 and 2.8 km/s.

***Conclusion*:** The winged air-space vehicle has a range that is greater by a minimum of 4.5–5 times than an optimal shot ballistic space vehicle. The variable wing improves the aerodynamic efficiency by 3–10% and also improves the range. An optimal variable wing requires a large wing area. If you do not need to increase the range, you may instead increase payload.

**B) Application to cannon wing projectiles**

Properties of a typical current cannons are shown in Table A4.1.

**Table A4.1. Properties of current typical Cannons.**

| Name | caliber, mm | Nozzle speed, m/s | Mass of projectile, kg | Range, km | RAP, km |
|---|---|---|---|---|---|
| M107 | 175 | 509–912 | 67 | 15–33 | |
| SD-203 | 203 | 960 | 110 | 37.5 | |
| 2S19 | 155 | 810 | 43.6 | 24.7 | |
| 2S1 | 122 | 690–740 | 21.6 | - | |
| S-23 | 180 | - | - | 30.4 | 43.8 |
| 2A36 | 152 | - | - | 17.1 | 24 |
| D-20 | 152 | 600–670 | 43.5–48.8 | 20 | |

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The computations using equation (A4.84)' for different $k$ and RAP with $dV$ = 270 m/s are presented in Figs. A4.20 and A4.21.

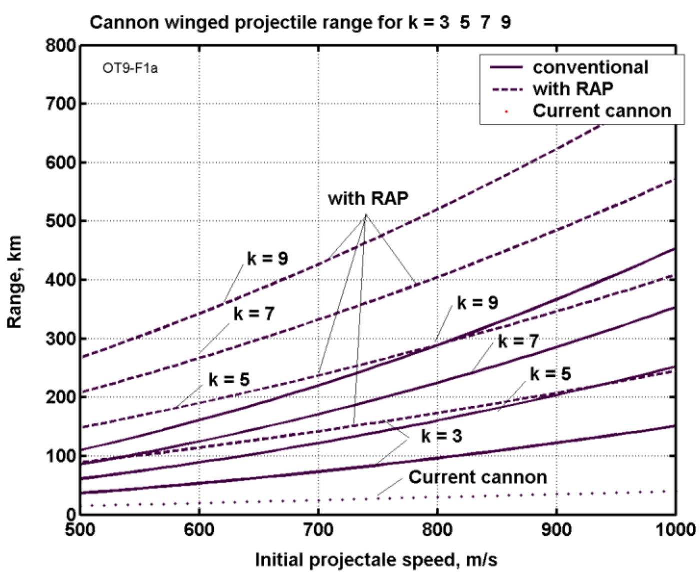


**Fig. A4.20.** Cannon winged projectile range for average aerodynamic efficiency $k$ = 3, 5, 7, 9.

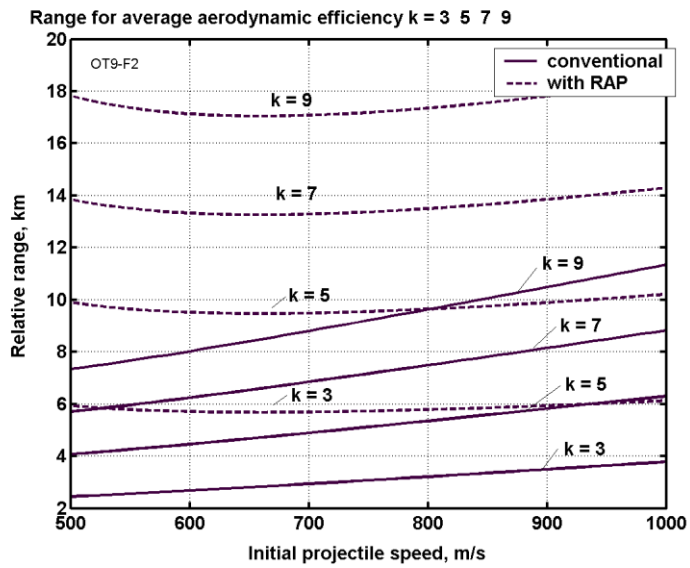


**Fig. A4.21**. Cannon winged projectile relative range for average aerodynamic efficiency $k$ = 3, 5, 7, 9.

***Conclusion.*** As you see (Figs. A4.20, A4.21), the winged projectile increase its range 3–9 times (from 35 up to 360 km, $k$ = 9). The projectile with RAP increases its range 5–14 (from 40 up to 620 km, $k$ = 9) . Winged shells have another important advantage: they do not need to rotate. We can use a barrel with a smooth internal channel. This allows for an increase in projectile nozzle speed of up to 2 km/s and in shell range of up to 1000 km ($k$ = 5).

**C) Application to current aircraft.**

We can use equations (A4.88) and (A4.89) for computations for typical passenger airplanes (Figs. A4.22, A4.23, A4.24, and A4.8), where all values are divided by the maximum range $R_m$ = 4381 km (for a fuel mass that is 20% of to vehicle mass) at a speed of $V$ = 240 m/s, and altitude $H$ = 12 km. The speed is limited by the critical Mach number ($V < M$ = 0.82), and the altitude is limited by the engine trust, when engine stability is such that it works in a cruise regime. Fig. A4.22 shows the typical long-range trajectory of aircraft.

***Conclusion:*** The best flight regime for a given air vehicle (closed to Boeing 737) is altitude $H$ = 12 km, speed $V$ = 240 m/s, specific fuel consumption $Cs$ = 0.00019 kg fuel/s/kg thrust. Any deviation from this flight regime significantly reduces the maximum range (by up to 10–50%). The vehicle with a variable wing area loses 50% less range than a vehicle with a fixed wing

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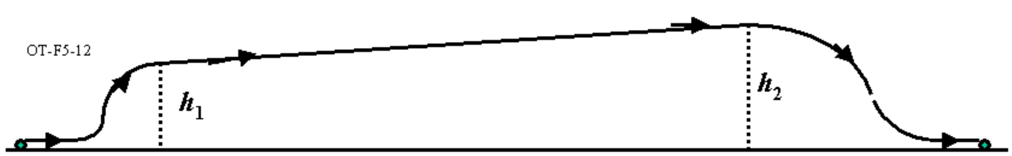


**Fig. A4.22.** Optimal trajectory of aircraft.

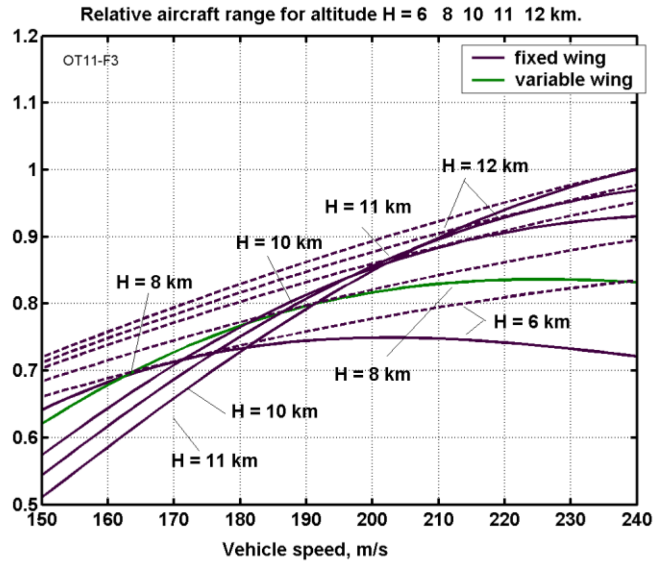


**Fig. A4.23.** Relative aircraft range for altitude $H$ = 6, 8, 10, 11 and 12 km, maximum range $R_m$ = 4381 km, relative fuel mass $M_r$ = 0.2, body drag coefficient $C_b$ = 0.08, wing drag coefficient $C_d$ = 0.02.

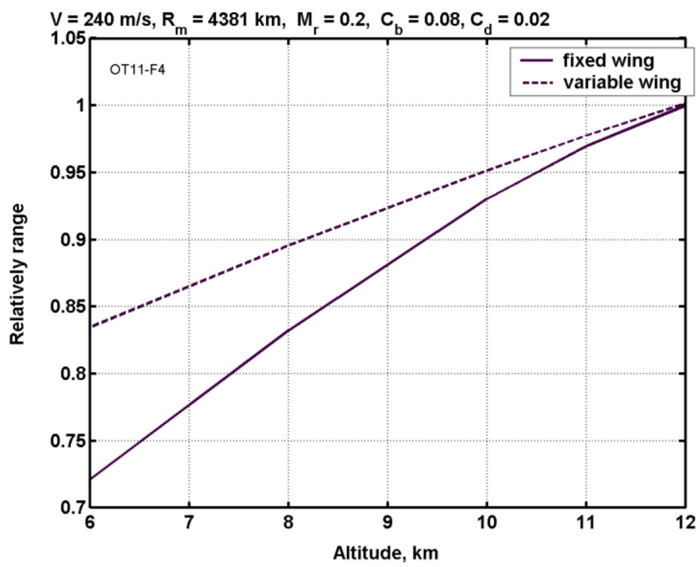


**Fig. A4.23.** Relative aircraft range for speed $V$ = 240 m/s, maximum range $R_m$ = 4381 km, relative fuel mass $M_r$ = 0.2, body drag coefficient $C_b$ = 0.08, wing drag coefficient $C_d$ = 0.02.

## General discussion and conclusion

a) The current space missiles were designed 30–40 years ago. In the past we did not have navigation satellites that allowed one to locate a missile (warhead) as close as 1 m to a target. Missile designers used inertial navigation systems for ballistic trajectories only. At the present time, we have a satellite navigation system and cheap devices, that enable aircraft, sea ships, cars, vehicles, and people to be located. If we exchange the conventional warhead for a warhead with a simple fixed wing with having a control and navigation system, we can increase the range of our old rockets 4.5–5 times (Fig. A4.13) or significantly increase the useful warhead weight (Fig. A4.19). We can also notably improve the precision of our aiming.

b) Current artillery projectiles for big guns and cannons were created many years ago. The designers assumed that the observer could see an aim point and correct the artillery. Now we have a satellite navigation system that allows one to determine the exact coordinates of targets and we have cheap and light navigation and control devices that can be placed in the cannon projectiles. If we replace our cannon ballistic projectiles with projectiles with a fixed wing, and a control and navigation system, we increase the range 3–9 times (from 35 km up to 360 km, see Fig. A4.20, A4.21). We can use a smooth barrel to increase the nozzle shell speed up to 2000 m/s and range up to 1000 km. These systems can guide the **winged** projectiles and significantly improving their aim. We can reach this result because we use all the **kinetic** energy of the projectile. A conventional projectile cannot remain in the atmosphere and drops at a very high speed. Most of its kinetic energy is wasted. In our case 70–85% of the projectile's kinetic energy is used for support of the moving projectile. This way the projectile range increases 3–9 times or more.

a) All aircraft are designed for only one optimal flight regime (speed, altitude, and fuel consumption). Any deviation from this regime decreases the aircraft range. For aircraft like to the Boeing 747 this regime is: altitude $H$ = 12 km, speed $V$ = 240 m/s, specific fuel consumption $Cs$ = 0.00019 kgf/s/kg thrust. If the speed is reduced from 240 m/s to 200 m/s, the range decreases by15% (Fig. A4.23). Application of the variable wing area reduces this loss from 15% to 10%. If the aircraft reduces its altitude from 12 km to 9 km, it loses 12% of its maximum range (Fig. A4.24). If it has a variable wing area, it loses only 7.5% of its maximum range. Civil air vehicles are forced to deviate from the optimal conditions by weather or a given flight air corridor. Military air vehicles sometimes have to make a very large deviation from the optimal conditions (for example, when they fly at low altitude, below the enemy radar system). A variable wing area may be very useful for them because it decreases the loss by approximately 50%, improves supersonic flight and taking off and landing lengths.

The author offers some fixed and variable wing designs for air vehicles (Fig. A4.25). Variants **a, b, c**, and **f** are for missiles and warheads, variants **d,** and **e** are for shells.

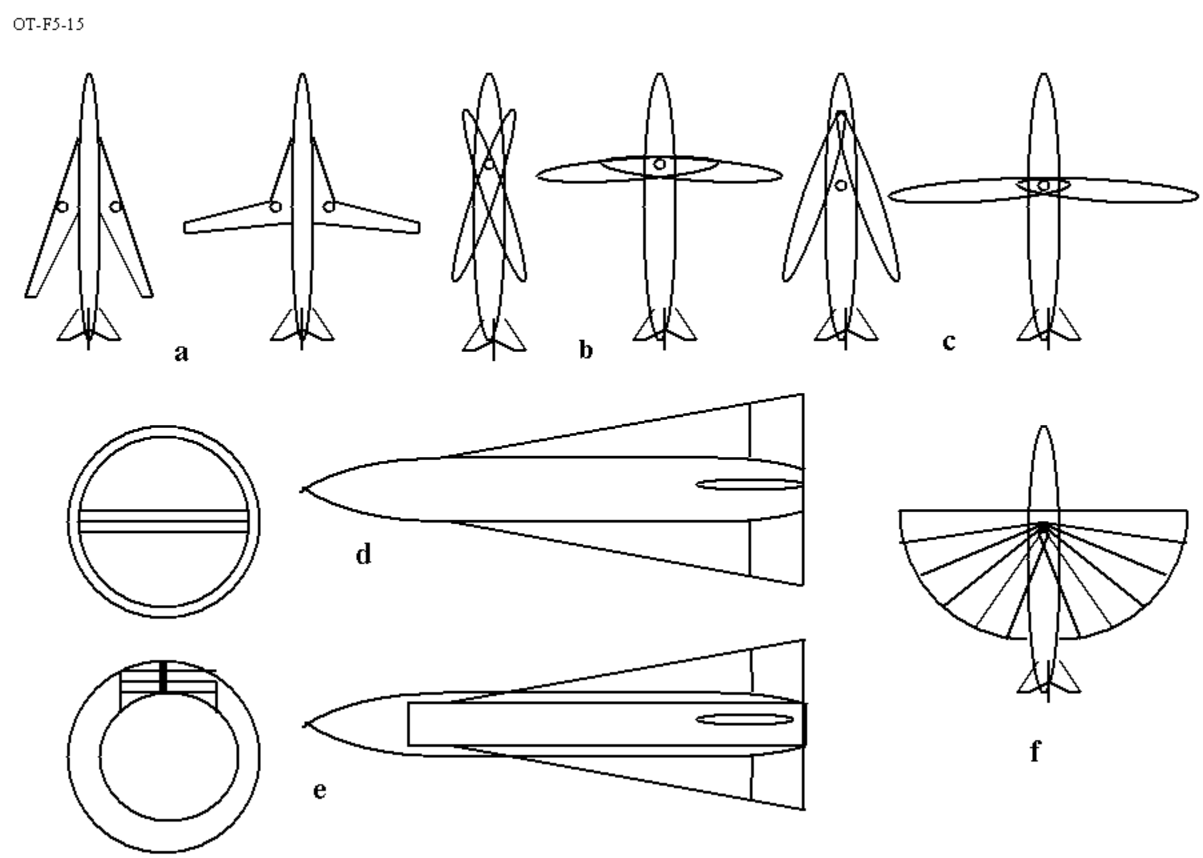


**Fig. A4.25.** Possible variants of variable wing designs: **a, b, c,** and **f** for aircraft; **d** and **e** for gun projectiles.

## *References for Attachment 4*


1. A. Miele, “General Variable Theory of the Flight Paths of Rocket-Powered Aircraft, Missile and Satellite Carries”, *Astronautica acta*, Vol. 4, pp. 256–288, 1958.
2. A.A. Bolonkin, “Special Extrema in Optimal Control Problems”, Akademiya Nauk, Izvestia, *Theknicheskaya Kibernetika*, No. 2, March–April, 1969, pp. 187–198. See also English translation in *Eng. Cybernetics*, No. 2, March–April 1969, pp. 170–183.
3. A.A. Bolonkin, *New Methods of Optimization and their Application*, Moscow, Bauman Technical University, 1972, (in Russian).
4. A.A. Bolonkin, “A New Approach to Finding a Global Optimum”, *New Americans Collected Scientific Report*s, Vol. 1, 1991, The Bnai Zion Scientists Division, New York.
5. A.A. Bolonkin, “Optimization of Trajectory of Multistage Flight Vehicles”, in collection *Researches of Flight Dynamics*, Masinobilding, Moscow, 1965, pp. 20–78 (in Russian).
6. A.A. Bolonkin, N.S. Khot, “Method for Finding a Global Minimum”, AIAA/NASA/USAF/SSMO Symposium on Multidisciplinary Analysis and Optimization, Panama City, Florida, USA, 7–9 Sept. 1994.
7. A.A. Bolonkin, “Solution Methods for Boundary-Value Problems of Optimal Control Theory”. 1973. Consultants Bureau, a division of Plenum Publishing Corporation, NY. Translation from *Prikladnaya Mekhanika*, Vol. 7, No 6, 1971, p. 639–650 (English).
8. A.A. Bolonkin, “Solution General Linear Optimal Problem with One Control”. Journal “*Pricladnaya Mechanica*”, Vol. 4, No 4, 1968, pps. 111–122, Moscow (in Russian).
9. A.A. Bolonkin, N. Khot, “Optimal Structural Control Design”, IAF-94-1.4.206, 45th Congress of the International Astronautical Federation, World Space Congress-1994, 9–14 October, 1994, Jerusalem, Israel.
10. A. A. Bolonkin, N. S. Khot, “Optimal Bounded Control Design for Vibration Suppression”, *Acta Astronautica*, Vol. 38, No. 10, pp. 803–813, 1996.
11. A.A. Bolonkin, D. Jeffcoat, “Optimal Thrust Angle of Aircraft”, 2002, see Attachment 6.
12. A.A. Bolonkin, R. Sierakowski, “Design of Optimal Regulators”, 2nd AIAA “Unmanned Unlimited” Systems, Technologies, and Operations – Aerospace, Land, and sea Conference and Workshop & Exhibit, San Diego, California, 15-18 Sep 2003, AIAA-2003-6638.
13. A.A. Bolonkin, Optimal Trajectories of Air and Space Vehicles”, *Aircraft Engineering and Aerospace Technology*, Vol.78, No.2, 2004, pp.192–214.
14. Bolonkin’s site: http://Bolonkin.narod.ru .