# Supersymmetrization of Quaternionic Quantum Mechanics

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**Abstract**

Keeping in view the application of SUSY and quaternion quantum mechanics, in this paper we have made an attempt to develop a complete theory for quaternionic quantum mechanics. We have discussed the N = 1, N = 2 and N = 4 supersymmetry in terms of one, two and four supercharges respectively and it has been shown that N=4 SUSY is the quaternionic extension of N = 2 complex SUSY.

Key Words: Supersymmetry, quaternion, quantum mechanics.

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## 1 INTRODUCTION:

Symmetries are one of the most powerful tools in the theoretical physics. Quaternions were discovered by Hamilton[1] in 1843 as an illustration of group structure and also applied to mechanics in three-dimensional space. Quaternions have the same properties as complex numbers with the difference that the commutative law is not valid in their case. It gave the importance to quaternions in terms of their possibility to understand the fundamental laws of physics. However, the relativistic quantum mechanics which is described as the theory of quantum mechanics consistent with the Einstein's theory of special relativity, gave the new birth to quantum field theory along with its further consequences towards the study of unification of fundemental interactions and search of new particles in high energy physics. In order to understand the fundamental laws of physics and the role of quaternions therein, quaternionic quantum mechanics has been extensively studied by Adler[2], while other authors [3, 4] revealed out the noble features of quaternionic quantum mechanics. Due to the non-commutativity of quaternions, the subject has not been considered widely instead of its advance algebraic structure. On the other hand, supersymmetric quantum mechanics [5] has been approved by quantum field theory as an application of SUSY super algebra to quantum mechanics. Accordingly, the attempts were made [6, 7, 8, 9, 10, 11, 12, 13] to analyse the supersymmetric quantum mechanics in an elegent way but the attention was taken particularly on one-dimensional SUSY quantum mechanics. Despite of the fact that supersymmetry is broken symmetry and supersymmetric quantum mechanics can explain these facts better, the attempts made earlier to generalize SUSY quantum mechanics into higher dimensions are lacking towards the consistent studies in a systematic manner. Although, Das et.al.[3] has made an attemt to describe the higher dimensional SUSY quantum mechanics in many contexts while Cooper et.al. [6] reviewed the theoretical formulation of quantum mechanics and discussed many problems therein. Supersymmetric quantum mechanics involves pairs of Hamiltonians which are called partner Hamiltonians and accordingly the potential energy terms occur in Hamiltonians are then called partner potentials. Accordingly, for every eigenstate of one Hamiltonian in partner Hamiltonians has a corresponding eigenstate with the same energy (except possible for zero energy eigenstates). Each boson would have a fermionic partner of eigen energy but in relativistic world energy and mass are interchangeable, so one can say that partner particles have equal mass. Supersymmetry has been applied to non-quantum statistical mechanics shown that even if the original inspiration of high-energy particle physics turns out to be a blind key, its investigation has brought almost many useful benefits . So, it is fruitful to use supersymmetric methods in quaternionic quantum mechanics. Attempts are made in this direction by Adler [1] and Davies [14] to study supersymmetric quaternionic quantum mechanics and are extensively studied recently by Rawat et al [15, 16, 17]. Keeping in view the utilities of SUSY and quaternion quantum mechanics, in this paper, we have made an attempt to revisit the quaternionic formulation of supersymmetric quantum mechanics for different (namely N=1, N=2 and N=4) dimensions. The N = 1 SUSY has been discussed in terms of Pauli Hamiltonian for a spin $\frac{1}{2}$ particle in an external magnetic field where we have obtained the self-adjoint supercharge and Dirac Hamiltonian in electromagnetic field for N = 1 supersymmetric quantum mechanics. The relation between Dirac Hamiltonian and Pauli Hamiltonian has also been established. It is shown that N = 2 SUSY consists an additional spin $\frac{1}{2}$ degrees of freedom [11]. Accordingly, the complex valued supercharges and corresponding Hamiltonian are discussed and the relation between them is established. It has been shown that the SUSY will be a good supersymmetry unless and until we impose the necessary condition showing that ground state energy must be vanishing. We have also discussed the Witten operator for N = 2 SUSY and it is shown that the operator satisfies the algebra of SUSY. It has also been shown that parity operator has positive and negative eigenstates for superpartner Hamiltonians. Positive eigenstate corresponds to $\hat{H}_{+}$ and negative eigenstate corresponds to $\hat{H}_{-}$. So Hilbert space is decomposed in two eigen space of $\hat{H}$ and corresponding SUSY transformations are obtained. We have extended N=2 SUSY to reformulate N = 4 SUSY quantum mechanics by extending the complex number to quaternion units and constructed the SUSY generators in the same manner as discussed by Hull[12]. Further more, it is concluded that quaternion quantum mechanics has the advantages over complex quantum mechanics and accordingly, the higher dimensional supersymmetry[3] could be extensively reorganized and explained better in quaternion quantum mechanics.

## 2 N=1, 2, 4 Supersymmetric Quantum Mechanics:

Let us analyze a quantum system, which is characterized by a Hamiltonian$\hat{H}$ acting on some Hilbert space$H$ and we postulate the existence of N self-adjoint operators $\hat{Q}_i = \hat{Q}_i^{\dagger}(i = 1, 2, 3, .....)$, which acts on $H$. Such quantum system is called supersymmetric if the following anticommutation relation is valid for $(i = 1, 2, 3, .....)$ i.e

$$\{Q_i,\, Q_j\} \quad = \hat{H} \quad \delta_{ij}, \quad (i = 1, 2, 3, .....). \tag{1}$$

The self-adjoint operators are called supercharges and the Hamiltonian is said to be SUSY Hamiltonian.

### 2.1 N=1 Supersymmetric Quantum Mechanics:

One-dimensional SUSY quantum mechanics is described by the graded algebra [3, 5, 6, 7, 8] and can be expressed as

$$< \Psi|H|\Psi > \quad = \quad < \Psi|\hat{Q}\hat{Q}^{\dagger}|\Psi > \; + \; < \Psi|\hat{Q}^{\dagger}\hat{Q}|\Psi >= |\hat{Q}|\Psi > |^2 + |\hat{Q}^{\dagger}|\Psi > |^2. \tag{2}$$

We may extend N=1 SUSY to the case of relativistic quantum mechanics where a system [18] is defined by a Pauli Hamiltonian for a spin 1/2 particle in an external magnetic field. Let us consider two quaternion gauge potentials $\vec{A}_{\mu}(x\,,\,t)$ and $\vec{B}_{\mu}(x\,,\,t)$ and two external quaternion gauge magnetic field given as

$$\vec{C} \quad = \quad \nabla \times \vec{A}_{\mu}(x\,,\,t) \;\; and \; \vec{C}' \; = \vec{\nabla} \vec{\times} \vec{B}_{\mu}(x\,,\,t) \tag{3}$$

where $\vec{A}_{\mu}(x\,,\,t)$ and $\vec{B}_{\mu}(x\,,\,t)$ are quaternion potentials, defined as

$$\vec{A}_{\mu}(x\,,\,t) \quad = e_l A_l \quad and \;\; \vec{B}_{\mu}(x\,,\,t) = e_l B_l \; (\forall\, l = 1, 2, 3). \tag{4}$$

Here $e_l$ are quaternion units which satisfy the quaternion multiplication rule [2, 16]. So, we may introduce [16] the self-adjoint supercharge in electromagnetic field system as

$$\hat{Q}_D \quad = \quad ie_l(p_l - ieA_l + ieB_l) = \hat{Q}_D^{\dagger} \tag{5}$$

Thus we may write the Pauli Hamiltonian as

$$\hat{H}_p = 2\hat{Q}_D^2 = 2i\left\{e_l\left(P_l - ie\vec{A} + ie\vec{B}\right)\right\}^2 \tag{6}$$

which is described as the Pauli Hamiltonian for spin- 1/2 particles. Accordingly, we may write Dirac Hamiltonian as

$$\hat{H}_D \quad = \textstyle\sum_{l=1}^{3} \alpha_l \quad \left(P_l - ie\vec{A} + ie\vec{B}\right) \; + \beta_m$$

$$\hat{H}_D = \begin{bmatrix} m & ie_l\left(P_l - ie\vec{A} + ie\vec{B}\right) \\ ie_l\left(P_l - ie\vec{A} + ie\vec{B}\right) & -m \end{bmatrix} = \begin{bmatrix} m & \hat{Q}_D^{\dagger} \\ \hat{Q}_D & -m \end{bmatrix} \tag{7}$$

where we have replaced Dirac $\alpha_l(\forall l = 1, 2, 3)$ and$\beta$ matrices with quaternions as

$$\alpha_l = \begin{bmatrix} 0 & ie_l \\ 0 & 0 \end{bmatrix} (\forall\, l = 1,2,3) \;\; \& \;\; \beta = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{8}$$

Squaring the Dirac Hamiltonian, we get

$$\begin{aligned} \hat{H}_D^2 &= \begin{bmatrix} \hat{Q}_D\hat{Q}_D^\dagger + m^2 & 0 \\ 0 & \hat{Q}_D\hat{Q}_D^\dagger + m^2 \end{bmatrix} \\ &= \begin{bmatrix} Q_D^2 + m^2 & 0 \\ 0 & Q_D^2 + m^2 \end{bmatrix} = \begin{bmatrix} \frac{\hat{H}_p}{2} + m^2 & 0 \\ 0 & \frac{\hat{H}_p}{2} + m^2 \end{bmatrix} \end{aligned} \tag{9}$$

Here we have assumed $m = \frac{1}{2}$ as initial value. Hence we may replace $m^2 = \frac{1}{4}$ and then get

$$\hat{H}_D^2 = \begin{bmatrix} \frac{\hat{H}_p}{2} + \frac{1}{4} & 0 \\ 0 & \frac{\hat{H}_p}{2} + \frac{1}{4} \end{bmatrix}$$

We also get

$$\left[\hat{Q}_s, \hat{H}_D\right] = 0 \;,\; \left[\hat{Q}, \hat{H}_p\right] = 0 \;, \left[\hat{Q}_D, \hat{Q}_D^{+}\right] = \hat{H}_p \tag{10}$$

## 2.2 N=2 Supersymmetric Quantum Mechanics:

N = 2 SUSY was discussed by Witten [5] as a simple model for SUSY quantum mechanics.. It consists of Cartesian degree of freedom which caries an additional spin $-1/2$ degree of freedom so that we may write two supercharges in terms of quaternion units $e_1$and $e_2$ in the following manner

$$\begin{aligned} \hat{Q}_1 &= \frac{1}{\sqrt{2}}\left(p \otimes ie_1 + \phi(x) \otimes ie_2\right); \\ \hat{Q}_2 &= \frac{1}{\sqrt{2}}\left(p \otimes ie_2 - \phi(x) \otimes ie_1\right). \end{aligned} \tag{11}$$

Then the Hamiltonian is given by

$$\hat{H} = 2\hat{Q}_1^2 = 2\hat{Q}_2^2 = p^2 + \phi^2 + \phi^{'}\sigma_3 \tag{12}$$

which may further be written as

$$\hat{H} = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix} = \begin{bmatrix} p^2 + \phi^2 + \phi^{'} & 0 \\ 0 & p^2 + \phi^2 - \phi^{'} \end{bmatrix} \tag{13}$$

For N = 2 SUSY there exists two complex supercharges $\hat{Q}_1$, $\hat{Q}_2$ and Hamiltonian $\widehat{H}$ which satisfy the following relations i.e

$$\begin{aligned} \hat{Q}_1\hat{Q}_2 &= -\hat{Q}_2\hat{Q}_1 \\ \widehat{H} &= 2\hat{Q}_1^2 = 2\hat{Q}_2^2 = \hat{Q}_1^2 + \hat{Q}_2^2 \end{aligned} \tag{14}$$

Let us define the two complex supercharges as

$$\hat{Q} = \frac{1}{\sqrt{2}} \left(\hat{Q}_1 + i\hat{Q}_2\right) \;\&\; \hat{Q}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{Q}_1 - i\hat{Q}_2\right) \tag{15}$$

where i is complex quantity and belongs to c(1,i) space. These supercharges and Hamiltonian H satisfy the SUSY algebra,b

$$\hat{Q}^2 = \left(\hat{Q}^{\dagger}\right)^2 = 0 \;\; and \quad \hat{H} \;= \left\{\hat{Q}\,,\,\hat{Q}^{\dagger}\right\}$$

Now to satisfy the relation $\hat{Q}_1^2 + \hat{Q}_2^{\dagger 2} = 0$ , the complex supercharges are given by nilpotent matrices, which are defined as follows

$$\hat{Q} \;=\; \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad \hat{Q}^{\dagger} = \begin{bmatrix} 0 & 0 \\ a^{\dagger} & 0 \end{bmatrix} \tag{16}$$

Hence the Hamiltonian becomes

$$\hat{H} = \left\{\hat{Q}\,,\,\hat{Q}^{\dagger}\right\} \;=\; \begin{bmatrix} aa^{\dagger} & 0 \\ 0 & a^{\dagger}a \end{bmatrix} \tag{17}$$

where $a$ and $a^{\dagger}$ a are defined as

$$a \;= \tfrac{1}{\sqrt{2\omega}} \quad (-ip + \omega q) \;\; \& \; a^{\dagger} = \frac{1}{\sqrt{2\omega}}\,(ip + \omega q) \tag{18}$$

Substituting $2\omega = 1$ and $\dfrac{\omega q}{2} = U(x)$, ($U(x)$ is real super potential), we can write e$a$ and $a^{\dagger}$ in following manner as

$$a \;=\; (-ip + U(x)) = -\frac{d}{dx} + U \;\&\; a^{\dagger} = (ip + U(x)) = \frac{d}{dx} + U \tag{19}$$

The supercharges for this case is given by follows

$$\hat{Q} \;=\; \begin{bmatrix} 0 & -\frac{d}{dx} + U \\ 0 & 0 \end{bmatrix} \quad \hat{Q}^{\dagger} = \begin{bmatrix} 0 & 0 \\ \frac{d}{dx} + U & 0 \end{bmatrix} \tag{20}$$

and the Hamiltonian is given by

$$\hat{H} = \begin{bmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{bmatrix} = \begin{bmatrix} -\frac{d^2}{dx^2} - U^{'} + U^2 & 0 \\ 0 & -\frac{d^2}{dx^2} + U^{'} + U^2 \end{bmatrix} \tag{21}$$

Here $\hat{H}, \hat{Q}\, and\, \hat{Q}^{\dagger}$ satisfy the SUSY algebra given by

$$\left[\hat{H}, \hat{Q}\right] = \left[\hat{H}, \hat{Q}^{\dagger}\right] = 0\;, \;\; \left\{\hat{Q}, \hat{Q}\right\} = \left\{\hat{Q}^{\dagger}, \hat{Q}^{\dagger}\right\} = 0 \; and \; \hat{H} = \left\{\hat{Q}, \hat{Q}^{\dagger}\right\} \tag{22}$$

We consider$\psi$ as a two-component spinor given by

$$\psi \;=\; \begin{bmatrix} \psi_a \\ \psi_b \end{bmatrix} \tag{23}$$

And the necessary condition for SUSY to be a good supersymmetry that supercharges annihilate the ground state

$$\hat{Q} \mid \Psi > \quad \hat{Q}^{\dagger} \mid \Psi > \;\; = 0 \tag{24}$$

In terms of energy, the condition for SUSY to be a good SUSY is that the ground state energy should be zero. Using equations (20 ) and (23) we get

$$-\Psi_{b}^{'}\;\;+U\Psi_{b}\;\;=0\;\;and\;\;\Psi_{a}^{'}+U\Psi_{a}=0 \tag{25}$$

where $U$ is given by

$$U\;\;=\;\;\frac{\Psi_{b}^{'}}{\Psi_{b}}=-\frac{\Psi_{a}^{'}}{\Psi_{a}}\;;\;\Psi_{b}=exp\left(-\int_{x_{0}}^{x}\vec{U}(s)d\vec{s}\right)\;\;and\;\;\Psi_{a}=exp\left(\int_{x_{0}}^{x}\vec{U}(s)d\vec{s}\right) \tag{26}$$

Let us describe Witten operator for N=2 SUSY. For N = 2 SUSY Witten's operator exp is a self adjoint operator and is defined as

$$\hat{W}\;\;=\;\;\begin{bmatrix} I & 0\\ 0 & -I \end{bmatrix} \tag{27}$$

It satisfies super algebra

$$\left[\hat{W},\hat{H}\right]=0\;,\;\;\left\{\hat{W},\hat{Q}\right\}=\left\{\hat{W},\hat{Q^{\dagger}}\right\}=0\;\hat{W}^{2}=1 \tag{28}$$

which shows that Witten operator is an unitary operator. Now we can define parity operator from Witten operator as follows

$$\hat{P^{\pm}}\;\;=\tfrac{1}{2}\;\;\left(I\pm\hat{W}\right),\;\;\hat{P^{+}}=\frac{1}{2}\left(I+\hat{W}\right)=\begin{bmatrix} I & 0\\ 0 & 0 \end{bmatrix}\;,\;\hat{P^{-}}=\frac{1}{2}\left(I-\hat{W}\right)=\begin{bmatrix} 0 & 0\\ 0 & I \end{bmatrix} \tag{29}$$

where $\hat{P^{+}}$corresponds to positive eigenvalue of $\hat{H}$ *i.e.* $\hat{H}_{+}$ and $\hat{P^{-}}$corresponds to negative eigenvalue of $\hat{H}$ *i.e.* $\hat{H}_{-}$. So Hilbert space is decomposed in two eigen space of and can be written as

$$\hat{H}\;\;=\;\;\hat{H_{+}}\oplus\hat{H_{-}} \tag{30}$$

Let us define the quaternion wave function as a two component complex spinors given by

$$\psi\;\;=\;\;\begin{bmatrix} \psi_{a}\\ \psi_{b} \end{bmatrix}=\begin{bmatrix} \psi^{+}\\ \psi^{-} \end{bmatrix} \tag{31}$$

where $\psi^{+}$and $\psi^{-}$are again two component spinors corresponding to upper and lower component of Dirac spinor in the following manner

$$\psi^{+}=\;\;\Psi_{0}+e_{1}\Psi_{1}\;\;=\begin{bmatrix} \phi^{+}\\ 0 \end{bmatrix},\psi^{-}=\Psi_{2}-e_{1}\Psi_{3}=\begin{bmatrix} 0\\ \phi^{-} \end{bmatrix} \tag{32}$$

Using equations (29) and (32) we get

$$P^{\pm}\psi^{\pm}\;\;=\;\;\psi^{\pm} \tag{33}$$

### 2.3 N=4 Supersymmetric Quantum Mechanics:

According to Hull[12], N = 4 SUSY QM can be formed from N = 2 SUSY QM by extending the complex number $i$ to three imaginary units $i$, $j$, $k$ . which can be described as quaternion units $e_{1},e_{2},e_{3}$ . Thus N = 4 SUSY QM can be obtained by replacing $i$ by $e$ in equation (18) and (19).

Then we get as

$$a \quad = \quad (-e.p+U) \quad \& \; a^{\dagger}=\left(e.p+U^{\dagger}\right) \tag{34}$$

Here we consider $U$ as the quaternionic super potential defined by

$$\hat{U}=\sum_{j=1}^{3}e_{j}w_{j}= \quad e_{1}w_{1}+e_{2}w_{2}+e_{3}w_{3} \;; \qquad \hat{U}=-\sum_{j=1}^{3}e_{j}w_{j}=-e_{1}w_{1}-e_{2}w_{2}-e_{3}w_{3} \tag{35}$$

Hence the supercharges for this case are deduced as

$$\hat{Q} \quad = \quad \begin{bmatrix} 0 & (-e.p+U) \\ 0 & 0 \end{bmatrix} \quad \hat{Q}^{\dagger}=\begin{bmatrix} 0 & 0 \\ (-e.p+U) & 0 \end{bmatrix} \tag{36}$$

and the Hamiltonian becomes

$$\hat{H} \quad = \quad \begin{bmatrix} (e.p+U)\left(-e.p+U^{\dagger}\right) & 0 \\ 0 & \left(-e.p+U^{\dagger}\right)(-e.p+U) \end{bmatrix} \tag{37}$$

and t and satisfy the usual SUSY algebra given by equation(18). Equation (36) reduces to the following expression of supercharges on using the value of $U$ from equation (35) i.e. as

$$\hat{Q} \quad = \quad \begin{bmatrix} 0 & e_{j}\left(-p_{j}+w_{j}\right) \\ 0 & 0 \end{bmatrix} \quad \& \quad \hat{Q}^{\dagger}=\begin{bmatrix} 0 & 0 \\ e_{j}\left(-p_{j}+w_{j}\right) & 0 \end{bmatrix} \tag{38}$$

where we have used $a=e_{j}\left(-p_{j}+w_{j}\right)$. If we substitute quaternion basisd $e_{j}$. elements by $e_{j}=\sigma$, then $a=\sigma\left(-p_{j}+w_{j}\right)$ , where $\sigma$ is Pauli spin matrix, showing that use of quaternion automatically suggests spin structure. Hence spin naturally occurs in quaternionic quantum mechanics, which is not possible in N $=2$ supersymmetric quantum mechanics. Replacing $\sigma\rightarrow e_{j}$ , we get following representation for Dirac matrices as follows

$$\gamma_{j} \quad = \quad \begin{bmatrix} 0 & e_{j} \\ e_{j} & 0 \end{bmatrix} \tag{39}$$

where

$$\gamma_{j}^{\dagger} \quad = - \quad \gamma_{j} \;, \; tr\gamma_{j}=0 \;\; \& \;\; \gamma_{j}\gamma_{k}+\gamma_{k}\gamma_{j}=-2\delta_{jk} \tag{40}$$

These are the matrices, which has been used by Rotelli [18] in formulating quaternionic Dirac equation.

# 3 Disscussion:

The N $=1$ SUSY has been discussed in terms of Pauli Hamiltonian for a spin $\frac{1}{2}$ particle in an external magnetic field. The self-adjoint supercharges and Dirac Hamiltonian in electromagnetic field for N $=1$ supersymmetric quantum mechanics are obtained. The relation between Dirac Hamiltonian and Pauli Hamiltonian has also been established. It has been shown that N $=2$ SUSY consists an additional spin $\frac{1}{2}$ degrees of freedom and accordingly supercharges and corresponding Hamiltonian are discussed. The complex supercharges and Hamiltonian is also discussed and the relation between them has been established. It has been shown that the SUSY will be a good supersymmetry unless and until we impose the necessary condition showing that ground state energy must be vanishing. We have described the Witten operator for N = 2 SUSY and shown that the operator satisfy the algebra of SUSY. It has been shown that parity operator has positive and negative eigenstates for superpartner Hamiltonians. Positive eigenstate corresponds to $\hat{H}_+$ and negative eigenstate corresponds to $\hat{H}_-$ . So Hilbert space is decomposed in two eigen space of $\hat{H}$ and corresponding SUSY transformations are obtained. We have also developed N = 4 SUSY quantum mechanics from N = 2 SUSY quantum mechanics by extending the complex number to quaternion units and constructed the SUSY generators in the same manner as discussed by Hull[12]. It has been concluded that quaternion quantum mechanics has the advantages over complex quantum mechanics and accordingly, the higher dimensional supersymmetry[2] could explained better in quaternion quantum mechanics.

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