An intuitive conceptualization of n! and its application to derive a well known result

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Abstract: n! is defined as the product 1.2.3………n and it popularly represents the number of ways of seating n people on n chairs. We conceptualize another way of describing n! using sequential cuts to an imaginary circle and derive the following well known result

(n!-1) =1.1!+2.2!+3.3!+4.4!+………..+(n-1){(n-1)!}

Results:

n! is defined as the product of n consecutive positive integers from 1 to n, each considered exactly once. It has been popularly described as the number of ways in which n people can be seated in n chairs (1).

We use another way to visualize the factorial function.

n!=1.2.3.4………..(n-2)(n-1)(n) = 2.3.4……….(n-2)(n-1)(n)

We can consider cutting a circle into n! pieces as a sequential step wise process.

Let us consider an intact circle.

In the first step let us cut it at two positions. This will give rise to 2 pieces.
(In this step we have used 2 cuts and end up with 2 arcs or pieces of the circle. Note that until this step we have used a total of 2 cuts).

In the second step let us cut each of the 2 pieces/arcs from the first step at (3-1)=2 positions. In the second step we use 2.(3-1)cuts and we end up with 2.3 pieces. Until now we have used 2+2.(3-1) cuts to end up with 2.3 pieces or arcs.

In the third step let us cut each 2.3 pieces from the second step at (4-1)=3 positions. In the third step we use 2.3.(4-1) cuts and we end up with 2.3.4 pieces. Until now we have used 2+2.(3-1)+2.3.(4-1) cuts to end up with 2.3.4 pieces or arcs.

In the fourth step let us cut each of the 2.3.4 pieces from the third step at (5-1)=4 positions. In the fourth step we use 2.3.4.(5-1) cuts and we end up with 2.3.4.5 pieces. Until now we have used 2+2.(3-1)+2.3.(4-1)+2.3.4.(5-1) cuts and end up with 2.3.4.5 pieces or arcs.

Proceeding in this manner……..

In the (n-1)th step we will cut each of the 2.3.4………(n-2).(n-1) pieces from previous step at (n-1) positions. Thus in the (n-1)th step we use 2.3.4….(n-2).(n-1).(n-1) cuts and we end up with 2.3.4………(n-2)(n-1)(n) pieces. Until now we have used

2+2.(3-1)+2.3.(4-1)+ 2.3.4.(5-1)+…………+2.3.4……(n-2).(n-1).(n-1) cuts to end up with 2.3.4………..(n-2).(n-1).(n) pieces or arcs.

Since a total of x-cuts to circle would result in x-pieces or arcs
therefore

2+2.(3-1)+2.3.(4-1)+ 2.3.4.(5-1)+….+2.3.4……(n-2).(n-1).(n-1)=

2.3.4…..(n-2).(n-1).(n)

Simplifying

2+2.2+2.3.3+2.3.4.4+…..+2.3.4….(n-2)(n-1)(n-1)=n!

2+2.2!+3.3!+4.4!+……..+(n-1){(n-1)!}=n!

or

1+1.1!+2.2!+…………+(n-1).{(n-1)!}=n!

or

(n!-1)=1.1!+2.2!+…………+(n-1).{(n-1)!}

The final equation is a well known result.

Reference:

1. The Factorial function and Generalization
   (Manjul Bhargava)
   The American Mathematical Monthly, Vol. 107, No. 9 (Nov., 2000), pp. 783-799