Two new warp drive equations based on parallel $3+1$ ADM formalisms in contravariant and covariant forms applied to the Natario spacetime geometry

Fernando Loup *
Residencia de Estudantes Universitas Lisboa Portugal

April 1, 2015

Abstract

Warp Drives are solutions of the Einstein Field Equations that allows superluminal travel within the framework of General Relativity. There are at the present moment two known solutions: The Alcubierre warp drive discovered in 1994 and the Natario warp drive discovered in 2001. However the major drawback concerning warp drives is the huge amount of negative energy density able to sustain the warp bubble.In order to perform an interstellar space travel to a "nearby" star at 20 light-years away in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!. With the correct form of the shape function the Natario warp drive can overcome this obstacle at least in theory.Other drawbacks that affects the warp drive geometry are the collisions with hazardous interstellar matter(asteroids,comets,interstellar dust etc)that will unavoidably occurs when a ship travels at superluminal speeds and the problem of the Horizons(causally disconnected portions of spacetime).The geometrical features of the Natario warp drive are the required ones to overcome these obstacles also at least in theory. Some years ago starting from 2012 to 2014 a set of works appeared in the current scientific literature covering the Natario warp drive with an equation intended to be the original Natario equation however this equation do not obeys the original $3+1$ Arnowitt-Dresner-Misner(ADM) formalism and hence this equation cannot be regarded as the original Natario warp drive equation.However this new equation satisfies the Natario criteria for a warp drive spacetime and as a matter of fact this equation must be analyzed under a new and parallel contravariant $3+1$ ADM formalism.In this work we introduce also a second new Natario equation but using a parallel covariant $3+1$ ADM formalism. We compare both the original and parallel $3+1$ ADM formalisms wether in contravariant or covariant form using the approach of Misner-Thorne-Wheeler(MTW) and Alcubierre and while in the $3+1$ spacetime the parallel equations differs radically from the original one when we reduce the equations to a $1+1$ spacetime all the equations becomes equivalent.We discuss the possibilities in General Relativity for these new equations.

*spacetimeshortcut@yahoo.com

1 Introduction:

The Warp Drive as a solution of the Einstein field equations of General Relativity that allows superluminal travel appeared first in 1994 due to the work of Alcubierre.([1]) The warp drive as conceived by Alcubierre worked with an expansion of the spacetime behind an object and contraction of the spacetime in front.The departure point is being moved away from the object and the destination point is being moved closer to the object.The object do not moves at all[1].It remains at the rest inside the so called warp bubble but an external observer would see the object passing by him at superluminal speeds(pg 8 in [1])(pg 1 in [2]).

Later on in 2001 another warp drive appeared due to the work of Natario.([2]).This do not expands or contracts spacetime but deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [2]). Imagine the object being a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium whose walls do not expand or contract. An observer in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.

However there are 3 major drawbacks that compromises the warp drive physical integrity as a viable tool for superluminal interstellar travel.

The first drawback is the quest of large negative energy requirements enough to sustain the warp bubble. In order to travel to a "nearby" star at 20 light-years at superluminal speeds in a reasonable amount of time a ship must attain a speed of about 200 times faster than light.However the negative energy density at such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!(see [7],[8] and [9]).

Another drawback that affects the warp drive is the quest of the interstellar navigation:Interstellar space is not empty and from a real point of view a ship at superluminal speeds would impact asteroids,comets,interstellar space dust and photons.(see [5],[7] and [8]).

The last drawback raised against the warp drive is the fact that inside the warp bubble an astronaut cannot send signals with the speed of the light to control the front of the bubble because an Horizon(causally disconnected portion of spacetime)is established between the astronaut and the warp bubble.(see [5],[7] and [8]).

We can demonstrate that the Natario warp drive can "easily" overcome these obstacles as a valid candidate for superluminal interstellar travel(see [7],[8] and [9]).

In this work we cover only the Natario warp drive and we avoid comparisons between the differences of the models proposed by Alcubierre and Natario since these differences were already deeply covered by the existing available literature.(see [5],[6] and [7])However we use the Alcubierre shape function to define its Natario counterpart.

[1] do not violates Relativity

Alcubierre([12]) used the so-called $3+1$ Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([11]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pgs [67(b)],[82(a)] in [12], see also eq 1 pg 3 in [1])[2][3] and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive spacetime.

Some years ago from 2012 to 2014 a set of works ([5],[6],[7],[8] and [10]) started to appear in the scientific literature covering the Natario warp drive spacetime using the following equation:

$$ds^2 = [1-(X^{rs})^2-(X^{\theta})^2]dt^2 + 2[X^{rs}drs + X^{\theta}rsd\theta]dt - drs^2 - rs^2d\theta^2 \tag{1}$$

The equation above appeared for the first time in the works pg 4 eq 1 in [5],pg 12 eq 50 in [6],pg 14 eq 38 in [7],pg 20 eq 80 in [8],pg 9 eq 12 in [10] and was intended to be the original Natario warp drive equation.However this equation do not obeys the original $3+1$ ADM formalism.The correct Natario warp drive equation that obeys the $3+1$ ADM formalism is given below:

$$ds^2 = (1-X_{rs}X^{rs}-X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2d\theta^2 \tag{2}$$

Indeed the equation presented in the works ([5],[6],[7],[8] and [10]) is a valid equation for the Natario warp drive spacetime but under the context of a new and parallel contravariant $3+1$ ADM formalism.

The $3+1$ original ADM formalism with signature $(-,+,+,+)$ is given by the equation (21.40) pg $[507(b)]$ $[534(a)]$ in [11]

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -\alpha^2dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{3}$$

The new $3+1$ parallel contravariant ADM formalism with signature $(-,+,+,+)$ is given by the equation:

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -\alpha^2dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{4}$$

Since we have a new Natario warp drive equation under a new $3+1$ parallel contravariant ADM formalism already presented in the works ([5],[6],[7],[8] and [10]) we examined the possibility of the existence of another new Natario warp drive equation but under another new $3+1$ parallel covariant ADM formalism.Such an equation also exists and can be written as shown below:

$$ds^2 = [1-(X_{rs})^2-(X_{\theta})^2]dt^2 + 2[X_{rs}drs + X_{\theta}rsd\theta]dt - drs^2 - rs^2d\theta^2 \tag{5}$$

Also the new $3+1$ parallel covariant ADM formalism with signature $(-,+,+,+)$ is given by the equation:

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -\alpha^2dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{6}$$

[2] see also Appendix E

[3] see the Remarks section on our system to quote pages in bibliographic references

In this work we study the validity of the new equations presented for the Natario warp drive spacetime using the new parallel $3+1$ ADM contravariant and covariant formalisms and we arrive at the conclusion that the new equations are valid solutions for the warp drive spacetime according to the Natario criteria.We also compare all the Natario warp drive equations in the original and parallel $3+1$ ADM formalisms wether contravariant or covariant and we arrive at two interesting conclusions:

- 1)-in the $3+1$ spacetime the parallel ADM formalisms wether contravariant or covariant differs radically from the original ADM formalism because while in the original formalism all the mathematical entities of General Relativity (eg:Christoffel symbols,Riemann and Ricci tensors,Ricci scalar,Einstein tensors,extrinsic curvature tensors) are cartographed and chartered these mathematical entities are completely unknown in the parallel formalisms and must be obtained by hand calculations in a all-the-way-round process starting from the covariant components of the $3+1$ spacetime metric and finishing with the Einstein tensor in a long and tedious sequence of calculations in tensor algebra liable of errors or can be obtained by computer programs like $Maple$ or $Mathematica$.

- 2)-A dimensional reduction from $3+1$ spacetime to a $1+1$ spacetime demonstrates that in a $1+1$ spacetime both the original and the parallel ADM formalisms wether contravariant or covariant are equivalent and since the works ([5],[6],[7],[8] and [10]) uses the dimensional reduction from a $3+1$ to a $1+1$ spacetime their conclusions are still valid.

For the study of the original ADM formalism we use the approaches of MTW([11]) and Alcubierre([12]) and we adopt the Alcubierre convention for notation of equations and scripts.

We adopt here the Geometrized system of units in which $c = G = 1$ for geometric purposes and the International System of units for energetic purposes.

This work is organized as follows:

- Section 2)-Introduces the Natario warp drive continuous shape function able to low the negative energy density requirements when a ship travels with a speed of 200 times faster than light.
The negative energy density for such a speed is directly proportional to the factor 10^{48} which is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!.

- Section 3)-presents the new equation for the Natario warp drive spacetime in the parallel contravariant $3+1$ ADM formalism in a rigorous mathematical fashion.We recommend the study of the Appendix B at the end of the work in order to fully understand the mathematical demonstrations. The dimensional reduction from a $3+1$ to a $1+1$ spacetime shows that the parallel contravariant ADM formalism in the $1+1$ spacetime is equal to the original ADM formalism in the $1+1$ spacetime.

- Section 4)-presents the new equation for the Natario warp drive spacetime in the parallel covariant $3+1$ ADM formalism in a rigorous mathematical fashion.We recommend the study of the Appendix C at the end of the work in order to fully understand the mathematical demonstrations. The dimensional reduction from a $3+1$ to a $1+1$ spacetime shows that the parallel covariant ADM formalism in the $1+1$ spacetime is equal to the original ADM formalism in the $1+1$ spacetime.

- Section 5)-presents the original equation for the Natario warp drive spacetime in the original $3+1$ ADM formalism in a rigorous mathematical fashion.We recommend the study of the Appendix E at the end of the work in order to fully understand the mathematical demonstrations The dimensional reduction from a $3+1$ to a $1+1$ spacetime shows that the original ADM formalism in the $1+1$ spacetime is equal to the parallel ADM formalism in the $1+1$ spacetime wether contravariant or covariant.

- Section 6)-compares both the original and both contravariant and covariant parallel formalisms and since in a $1+1$ spacetime all these formalisms are equivalent the shape function used to lower the negative energy density requirements in the original equation is valid also for the new equations so these new Natario warp drives are also affordable from the point of view of negative energy densities in a $1+1$ spacetime.For a better description about how the Natario shape function can lower the negative energy density requirements in the Natario warp drive see [8] and [9] .Also when we reduce the original $3+1$ ADM formalism to a $1+1$ original ADM formalism the zero expansion behavior of the Natario warp drive is maintained in the original equation and since the $1+1$ parallel ADM formalisms are equivalent to the originasl one then at least in a $1+1$ dimensions the new equations for the Natario warp drive also retains the zero expansion behavior.Another important thing is the fact that even in the $1+1$ spacetime all the warp drive equations possesses the negative energy density in the warp bubble in front of the ship[4] and the repulsive behavior of the negative energy density in the bubble can protect the ship against incoming highly energetic Doppler blueshifted photons or interstellar hazardous matter (eg:space dust,gas clouds,supernova remmants,asteroids comets etc) a ship would encounter in a realistic interstellar spaceflight at superluminal speeds.Also this negative energy density in front of the ship protects the ship against the so-called infinite Doppler Blueshifts in the Horizon.For more about how the Natario warp drive deals with collisions with interstellar matter or infinite Doppler blueshifts see [5] , [7] and [8]

Although this work was designed to be independent consistent and self-contained concerning ADM formalisms it can be regarded as a companion work to our work in [15]

[4]the negative energy density do not vanish even in a $1+1$ spacetime

2 The Natario warp drive continuous shape function

Introducing here $f(rs)$ as the Alcubierre shape function that defines the Alcubierre warp drive spacetime we can construct the Natario shape function $n(rs)$ that defines the Natario warp drive spacetime using its Alcubierre counterpart.Below is presented the equation of the Alcubierre shape function.[5].

$$f(rs) = \frac{1}{2}[1 - tanh[@(rs - R)] \tag{7}$$

$$rs = \sqrt{(x - xs)^2 + y^2 + z^2} \tag{8}$$

According with Alcubierre any function $f(rs)$ that gives 1 inside the bubble and 0 outside the bubble while being $1 > f(rs) > 0$ in the Alcubierre warped region is a valid shape function for the Alcubierre warp drive.(see eqs 6 and 7 pg 4 in [1] or top of pg 4 in [2]).

In the Alcubierre shape function xs is the center of the warp bubble where the ship resides. R is the radius of the warp bubble and @ is the Alcubierre parameter related to the thickness.According to Alcubierre these can have arbitrary values.We outline here the fact that according to pg 4 in [1] the parameter @ can have arbitrary values.rs is the path of the so-called Eulerian observer that starts at the center of the bubble $xs = R = rs = 0$ and ends up outside the warp bubble $rs > R$.

According to Natario(pg 5 in [2]) any function that gives 0 inside the bubble and $\frac{1}{2}$ outside the bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region is a valid shape function for the Natario warp drive.

The Natario warp drive continuous shape function can be defined by:

$$n(rs) = \frac{1}{2}[1 - f(rs)] \tag{9}$$

$$n(rs) = \frac{1}{2}[1 - [\frac{1}{2}[1 - tanh[@(rs - R)]]]] \tag{10}$$

This shape function gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region.

Note that the Alcubierre shape function is being used to define its Natario shape function counterpart.

For the Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

[5] $tanh[@(rs + R)] = 1$,$tanh(@R) = 1$ for very high values of the Alcubierre thickness parameter $@ >> |R|$

Another Natario warp drive valid shape function can be given by:

$$n(rs) = [\frac{1}{2}][1 - f(rs)^{WF}]^{WF} \tag{11}$$

Its derivative square is :

$$n'(rs)^2 = [\frac{1}{4}]WF^4[1 - f(rs)^{WF}]^{2(WF-1)}[f(rs)^{2(WF-1)}]f'(rs)^2 \tag{12}$$

The shape function above also gives the result of $n(rs) = 0$ inside the warp bubble and $n(rs) = \frac{1}{2}$ outside the warp bubble while being $0 < n(rs) < \frac{1}{2}$ in the Natario warped region(see pg 5 in [2]).

Note that like in the previous case the Alcubierre shape function is being used to define its Natario shape function counterpart. The term WF in the Natario shape function is dimensionless too:it is the warp factor.It is important to outline that the warp factor $WF >> |R|$ is much greater than the modulus of the bubble radius.

For the second Natario shape function introduced above it is easy to figure out when $f(rs) = 1$(interior of the Alcubierre bubble) then $n(rs) = 0$(interior of the Natario bubble) and when $f(rs) = 0$(exterior of the Alcubierre bubble)then $n(rs) = \frac{1}{2}$(exterior of the Natario bubble).

- Numerical plot for the second shape function with @ = 50000 and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99970000000E+001$	1	0	$2,650396620740E-251$	0
$9,99980000000E+001$	1	0	$1,915169647489E-164$	0
$9,99990000000E+001$	1	0	$1,383896564748E-077$	0
$1,00000000000E+002$	0,5	0,5	$6,250000000000E+008$	$3,872591914849E-103$
$1,00001000000E+002$	0	0,5	$1,383896486082E-077$	0
$1,00002000000E+002$	0	0,5	$1,915169538624E-164$	0
$1,00003000000E+002$	0	0,5	$2,650396470082E-251$	0

- Numerical plot for the second shape function with @ = 75000 and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99980000000E+001$	1	0	$5,963392481410E-251$	0
$9,99990000000E+001$	1	0	$1,158345097767E-120$	0
$1,00000000000E+002$	0,5	0,5	$1,406250000000E+009$	$8,713331808411E-103$
$1,00001000000E+002$	0	0,5	$1,158344999000E-120$	0
$1,00002000000E+002$	0	0,5	$5,963391972940E-251$	0

- Numerical plot for the second shape function with @ = 100000 and warp factor with a value $WF = 200$

rs	$f(rs)$	$n(rs)$	$f'(rs)^2$	$n'(rs)^2$
$9,99990000000E+001$	1	0	$7,660678807684E-164$	0
$1,00000000000E+002$	0,5	0,5	$2,500000000000E+009$	$1,549036765940E-102$
$1,00001000000E+002$	0	0,5	$7,660677936765E-164$	0

The plots in the previous page demonstrate the important role of the thickness parameter @ in the warp bubble geometry wether in both Alcubierre or Natario warp drive spacetimes.For a bubble of 100 meters radius $R = 100$ the regions where $1 > f(rs) > 0$(Alcubierre warped region) and $0 < n(rs) < \frac{1}{2}$(Natario warped region) becomes thicker or thinner as @ becomes higher.

Then the geometric position where both Alcubierre and Natario warped regions begins with respect to R the bubble radius is $rs = R - \epsilon < R$ and the geometric position where both Alcubierre and Natario warped regions ends with respect to R the bubble radius is $rs = R + \epsilon > R$

As large as @ becomes as smaller ϵ becomes too.

Note from the plots of the previous page that we really have two warped regions:

- 1)-The geometrized warped region where $1 > f(rs) > 0$(Alcubierre warped region) and $0 < n(rs) < \frac{1}{2}$ (Natario warped region).
- 2)-The energized warped region where the derivative squares of both Alcubierre and Natario shape functions are not zero.

The parameter @ affects both energized warped regions wether in Alcubierre or Natario cases but is more visible for the Alcubierre shape function because the warp factor WF in the Natario shape functions squeezes the energized warped region into a very small thickness.

The negative energy density for the Natario warp drive is given by(see pg 5 in [2])

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{13}$$

Converting from the Geometrized System of Units to the International System we should expect for the following expression:

$$\rho = -\frac{c^2}{G}\frac{vs^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{rs}{2}n''(rs)\right)^2\sin^2\theta\right]. \tag{14}$$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for[6]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{15}$$

[6]see Appendix D

In the equatorial plane($1+1$ dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$):

$$\rho = T_{\mu\nu} u^{\mu} u^{\nu} = -\frac{c^2}{G} \frac{v_s^2}{8\pi} \left[3(n'(rs))^2 \right] \quad (16)$$

Note that in the above expressions the warp drive speed vs appears raised to a power of 2. Considering our Natario warp drive moving with $vs = 200$ which means to say 200 times light speed in order to make a round trip from Earth to a nearby star at 20 light-years away in a reasonable amount of time(in months not in years) we would get in the expression of the negative energy the factor $c^2 = (3 \times 10^8)^2 = 9 \times 10^{16}$ being divided by $6,67 \times 10^{-11}$ giving $1,35 \times 10^{27}$ and this is multiplied by $(6 \times 10^{10})^2 = 36 \times 10^{20}$ coming from the term $vs = 200$ giving $1,35 \times 10^{27} \times 36 \times 10^{20} = 1,35 \times 10^{27} \times 3,6 \times 10^{21} = 4,86 \times 10^{48}$!!!

A number with 48 zeros!!!The planet Earth have a mass[7] of about $6 \times 10^{24} kg$

This term is 1.000.000.000.000.000.000.000.000 times bigger in magnitude than the mass of the planet Earth!!!or better:The amount of negative energy density needed to sustain a warp bubble at a speed of 200 times faster than light requires the magnitude of the masses of 1.000.000.000.000.000.000.000.000 planet Earths!!!

Note that if the negative energy density is proportional to 10^{48} this would render the warp drive impossible but fortunately the square derivative of the Natario shape function possesses values of 10^{-102} ameliorating the factor 10^{48} making the warp drive negative energy density more "affordable".

[7]see Wikipedia:The free Encyclopedia

3 The equation of the Natario warp drive spacetime metric in the parallel contravariant $3+1$ ADM formalism

The warp drive spacetime according to Natario for the coordinates rs and θ in the parallel contravariant $3+1$ ADM formalism is defined by the following equation:(see Appendix B for details)

$$ds^2 = [1-(X^{rs})^2-(X^{\theta})^2]dt^2 + 2[X^{rs}drs + X^{\theta}rsd\theta]dt - drs^2 - rs^2d\theta^2 \tag{17}$$

The expressions for X^{rs} and X^{θ} are given by:(see pg 5 in [2],see also Appendix A for details)

$$X^{rs} = -2v_s n(rs)\cos\theta \tag{18}$$

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{19}$$

$$X^{\theta} = v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{20}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{21}$$

Looking both the equation of the Natario warp drive and the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$ds^2 = [1-(X^{rs})^2-(X^{\theta})^2]dt^2 + 2[X^{rs}drs + X^{\theta}rsd\theta]dt - drs^2 - rs^2d\theta^2 \tag{22}$$

$$nX = X^{rs}drs + X^{\theta}rsd\theta \tag{23}$$

We can see that the Natario vector is completely inserted twice in the non-diagonalized components of the metric of the Nayario warp drive equation which gives:

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs)\cos\theta \tag{24}$$

$$g_{02} = g_{20} = X^{\theta}rs = -v_s(2n(rs) + (rs)n'(rs))rs\sin\theta \tag{25}$$

Since we have two sets of non-diagonalized components in the Natario warp drive equation and each set possesses equal components of the Natario vector nX this is the reason why the Natario vector nX appears twice in the Natario warp drive equation.

The diagonalized components of the metric of the Natario warp drive equation are given by:

$$g_{00} = 1-(X^{rs})^2-(X^{\theta})^2 = 1-(2v_s n(rs)\cos\theta)^2 - (-v_s(2n(rs)+(rs)n'(rs))\sin\theta)^2 \tag{26}$$

The term $(-v_s(2n(rs)+(rs)n'(rs))\sin\theta)^2 = (v_s(2n(rs)+(rs)n'(rs))\sin\theta)^2$

$$g_{00} = 1-(X^{rs})^2-(X^{\theta})^2 = 1-(2v_s n(rs)\cos\theta)^2 - (v_s(2n(rs)+(rs)n'(rs))\sin\theta)^2 \tag{27}$$

$$g_{11} = -1 \tag{28}$$

$$g_{22} = -rs^2 \tag{29}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We can see that the Natario warp drive equation given in the previous page satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = -vs(t)dx$ or $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

The statement above can be explained in the following way:

Consider again the Natario vector nX(pg 2 and 5 in [2]) defined below as:

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{30}$$

The components of the Natario vector nX are X^{rs} and X^{θ}.These are the shift vectors.Then a Natario vector is constituted by one or more shift vectors.

When the Natario shape function $n(rs) = 0$ inside the bubble then $X^{rs} = 2v_s n(rs)\cos\theta = 0$ and $X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta = 0$.Then inside the bubble both shift vectors are zero resulting in a zero Natario vector.

When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then $X^{rs} = 2v_s n(rs)\cos\theta = v_s\cos\theta$ and $X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta = -v_s\sin\theta$.Then outside the bubble both shift vectors are not zero resulting in a not zero Natario vector.

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

The Natario warp drive equation and the Natario vector nX in the equatorial plane $1 + 1$ spacetime now becomes:

$$ds^2 = [1 - (X^{rs})^2]dt^2 + 2[X^{rs}drs]dt - drs^2 \tag{31}$$

$$nX = X^{rs} drs \tag{32}$$

Note that the Natario vector nX is still inserted twice in the Natario warp drive equation due to the 2 remaining non-diagonalized components which are:

$$g_{01} = g_{10} = X^{rs} = 2v_s n(rs) \tag{33}$$

When the Natario shape function $n(rs) = 0$ inside the bubble then the shift vector $X^{rs} = 2v_s n(rs) = 0$ .Then inside the bubble the shift vector $X^{rs} = 0$ is zero resulting in a zero Natario vector.

When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then the shift vector $X^{rs} = 2v_s n(rs) = v_s$ .Then outside the bubble both shift and Natario vectors are not zero and the shift vector is equal to the bubble speed vs $X^{rs} = vs$.

The above statements explain the Natario affirmation of $X = 0$ inside the bubble and $X = vs$ outside the bubble.(pg 4 in [2])

The diagonalized components of the metric of the Natario warp drive equation are given by:

$$g_{00} = 1 - (X^{rs})^2 = 1 - (2v_s n(rs))^2 \tag{34}$$

$$g_{11} = -1 \tag{35}$$

4 The equation of the Natario warp drive spacetime metric in the parallel covariant $3+1$ ADM formalism

The warp drive spacetime according to Natario for the coordinates rs and θ in the parallel covariant $3+1$ ADM formalism is defined by the following equation:(see Appendix C for details).

$$ds^2 = [1 - (X_{rs})^2 - (X_\theta)^2]dt^2 + 2[X_{rs}drs + X_\theta rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{36}$$

Looking to the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs}drs + X^\theta rsd\theta \tag{37}$$

With the contravariant shift vector components X^{rs} and X^θ given by:(see pg 5 in [2],see also Appendix A for details):

$$X^{rs} = -2v_s n(rs)\cos\theta \tag{38}$$

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{39}$$

$$X^\theta = v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{40}$$

$$X^\theta = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{41}$$

But remember that $dl^2 = \gamma_{ij}dx^i dx^j = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$,$\gamma_{\theta\theta} = r^2$ $\sqrt{\gamma_{rr}} = 1$ $\sqrt{\gamma_{\theta\theta}} = r$ and $r = rs$.Then the covariant shift vector components X_{rs} and X_θ with $r = rs$ are given by:

$$X_i = \gamma_{ii}X^i \tag{42}$$

$$X_r = \gamma_{rr}X^r = X_{rs} = \gamma_{rsrs}X^{rs} = 2v_s n(rs)\cos\theta = X^r = X^{rs} \tag{43}$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = rs^2 X^\theta = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{44}$$

It is possible to construct a covariant form for the Natario vector nX defined as n_cX as follows:

$$n_cX = X_{rs}drs + X_\theta rsd\theta \tag{45}$$

With the covariant shift vector components X_{rs} and X_θ defined as shown above:

Looking both the equation of the Natario warp drive and the equation of the covariant Natario vector n_cX;

$$ds^2 = [1 - (X_{rs})^2 - (X_\theta)^2]dt^2 + 2[X_{rs}drs + X_\theta rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{46}$$

$$n_cX = X_{rs}drs + X_\theta rsd\theta \tag{47}$$

We can see that the covariant Natario vector is completely inserted twice in the non-diagonalized components of the metric of the Nayario warp drive equation which gives:

$$g_{01} = g_{10} = X_{rs} = 2v_s n(rs)\cos\theta = X^r = X^{rs} \tag{48}$$

$$g_{02} = g_{20} = X_\theta rs = rs^3 X^\theta = -rs^3 v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{49}$$

Since we have two sets of non-diagonalized components in the Natario warp drive equation and each set possesses equal components of the covariant Natario vector $n_c X$ this is the reason why the Natario vector $n_c X$ appears twice in the Natario warp drive equation.

The diagonalized components of the metric of the Natario warp drive equation are given by:

$$g_{00} = 1 - (X_{rs})^2 - (X_\theta)^2 = 1 - (2v_s n(rs)\cos\theta)^2 - (-rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta)^2 \tag{50}$$

The term $(-rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta)^2 = (rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta)^2$

$$g_{00} = 1 - (X_{rs})^2 - (X_\theta)^2 = 1 - (2v_s n(rs)\cos\theta)^2 - (rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta)^2 \tag{51}$$

$$g_{11} = -1 \tag{52}$$

$$g_{22} = -rs^2 \tag{53}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We can see that the Natario warp drive equation given in the previous page satisfies the Natario requirements for a warp bubble defined by:

any covariant Natario vector $n_c X$ generates a warp drive spacetime if $n_c X = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $n_c X = -vs(t)dx$ or $n_c X = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

The statement above can be explained in the following way:

Consider again the covariant Natario vector $n_c X$ defined below as:

$$n_c X = X_{rs} drs + X_\theta rs d\theta \tag{54}$$

The covariant components of the Natario vector $n_c X$ are X_{rs} and X_θ.These are the covariant shift vectors.Then a covariant Natario vector is constituted by one or more covariant shift vectors.

When the Natario shape function $n(rs) = 0$ inside the bubble then $X_{rs} = 2v_s n(rs)\cos\theta = 0$ and $X_\theta = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta = 0$.Then inside the bubble both covariant shift vectors are zero resulting in a zero covariant Natario vector.

When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then $X_{rs} = 2v_s n(rs)\cos\theta = v_s \cos\theta$ and $X_\theta = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta = -rs^2 v_s \sin\theta$.Then outside the bubble both covariant shift vectors are not zero resulting in a not zero covariant Natario vector.

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

The Natario warp drive equation and the covariant Natario vector $n_c X$ in the equatorial plane $1 + 1$ spacetime now becomes:

$$ds^2 = [1 - (X_{rs})^2]dt^2 + 2[X_{rs}drs]dt - drs^2 \tag{55}$$

$$n_c X = X_{rs}drs \tag{56}$$

Note that the covariant Natario vector $n_c X$ is still inserted twice in the Natario warp drive equation due to the 2 remaining non-diagonalized components which are:

$$g_{01} = g_{10} = X_{rs} = 2v_s n(rs) \tag{57}$$

When the Natario shape function $n(rs) = 0$ inside the bubble then the covariant shift vector $X_{rs} = 2v_s n(rs) = 0$ .Then inside the bubble the covariant shift vector $X_{rs} = 0$ is zero resulting in a zero covariant Natario vector.

When the Natario shape function $n(rs) = \frac{1}{2}$ outside the bubble then the covariant shift vector $X_{rs} = 2v_s n(rs) = v_s$ .Then outside the bubble both covariant shift and Natario vectors are not zero and the covariant shift vector is equal to the bubble speed vs $X_{rs} = vs$.

The above statements explain the Natario affirmation of $X = 0$ inside the bubble and $X = vs$ outside the bubble.(pg 4 in [2])

The diagonalized components of the metric of the Natario warp drive equation are given by:

$$g_{00} = 1 - (X_{rs})^2 = 1 - (2v_s n(rs))^2 \tag{58}$$

$$g_{11} = -1 \tag{59}$$

5 The equation of the Natario warp drive spacetime metric in the original $3+1$ ADM formalism

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:(see Appendix E for details)

$$ds^2 = (1 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{60}$$

The equation of the Natario vector nX(pg 2 and 5 in [2]) is given by:

$$nX = X^{rs}drs + X^{\theta}rsd\theta \tag{61}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2])(see also Appendix A for details)

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{62}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{63}$$

The covariant shift vector components X_{rs} and X_{θ} are given by:

$$X_{rs} = X^{rs} = 2v_s n(rs)\cos\theta \tag{64}$$

$$X_{\theta} = rs^2 X^{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{65}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [2]):

We can see that the Natario warp drive equation given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [2])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x-axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [2]).

In a $1+1$ spacetime the equatorial plane we get:

$$ds^2 = (1 - X_{rs}X^{rs})dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{66}$$

In a $1+1$ spacetime in the equatorial plane the equation in the original ADM formalism can be written as:

$$ds^2 = (1 - X_{rs}X^{rs})dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{67}$$

But since $X_{rs} = X^{rs}$ the equation can be written as given below:

- 1)-In contravariant form:

$$ds^2 = (1 - [X^{rs}]^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{68}$$

- 2)-In covariant form:

$$ds^2 = (1 - [X_{rs}]^2)dt^2 + 2(X_{rs}drs)dt - drs^2 \tag{69}$$

The first equation above is the equation in the $1+1$ spacetime for the parallel contravariant ADM formalism while the second is the equation in the $1+1$ spacetime for the parallel covariant ADM formalism.

All the 3 ADM formalisms wether original parallel contravariant or parallel covariant are mathematically equivalent between each other in a $1+1$ spacetime.

6 Differences and resemblances between the original $3+1$ ADM formalism when compared to both parallel contravariant and parallel covariant $3+1$ ADM formalisms for the Natario warp drive spacetime

The warp drive spacetime according to Natario for the coordinates rs and θ in the parallel contravariant $3+1$ ADM formalism is defined by the following equation:(see Appendix B for details)

$$ds^2 = [1 - (X^{rs})^2 - (X^{\theta})^2]dt^2 + 2[X^{rs}drs + X^{\theta}rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{70}$$

The warp drive spacetime according to Natario for the coordinates rs and θ in the parallel covariant $3+1$ ADM formalism is defined by the following equation:(see Appendix C for details)

$$ds^2 = [1 - (X_{rs})^2 - (X_{\theta})^2]dt^2 + 2[X_{rs}drs + X_{\theta}rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{71}$$

The equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:(see Appendix E for details)

$$ds^2 = (1 - X_{rs}X^{rs} - X_{\theta}X^{\theta})dt^2 + 2(X_{rs}drs + X_{\theta}d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{72}$$

Note that the first equation of the parallel contravariant $3+1$ ADM formalism have the Natario vector nX inserted twice in the non-diagonalized components.This Natario vector nX is given in contravariant form (pg 2 and 5 in [2]):

$$nX = X^{rs}drs + X^{\theta}rsd\theta \tag{73}$$

Note that the second equation of the parallel covariant $3+1$ ADM formalism have the Natario vector n_cX inserted twice in the non-diagonalized components.This Natario vector n_cX is given in covariant form:

$$n_cX = X_{rs}drs + X_{\theta}rsd\theta \tag{74}$$

A pseudo-"covariant" form of the Natario vector cX can be given by:[8]

$$cX = X_{rs}drs + X_{\theta}d\theta \tag{75}$$

Note that the third equation of the original $3+1$ ADM formalism have the pseudo-"covariant" Natario vector cX inserted twice in the non-diagonalized components.

The difference between all these equations in the $3+1$ spacetime is precisely the fact that one of these equations have the Natario vector nX in contravariant form (parallel contravariant ADM formalism) while other equation have the Natario vector n_cX in covariant form (parallel covariant ADM formalism) and another equation have the Natario vector cX in pseudo-"covariant" form (original ADM formalism).Also one of the equations uses exclusively contravariant components (parallel contravariant ADM formalism) while other equation uses exclusively covariant components (parallel covariant ADM formalism) and another equation uses both mixed contravariant and covariant components (original ADM formalism).

[8]all the shift vectors are covariant in this expression

But in the $1+1$ spacetime all these equations are equal due to the equivalence between the contravariant and covariant shift vector components $X_{rs} = X^{rs}$ of both Natario vectors nX and n_cX together with cX:

Alcubierre used the original $3+1$ ADM formalism in his warp drive(see eq 1 pg 3 in [1])[9] and we have reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to derive the original Natario warp drive equation:

$$ds^2 = (1 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{76}$$

The negative energy density for the Natario warp drive in the original $3+1$ ADM formalism is given by(see pg 5 in [2])

$$\rho = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{77}$$

In the equatorial plane($1+1$ dimensional spacetime with $rs = x - xs$,$y = 0$ and center of the bubble $xs = 0$):[10]

$$\rho = T_{\mu\nu}u^\mu u^\nu = -\frac{c^2}{G}\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{78}$$

But for the Natario warp drive equation in the parallel contravariant $3+1$ ADM formalism

$$ds^2 = [1 - (X^{rs})^2 - (X^\theta)^2]dt^2 + 2[X^{rs}drs + X^\theta rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{79}$$

or for the Natario warp drive equation in the parallel covariant $3+1$ ADM formalism

$$ds^2 = [1 - (X_{rs})^2 - (X_\theta)^2]dt^2 + 2[X_{rs}drs + X_\theta rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{80}$$

We can say nothing about the negative energy density at first sight and we need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors.

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pgs $[342(b)]$ or $[369(a)]$ in [11], pgs $[276(b)]$ or $[294(a)]$ in [13],pgs $[454, 457, 560(b)]$ or $[465, 468, 567(a)]$ in [14]).

Appendix C pgs $[551 - 555(b)]$ or $[559 - 563(a)]$ in [14] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3+1$ spacetime metric using $Mathematica$.

But since the $1+1$ equation for the parallel ADM formalism wether in contravariant or covariant form is equal to the $1+1$ equation for the original ADM formalism the negative energy density in $1+1$ spacetime is the same for all these equations.

[9] see Appendix E

[10] see Appendix D

Also in the geometry of the original $3+1$ ADM formalism Natario warp drive the spacetime contraction in one direction(radial) is balanced by the spacetime expansion in the remaining direction(perpendicular).

Remember also that the expansion of the normal volume elements in the original $3+1$ ADM formalism for the Natario warp drive is given by the following expressions(pg 5 in [2]). :

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r)\cos\theta \tag{81}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = v_s n'(r)\cos\theta; \tag{82}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = v_s n'(r)\cos\theta \tag{83}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{84}$$

If we expand the radial direction the perpendicular direction contracts to keep the expansion of the normal volume elements equal to zero resulting in a warp drive with zero expansion.

Note also that even in a $1+1$ dimensional spacetime the original $3+1$ ADM formalism for the Natario warp drive when reduced to a $1+1$ dimensions retains the zero expansion behavior:

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s n'(r)\cos\theta \tag{85}$$

$$K_{\theta\theta} = \frac{X^r}{r} = v_s n'(r)\cos\theta; \tag{86}$$

$$K_{\varphi\varphi} = \frac{X^r}{r} = v_s n'(r)\cos\theta \tag{87}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{88}$$

So we cannot say anything about the geometry of the parallel $3+1$ ADM formalisms wether in contravariant or covariant form concerning the expansion of the normal volume elements without the computation of the extrinsic curvatures but at least in a $1+1$ spacetime the parallel contravariant or the parallel covariant $1+1$ ADM formalism are equivalent to the original $1+1$ ADM formalism which gives also a warp drive with zero expansion.

7 Conclusion:

In this work we demonstrated the existence of two alternative equations for the warp drive spacetime according to Natario in two parallel $3 + 1$ ADM formalisms(contravariant and covariant) beyond the original $3 + 1$ ADM formalism used by both Alcubierre and Natario.

- 1)-equation of the Natario warp drive given in the parallel contravariant $3 + 1$ ADM formalism.

$$ds^2 = [1 - (X^{rs})^2 - (X^\theta)^2]dt^2 + 2[X^{rs}drs + X^\theta rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{89}$$

This equation appeared for the first time some years ago from 2012 to 2014 in the works pg 4 eq 1 in [5],pg 12 eq 50 in [6],pg 14 eq 38 in [7],pg 20 eq 80 in [8],pg 9 eq 12 in [10]

Note that all the shift vectors X^{rs} and X^θ which composes the Natario vector nX are given in contravariant form and the Natario vector nX is also written in contravariant form (pg 2 and 5 in [2]):

$$nX = X^{rs}drs + X^\theta rsd\theta \tag{90}$$

- 2)-equation of the Natario warp drive given in the parallel covariant $3 + 1$ ADM formalism.

$$ds^2 = [1 - (X_{rs})^2 - (X_\theta)^2]dt^2 + 2[X_{rs}drs + X_\theta rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{91}$$

Since the Natario warp drive can be written using an alternative equation in the parallel contravariant $3 + 1$ ADM formalism we examined the possibility of the existence of even another alternative equation for the Natario warp drive but written in the parallel covariant $3 + 1$ ADM formalism.Such equation is depicted above.

Note that all the shift vectors X_{rs} and X_θ which composes the Natario vector n_cX are given in covariant form and the Natario vector n_cX is also written in covariant form

$$n_cX = X_{rs}drs + X_\theta rsd\theta \tag{92}$$

- 3)-equation of the Natario warp drive given in the original $3 + 1$ ADM formalism.

$$ds^2 = (1 - X_{rs}X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs}drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{93}$$

Alcubierre used the original $3 + 1$ ADM formalism in his warp drive(see eq 1 pg 3 in [1]) and we have reasons to believe that Natario which followed the Alcubierre steps also used the original $3 + 1$ ADM formalism to derive the original Natario warp drive equation depicted above.

Note that this equation have both contravariant and covariant shift vectors in the g_{00} component and the pseudo Natario vector with all the shift vectors in covariant form cX can be given by:

$$cX = X_{rs}drs + X_\theta d\theta \tag{94}$$

But in the $1+1$ spacetime all these equations are mathematically equal due to the equivalence between the contravariant and covariant shift vector components $X_{rs} = X^{rs}$:

$$ds^2 = (1 - [X^{rs}]^2)dt^2 + 2(X^{rs}drs)dt - drs^2 \tag{95}$$

So at least in a $1+1$ spacetime the parallel $1+1$ ADM formalism wether contravariant or covariant coincides with the original $1+1$ ADM formalism and since the works [5],[6],[7],[8] and [10] uses the dimensional reduction from a $3+1$ spacetime to a $1+1$ spacetime the conclusions of these works remains correct.

In section 2 we presented two Natario shape functions and while one of them makes the Natario warp drive impossible to be physically achieved due to high negative energy density requirements the other makes the Natario warp drive perfectly possible to be achieved because this shape function have a form that allows low and "affordable" negative energy density requirements.Then the form of the shape functions affects the behavior of the Natario warp drive spacetime specially in the Natario warped region.For a better description about how the second Natario shape function reduces the negative energy density requirements in the Natario warp drive see [8] and [9].

In section 3 we presented the detailed mathematical structure of the new equation for the Natario warp drive spacetime metric in the parallel contravariant $3+1$ ADM formalism and we verified that this equation satisfies the Natario requirements for a warp drive spacetime.

In section 4 we presented the detailed mathematical structure of the new equation for the Natario warp drive spacetime metric in the parallel covariant $3+1$ ADM formalism and we verified that this equation also satisfies the Natario requirements for a warp drive spacetime.

In section 5 we presented the detailed mathematical structure of the equation for the Natario warp drive spacetime metric in the original $3+1$ ADM formalism using the approaches of MTW([11]) and Alcubierre([12]).We also verified that this equation satisfies the Natario requirements for a warp drive spacetime.

In section 6 we compared the original $3+1$ ADM formalism with the parallel contravariant and covariant $3+1$ ADM formalisms for all the Natario warp drive equations and while the equation in the original formalism have the spacetime geometry completely known(eq:Christoffel symbols,Riemann and Ricci tensors,Ricci scalar,Einstein tensor,stress-energy-momentum tensor for negative energy densities,extrinsic curvatures etc) the same mathematical entities for the new equations in the parallel formalisms remains unknown and must be calculated in a "all-the-way-round" hand by hand or can be obtained using computer programs like $Maple$ or $Mathematica$.

Still in section 6 we can see that in the $1+1$ spacetime all these ADM formalisms are identical and the new Natario warp drive equations have the same negative energy density requirements of the original one so the shape function used to lower the negative energy density to "affordable" levels in the original equation is valid also in the new ones.

Also in section 6 we demonstrated that the zero expansion behavior of the original Natario warp drive equation in the original $3+1$ ADM formalism is maintained when we reduce the dimensions to a original $1+1$ ADM formalism and since the parallel $1+1$ ADM formalisms wether contravariant or covariant are equivalent to the original one then we can say that at least in a $1+1$ spacetime the new equations have also a zero expansion behavior.

Another important thing is the fact that all these equations possesses negative energy density in the warp bubble in front of the ship even in a $1+1$ spacetime[11]and the repulsive behavior of the negative energy density can protect the ship against Doppler blueshifted photons or collisions with hazardous interstellar matter(space dust,debris,asteroids,comets etc) a ship would encounter in a superluminal interstellar spaceflight in a real fashion.Also the negative energy density in front of the ship can protect the ship against the infinite Doppler blueshifts in the Horizon.For more about collisions with interstellar matter and infinite Doppler blueshifts see [5],[7] and [8].

The Natario warp drive spacetime is a very rich environment to study the superluminal features of General Relativity because now we have three spacetime metrics and not only one and the geometry of the new equations in the $3+1$ spacetime is still unknown and needs to be cartographed.

The $3+1$ original ADM formalism with signature $(-,+,+,+)$ is given by the equation (21.40) pg [507(b)] [534(a)] in [11]

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{96}$$

The $3+1$ parallel contravariant ADM formalism with signature $(-,+,+,+)$ is given by the equation:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{97}$$

The $3+1$ parallel covariant ADM formalism with signature $(-,+,+,+)$ is given by the equation:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{98}$$

While the Christoffel symbols,Riemann and Ricci tensors,Ricci scalar,Einstein tensors or extrinsic curvature tensors are completely known and chartered for the original $3+1$ ADM formalism these mathematical entities are completely unknown for the parallel $3+1$ ADM formalisms and this can open new avenues of research in General Relativity.

In this work we developed the parallel contravariant and covariant $3+1$ ADM formalisms exclusively for the Natario warp drive spacetime but it can also be applied to other spacetime metrics.

[11]the negative energy density do not vanish in front of the ship even in a $1+1$ spacetime

But unfortunately although we can discuss mathematically how to reduce the negative energy density requirements to sustain a warp drive we dont know how to generate the shape function that distorts the spacetime geometry creating the warp drive effect.So unfortunately all the discussions about warp drives are still under the domain of the mathematical conjectures.

However we are confident to affirm that the Natario warp drive will survive the passage of the Century XXI and will arrive to the Future.The Natario warp drive as a valid candidate for faster than light interstellar space travel will arrive to the the Century $XXIV$ on-board the future starships up there in the middle of the stars helping the human race to give his first steps in the exploration of our Galaxy

Live Long And Prosper

8 Appendix A:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{99}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (r \sin\theta d\varphi) \wedge dr \sim r \sin\theta (d\varphi \wedge dr) \tag{100}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{101}$$

From above we get the following results

$$dr \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{102}$$

$$rd\theta \sim r\sin\theta (d\varphi \wedge dr) \tag{103}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{104}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(pg 8 in [4]):

$$*dr = r^2 \sin\theta (d\theta \wedge d\varphi) \tag{105}$$

$$*rd\theta = r\sin\theta (d\varphi \wedge dr) \tag{106}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{107}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [2]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{108}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{109}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{110}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{111}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{112}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{113}$$

We know that the following expression holds true(see pg 9 in [3]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{114}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{115}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [2]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{116}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{117}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{118}$$

According to pg 10 in [3] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{119}$$

Because and according to pg 10 in [3]:

$$d(\alpha+\beta) = d\alpha + d\beta \tag{120}$$

$$d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{121}$$

$$d(dx) = d(dy) = d(dz) = 0 \tag{122}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{123}$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \tag{124}$$

And then we derived again the Natario result of pg 5 in [2]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \tag{125}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [2] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \tag{126}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{127}$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{128}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{129}$$

Comparing the above expressions with the Natario definitions of pg 4 in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta)\wedge(r\sin\theta d\varphi) \sim r^2\sin\theta(d\theta\wedge d\varphi) \tag{130}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi)\wedge dr \sim r\sin\theta(d\varphi\wedge dr) \sim -r\sin\theta(dr\wedge d\varphi) \tag{131}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr\wedge(rd\theta) \sim r(dr\wedge d\theta) \tag{132}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta\wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dr\wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dr\wedge d\varphi)] \tag{133}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{134}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{135}$$

Defining the Natario Vector as in pg 5 in [2] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{136}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{137}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [2]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{138}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{139}$$

With our pedagogical approaches

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{140}$$

$$nX = -2vs(t)f(r)\ cos\theta dr + vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{141}$$

9 Appendix B:The Natario warp drive and the parallel contravariant $3+1$ ADM Formalism

A $3+1$ ADM contravariant formalism parallel to the original $3+1$ ADM formalism according with the equation (21.40) pg [507(b)] [534(a)] in [11]

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{142}$$

using the signature $(-,+,+,+)$ can be given by:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)(\sqrt{\gamma_{jj}}dx^j + \beta^j dt) \tag{143}$$

Note that in the equation above all the essential 3 elements of the original $3+1$ ADM formalism are also present[12].These elements are:

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ that measures the proper distance between two points inside each hypersurface.In this case $dl = \sqrt{\gamma_{ij}dx^i dx^j}$.
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.
- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(\sqrt{\gamma_{ii}}dx^i + \beta^i dt)$.$\beta^i$ is known as the contravariant shift vector.

But since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ $dl = \sqrt{\gamma_{ii}}dx^i$ and we have for the $3+1$ spacetime metric the following result:

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta^i dt)^2 \tag{144}$$

$$(\sqrt{\gamma_{ii}}dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + (\beta^i dt)^2 \tag{145}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i)^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + (\beta^i dt)^2 \tag{146}$$

$$ds^2 = -\alpha^2 dt^2 + (\beta^i dt)^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}(dx^i)^2 \tag{147}$$

$$ds^2 = (-\alpha^2 + [\beta^i]^2)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{148}$$

$$ds^2 = (-\alpha^2 + \beta^i\beta^i)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{149}$$

[12]see Appendix E on the original $3+1$ ADM formalism

Then the equations of the Natario warp drive in the parallel contravariant $3+1$ ADM formalism are given by:

$$ds^2 = (-\alpha^2 + \beta^i\beta^i)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{150}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta^i\beta^i & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & \gamma_{ii} \end{pmatrix} \tag{151}$$

The components of the inverse metric are given by the matrix inverse :[13]

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{152}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([-\alpha^2 + \beta^i\beta^i] \times \gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta^i \times \sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} \gamma_{ii} & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -\alpha^2 + \beta^i\beta^i \end{pmatrix} \tag{153}$$

Suppressing the lapse function $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta^i\beta^i)dt^2 + 2\sqrt{\gamma_{ii}}\beta^i dx^i dt + \gamma_{ii}dx^i dx^i \tag{154}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta^i\beta^i & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & \gamma_{ii} \end{pmatrix} \tag{155}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{156}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([-1 + \beta^i\beta^i] \times \gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta^i \times \sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} \gamma_{ii} & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -1 + \beta^i\beta^i \end{pmatrix} \tag{157}$$

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ we should expect for:

$$ds^2 = (1 - \beta^i\beta^i)dt^2 - 2\sqrt{\gamma_{ii}}\beta^i dx^i dt - \gamma_{ii}dx^i dx^i \tag{158}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta^i\beta^i & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -\gamma_{ii} \end{pmatrix} \tag{159}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{160}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta^i\beta^i] \times -\gamma_{ii}) - (-\sqrt{\gamma_{ii}}\beta^i \times -\sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & 1 - \beta^i\beta^i \end{pmatrix} \tag{161}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta^i\beta^i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta^i \times \sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & 1 - \beta^i\beta^i \end{pmatrix} \tag{162}$$

[13] see Wikipedia:the free Encyclopedia on inverse or invertible matrices

The equations of the Natario warp drive in the parallel contravariant $3+1$ ADM formalism given by:

$$ds^2 = (1 - \beta^i \beta^i)dt^2 - 2\sqrt{\gamma_{ii}}\beta^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{163}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta^i \beta^i & -\sqrt{\gamma_{ii}}\beta^i \\ -\sqrt{\gamma_{ii}}\beta^i & -\gamma_{ii} \end{pmatrix} \tag{164}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{165}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta^i \beta^i] \times -\gamma_{ii}) - (-\sqrt{\gamma_{ii}}\beta^i \times -\sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & 1 - \beta^i \beta^i \end{pmatrix} \tag{166}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta^i \beta^i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta^i \times \sqrt{\gamma_{ii}}\beta^i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta^i \\ \sqrt{\gamma_{ii}}\beta^i & 1 - \beta^i \beta^i \end{pmatrix} \tag{167}$$

obeys the generic equation of a warp drive in the parallel contravariant $3+1$ ADM formalism:

$$ds^2 = dt^2 - (\sqrt{\gamma_{ii}} dx^i + \beta^i dt)^2 \tag{168}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3} (dx^i - X^i dt)^2 \tag{169}$$

The Natario equation above gicen in oontravariant form is valid only in cartezian coordinates.For a generic coordinates system in contravariant form we must employ the equation given by the parallel contravariant $3+1$ ADM formalism as being:

$$ds^2 = dt^2 - \sum_{i=1}^{3} (\sqrt{\gamma_{ii}} dx^i - X^i dt)^2 \tag{170}$$

Note that $\beta^i = -X^i$ and $\beta^i \beta^i = X^i X^i$ with X^i being the Natario contravariant shift vectors. Hence we have:

$$ds^2 = (1 - X^i X^i)dt^2 + 2\sqrt{\gamma_{ii}} X^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{171}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X^i X^i & \sqrt{\gamma_{ii}} X^i \\ \sqrt{\gamma_{ii}} X^i & -\gamma_{ii} \end{pmatrix} \tag{172}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{173}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - X^i X^i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}} X^i \times \sqrt{\gamma_{ii}} X^i)} \begin{pmatrix} -\gamma_{ii} & -\sqrt{\gamma_{ii}} X^i \\ -\sqrt{\gamma_{ii}} X^i & 1 - X^i X^i \end{pmatrix} \tag{174}$$

For the equations of the Natario warp drive in the parallel contravariant $3+1$ ADM formalism:

$$ds^2 = (1 - X^i X^i)dt^2 + 2\sqrt{\gamma_{ii}}X^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{175}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X^i X^i & \sqrt{\gamma_{ii}}X^i \\ \sqrt{\gamma_{ii}}X^i & -\gamma_{ii} \end{pmatrix} \tag{176}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{177}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - X^i X^i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}X^i \times \sqrt{\gamma_{ii}}X^i)} \begin{pmatrix} -\gamma_{ii} & -\sqrt{\gamma_{ii}}X^i \\ -\sqrt{\gamma_{ii}}X^i & 1 - X^i X^i \end{pmatrix} \tag{178}$$

And looking to the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{179}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2]):

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{180}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{181}$$

But remember that $dl^2 = \gamma_{ij} dx^i dx^j = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$,$\gamma_{\theta\theta} = r^2$ $\sqrt{\gamma_{rr}} = 1$ $\sqrt{\gamma_{\theta\theta}} = r$ and $r = rs$.Then the equation of the Natario warp drive in the parallel contravariant $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X^i X^i)dt^2 + 2\sqrt{\gamma_{ii}}X^i dx^i dt - \gamma_{ii} dx^i dx^i \tag{182}$$

$$ds^2 = (1 - X^{rs}X^{rs} - X^{\theta}X^{\theta})dt^2 + 2(X^{rs} drs dt + X^{\theta} rs d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{183}$$

$$ds^2 = (1 - X^{rs}X^{rs} - X^{\theta}X^{\theta})dt^2 + 2(X^{rs} drs + X^{\theta} rs d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{184}$$

$$ds^2 = [1 - (X^{rs})^2 - (X^{\theta})^2]dt^2 + 2[X^{rs} drs + X^{\theta} rs d\theta]dt - drs^2 - rs^2 d\theta^2 \tag{185}$$

Note that the equation of the Natario vector nX(pg 2 and 5 in [2]) appears twice in the equation above due to the non-diagonalized shift components:

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{186}$$

As a matter of fact expanding the term

$$\sqrt{\gamma_{ii}}X^i dx^i = X^{rs} drs + X^{\theta} rs d\theta \tag{187}$$

we recover again the Natario vector since $\gamma_{rr} = 1$,$\gamma_{\theta\theta} = rs^2$ $\sqrt{\gamma_{rr}} = 1$ $\sqrt{\gamma_{\theta\theta}} = rs$

10 Appendix C:The Natario warp drive and the parallel covariant $3+1$ ADM Formalism

A $3+1$ ADM covariant formalism parallel to the original $3+1$ ADM formalism according with the equation (21.40) pg [507(b)] [534(a)] in [11]

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{188}$$

using the signature $(-,+,+,+)$ can be given by:

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)(\sqrt{\gamma_{jj}}dx^j + \beta_j dt) \tag{189}$$

Note that in the equation above all the essential 3 elements of the original $3+1$ ADM formalism are also present[14].These elements are:

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ that measures the proper distance between two points inside each hypersurface.In this case $dl = \sqrt{\gamma_{ij}dx^i dx^j}$.
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.
- 3)-the relative velocity β_i between Eulerian observers and the lines of constant spatial coordinates $(\sqrt{\gamma_{ii}}dx^i + \beta_i dt)$.$\beta_i$ is known as the covariant shift vector defined as : $\beta_i = \gamma_{ij}\beta^j$.

But since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ $dl = \sqrt{\gamma_{ii}}dx^i$ and we have for the $3+1$ spacetime metric the following result:

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{190}$$

$$(\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + (\beta_i dt)^2 \tag{191}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i)^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + (\beta_i dt)^2 \tag{192}$$

$$ds^2 = -\alpha^2 dt^2 + (\beta_i dt)^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}(dx^i)^2 \tag{193}$$

$$ds^2 = (-\alpha^2 + [\beta_i]^2)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{194}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{195}$$

[14]see Appendix E on the original $3+1$ ADM formalism

Then the equations of the Natario warp drive in the parallel covariant $3+1$ ADM formalism are given by:

$$ds^2 = (-\alpha^2 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{196}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta_i & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & \gamma_{ii} \end{pmatrix} \tag{197}$$

The components of the inverse metric are given by the matrix inverse :[15]

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{198}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([-\alpha^2 + \beta_i\beta_i] \times \gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta_i \times \sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} \gamma_{ii} & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\alpha^2 + \beta_i\beta_i \end{pmatrix} \tag{199}$$

Suppressing the lapse function $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i\beta_i)dt^2 + 2\sqrt{\gamma_{ii}}\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{200}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta_i & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & \gamma_{ii} \end{pmatrix} \tag{201}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{202}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([-1 + \beta_i\beta_i] \times \gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta_i \times \sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} \gamma_{ii} & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -1 + \beta_i\beta_i \end{pmatrix} \tag{203}$$

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ we should expect for:

$$ds^2 = (1 - \beta_i\beta_i)dt^2 - 2\sqrt{\gamma_{ii}}\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{204}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta_i & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\gamma_{ii} \end{pmatrix} \tag{205}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{206}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta_i\beta_i] \times -\gamma_{ii}) - (-\sqrt{\gamma_{ii}}\beta_i \times -\sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & 1 - \beta_i\beta_i \end{pmatrix} \tag{207}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - \beta_i\beta_i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta_i \times \sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & 1 - \beta_i\beta_i \end{pmatrix} \tag{208}$$

[15]see Wikipedia:the free Encyclopedia on inverse or invertible matrices

The equations of the Natario warp drive in the parallel covariant $3+1$ ADM formalism given by:

$$ds^2 = (1-\beta_i\beta_i)dt^2 - 2\sqrt{\gamma_{ii}}\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{209}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1-\beta_i\beta_i & -\sqrt{\gamma_{ii}}\beta_i \\ -\sqrt{\gamma_{ii}}\beta_i & -\gamma_{ii} \end{pmatrix} \tag{210}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{211}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1-\beta_i\beta_i] \times -\gamma_{ii}) - (-\sqrt{\gamma_{ii}}\beta_i \times -\sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & 1-\beta_i\beta_i \end{pmatrix} \tag{212}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1-\beta_i\beta_i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}\beta_i \times \sqrt{\gamma_{ii}}\beta_i)} \begin{pmatrix} -\gamma_{ii} & \sqrt{\gamma_{ii}}\beta_i \\ \sqrt{\gamma_{ii}}\beta_i & 1-\beta_i\beta_i \end{pmatrix} \tag{213}$$

obeys the generic equation of a warp drive in the parallel covariant $3+1$ ADM formalism:

$$ds^2 = dt^2 - (\sqrt{\gamma_{ii}}dx^i + \beta_i dt)^2 \tag{214}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{215}$$

The Natario equation above given in contravariant form is valid only in cartezian coordinates.For a generic coordinates system in covariant form we must employ the equation given by the parallel covariant $3+1$ ADM formalism as being:

$$ds^2 = dt^2 - \sum_{i=1}^{3}(\sqrt{\gamma_{ii}}dx^i - X_i dt)^2 \tag{216}$$

with $X_i = \gamma_{ii}X^i$

Note that $\beta_i = -X_i$ and $\beta_i\beta_i = X_iX_i$ with X_i being the covariant Natario shift vectors. Hence we have:

$$ds^2 = (1-X_iX_i)dt^2 + 2\sqrt{\gamma_{ii}}X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{217}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1-X_iX_i & \sqrt{\gamma_{ii}}X_i \\ \sqrt{\gamma_{ii}}X_i & -\gamma_{ii} \end{pmatrix} \tag{218}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{219}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1-X_iX_i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}}X_i \times \sqrt{\gamma_{ii}}X_i)} \begin{pmatrix} -\gamma_{ii} & -\sqrt{\gamma_{ii}}X_i \\ -\sqrt{\gamma_{ii}}X_i & 1-X_iX_i \end{pmatrix} \tag{220}$$

For the equations of the Natario warp drive in the parallel covariant $3+1$ ADM formalism:

$$ds^2 = (1 - X_i X_i)dt^2 + 2\sqrt{\gamma_{ii}} X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{221}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X_i & \sqrt{\gamma_{ii}} X_i \\ \sqrt{\gamma_{ii}} X_i & -\gamma_{ii} \end{pmatrix} \tag{222}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{(g_{00} \times g_{ii}) - (g_{i0} \times g_{0i})} \begin{pmatrix} g_{ii} & -g_{0i} \\ -g_{i0} & g_{00} \end{pmatrix} \tag{223}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \frac{1}{([1 - X_i X_i] \times -\gamma_{ii}) - (\sqrt{\gamma_{ii}} X_i \times \sqrt{\gamma_{ii}} X_i)} \begin{pmatrix} -\gamma_{ii} & -\sqrt{\gamma_{ii}} X_i \\ -\sqrt{\gamma_{ii}} X_i & 1 - X_i X_i \end{pmatrix} \tag{224}$$

And looking to the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{225}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2]):

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{226}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{227}$$

But remember that $dl^2 = \gamma_{ij} dx^i dx^j = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$,$\gamma_{\theta\theta} = r^2$ $\sqrt{\gamma_{rr}} = 1$ $\sqrt{\gamma_{\theta\theta}} = r$ and $r = rs$.Then the covariant shift vector components X_{rs} and X_{θ} with $r = rs$ are given by:

$$X_i = \gamma_{ii} X^i \tag{228}$$

$$X_r = \gamma_{rr} X^r = X_{rs} = \gamma_{rsrs} X^{rs} = 2v_s n(rs) \cos\theta = X^r = X^{rs} \tag{229}$$

$$X_{\theta} = \gamma_{\theta\theta} X^{\theta} = rs^2 X^{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{230}$$

It is possible to construct a covariant form for the Natario vector nX defined as $n_c X$ as follows:

$$n_c X = X_{rs} drs + X_{\theta} rs d\theta \tag{231}$$

With the covariant shift vector components X_{rs} and X_{θ} defined as shown above:

The equation of the Natario warp drive in the parallel covariant $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X_i)dt^2 + 2\sqrt{\gamma_{ii}} X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{232}$$

$$ds^2 = (1 - X_{rs}X_{rs} - X_\theta X_\theta)dt^2 + 2(X_{rs}drsdt + X_\theta rsd\theta dt) - drs^2 - rs^2 d\theta^2 \tag{233}$$

$$ds^2 = (1 - X_{rs}X_{rs} - X_\theta X_\theta)dt^2 + 2(X_{rs}drs + X_\theta rsd\theta)dt - drs^2 - rs^2 d\theta^2 \tag{234}$$

$$ds^2 = [1 - (X_{rs})^2 - (X_\theta)^2]dt^2 + 2[X_{rs}drs + X_\theta rsd\theta]dt - drs^2 - rs^2 d\theta^2 \tag{235}$$

Note that the equation of the covariant Natario vector $n_c X$ appears twice in the equation above due to the non-diagonalized shift components:

$$n_c X = X_{rs}drs + X_\theta rsd\theta \tag{236}$$

As a matter of fact expanding the term

$$\sqrt{\gamma_{ii}} X_i dx^i = X_{rs}drs + X_\theta rsd\theta \tag{237}$$

we recover again the covariant form of the Natario vector since $\gamma_{rr} = 1$,$\gamma_{\theta\theta} = rs^2$ $\sqrt{\gamma_{rr}} = 1$ $\sqrt{\gamma_{\theta\theta}} = rs$

11 Appendix D:The Natario warp drive negative energy density in Cartezian coordinates

The negative energy density according to Natario is given by(see pg 5 in [2])[16]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\cos^2\theta + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2\sin^2\theta\right] \tag{238}$$

In the bottom of pg 4 in [2] Natario defined the x-axis as the polar axis.In the top of page 5 we can see that $x = rs\cos(\theta)$ implying in $\cos(\theta) = \frac{x}{rs}$ and in $\sin(\theta) = \frac{y}{rs}$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2(\frac{x}{rs})^2 + \left(n'(rs) + \frac{r}{2}n''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{239}$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2 + z^2] = 0$ and $rs^2 = [(x - xs)^2]$ and making $xs = 0$ the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $rs^2 = x^2$ because in the equatorial plane $y = z = 0$.

Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(n'(rs))^2\right] \tag{240}$$

[16] $n(rs)$ is the Natario shape function.Equation written in the Geometrized System of Units $c = G = 1$

12 Appendix E:mathematical demonstration of the Natario warp drive equation for a constant speed vs in the original $3+1$ ADM Formalism according to MTW and Alcubierre

General Relativity describes the gravitational field in a fully covariant way using the geometrical line element of a given generic spacetime metric $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ where do not exists a clear difference between space and time.This generical form of the equations using tensor algebra is useful for differential geometry where we can handle the spacetime metric tensor $g_{\mu\nu}$ in a way that keeps both space and time integrated in the same mathematical entity (the metric tensor) and all the mathematical operations do not distinguish space from time under the context of tensor algebra handling mathematically space and time exactly in the same way.

However there are situations in which we need to recover the difference between space and time as for example the evolution in time of an astrophysical system given its initial conditions.

The $3+1$ ADM formalism allows ourselves to separate from the generic equation $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ of a given spacetime the 3 dimensions of space and the time dimension.(see pg [64(b)] [79(a)] in [12])

Consider a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg [65(b)] [80(a)] in [12].

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg [65(b)] [80(a)] in [12])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg [66(b)] [81(a)] in [12])
(see also fig 21.2 pg [506(b)] [533(a)] in [11] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg [507(b)] [534(a)] in [11])[17]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface

- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.

- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

[17] we adopt the Alcubierre notation here

Combining the eqs (21.40),(21.42) and (21.44) pgs $[507, 508(b)]$ $[534, 535(a)]$ in [11] with the eqs (2.2.5) and (2.2.6) pgs $[67(b)]$ $[82(a)]$ in [12] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{241}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{242}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{pmatrix} \tag{243}$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{244}$$

But since $dl^2 = \gamma_{ij} dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii} dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{245}$$

$$(dx^i + \beta^i dt)^2 = (dx^i)^2 + 2\beta^i dx^i dt + (\beta^i dt)^2 \tag{246}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\gamma_{ii}\beta^i dx^i dt + \gamma_{ii}(\beta^i dt)^2 \tag{247}$$

$$\beta_i = \gamma_{ii}\beta^i \tag{248}$$

$$\gamma_{ii}(\beta^i dt)^2 = \gamma_{ii}\beta^i \beta^i dt^2 = \beta_i \beta^i dt^2 \tag{249}$$

$$(dx^i)^2 = dx^i dx^i \tag{250}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i \beta^i dt^2 \tag{251}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i \beta^i dt^2 \tag{252}$$

$$ds^2 = (-\alpha^2 + \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{253}$$

Note that the expression above is exactly the eq (2.2.4) pgs $[67(b)]$ $[82(a)]$ in [12].It also appears as eq 1 pg 3 in [1].

With the original equations of the $3+1$ ADM formalism given below:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{254}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{255}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ii} - \frac{\beta^i\beta^i}{\alpha^2} \end{pmatrix} \tag{256}$$

and suppressing the lapse function making $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{257}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{258}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -1 & \beta^i \\ \beta^i & \gamma^{ii} - \beta^i\beta^i \end{pmatrix} \tag{259}$$

changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-1 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{260}$$

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{261}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{262}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{263}$$

Remember that the equations given above corresponds to the generic warp drive metric given below:

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{264}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [2])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{265}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{266}$$

Comparing all these equations

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{267}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{268}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{269}$$

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{270}$$

With

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{271}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i\beta^i = X_iX^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have:

$$ds^2 = (1 - X_iX^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{272}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_iX^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{273}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^iX^i \end{pmatrix} \tag{274}$$

Looking to the equation of the Natario vector nX(pg 2 and 5 in [2]):

$$nX = X^{rs} drs + X^{\theta} rs d\theta \tag{275}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [2]):

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{276}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{277}$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_{rs} and X_{θ} with $r = rs$ are given by:

$$X_i = \gamma_{ii} X^i \tag{278}$$

$$X_r = \gamma_{rr} X^r = X_{rs} = \gamma_{rsrs} X^{rs} = 2v_s n(rs)\cos\theta = X^r = X^{rs} \tag{279}$$

$$X_{\theta} = \gamma_{\theta\theta} X^{\theta} = rs^2 X^{\theta} = -rs^2 v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{280}$$

The equations of the Natario warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{281}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{282}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{283}$$

The matrix components 2×2 evaluated separately for rs and θ gives the following results:[18]

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0r} \\ g_{r0} & g_{rr} \end{pmatrix} = \begin{pmatrix} 1 - X_r X^r & X_r \\ X_r & -\gamma_{rr} \end{pmatrix} \tag{284}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0r} \\ g^{r0} & g^{rr} \end{pmatrix} = \begin{pmatrix} 1 & X^r \\ X^r & -\gamma^{rr} + X^r X^r \end{pmatrix} \tag{285}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0\theta} \\ g_{\theta 0} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 - X_\theta X^\theta & X_\theta \\ X_\theta & -\gamma_{\theta\theta} \end{pmatrix} \tag{286}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0\theta} \\ g^{\theta 0} & g^{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & X^\theta \\ X^\theta & -\gamma^{\theta\theta} + X^\theta X^\theta \end{pmatrix} \tag{287}$$

Then the equation of the Natario warp drive spacetime in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{288}$$

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drsdt + X_\theta d\theta dt) - drs^2 - rs^2 d\theta^2 \tag{289}$$

$$ds^2 = (1 - X_{rs} X^{rs} - X_\theta X^\theta)dt^2 + 2(X_{rs} drs + X_\theta d\theta)dt - drs^2 - rs^2 d\theta^2 \tag{290}$$

[18]Actually we know that the real matrix is a 3×3 matrix with dimensions t rs and θ.Our 2×2 approach is a simplification

13 Remarks

References [11],[12],[13] and [14] are standard textbooks used to study General Relativity and these books are available or in paper editions or in electronic editions all in Adobe PDF Acrobat Reader.

We have the electronic editions of all these books

In order to make easy the reference cross-check of pages or equations specially for the readers of the paper version of the books we adopt the following convention:when we refer for example the pages $[507, 508(b)]$ or the pages $[534, 535(a)]$ in [11] the (b) stands for the number of the pages in the paper edition while the (a) stands for the number of the same pages in the electronic edition displayed in the bottom line of the Adobe PDF Acrobat Reader

14 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[19]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[20][21]

[19] special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[20] "Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[21] appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

References

[1] *Alcubierre M., (*1994*). Classical and Quantum Gravity. 11 L73-L77,arXiv gr-qc/0009013*

[2] *Natario J.,(*2002*). Classical and Quantum Gravity. 19 1157-1166,arXiv gr-qc/0110086*

[3] *Introduction to Differential Forms,Arapura D.,*2010

[4] *Teaching Electromagnetic Field Theory Using Differential Forms,Warnick K.F. Selfridge R. H.,Arnold D. V.,IEEE Transactions On Education Vol* 40 *Num* 1 *Feb* 1997

[5] *Loup F.,(*2012*).,HAL-00711861*

[6] *Loup F.,(*2013*).,HAL-00852077*

[7] *Loup F.,(*2013*).,HAL-00879381*

[8] *Loup F.,(*2014*).,HAL-00937933*

[9] *Loup F.,(*2014*).,HAL-01021924*

[10] *Loup F.,(*2014*).,HAL-01075682*

[11] *Misner C.W.,Thorne K.S.,Wheeler J.A,(Gravitation) (W.H.Freeman* 1973*)*

[12] *Alcubierre M.,(Introduction to* $3+1$ *Numerical Relativity) (Oxford University Press* 2008*)*

[13] *Schutz B.F.,(A First Course in General Relativity . Second Edition) (Cambridge University Press* 2009*)*

[14] *Hartle J.B.,(Gravity:An Introduction to Einstein General Relativity) (Pearson Education Inc. and Addison Wesley* 2003*)*

[15] *Loup F.,(*2015*).,HAL-01101134*

.