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March 23, 2015

Consider:

$$D_{h,x}f = a_1 f + a_2 f^2$$

Let

$$f = \Omega u$$

Then:

$$D_{h,x}f = D_{h,x}u\Omega + uD_{h,x}\Omega + hD_{h,x}uD_{h,x}\Omega$$

Thus we have:

$$D_{h,x}u\Omega + uD_{h,x}\Omega + hD_{h,x}uD_{h,x}\Omega = \mathrm{a}_1\Omega\mathrm{u} + \mathrm{a}_2\Omega^2\mathrm{u}^2$$

If we let:

$$\frac{a_1\Omega - D_{h,x}\Omega}{\Omega + hD_{h,x}\Omega} = \kappa$$

For a constant $\kappa$.

This yields

$$D_{h,x}\Omega - a_1\Omega = k\Omega + khD_{h,x}\Omega$$

Giving us:

$$0 = (k + a_1)\Omega + (kh - 1)D_{h,x}\Omega$$

Which is a linear Difference equation we can solve for Ω.

Note that the function in equation

$$D_{h,x}f = a_1 f + a_2 f^2$$

After finding a suitable Ω can be reduced to

$$D_{h,x}u = \kappa u + b(x)u^2$$

Meaning that

$$D_{h,x}u = u(b(x)u + \kappa)$$

Now suppose we assume:

$$u = (1+h)^{\frac{q}{h}} \rightarrow D_{h,x}u = (1+h)^{\frac{q}{h}}\left(\frac{(1+h)^{Dq}}{h} - \frac{1}{h}\right)$$

If we can resolve:

$$\frac{(1+h)^{D_{h,x}q}}{h} - \frac{1}{h} = b(x)u + \kappa$$

We're done. But some care is needed since if kh = 1 we get a problem.

$$0 = (k + a_1)\Omega + (kh - 1)D_{h,x}\Omega$$

Then declare that

$$-\frac{1}{h} = \kappa$$

Thus we are left with the question. Find q if:

$$\frac{(1+h)^{D_{h,x}q}}{h} = b(x)u \rightarrow \frac{(1+h)^{D_{h,x}q}}{h} = b(x)(1+h)^{\frac{q}{h}}$$

This gives us

$$(1+h)^{D_{h,x}q} = hb(x)(1+h)^{\frac{q}{h}} \rightarrow$$

$$D_{h,x}q = \log_{1+h} hb(x) + \frac{q}{h} \rightarrow$$

$$D_{h,x}q - \frac{1}{h}q = \log_{1+h} hb(x)$$

We solve this by the technique of coupled integration factors. We declare:

$$\begin{Bmatrix} \lambda_2 \\ \lambda_1 \end{Bmatrix}$$

Such that:

$$\lambda_1 + hD_{h,x}\lambda_1 = \lambda_2$$
$$D_{h,x}\lambda_1 = -\frac{1}{h}\lambda_2$$

From which it follows

$$\begin{Bmatrix} \lambda_1 = 2\lambda_2 \\ D_{h,x}\lambda_1 = -\frac{1}{2h}\lambda_1 \end{Bmatrix}$$

The latter yields us a solution assuming exponential ansatz:

$$\lambda_1 = (1+h)^{\frac{m(x)}{h}} \to D_{h,x}\lambda_1 = \lambda_1 \frac{(1+h)^{D_{h,x}m} - 1}{h} = -\frac{1}{2h}\lambda_1$$

Thus:

$$\frac{(1+h)^{D_{h,x}m} - 1}{h} = -\frac{1}{2h} \to (1+h)^{D_{h,x}m} = \frac{1}{2} \to m = \log_{1+h}\left(\frac{1}{2}\right) x$$

Thus:

$$\lambda_1 = \left(\frac{1}{2}\right)^{\frac{x}{h}} \to 2^{-\frac{x}{h}}, \lambda_2 = \frac{1}{2}2^{-\frac{x}{h}}$$

From here it follows:

$$\left(\frac{1}{2}\right)^{\frac{x}{h}} q = D_{h,x}^{-1}\left[2^{-\frac{x+h}{h}} \log_{1+h} hb(x)\right]$$

And therefore

$$q = 2^{\frac{x}{h}} D_{h,x}^{-1}\left[2^{-\frac{x+h}{h}} \log_{1+h} hb(x)\right]$$

And from here we find that:

$$u = (1+h)^{\frac{q}{h}} \to u = (1+h)^{\frac{1}{h}2^{\frac{x}{h}} D_{h,x}^{-1}\left[2^{-\frac{x+h}{h}} \log_{1+h} hb(x)\right]}$$

We now solve the Omega equation which was:

$$0 = (k + a_1)\Omega + (kh - 1)D_{h,x}\Omega$$

Meaning:

$$0 = \left(-\frac{1}{h} + a_1\right)\Omega + (-1 - 1)D_{h,x}\Omega$$

Which breaks down to

$$0 = \left(a_1 - \frac{1}{h}\right)\Omega - 2D_{h,x}\Omega$$

We can solve this like any classical difference equation by resolving terms:

$$0 = -\frac{1}{2}\left(a_1 - \frac{1}{h}\right)\Omega + D_{h,x}\Omega$$

Now we use the technique of dual-integration factors. We declare that

$$D_{h,x}(\lambda_1\Omega) = -\frac{1}{2}\left(a_1 - \frac{1}{h}\right)\Omega\lambda_2 + D_{h,x}\Omega\lambda_2$$

From which it arises that

$$D_{h,x}\lambda_1\Omega + \lambda_1 D_{h,x}\Omega + hD_{h,x}\lambda_1 D_{h,x}\Omega = -\frac{1}{2}\left(a_1 - \frac{1}{h}\right)\Omega\lambda_2 + D_{h,x}\Omega\lambda_2$$

Thus we create the system

$$\left\{\begin{matrix} \lambda_1 + hD_{h,x}\lambda_1 = \lambda_2 \\ D_{h,x}\lambda_1 = -\frac{1}{2}\left(a_1 - \frac{1}{h}\right)\lambda_2 \end{matrix}\right\}$$

From here it becomes clear that

$$\lambda_1 - h\frac{1}{2}\left(a_1 - \frac{1}{h}\right)\lambda_2 = \lambda_2$$

Which yields

$$\left(1 + \frac{h}{2}\left(a_1 - \frac{1}{h}\right)\right)\lambda_2 = \lambda_1 \rightarrow \lambda_1 = \left(1 + \frac{a_1 h^2 - h}{2h}\right)\lambda_2$$

And yet

$$D_{h,x}\lambda_1 = -\frac{1}{2}\left(a_1 - \frac{1}{h}\right)\lambda_2$$

Therefore

$$D_{h,x}\lambda_1 = -\frac{\frac{1}{2}\left(a_1 - \frac{1}{h}\right)}{\left(1 + \frac{a_1 h^2 - h}{2h}\right)}\lambda_1$$

To avoid ballooning our formula we simplify here and then solve this as a classical example of the exponential.

$$D_{h,x}\lambda_1 = -\frac{\frac{(a_1h-1)}{2h}}{\left(\frac{2h+a_1h^2-h}{2h}\right)}\lambda_1 \to D_{h,x}\lambda_1 = \frac{a_1h-1}{a_1h^2+h}\lambda_1$$

Now consider the function

$$\lambda_1(x) = (1+h)^{\frac{y}{h}}$$

For a function y. Then it trivially follows that

$$D_{h,x}\lambda_1 = \lambda_1\left(\frac{(1+h)^{D_{h,x}y}-1}{h}\right) = \frac{a_1h-1}{a_1h^2+h}\lambda_1$$

And therefore:

$$y = D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1h}{a_1h+1}\right)\right]$$

We thus recall that:

$$0 = -\frac{1}{2}\left(a_1 - \frac{1}{h}\right)\Omega + D_{h,x}\Omega$$

Multiply both sides by the given $\lambda_2$ that can be derived from our $\lambda_1$ and then integrate to conclude

$$C_1 = \Omega(1+\mathrm{h})^{\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1h}{a_1h+1}\right)\right]}$$

And therefore

$$C_1(1+\mathrm{h})^{-\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1h}{a_1h+1}\right)\right]} = \Omega$$

Now recall the original:

$$D_{h,x}f = a_1f + a_2f^2$$

We made the substitution:

$$f = \Omega u$$

To yield:

$$D_{h,x}u = -\kappa u + \frac{a_2}{\Omega + D_{h,x}\Omega}u^2$$

We note that this yields

$$D_{h,x}u = -\frac{1}{h}u + b_0u^2$$

Whereas:

$$b_0 = \frac{a_2}{C_1(1+\mathrm{h})^{-\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1h}{a_1h+1}\right)\right]} + C_1(1+\mathrm{h})^{-\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1h}{a_1h+1}\right)\right]}\left(\frac{1-a_1h}{2a_1h^2}\right)}$$

Which (due to the generality of the constant C) gives us

$$b_0 = \frac{C_1(1+\mathrm{h})^{\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1h}{a_1h+1}\right)\right]}a_2}{1+\frac{1-a_1h}{2a_1h^2}} \rightarrow b_0 = \frac{a_1a_2C_1(1+\mathrm{h})^{\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1h}{a_1h+1}\right)\right]}}{2a_1h^2 - a_1h + 1}$$

Note that we already have that :

$$u = (1+h)^{\frac{1}{h}2^{\frac{x}{h}}D_{h,x}^{-1}\left[2^{-\frac{x+h}{h}}\log_{1+h} hb(x)\right]}$$

Therefore:

$$u = (1+h)^{\frac{1}{h}2^{\frac{x}{h}}D_{h,x}^{-1}\left[2^{-\frac{x+h}{h}}\log_{1+h}\left(\frac{a_1a_2C_1(1+\mathrm{h})^{\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1h}{a_1h+1}\right)\right]}}{2a_1h^2-a_1h+1}\right)\right]}$$

And therefore since:

$$f = \Omega u$$

We have

$$f = C_1(1+\mathrm{h})^{-\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1h}{a_1h+1}\right)\right]}(1+h)^{\frac{1}{h}2^{\frac{x}{h}}D_{h,x}^{-1}\left[2^{-\frac{x+h}{h}}\log_{1+h}\left(\frac{a_1a_2C_1(1+\mathrm{h})^{\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1h}{a_1h+1}\right)\right]}}{2a_1h^2-a_1h+1}\right)\right]}$$

As the general solution to the constant-depressed: Riccati like equation. We now consider the radical formula:

$$D_{h,x}f = a_1f + a_2f^2$$

$$0 = -f(x+h) + (1+ha_1)f + ha_2f^2$$

$$f = -\frac{1+ha_1}{2ha_2} + \frac{1}{2ha_2}\sqrt{(1+ha_1)^2 + 4ha_2f(x+h)}$$

This yields:

$$2ha_2 f + 1 + ha_1 = \sqrt{(1+ha_1)^2 + 4ha_2(-\frac{1+ha_1(x+h)}{2ha_2(x+h)} + \frac{1}{2ha_2(x+h)}\sqrt{(1+ha_1(x+h))^2 + 4ha_2(x+h)\sqrt{\ldots}}}$$

In general if we have :

$$\sqrt{a(x) + \ b(x)\sqrt{a(x+h) + \ b(x+h)\sqrt{a(x+2h) + \ b(x+2h)\sqrt{\ldots}}}}$$

We have that

$$b(x) = \ \frac{1}{2ha_2(x+h)}$$

$$a(x) = \ (1+ha_1)^2 - \ 4ha_2\frac{1+ha_1(x+h)}{2ha_2(x+h)}$$

We can solve for $a_1, a_2$ in terms of a and b

$$a_2(x+h) = \frac{1}{2hb(x)}$$

$$D_{h,x}a_2 + \frac{1}{h}a_2 = \frac{1}{2h^2 b(x)}$$

And now we can resolve that using coupled integration factors

$$\begin{Bmatrix} \lambda_1 + \ D_{h,x}\lambda_1 = \lambda_2 \\ D_{h,x}\lambda_1 = \frac{1}{h}\lambda_2 \end{Bmatrix}$$

$$\lambda_1 = \left(1 - \frac{1}{h}\right)\lambda_2$$

$$D_{h,x}\lambda_1 = \frac{1}{h\left(1-\frac{1}{h}\right)}\lambda_1 = \frac{1}{h-1}\lambda_1$$

Naturally we assume exponential form:

$$\lambda_1 = (1+h)^{\frac{w(x)}{h}} \rightarrow \ D_{h,x}\lambda_1 = \lambda_1\left(\frac{(1+h)^{D_{h,x}w} - 1}{h}\right) = \ \frac{1}{h-1}\lambda_1 \rightarrow$$

$$w = D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2h-1}{h-1}\right)\right] = \log_{1+h}\left(\frac{2h-1}{h-1}\right)x$$

$$\lambda_1 = \left(\frac{2h-1}{h-1}\right)^{\frac{x}{h}}, \lambda_2 = \frac{h-1}{h}\left(\frac{2h-1}{h-1}\right)^{\frac{x}{h}}$$

Thus it follows:

$$a_2\left(\frac{2h-1}{h-1}\right)^{\frac{x}{h}} = D_{h,x}^{-1}\left[\frac{h-1}{2h^3 b(x)}\left(\frac{2h-1}{h-1}\right)^{\frac{x}{h}}\right] + C_2$$

$$a_2 = \left(\frac{2h-1}{h-1}\right)^{-\frac{x}{h}}\left(D_{h,x}^{-1}\left[\frac{h-1}{2h^3 b(x)}\left(\frac{2h-1}{h-1}\right)^{\frac{x}{h}}\right] + C_2\right)$$

Now we return to the second half of the original challenge:

$$a(x) = (1 + ha_1)^2 - 4ha_2\frac{1 + ha_1(x+h)}{2ha_2(x+h)}$$

Which simplifies to:

$$a(x) = (1 + ha_1)^2 - 2\big(1 + ha_1(x+h)\big)$$

Yielding

$$a(x) = 1 + 2ha_1(x) + a_1^2 - 2 - 2ha_1(x+h)$$

$$a(x) + 1 = 2ha_1(x) - 2ha_1(x+h) + a_1^2$$

$$1 + a(x) = -2h^2 D_{h,x}a_1 + a_1^2$$

$$D_{h,x}a_1 = \frac{1 + a(x)}{2h^2} + \frac{1}{2h^2}a_1^2$$

This is not yet something we know how to solve, but if a single test solution:

$$\theta(x)$$

We can reduce this equation to something we do know how to solve.

$$a_1 = E + \theta \rightarrow$$

$$D_{h,x}E + D_{h,x}\theta = \frac{1 + a(x)}{2h^2} + \frac{1}{2h^2}(E^2 + 2E\theta + \theta^2)$$

$$D_{h,x}E = \frac{\theta}{2}E + \frac{1}{2h^2}E^2$$

This has the solution we generated:

$$E = C_3(1+\mathrm{h})^{-\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{\theta h}{\frac{\theta}{2}h+1}\right)\right]}(1+h)^{\frac{1}{h}2^{\frac{x}{h}}D_{h,x}^{-1}\left[2^{-\frac{x+h}{h}}\log_{1+h}\left(\frac{\theta C_3(1+\mathrm{h})^{\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{\theta h}{\frac{\theta}{2}h+1}\right)\right]}}{\theta h^2-\frac{\theta}{2}h+1}\right)\right]}$$

Therefore:

$$a_1 = \theta + C_3(1+\mathrm{h})^{-\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{\theta h}{\frac{\theta}{2}h+1}\right)\right]}(1+h)^{\frac{1}{h}2^{\frac{x}{h}}D_{h,x}^{-1}\left[2^{-\frac{x+h}{h}}\log_{1+h}\left(\frac{\theta C_3(1+\mathrm{h})^{\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{\theta h}{\frac{\theta}{2}h+1}\right)\right]}}{\theta h^2-\frac{\theta}{2}h+1}\right)\right]}$$

And we already derived

$$a_2 = \left(\frac{2h-1}{h-1}\right)^{-\frac{x}{h}}\left(D_{h,x}^{-1}\left[\frac{h-1}{2h^3 b(x)}\left(\frac{2h-1}{h-1}\right)^{\frac{x}{h}}\right] + C_2\right)$$

Furthermore let:

$$f = C_1(1+\mathrm{h})^{-\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1 h}{a_1 h+1}\right)\right]}(1+h)^{\frac{1}{h}2^{\frac{x}{h}}D_{h,x}^{-1}\left[2^{-\frac{x+h}{h}}\log_{1+h}\left(\frac{a_1 a_2 C_1(1+\mathrm{h})^{\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1 h}{a_1 h+1}\right)\right]}}{2a_1h^2-a_1h+1}\right)\right]}$$

Then:

$$2ha_2 f + 1 + ha_1$$

Is the solution to the infinite radical, once the constants have been appropriate set to account for the radical's roots of unities. Note that if:

$$a(x) = -1 \textit{ then it follows that}$$

$$a_1 = \lim_{\substack{a_1\to 0 \\ a_2\to\frac{1}{2h^2}}} C_1(1+\mathrm{h})^{-\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1 h}{a_1 h+1}\right)\right]}(1+h)^{\frac{1}{h}2^{\frac{x}{h}}D_{h,x}^{-1}\left[2^{-\frac{x+h}{h}}\log_{1+h}\left(\frac{a_1 a_2 C_1(1+\mathrm{h})^{\frac{1}{h}D_{h,x}^{-1}\left[\log_{1+h}\left(\frac{2a_1 h}{a_1 h+1}\right)\right]}}{2a_1h^2-a_1h+1}\right)\right]}$$

And thus we have a closed form.

Furthermore note that:

$$D_{h,x}f = a_1 f + a_2 f^2$$

Can be split up:

$$Du + Dv = a_1 u + \ a_1 v + a_2 u^2 + 2a_2 uv + a_2 v^2$$

Such that:

$$v = -\frac{a_1}{2a_2}$$

Yields:

$$Du = (a_1 v + a_2 v^2 - Dv) + \ a_2 u^2$$

Which itself shows a mapping exists from our class to the second, finding the inverse mapping is the next challenge, whose completion will resolve this area and ready us for braver generalizations.