Fractal Geometry a Possible Explanation to the Accelerating Growth Rate of Trees

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Abstract

In a recent publication it was discovered trees growth rate accelerates with age. Trees are described as being clear examples of natural fractals. Do fractals offer insight to the accelerating expansion?

In this investigation the classical (Koch snowflake) fractal was inverted to model the growth of a fractals seen from a fixed – new growth – perspective. New triangle area sizes represented new branch volume; these new triangles were held constant allowing earlier triangles in the set to expand as the fractal set iterated (grew) through time.

Velocities and accelerations were calculated for both the area of the total fractal, and the distance between points within the fractal set using classical kinematic equations.

It was discovered that the area(s) of earlier triangles expanded exponentially, and as a consequence the total snowflake area grew exponentially. Distances between points (nodes) – from any location within the fractal set – receded away at exponentially increasing velocities and accelerations. For trees, if the new growth branch volume size remains constant through time, its supporting branches volumes will grow exponentially to support their mass. This property of fractals may account for the accelerating volumetric growth rates of trees. A trees age can be measured not only by its annual (growth ring) age, but also by its iteration age: the amount of iterations from trunk to new growth branch.

Thought the findings have obvious relevance to the study of trees directly, they may also offer insight into the recently discovered observation of the accelerating growth rate of the universe.

Key Words: Accelerating growth rate, L- systems, fractals, trees, systems biology, mathematical biology, plant morphology

1 INTRODUCTION

In a recent publication[1] trees were found to be growing at an accelerating rate. The study measured up to 80 years of tree growth, on more than 600,000 trees, over 6 continents and found that the growth of 97 percent of the trees was accelerating with age. This accelerated growth rate with time is a mystery to biologists.

This publication is developed from of my previous publication[2] where I experimented and proposed the accelerating expansion (the dark energy) of the universe may be due to a property of fractal geometry.

A tree's growth is generally described as being of a 'natural' fractal geometry (or L system [3]). Is the accelerating growth rate of the tree, and of trees in general, as a result of a general property of fractal geometry? Do fractals expand at accelerating rates?

For this to so, the fractal will have to demonstrate accelerating expansion (section 4.2).

Tree plant growth models appear to agree with the fractal branching structure of trees[4],[5],[6], but do not appear to pick up on the exponential expansion of volume with growth; indeed, until this accelerating growth rate discovery, trees were assumed to grow at a decreasing rate with age, or with an S shape [7] like growth pattern.

This investigation was an applied mathematics experiment, analysing of the growth behaviour of the fractal attractor.

To model and measure the tree fractal growth properties, the Koch snowflake fractal (figure 1 B below) was chosen for its quantitative regularity. The Koch snowflake was inverted, and areas recorded as the fractal iterated. Measurements were taken as from a fixed reference, perspective or position within the iterating set – simulating the new growth branch of the tree.

1.1 The Classical Fractal

Fractals are described as emergent objects from iteration, possessing regular irregularity (same but different) at all scales, and is classically demonstrated by the original Mandelbrot Set (Figure 1 A below).

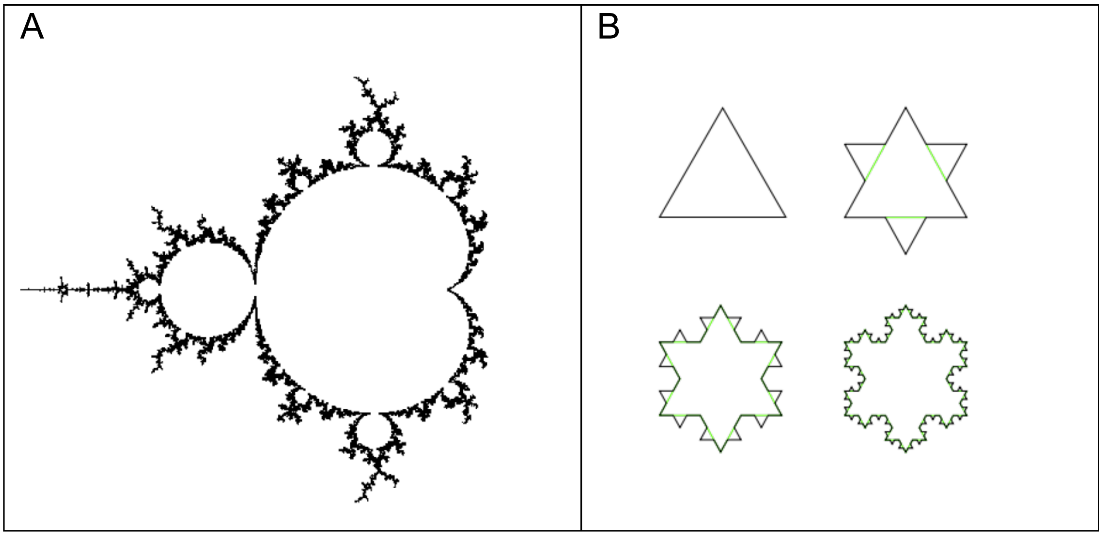

Figure 1. (Classical) Fractals: (A) boundary of the Mandelbrot set; (B) The Koch Snowflake fractal from iteration 0 to 3; (C) fractal tree after 6 iterations; (D) fully developed fractal tree with changing branch size. Reference: (A) [8]; (B) [9].

The classical fractal shape – as demonstrated in the Koch Snowflake – emerges as a result of the iteration of a simple rule: the repeating the process of adding triangles in the case of the Koch Snowflake. The complete emergent structure is at shape equilibrium (where no more detail can be observed – with additional iterations – to an observer of fixed position) at or around four to seven iterations. Trees also demonstrate this equilibrium – the number of nodes from trunk to new growth branch.

1.2 Fractal Tree Growth

As with the triangles forming a snowflake, iterated branches will form an emergent tree (figure 2A and 2B). Follow the first (new growth) stem size – keeping this stem/branch size at a constant size – as the rest of the tree grows. To grow more branches, the volume of the earlier/older branches must expand. Now think of sitting on one the branches of a tree that is infinitely large, infinitely growing. What would you see in front? What would you see behind?

If an observer were to remain at this constant static position (or alternatively change position by zooming forward into the structure) they would experience – according to the principles of the iterating fractal, as demonstrated in Figure 2 (A) – an infinity of self-similar Koch Snowflake like structure would be seen ahead of them, at never see triangles more than four or five iteration/sizes. There will always be (classical) fractal shape ahead, and looking back the observer would see expansion.

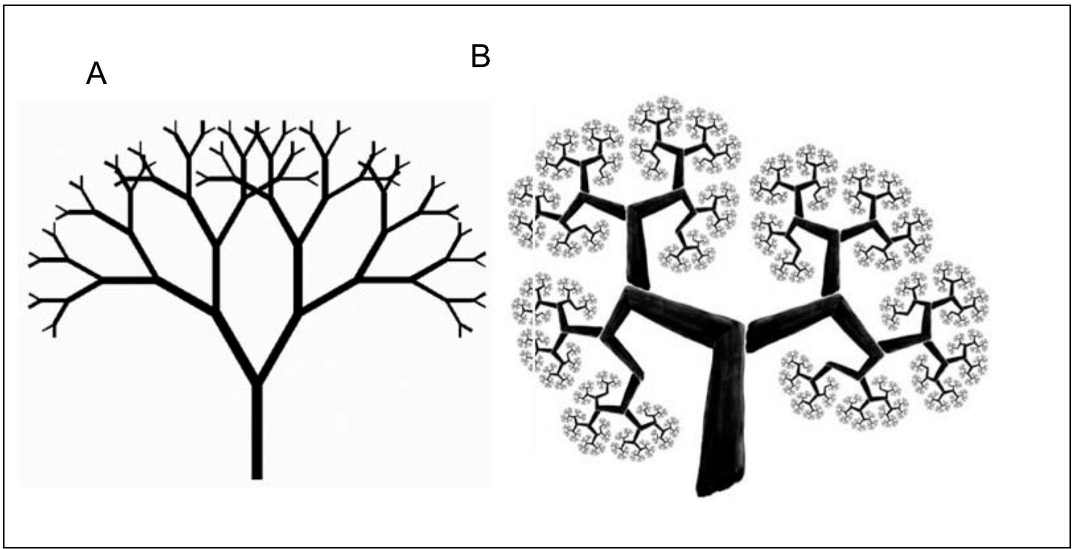

Figure 2. (Classical) Tree Fractals: (A) fractal tree after 6 iterations; (B) fully developed fractal tree with changing branch size. Reference: [10].

1.3 Fractspansion – The Fractal Viewed From Within

To simulate observations from a position or perspective within the fractal set, the fractal was (simply) inverted. By doing this, the focus is placed on the newly added triangle (branch of the tree), holding its size constant, and allowing the previous triangle sizes to expand – rather than diminish as with the classic fractal. The inverted fractal reveals this fractal expansion – termed fractspansion as demonstrated in Figure 2 (B). Colours (red, blue, black followed by purple) and numbers are used to demonstrate the expansion.

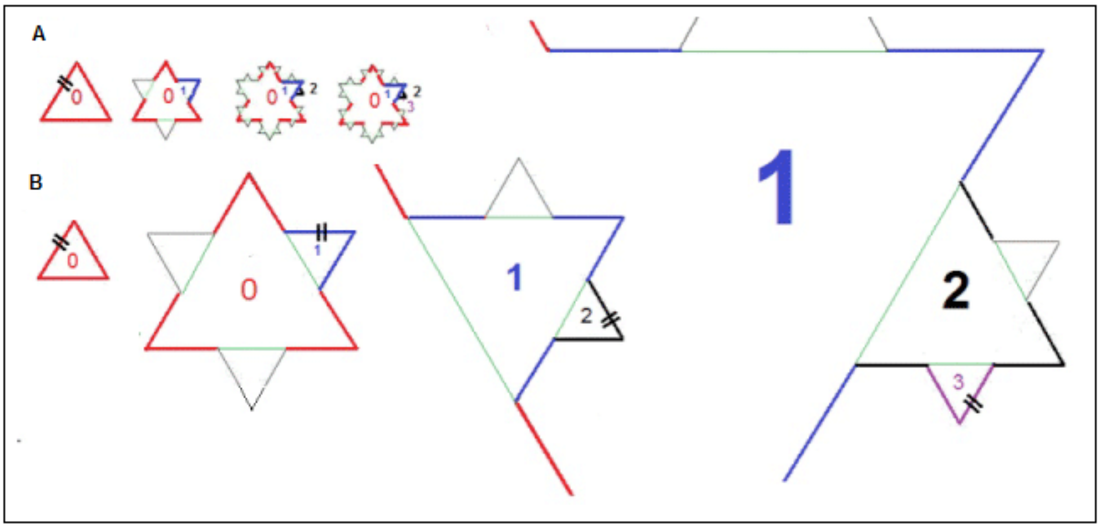

Figure 3. Expansion of the inverted Koch Snowflake fractal (fractspansion): The schematics above demonstrate fractal development by (A) the classical snowflake perspective, where the standard sized thatched (iteration '0') is the focus, and the following triangles diminish in size from colour red iteration 0 to colour purple iteration 3; and (B) the inverted, fractspanding perspective where the new (thatched) triangle is the focus and held at standard size while the original red iteration 0 triangle expands in area – as the fractal iterates.

The size of the initial red iteration 0 triangle, with fractspansion, expands relative to the new.

2 METHODS

To create a quantitative data series for analysis of the inverted fractal, the classical Koch Snowflake area equations were adapted to account for this perspective, and a spreadsheet model [11] was developed to trace area expansion with iteration.

2.1 Iteration Time

For the purposes of this investigation the iteration count was assumed to be equal to time, called: iteration time, and denoted ***t***.

2.2 Area (volume)

The scope of this investigation was limited to the two-dimensional; three-dimensional tree volume and mass can be inferred from this initial assumption. Changes in the areas of triangles, and distances between points in the fractal set were measured and analysed to determine whether the fractal area and distance between points expand.

A data table was produced (Table 1) to calculate the area growth at each, and every iteration of a single triangle. Area was calculated from the following formula (1) measured in standard (arbitrary) units (***u***)

$$A = \frac{l^2\sqrt{3}}{4} \tag{1}$$

where (***A***) is the area of a single triangle, and where *l* is the triangle's base length. *l* was placed in Table 1 and was set to 1.51967128766173 ***u*** so that the area of the first triangle (t_0) approximated an arbitrary area of 1 u^2. To expand the triangle with iteration the base length was multiplied by a factor of 3. The iteration number was placed in a column, followed by the base length of the equilateral triangle, and in the final column the formula to calculate the area of the triangle. Calculations were made to the 10th iteration, and the results graphed.

2.3 Distance and Displacement

To measure and analyse the changes in position of points (the distance between points in the set after iteration) a second data table (table 2) was developed on the spreadsheet. The triangle's geometric centre points were chosen as the points to measure. Formula (2) below calculated the inscribed radius of an equilateral triangle. Distance between points was calculated by adding the inscribed radius of the first triangle (t_0) to the inscribed radius of the next expanded triangle (t_1) described by

$$r = \frac{\sqrt{3}}{6} l \,. \tag{2}$$

From the radius distance measurements; displacement, displacement expansion ratio, velocity, acceleration, and expansion acceleration ratio for each and every iteration time were calculated using classical mechanics equations.

The change in distance between points was recorded, as was the change in displacement (distance from t_0).

2.4 Area Expansion of the Total Inverted Fractal

With iteration, new triangles are (in discrete quantities) introduced into the set – at an exponential rate. While the areas of new triangles remain constant, the earlier triangles expand, and by this the total fractal set expands. To calculate the area change of a total inverted fractal (as it iterated), the area of the single triangle (at each iteration time) was multiplied by its corresponding quantity of triangles (at each iteration time).

Two data tables (tables 3 and 4 in the spreadsheet file) were developed. Table 3 columns were filled with the calculated triangle areas at each of the corresponding iteration time – beginning with the birth of the triangle and continuing to iteration ten.

Table 4 triangle areas of table 3 were multiplied by the number of triangles in the series corresponding with their iteration time.

Values calculated in table 3 and 4 were totalled and analysed in a new table (table 5). Analysed were: total area expansion per iteration, expansion ratio, expansion velocity, expansion acceleration, and expansion acceleration ratio. Calculations in the columns used kinematic equations developed below.

2.5 Kinematics

Classical physics equations were used to calculate velocity and acceleration of: the receding points (table 2) and the increasing area (table 5).

$$v = \frac{\Delta d}{\Delta t} \qquad (3)$$

2.5.1 Velocity

Velocity ($\boldsymbol{v}$) was calculated by the following equation

where classical time was exchanged for iteration time ($\boldsymbol{t}$). Velocity is measured in standard units per iteration. $u^{-1}\,t^{-1}$ for receding points and $u^{-2}\,t^{-1}$ for increasing area.

2.5.2 Acceleration

Acceleration ($\boldsymbol{a}$) was calculated by the following equation

$$a = \frac{\Delta v}{\Delta t}. \qquad (4)$$

Acceleration is measured in standard units per iteration $u^{-1}\,t^{-2}$ and $u^{-2}\,t^{-2}$.

3 RESULTS

Figures 3 to 6 show graphically the results of the experiment.

3.1 Expansion

The area of the initial triangle of the inverted Koch Snowflake fractal increased exponentially – shown below in Figure 3.

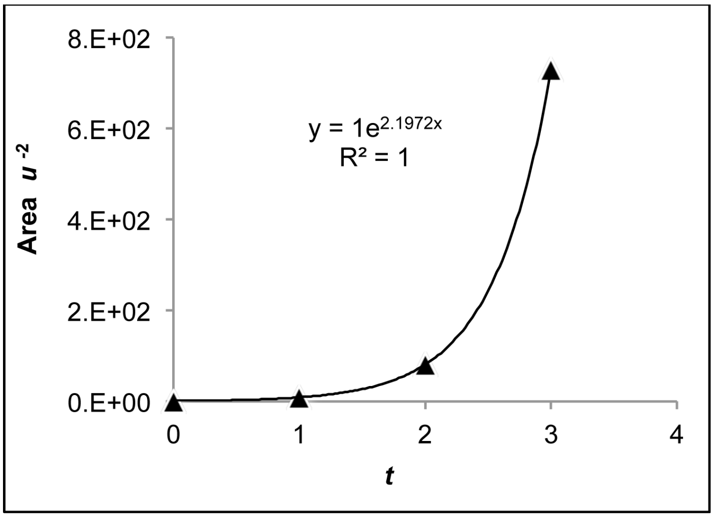

Figure 3. Area Expansion of a single triangle in the inverted Koch Snowflake fractal by iteration time (t). u = arbitrary length.

This expansion with respect to iteration time is written as

$$A = 1e^{2.1972t}. \qquad (5)$$

The area of the total fractal (Figure 4A) and the distance between points (Figure 4B) of the inverted fractal also expanded exponentially.

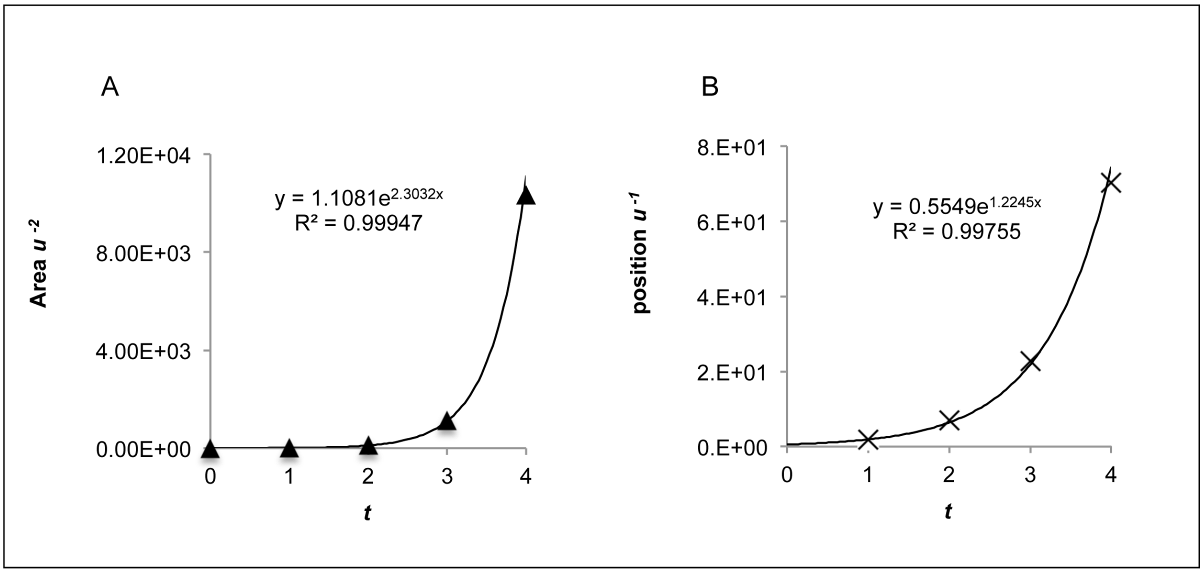

Figure 4. Inverted Koch Snowflake fractal expansion per iteration time (t). (A) total area expansion and (B) distance between points. u = arbitrary length.

The expansion of area is described as

$$A^T = 1.1081e^{2.3032t} \quad (6)$$

where A^T is the total area.

The expansion of distance between points is described by the equation

$$D = 0.5549e^{1.2245t} \quad (7)$$

where D is the distance between points.

3.2 Velocity

The (recession) velocities for both total area and distance between points (Figures 6A and 6B respectively) increased exponentially per iteration time.

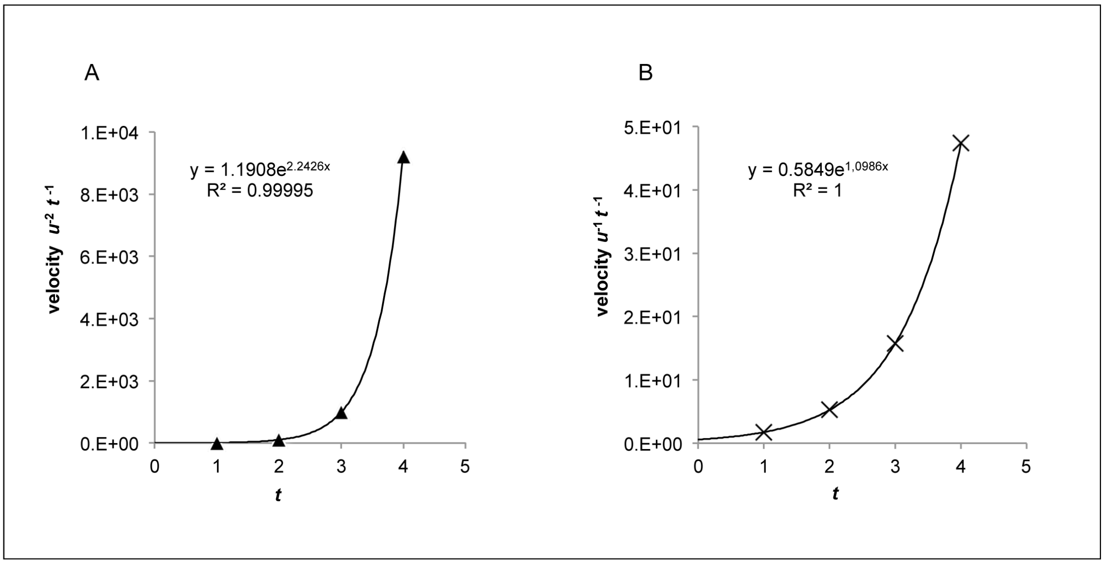

Figure 5. Inverted Koch Snowflake fractal (expansion) velocity. Expansion velocity of the inverted fractal at each corresponding iteration time (t): (A) expansion of total area, and (B) distance between points. u = arbitrary length.

Velocity is described by the following equations respectively

$$v = 1.1908e^{2.3032t} \tag{8}$$

$$v^T = 0.5549e^{1.0986t} \tag{9}$$

where $\boldsymbol{v^T}$ is the (recession) velocity of the total area; and $\boldsymbol{v}$ the (recession) velocity of distance between points.

3.3 Acceleration of Area and Distance Between points

The growth accelerations for both total area and distance between points (Figure 6A and 6B respectively) increased exponentially per iteration time.

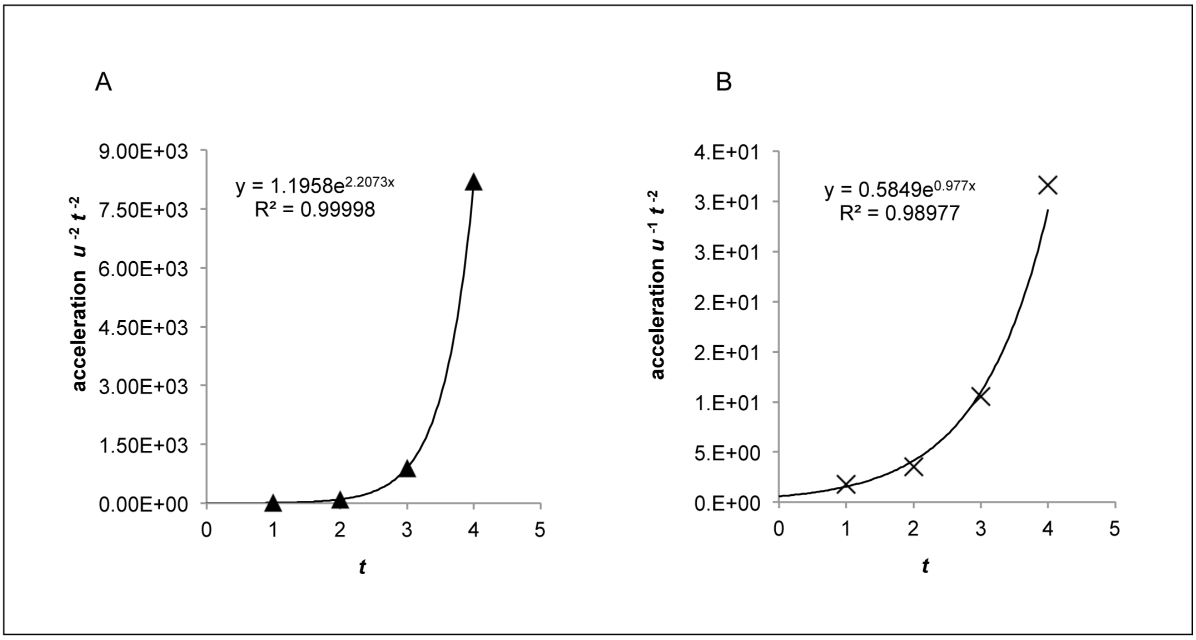

Figure 6. Inverted Koch Snowflake fractal (expansion) acceleration. Acceleration of the inverted fractal at each corresponding iteration time (t): (A) expansion of total area, and (B) distance between points. u = arbitrary length.

Acceleration is described by the following equations respectively

$$a^T = 1.1958e^{2.2073t} \tag{10}$$

$$a = 0.5849e^{0.977t} \tag{11}$$

where $\boldsymbol{a^T}$ is the (recession) acceleration of the total area; $\boldsymbol{a}$ the (recession) acceleration of distance between points.

4 DISCUSSIONS

4.1 Trunk Growth

The single inverted triangle expansion (Figure 3) demonstrates the expansion of the trunk of the tree over time. Its area begins arbitrary small (it could be set to any size value, one akin to a tree seedling or new growth branch), and is followed by exponential area expansion as (iteration) time passes. The acceleration between points (nodes on the tree) with respect to time (from equation 5) is described as

$$a = a_0 e^{F\lambda t} \tag{12}$$

where the constant ***F* λ** may be interpreted as a fractal 'Growth Constant' with respect to point acceleration and iteration time.

4.2 Accelerating Tree Growth

If the productive leafy stem of the emergent tree becomes the focus of the tree growth, and held constant in size – just as with the standard triangle size is to the fractspanding Koch snowflake – then the older branches and the load bearing trunk of the tree will grow exponentially with iteration time. This is to say: the tree grows in terms of iteration time, and not solar time. As trees grow they lay down tree rings, these rings do not show exponential growth. Trees can generally – by counting the tree rings – age several hundreds of years old, but in terms of fractal age, may only be some 4 to 7 iteration times old. One can imagine that more iteration times would result in an exponentially growing, exponentially large base trunk.

With entry (or birth) of new triangles (branches) into the fractal set the total tree volume (Figure 6 above) growths exponentially. The total area expansion with respect to time is described by the function

$$A^T = A_0 e^{F\Lambda t} \tag{13}$$

where ***F*Λ** is a fractal constant with respect to total area expansion and time.

While results from this investigation point immediately to plant growth modelling, owing to the scale invariant universality of the fractal, the findings are relevant to all things fractal, able to be observed or experienced, in principle, throughout.

5 CONCLUSIONS

This investigation it was found the inverted iterating Koch snowflake fractal expands exponentially, while points between triangles recede away both with exponential velocity and acceleration. This expansion, revealed by the (unrealistic) regular, Koch snowflake – termed fractspansion – is a property unique to fractals, and is a property shared in all (irregular) fractal objects. Fractspansion addresses and demonstrates and models problems directly associated with recently discovered accelerating tree growth rates.

Trees grow by fractal branching, iteration by iteration. The annual growth rings a measure of growth per solar year and have to do with growth, they have nothing to do with the acceleration observed in almost all trees. The amount of iterations is the fractal age of the tree. The more the amount of iteration, branch nodes, the older the tree. Fractals tend to around 7 plus or minus 2 iterations.

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