INFINITE QUATERNION PSEUDO RINGS USING [0, N)

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CONTENTS

PREFACE

In this book authors study the properties of finite real quaternion ring which was introduced in [2000]. Here a complete study of these finite quaternion rings are made. Also polynomial quaternion rings are defined, they happen to behave in a very different way. In the first place the fundamental theorem of algebra, “a nth degree polynomial has n and only n roots”, n is untrue in case of polynomial in polynomial quaternion rings in general. Further the very concept of derivative and integrals of these polynomials are untrue.

Finally interval pseudo quaternion rings also behave in an erratic way. Not only finite real quaternion rings are studied, but also finite complex modulo integer quaternion rings, neutrosophic finite quaternion rings, complex neutrosophic quaternion rings for the first time are introduced and analysed. All these rings behave in a very unique way. This book contains several open problems which will be a boon to any researcher.

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Chapter One

ON FINITE QUATERNION RINGS

In 2000 [126] the finite quaternion ring was defined for the first time which is as follows:

DEFINITION 1.1: *Let* $P = \{p_0 + p_1i + p_2j + p_3k \mid p_0, p_1, p_2, p_3 \in Z_n$, *n a finite number,* $n > 2$; $+, \times\}$ *with usual addition and multiplication modulo n defined in the following way is defined as the ring of real quaternions of characteristic n; n a finite positive integer where*

$i^2 = j^2 = k^2 = (n–1) = ijk$ *and* $ij = (n–1)\, ji = k$, $jk = (n–1)\, kj = i$,

$=k_i(n–1)ik = j$;

$0 = 0i + 0j + 0k$ *is the additive identity of P and*

$1 = 1 + 0i + 0j + 0k$ *is the multiplicative identity of P.*

This will also be known as finite quarternion ring.

We just illustrate how we define '+' and '." on P.

Let $x = p_0 + p_1i + p_2j + p_3k$
and $y = q_0 + q_1i + q_2j + q_3k \in P$;
where $p_i, q_i \in Z_n$; $0 \leq i \leq 3$.

$x + y = (p_0 + p_1i + p_2j + p_3k) + (q_0 + q_1i + q_2j + q_3k)$
$= (p_0 + q_0) + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3)k$
(where $p_i + q_i \equiv r_i \pmod n$, $0 \leq i \leq 3$)

$= r_0 + r_1i + r_2j + r_3k; r_i \in Z_n, 0 \leq i \leq 3.$

This is the way addition of defined.

$x.y = (p_0 + p_1i + p_2j + p_3k)(q_0 + q_1i + q_2j + q_3k)$
$= p_0q_0 + p_1q_0i + p_2q_0j + p_3q_0k + p_0q_1i + p_1q_1i^2 + p_2q_1ji + p_3q_1ki + p_0q_2j + p_1q_2ij + p_2q_2j^2 + p_3q_2kj + p_0q_3k + p_1q_3ik + p_2q_3jk + p_3q_3k^2$

$= (p_0q_0 + (n-1)p_1q_1 + (n-1)p_2q_2 + (n-1)p_3q_3) + (p_0q_1 + p_1q_0 + p_2q_3 + (n-1)p_3q_2)\, i + (p_0q_2 + q_0 p_2 + p_3q_1 + (n-1)p_1q_3)j + (p_0q_3 + q_0p_3 + (n-1)p_2q_1 + p_1q_2)\, k \in P.$

This is the way + and '.' operations are performed on P and P is a non commutative finite ring.

We will illustrate this situation by some examples.

Example 1.1: Let $P = \{p_0 + p_1i + p_2j + p_3k \mid p_i \in Z_3 = \{0, 1, 2\}, 0 \leq i \leq 3; i^2 = j^2 = k^2 = 2 = ijk, ij = 2ji = k, jk = 2kj = i, ki = 2ik = j, +, \times\}$ be the ring of order 81 under + and ×.

P is a non communicative finite quaternion ring. It is important to observe P has zero divisors. For $x = i + j + k \in P$ is such that $x^2 = 0$. Thus at this point the first author theorem 2 [126] is not correct for it is made under the assumption $\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2$ has inverse which is not correct. Thus the statement of theorem 2 of [126] is not valid.

We will prove this by examples.

Example 1.2 : Let $P = \{p_0 + p_1i + p_2j + p_3k \mid p_i \in Z_{11}, 0 \leq i \leq 3, i^2 = j^2 = k^2 = ijk = 10, ij = 10ji = k, jk = 10kj = i, ki = 10ik = j, +, \times\}$ be the ring of finite quaternions. P has zero divisors.

Consider $x = i + j + k \in P$.

$$x^2 = (i + j + k)^2$$
$$= i^2 + ji + ki + ij + j^2 + kj + ik + jk + k^2$$
$$= 10 + 10k + j + k + 10 + 10i + i + 10j + 10$$
$$= 30 \pmod{11}.$$
$$= 8 \neq 0.$$

But $y = 3i + 3j + 2k \in P$ is such that $y^2 = 0$ we see $3^2 + 3^2 + 2^2 = 9 + 9 + 4 = 0 \pmod{11}$.

Consider $z = 4i + 4j + k \in P$, clearly $z^2 = 0$ we see $4^2 + 4^2 + 1^2 = 0 \pmod{11}$.

That is we see sum of the square of the coefficients of i, j and k is 0 in Z_{11} then that element is nilpotent of order two. Thus P is only a ring and has zero divisors.

Thus we show this by a simple theorem.

THEOREM 1.1: *Let $P = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_n;\ 0 \leq i \leq 3.\ i^2 = j^2 = k^2 = (n-1) = ijk,\ ij = (n-1)ji = k,\ jk = (n-1)kj = i,\ ki = (n-1)ik = j, +, \times\}$ be the finite ring of quaternions has zero divisors.*

Proof: Let $x = ti + sj + mk \in P$; x is a nilpotent element of order two and $t, s, m \in Z_n$ is such that $t^2 + s^2 + m^2 = 0$. That is P has zero divisor for $x^2 = (ti + sj + mk)^2 = t^2i^2 + tsji + tmki + stij + s^2j^2 + smkj + tmik + msjk + m^2k^2$

$= (n-1)t^2 + ts\,(n-1)k + tmj + stk + (n-1)s^2 + sm\,(n-1)i + tm\,(n-1)j + msi + m^2\,(n-1)$

$= (n-1)\,(t^2 + m^2 + s^2) + tsk\,((n-1) + 1) + tmj\,((n-1) + 1) + smi\,((n-1) + 1)$

$= 0$ as given $t^2 + m^2 + s^2 \equiv 0 \pmod{n}$ and $t, s, m \in Z_n$ so $ts = st$, $sm = ms$ and $tm = mt$

and $n-1 + 1 \equiv 0 \pmod{n}$.

Hence the claim.

Now we will find all the zero divisors of the form $(ai + bj + ck) \in P$ where P is built using Z_3.

$X = i + j + k$ is a zero divisor of P.
$Y = 2i + j + k \in P$ is such that $Y^2 = 0$.
$Z = i + 2j + k \in P$ is a zero divisor, $T = i + j + 2k \in P$ is also a zero divisor.

$U = 2i + 2j + 2k \in P$ is a zero divisor of P.
$M = 2i + 2j + k$ is such that $M^2 = 0$, $N = 2i + j + 2k$ and $R = 2j + 2k + i$ are all zero divisors.

Infact P has 8 zero divisors of this form.

Now it is left as an open conjecture to find number of zero divisors of the form $ai + bj + ck \in P$ over any Z_n.

Example 1.3: Let $P = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_4, 0 \leq i \leq 3, i^2 = j^2 = k^2 = 3 = ijk; ij = 3ji = k, jk = 3kj = i, ki = 3ik = j; +, \times\}$ be the finite ring of real quaternions.

$(2 + 2i)$, $(2 + 2k)$, $2k + 2j$, $2 + 2j$, $2i + 2j$, $2i + 2k$, $2 + 2i + 2j$, $2 + 2j + 2k$, $2 + 2j + 2k$, $2i + 2j + 2k$, $2 + 2i + 2j + 2k$ are all zero divisors different from other zero divisors.

$x = 3 + i + j + k \in P$ is such that $x^2 \neq 0$ that is x is not a zero divisor in P.

Can we have $x = ai + bj + ck \in P$ with $a, b, c \in Z_4 \setminus \{2\}$ such that $x^2 = 0$?

This remains an open problem for any Z_n (n not a prime).

Example 1.4: Let $P = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_6; 0 \leq i \leq 3, i^2 = kj^2 = k^2 = 5 = ijk, ij = 5ji = k, jk = 5kj = i, ki = 5ik = j\}$ be the finite quaternion ring.

Consider $x = 2i + 2j + 2k \in P$.

$x^2 = 4 \times 5 + 4 \times 5 + 4 \times 5 + 4ij + 4jk + 4ki + 4ji + 4kj + 4ik$

$= 20 + 20 + 20 + 4k + 20k + 4i + 20i + 4j + 20j$
$= 0$ is a zero divisor.

$y = 4i + 4j + 4k \in P$; we find $y^2 = 0$ so is zero divisor.

$z = i + 2j + k \in P$ is such that $z^2 = 0$ so is zero divisor.

$t = 2i + j + k \in P$ is such that $t^2 = 0$ is a zero divisor.

$s = i + j + 2k \in P$ is such that $s^2 = 0$ is a zero divisor.

Now $m = 4i + 2j + 2k \in P$ is a zero divisor in P.

Now $a = 2i + 4j + 2k$ and $b = 2i + 2j + 4k \in P$ are zero divisors in P.

Let $x = 4i + 2j + 2k \in P$.

Consider $x^2 = (4i + 2j + 2k)^2$
$= 16 \times 5 + 4 \times 5 + 4 \times 5 + 8ij + 8ik + 8ji + 8ki + 4jk + 4kj$
$= 80 + 20 + 20 + 8k + 8 \times 5k + 8 \times 5j + 8j + 4 \times 5j + 4i$
$= 0 \pmod 6$ is a zero divisor and $4^2 + 2^2 + 2^2 = 0 \pmod 6$.

Likewise $y = 2i + 4j + 2k$ and $z = 2i + 2k + 4k$ are all zero divisors of P.

$x = 5i + 2j + k \in P$. We see $x^2 = (5i + 2j + k)^2 = 0$ is a zero divisor of P.

$y = 2i + 5j + k$ is also a zero divisor.

$5i + j + 2k$ is a zero divisor $2i + 5k + j$ is again a zero divisor.

We see $5^2 + 2^2 + 1 = 25 + 4 + 1 \equiv 0 \pmod 6$.

$a_1 = 5i + 5j + 2k$, $a_2 = 5j + 2i + 5k$ and $a_3 = 2j + 5i + 5k$ are zero divisors and $5^2 + 5^2 + 2^2 = 0 \pmod 6$.

Thus we see $x = a_0i + a_1j + a_2k$ is a zero divisor only if $a_0^2 + a_1^2 + a_2^2 = 0$ where $a_0, a_1, a_2 \in Z_6$.

Now we see in Z_6 all element $x = a_0i + a_1j + a_2k \in P$ with $a_0^2 + a_1^2 + a_2^2 = 0 \pmod 6$ are such that $x^2 = 0$.

Example 1.5: Let $P = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_8; 0 \leq i \leq 3, i^2 = j^2 = k^2 = 7 = ijk, ij = 7\ ji = k, jk = 7kj = i, ki = 7ik = j; +, \times\}$ be the finite ring of quaternions.

Let $x = 4i + 2j + 2k \in P$.

$$\begin{aligned} x^2 &= 16 \times 7 + 4 \times 7 + 4 \times 7 + 8k + 8 \times 7 \times k + 4i + \\ &\quad 4 \times 7 \times I + 8j + 8 \times 7 \times j \\ &= 112 + 28 + 28 + 64k + 64j + 32i \\ &= 0 \pmod 8 \text{ is a zero divisor.} \end{aligned}$$

Thus P has zero divisors.

All elements of the form $x = a_0i + a_1j + a_2k$; $0 \leq i \leq 2$, $a_i \in Z_8$ are zero divisors provided $a_0^2 + a_1^2 + a_2^2 = 0 \pmod 8$.

We see P has also other zero divisors of the form

$(4i + 2j)\,(2i + 4j) = 0$,
$(2k + 4j + 4i + 2) \times (2 + 4j) = 0$ and so on.

$x = (6i + 2j) \in P$ is a zero divisor for $y = (4j + 4k) \in P$ is such that $xy = 0$. This ring has different types of zero divisors.

$x = 4i$ and $y = (6j + 6k + 2i + 4) \in P$ is such that $x.y = 0$.

Example 1.6: Let P = $\{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_{12}; i^2 = j^2 = k^2 = 11 = ijk, ij = 11, (ji) = k, jk = 11, (kj) = i, ki = 11, (ik) = j, +, \times; 0 \leq t \leq 3\}$ be the ring of finite quaternions.

P has several zero divisors. P has subrings which are not ideals.

For $S_1 = \{ai + b \mid b, a \in Z_{12}\}$ is a subring and is not an ideal.

$S_2 = \{a + bj \mid a, b \in Z_{12}\}$ is again a subring and is not an ideal.

$S_3 = \{a + b_k \mid a, b \in Z_{12}\}$ is a subring and is not an ideal.

It is little difficult to find ideals but P has ideals.

Consider S = $\{a_0 + a_1i + a_2j + a_3k \mid a_i \in \{0, 2, 4, 6, 8, 10\}; 0 \leq i \leq 3; i^2 = j^2 = k^2 = 11 = ijk, ij = 11(ji) = k, jk = 11(kj) = i, ki = 11(ik) = j\} \subseteq P$ is a ideal of P.

M = $\{a_0 + a_1i + a_2j + a_3k \mid a_t \in \{0, 3, 6, 9\}; 0 \leq t \leq 3\} \subseteq P$ is an ideal of finite order.

$M \cap S$ is an ideal of P, however $M \cup S$ is not an ideal of P.

We can find also idempotents in P.

Let $x = 4i + 4j \in P$.

$$\begin{aligned} x^2 = (4i + 4j)^2 &= 16i^2 + 16j^2 + 16k + 16 \times 11 \times k \\ &= 4 \times 11 + 4 \times 11 + 4k + 8k \\ &= 4. \end{aligned}$$

Let $x = (9i + 9j + 9k) \in P$.

$$\begin{aligned} x^2 &= 81i^2 + 81j^2 + 81k^2 + 81k + 81 \times 11 \times k + 81k + \\ &\quad 81 \times 11 \times I + 81j + 81 \times 11 \times j \\ &= 9 + 9 + 9 \\ &= 9. \end{aligned}$$

$$y = (i + j + k) \in P.$$

$$y^2 = i^2 + j^2 + k^2 + k + 11k + j + 11j + i + 11i = 33 = 9.$$

Thus we see one can get the square of the sum of real quaternion is an element in Z_{12}.

It is an interesting aspect to study units of P.

$x = 11i$ is a unit for $y = i \in P$ is such that $xy = 11i \times i = 11 \times 11 = 1$.

Certainly every ring of finite quaternions has units.

THEOREM 1.2: *Let $P = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_n, 0 \leq i \leq 3, i^2 = j^2 = k^2 = ijk = n–1, ij = (n–1)ji = k, jk = (n–1)kj = i, ki = (n–1)ik = j; +, \times\}$ be the ring of real quaternions. P has units.*

The proof is direct and hence left as an exercise to the reader. However $x = (n–1)i$, $y = (n–1)j$ and $z = (n–1)k$ are units in P. For $x_1 = i$ is such that $xx_1 = 1$ and $y_1 = j$ is such that $yy_1 = 1$ and $z_1 = k$ is such that $zz_1 = 1$.

Example 1.7: Let $P = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_{11}; i^2 = j^2 = k^2 = 10 = ijk, ij = 10(ji) = k, jk = 10(kj) = i, ki = 10(ik) = j; \times, +\}$ be the finite ring of quaternions. P has zero divisors. P is a Smarandache ring as $Z_{11} \subseteq P$ is a field. Hence P is a S-ring.

However P is not a field every element $x = a_0i + a_1j + a_2k$ with $a_0^2 + a_1^2 + a_2^2 = 0 \pmod{11}$ is a nilpotent element of order two.

For instace $x_1 = 3i + j + k$, $x_2 = i + 3j + k$ and $x_3 = i + j + 3k$ are all nilpotent elements of order two.

We have subrings of finite order.

However we do not know whether they have ideals?

THEOREM 1.3: *Let $P = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_n; 0 \leq i \leq 3, i^2 = j^2 = k^2 = n\text{-}1 = ijk, ij = (n–1)(ji) = k, jk = (n–1)(kj) = i, ki = (n\text{-}1)(ik) = j; +, \times\}$ be the ring of finite quaternions. If Z_n is S-ring then P is a S-ring.*

Proof is direct and hence left as an exercise to the reader.

Now we can using the finite ring of quaternions built vector space of the real quaternions. This would be useful to us for we can have also eigen values to be the finite real quaternions. However the linear algebra of finite real quaternions may not be always a commutative linear algebra.

We can also in case of finite ring of quaternions get all the properties associated with finite non commutative ring.

We also have finite complex number of real quaternions which is defined as follows:

DEFINITION 1.2: *Let*
$C(Z_n) = \{a + bi_F$ where $a, b \in Z_n$, $i_F^2 = n–1\}$ be the finite ring of complex modulo integers. $P_C = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in C(Z_n); 0 \leq i \leq 3, i^2 = j^2 = k^2 = n–1 = ijk, ij = (n–1)(ji) = k, jk = (n–1)(kj) = i, ki = (n–1)(ik) = j; +, \times\}$ under + and $\times$ is a ring of finite order. We define P_C to be the finite complex modulo integer ring of quaternions.

We just show has sum and product are defined on P_C.

Let $x = (a_0 + b_0i_F) + (a_1 + b_1i_F)i + (a_2 + b_2i_F)j + (a_3 + b_3i_F)k$

and $y = (c_0 + d_0i_F) + (c_1 + d_1i_F)i + (c_2 + d_2i_F)j + (c_3 + d_3i_F)k \in P_C$

We define $x + y = ((a_0 + c_0 + (b_0+ d_0)i_F) + ((a_1 + c_1) + (b_1 + d_1)i_F)i + ((a_2 + c_2) + (b_2+ d_2)i_F)j + (a_3 + c_3 + (b_3 + d_3)i_F)k \in P_C$.

Clearly x + y = y + x as

$(a + bi_F) + (c + di_F) = (c + di_F) + (a + bi_F)$ for every $a + bi_F$ and $c + di_F \in C(Z_n)$.

Consider $x \times y$

$$= [(a_0 + b_0i_F) \ (a_1 + b_1i_F)i + (a_2 + b_2i_F)j + (a_3 + b_3i_F)k] \times [(c_0 + d_0i_F) + (c_1 + d_1i_F)i + (c_2 + d_2i_F)j + (c_3 + d_3i_F)k]$$

$$= (a_0 + b_0i_F)(c_0 + d_0i_F) +[(a_1 + b_1i_F)(c_0 + d_0i_F)]i + [(a_2 + b_2i_F)(c_0 + d_0i_F)]j + [(a_3 + b_3i_F)(c_0 + d_0i_F)]k + (a_0 + b_0i_F)(c_1 + d_1i_F)i + [(a_1 + b_1i_F)(c_1 + d_1i_F)](n-1) + [(a_2 + b_2i_F)(c_1 + d_1i_F)(n-1)k + [(a_3 + b_3i_F)(c_1 + d_1i_F)]j + (a_0 + b_0i_F)(c_2 + d_2i_F)j + (a_1 + b_1i_F)(c_2 + d_2i_F)k + (a_2 + b_2i_F)(c_2 + d_2i_F)(n-1) + (a_3 + b_3i_F)(c_2 + d_2i_F)(n-1)i + (a_0 + b_0i_F)(c_3 + d_3i_F)k + (a_1 + b_1i_F)(c_3 + d_3i_F)(n-1)j + (a_2 + b_2i_F)(c_3 + d_3i_F)i + (a_3 + b_3i_F)(c_3 + d_3i_F)(n-1).$$

Collecting the elements of $C(Z_n)$ + coefficient of i + coefficient of j + coefficient of k; we get the product to be in P_C.

We can easily verify 0 = 0 + 0i + 0j + 0k is the additive identity of P_C and 1 = 1 + 0i + 0j + 0k acts as the multiplicative identity.

We will give some examples of them.

Example 1.8: Let $P_C = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in C(Z_4); 0 \leq i \leq 3, i^2 = j^2 = k^2 = 3 = ijk, ij = 3(ji) = k, jk = 3(kj) = i, ki = 3(ik) = j; +, \times\}$ be the finite complex modulo integer of real quaternions. P_C has zero divisors.

For $x = 2 + (2 + 2i_F)i$ and $y = (2 + 2i_F)k + (2 + 2i_F)j$ in P_C are such that $x \times y = 0$ is a zero divisor. Let $x = 3i \in P_C$, we have $y = i$, such that $xy = 3 \times i^2 = 3 \times 3 = 1$; a unit.

Thus P_C is a ring with units and also has zero divisors. It is easily verified P_C has ideals and subrings which are not ideals.

Now if $x = (2 + i_F) + (3 + 2i_F)i$ and
$y = (i_F + 1)k + (3i_F + 2)j$ are in P_C then
$x + y = (2 + i_F) + (3 + 2i_F)i +(i_F + 1)k + (3i_F + 2)j$
$x \times y = [(2 + i_F) + (3 + 2i_F)i] \times [(i_F + 1)k + (3i_F + 2)j]$
$= (2 + i_F)(i_F + 1)k + (3 + 2i_F)i(i_F + 1)k + (2 + i_F)(3i_F + 2)j$
$+ (3 + 2i_F)(3i_F + 2)ij$
$= (2 + i_F + 2i_F + 3)k + (3 + 3i_F + 2i_F + 2 \times 3)3j + (4 + 2i_F$
$+ 6i_F + 3 \times 3)j + (9i_F + 6 + 6 \times 3 + 4i_F)k$
$= (1 + 3i_F)k + (3 + 3i_F)j + jj + i_Fk \in P_C$.

This is the way sum and product are got in P_C.

P is a finite ring.

Example 1.9: Let $P_C = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in C(Z_{11});\ 0 \leq i \leq 3,\ i^2 = j^2 = k^2 = 10 = ijk,\ ij = 10(ji) = k,\ jk = 10(kj) = i,\ ki = 10(ik) = j;\ +, \times\}$ be the finite complex modulo integer ring of real quaternios.

$o(P_C) < \infty$. P_C is non commutative has units and zero divisors.

Clearly $Z_{11} \subseteq P_C$ so P_C is a Smarandache ring.

Example 1.10: Let $P_C = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in C(Z_{12});\ 0 \leq t \leq 3,\ i^2 = j^2 = k^2 = 11 = ijk,\ ij = 11(ji) = k,\ jk = 11(kj) = i,\ ki = 11(ik) = j;\ +, \times\}$ be the finite complex modulo integer ring of finite real quaternions.

P_C has zero divisors, units, ideals and is non commutative.

Interested reader and can develop all the properties associated with finite non commutative rings.

Now we can develop the notion of neutrosophic ring of finite real quaternions.

DEFINITION 1.3: *Let $N(Z_n) = \langle Z_n \cup I \rangle$ be the ring of neutrosophic modulo integers.*

$N(Z_n) = \{a + bI \mid a, b \in Z_n$ and $I^2 = I$, I is an interminate$\}$.

Consider $P_N = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in \langle Z_n \cup I\rangle$ where $iI = Ii$, $jI = Ij$, $Ik = kI$, $0 \leq t \leq 3$, $i^2 = j^2 = k^2 = (n–1) = ijk$, $ij = (n–1)(ji) = k$, $jk = (n–1)(kj) = i$, $ki = (n–1)(ik) = j$, $I^2 = I$; $+, \times\}$. P_N under + and $\times$ is a ring defined as the neutrosophic modulo integer ring of real finite quaternions.

We will illustrate this by a few examples.

Example 1.11: Let $P_N = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in \langle Z_9 \cup I\rangle\ 0 \leq t \leq 3$, $i^2 = j^2 = k^2 = ijk$, $ij = 8(ji) = k$, $jk = 8(kj) = i$, $ki = 8(ik) = j$; $+, \times\}$ be the finite ring of neutrosophic real quaternions. P_N is non commutative and has zero divisors, idempotents and units further $|P_N| < \infty$.

Example 1.12: Let $P_N = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in \langle Z_{13} \cup I\rangle$ $0 \leq t \leq 3$, $i^2 = j^2 = k^2 = ijk$, $ij = 12(ji) = k$, $jk = 12(kj) = i$, $ki = 12(ik) = j$; $+, \times\}$ be the finite neutrosophic rings of real quaternions.

Let $x = (3I + 2) + (4I + 1)i + (5I + 3)j$ and
$y = (5 + 10I)k +(4I + 2)j \in P_N$.

$$x \times y = ((3I + 2) + (4I + 1)i + (5I + 3)j \times ((5 + 10I)k + (4I + 2)j)$$

$$= (3I + 2)(5 + 10I)k + (4I + 1)(5 + 10I) \times 12j + (3I + 2)(2 + 4I)j + (4I + 1)(2 + 4I)ij$$

$$= (15I + 10 + 20I + 30I)k + (20I + 5 + 10I+ 40I)12j + (6I + 4 + 8I + 12I)j + (8I + 2 + 16I + 4I)k$$

$$= 10k + (8 + 8I)j + 4j + (2I + 2)k$$
$$= (12 + 2I)k + (12 + 8I)j \in P_N.$$

$$y \times x = ((5 + 10I)k +(4I + 2)j) \times [(3I + 2) + (4I + 1)i + (5I + 3)j]$$

$$= \quad (5 + 10I)\,(3I + 2)k + (2 + 4I)(3I + 2)j + (5 + 10I)\,(4I + 1)j + (2 + 4I)\,(4I + 1)\,12k$$

$$= \quad (15I + 20I + 30I + 10)k + (6I + 12I + 4 + 8I)j + (20I + 40I + 5 + 10I)\,j + (8I + 4I + 16I + 2)\,12k$$

$$= \quad 10k + 4j + (5I + 5)j + (2I + 2)\,12k$$

$$= \quad 10k + 4j + (5I + 5)j + (11I + 11)k$$
$$= \quad (8 + 11I)k + (9 + 5I)j.$$

Clearly $x \times y = y \times x$ so P_N is a non commutative ring. P_N is a Smarandache ring as $Z_{13} \subseteq P_N$.

P_N has subrings which are not ideals as well as P_N has subrings which are ideals. All of them are finite order.

Example 1.13: Let $P_N = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in \langle Z_{12} \cup I\rangle$ $0 \leq t \leq 3$, $i^2 = j^2 = k^2 = ijk$, $ij = 11(ji) = k$, $jk = 11(kj) = i$, $ki = 11(ik) = i$, $I^2 = I$; $+, \times\}$ be the finite real quaternion neutrosophic ring of modulo integers. P_N has zero divisors units and idempotents.

All properties of finite non commutative rings can be derived for these rings also in a systematic way without any difficulty.

Next we define the notion of neutrosophic finite complex modulo integer real quaternions ring.

DEFINITION 1.4: *Let $C(\langle Z_n \cup I\rangle) = \{a + bI + ci_F + dIi_F \mid a, b, c, d \in Z_n, I^2 = I, i_F^2 = n{-}1, (Ii_F)^2 = (n{-}1)I; +, \times\}$ be the finite neutrosophic complex modulo integer ring. $P_{NC} = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle Z_n \cup I\rangle); 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = (n - 1)$, $ij = (n{-}1)(ji) = k$, $jk = (n{-}1)(kj) = i$, $ki = (n{-}1)(ik) = j$; $iI = Ii$, $ii_F = i_Fi$, $jI = Ij$, $ji_F = i_Fj$, $ki_F = i_Fk$, $kI = Ik\}$ be the ring under + and '.'. P_{NC} is defined as the finite neutrosophic complex* *modulo integer ring of real quaternions.* $|P_{NC}| < \infty$ *and* P_{NC} *is a non commutative ring.*

We will illustrate this by examples.

Example 1.14: Let $P_{NC} = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_i \in \langle Z_3 \cup I\rangle, 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk, ij = 2(ji) = k, jk = 2(kj) = i, ki = 2(ik) = j, I^2 = I; +, \times\}$ be the finite neutrosophic complex modulo integer real quaternion ring of finite order.

Let $x = (2 + i_F + I) + (1 + i_F + 2I)\, i + (2i_FI + i_F)j$ and
$y = (2 + i_F + I + 2i_FI)k \in P_{NC}$.

$$x + y = [(2 + i_F + I) + (1 + i_F + 2I)\, i + (2i_FI + i_F)j] + [(2 + i_F + I + 2i_FI)k].$$

$$\begin{aligned}
x \times y &= [(2 + i_F + I) + (1 + i_F + 2I)\, i + (2i_FI + i_F)j] \times [(2 + i_F + I + 2i_FI)k] \\
&= (2 + i_F + I) \times (2 + i_F + I + 2i_FI)k + (i_F + 1 + 2I)(2 + i_F + I + 2Ii_F)2j + (2i_FI + i_F)(2 + i_F + I + 2i_FI)i \\
&= (4 + 2i_F + 2I + 2i_F + 2 + Ii_F + 2I + Ii_F + I + 4i_FI + 4I + 2Ii_F)k + (2i_F + 2 + 4I + 2 + i_F + 2Ii_F + Ii_F + I + 2I + 4Ii_F + 2Ii_F + 4I)\, 2j + (4i_FI + 2i_F + 4I + 2 + 2i_FI + i_F + 8I + 4I)i \\
&= (i_F + 2I + 2i_FI)k + (2 + I)j + (I + 2)i.
\end{aligned}$$

This is the way product is performed on P_{NC}.

Clearly P_{NC} is only a ring for $x = i + j + k$ in P_{NC} is such that $x^2 = 0$.

Thus P_{NC} has zero divisors, units and has subrings.

$Z_3 \subseteq P_{NC}$ so P_{NC} is a Smarandache ring.

Example 1.15: Let $P_{NC} = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in C(\langle Z_{12} \cup I\rangle, 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 11, ij = 11(ji) = k, jk = 11(kj) = i, ki = 11(ik) = j, I^2 = I, i_F^2 = 11, (i_FI)^2 = 11I; +, \times\}$ be the finite neutrosophic complex modulo integer ring of real quaternions.

$o(P_{NC}) < \infty$. P_{NC} has units zero divisors, subrings and ideals. P_{NC} is a non commutative ring and all properties of non commutative rings can be obtained as a matter of routine.

$Z_{12} \subseteq P_{NC}$ is a subring of P_{NC} and is not an ideal of P_{NC}. $\langle Z_{12} \cup I\rangle \subseteq P_{NC}$ is only a subring and not an ideal of P_{NC}.

$C(\langle Z_{12} \cup I\rangle)$ is subring of P_{NC} and is not an ideal of P_{NC}.

$S_i = \{a + bi \mid a, b \in Z_{12}, i^2 = 11\} \subseteq P_{NC}$ is a subring and not an ideal of P_{NC}.

$S_j = \{a + bj \mid a, b \in Z_{12}, j^2 = 11\}$ is again only a subring of P_{NC}.

$S_k = \{a + bk \mid a, b \in Z_{12}, k^2 = 11\} \subseteq P_{NC}$ is not an ideal of P_{NC} only a subring.

$T_i = \{a + bi \mid a, b \in C(Z_{12}), (ii_F)^2 = 1, i^2 = 11\}$ is a subring of P_{NC}.

$T_j = \{a + bj \mid a, b \in C(Z_{12}), i_F^2 = 11, j^2 = 11, ji_F = i_Fj$ and $(ji_F)^2 = 1\}$ is a subring of P_{NC} and not an ideal.

$T_k = \{a + bk \mid a, b \in C(Z_{12}), k^2 = 11, i_F^2 = 11, (i_Fk)^2 = 1\} \subseteq P_{NC}$ is a subring of P_{NC} and not an ideal of P_{NC}.

$M_i = \{a + bi \mid a, b \in \langle Z_{12} \cup I\rangle, i^2 = 11, I^2 = I, (Ii)^2 = 11I\} \subseteq P_{NC}$ is not an ideal of P_{NC} only a subring.

$M_j = \{a + bj \mid a, b \in \langle Z_{12} \cup I\rangle, j^2 = 11, I^2 = I, (Ij)^2 = 11I\}$ is a subring and is not an ideal.

Now

$M_k = \{a + bk \mid a, b \in \langle Z_{12} \cup I\rangle, k^2 = 11, I^2 = I, (Ik)^2 = 11I\} \subseteq P_{NC}$ is not an ideal of P_{NC} only a subring of P_{NC}.

We can have several subrings which are not ideals.

Clearly P_{NC} has units and zero divisors. Further P_{NC} also has idempotents.

Example 1.16: Let $P_{NC} = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle Z_{19} \cup I\rangle, 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 18, ij = 18(ji) = k, jk = 18(kj) = i, ki = 18(ik) = j, I^2 = I, i_F^2 = 18, (Ii_F)^2 = 18I, (Iji_F)^2 = I, (Iii_F)^2 = I$ and so on; $+, \times\}$ be the finite complex modulo integer ring of real quaternions. P_{NC} has zero divisors and units. P_{NC} is a Smarandache ring as $Z_{19} \subseteq P_{NC}$ is a field. Infact P_{NC} has subrings which are S-subrings.

For $L_i = \{\langle Z_{19} \cup I\rangle \mid i^2 = 18\}$ is a subring which is a S-subring of P_{NC}.

Similarly $L_j = \{a + bj \mid a, b \in Z_{19}, j^2 = 18\} \subseteq P_{NC}$ is also a subring which is a S-subring and

$L_k = \{a + bk \mid a, b \in Z_{19}, k^2 = 18\} \subseteq P_{NC}$ is a S-subring of P_{NC}. None of these S-subrings are ideals or S-ideals of P_{NC}.

Interested reader can find S-ideals if any in P_{NC}.

Example 1.17: Let $P_{NC} = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle Z_{24} \cup I\rangle, 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 23, ij = 23(ji) = k, jk = 23(kj) = i, ki = 23(ik) = j, (Ik)^2 = 23I, I^2 = I^2, (Ii)^2 = 23I, (Ij)^2 = 23I, (Ii_F)^2 = 23I, (ii_F)^2 = 1, (i_Fj)^2 = 1, (i_Fk) = 1$ and so on; $+, \times\}$ be the finite neutrosophic complex modulo integer ring of real quaternions.

P_{NC} has ideals zero divisors, units and idempotents.

We see clearly

$$Z_n \underset{\neq}{\subset} P \underset{\neq}{\subset} P_C \underset{\neq}{\subset} P_{NC}$$

as rings and the containment relation is strict.

$$Z_n \underset{\neq}{\subset} P \underset{\neq}{\subset} P_N \underset{\neq}{\subset} P_{NC}$$

This relation is also a strict containment relation.

Now we know Z_n, P, P_N, P_C and P_{NC} are groups under addition.

In the following we define the notion of vector space (linear algebra) of real finite quaternions and S-vector spaces (S-linear algebras) of real quaternions.

DEFINITION 1.5: *Let $P = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle Z_{24} \cup I\rangle, 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = p–1, ij = (p–1)(ji) = k, jk = (p–1)kj = i, ki = (p–1)ik = j, +\}$ be the additive abelian group. P is a vector space over the field Z_p called the vector space of finite real quaternions. P is finite dimensional over Z_p. Infact P is a non commutative linear algebra over Z_p. P is known as the linear algebra of finite real quaternions over Z_p.*

We see $|P| < \infty$ and infact P is finite dimensional real quaternion vector space (linear algebra) over Z_p.

We will first illustrate this situation by some examples.

Example 1.18: Let $P = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in Z_3, i^2 = j^2 = k^2 = ijk = 2, ij = 2(ji) = k, jk = 2(kj) = i, ki = 2(ik) = j, +\}$ be the finite real quaternion vector space over the field Z_3.

Example 1.19: Let $P = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in Z_{19}, 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 18, ij = 18(ji) = k, jk = 18(kj) = i, ki = 18(ik) = j; +, \times\}$ be the finite real quaternion non commutative linear algebra of finite dimension over Z_{19}. B = {i, j} is a basis.

For $\langle B\rangle = \{i, j, i + j, 2i, 3i, \ldots, 18i, j, 2j, \ldots, 18j, ai + bj, a, b \in Z_{19}, i^2 = 18, 1, ji = 18k, k, \ldots\}$. So B = {i, j} is a basis.

Similarly C = {j, k} and D = {i, k} are basis of P over Z_{19}.

Example 1.20: Let P = $\{a_0 + a_1i + a_2j + a_3k \mid a_t \in Z_5, 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 4, ij = 4(ji) = k, jk = 4(kj) = i, ki = 4(ik) = j; +, \times\}$ be the linear algebra of finite real quaternions over the field Z_5.

We see B = {i, j} is a basis of P over Z_5.

For $\langle B \rangle = \{i, j, 4, 1, 2, 3, (i^2 = 4), k, ji = 4k$ hence $a_0 + a_1i + a_2j + a_3k$ with $a_t \in Z_5\} = P$. Thus B is a basis. Hence dimension of P over Z_5 is two.

Inview of this we can have the following theorem.

THEOREM 1.4: *Let P = $\{a_0 + a_1i + a_2j + a_3k \mid a_t \in Z_p, i^2 = j^2 = k^2 = (p–1) = ijk, ij = (p–1)(ji) = k, jk = (p–1)(kj) = i, ki = (p–1)(ik) = j, +, \times; 0 \leq t \leq 3\}$ be a linear algebra of finite real quaternions over Z_p. P is of dimension two over Z_p.*

Proof is direct and hence left as an exercise to the reader.

THEOREM 1.5: *Let P = $\{a_0 + a_1i + a_2j + a_3k \mid a_t \in Z_p, 0 \leq t \leq 3, i^2 = j^2 = k^2 = (p–1) = ijk, ij = (p–1)(ji) = k, jk = (p–1)(kj) = i, ki = (p–1)(ik) = j, +\}$ be the finite real quaternion vector space over the field Z_p. Dimension of P over Z_p is four.*

Proof: Take B ={1, i, j, k} $\subseteq$ P, clearly B is a basis of P over Z_p and dimension of P over Z_p is four.

Example 1.21: Let P = $\{a_0 + a_1i + a_2j + a_3k \mid a_t \in Z_7, 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 6, ij = 6ji = k, jk = 6kj = i, ki = 6ik = j, +\}$ be a vector space of finite real quaternions over the field Z_7.

B_1 = {1, i, j, k}, B_2 = {2, i, j, k}, B_3 = {3, i, j, k},
B_4 = {4, i, j, k} and so on are all basis of P over Z_7.

T = {6, 3i, 4j, 5k} is also a basis of P over Z_7.

We can using the real quaternion groups to built matrices and polynomials.

Example 1.22: Let M = $\{(x_1, x_2, x_3, x_4) \mid x_i \in P = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in Z_{13}, 0 \leq t \leq 3, +\}\ 1 \leq i \leq 4\}$ be a finite real quaternion vector space of row matrices over the field Z_{13}. M is finite dimensional over Z_{13}.

P ={(1, 0, 0, 0), (i, 0, 0, 0), (j, 0, 0, 0), (k, 0, 0, 0), (0, 1, 0, 0), (0, i, 0, 0), (0, j, 0, 0), (0, k, 0, 0), (0, 0, 1, 0), (0, 0, i, 0), (0, 0, j, 0), (0, 0, k, 0), (0, 0, 0, 1), (0, 0, 0, i), (0, 0, 0, j), (0, 0, 0, k)} is a basis of M over Z_{13}.

Clearly dimension of M over Z_{13} is 16.

Example 1.23: Let M = $\{(x_1, x_2, \ldots, x_{36}) \mid x_i \in P = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in Z_{43}, 1 \leq i \leq 36, +\}$ be the vector space of row matrices of real quaternions over the field Z_{43}.

We see T has a finite basis given by
B = {(1, 0, …, 0), (i, 0, …, 0), (j, 0, …, 0), (k, 0, …, 0), (0, 1, 0, …, 0), (0, i, 0, …, 0), …, (0, 0, …, k)}.

Example 1.24: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \ a_i \in P = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in Z_{17}; 0 \leq t \leq 3\}, \right.$$

$$\left. 1 \leq i \leq 6, + \right\}$$

be the column matrix of vector space of finite real quaternions over the field Z_{17}.

B is finite dimensional over Z_{17}.

Example 1.25: Let

$$L = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right| a_i \in P = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in Z_{11};$$

$i^2 = j^2 = k^2 = ijk = 10$, $ij = 10ji = k$, $jk = 10kj = i$, $ki = 10ik = j$, $0 \leq t \leq 3\}$, $1 \leq i \leq 4$, $+, \times_n\}$ be the linear algebra of column matrix of real quaternions over the field Z_{11}.

L is finite dimensional over Z_{11}. A basis of L over Z_{11} is

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} j \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 \\ i \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ j \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ i \\ 0 \end{bmatrix},$$

$$\left. \begin{bmatrix} 0 \\ 0 \\ j \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ i \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ j \end{bmatrix} \right\}$$

The dimension of L over Z_{11} is 12 as a linear algebra.

If L is considered only as a vector space over Z_{11} dimension of L over Z_{11} is 16.

Example 1.26: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in Z_{41}; \; 0 \le t \le 3\}, 1 \le i \le 15, +, \times_n \right\}$$

be a finite real quaternion vector space linear algebra over the field Z_{41}. This has finite basis.

Likewise we can use also define super matrices and obtain the finite real quaternion vector space over the P and find a basis.

Further on similar lines we can build using P_C the finite complex modulo integer of real quaternion vector spaces over the field Z_p.

We will just illustrate this situation by an example or two.

Example 1.27: Let
$V = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_t \in C(Z_{19}), 0 \le t \le 3, \; i_F^2 = 18, +\}$ be the vector space of complex finite modulo integer of finite real quaternions over the field Z_{19}. V is a finite dimensional complex modulo integer quaternion vector space over Z_{19}.

On V we can define product so that V can be made into a linear algebra of complex modulo integer real quaternions over Z_{19}.

Example 1.28: Let

$$V = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_t \in C(Z_{23}), 0 \le t \le 3, \; +, \times\}$$

be a finite complex number of real finite quaternions. This has finite basis both as a vector space as well as a linear non commutative algebra over the field Z_{23}.

Example 1.29: Let M = {P_C, +, × | $P_C = a_0 + a_1i + a_2j + a_3k$ | $a_t \in C(Z_{29})$, $0 \leq t \leq 3$, $i_F^2 = 28$, $i^2 = j^2 = k^2 = ijk = 28$, $ij = 28ji = k$, $jk = 28kj = i$, $ki = 28ik = j$, +, ×} be the finite complex modulo integer vector space (linear algebra) of finite real quaternions over the field Z_{29}. M is infinite dimensional over Z_{29}.

As in case of real quaternions here also we can build using matrices with entries from P_C.

All these will be illustrated by examples.

Example 1.30: Let M = {(d_1, d_2, d_3, d_4) | $d_i \in P_C$ = {$a_0 + a_1i + a_2j + a_3k$ | $a_t \in C(Z_{47})$, $0 \leq t \leq 3$, $i^2 = j^2 = k^2 = ijk = 46$, $ij = 46ji = k$, $jk = 46kj = i$, $ki = 46ik = j$, $i_F^2 = 46$, $(i_Fj)^2 = 1$, $(i_F1)^2 = 1$, $(i_Fk)^2 = 1$, $(i_Fk)^2 = 1$}, $1 \leq i \leq 4$, +, ×} be a finite complex modulo integer vector space (linear algebra) of finite real quaternions over the field Z_{47}. M is finite dimensional over Z_{47}.

Example 1.31: Let N = {($x_1, x_2, \ldots, x_{20}$) | $x_i \in P_C$ = {$a_0 + a_1i + a_2j + a_3k$ | $a_t \in C(Z_7)$, $0 \leq t \leq 3$}, $1 \leq i \leq 20$, +, ×} be the finite complex modulo integer finite real quaternion row matrix vector space (linear algebra) over the field Z_7.

N is finite dimensional over Z_7.

Example 1.32: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_i \in P_C = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in C(Z_{53}); \right.$$

$$\left. 0 \leq t \leq 3\}, 1 \leq i \leq 10, +, \times_n \right\}$$

be the finite complex modulo integer column matrix vector space (linear algebra under natural product) of finite real quaternions over the field Z_{53}. T is finite dimensional over the field Z_{53}.

Example 1.33: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix} \middle| \; a_i \in P_C = \{b_0 + b_1i + b_2j + b_3k \mid \right.$$

$$b_t \in C(Z_3);\ 0 \leq t \leq 3\},\ 1 \leq i \leq 40,\ +,\ \times_n\}$$

be the finite complex modulo integer column matrix vector space (linear algebra under natural product $\times_n$) of finite real quaternions over Z_3. S is finite dimensional over Z_3.

S is finite dimensional over Z_3 both as a vector space as well as the linear algebra over Z_3.

We can also define the notion of super matrix finite complex modulo integer real quaternion vector spaces and linear algebras. This is a matter of routine hence left as an exercise to the reader.

Next we define the notion of neutrosophic finite real quaternion vector spaces and linear algebras over the field Z_p. This situation is exhibited by the following examples.

Example 1.34: Let $S = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in P_N, \{b_0 + b_1i + b_2j + b_3k \mid b_t \in \langle Z_{11} \cup I\rangle, 0 \leq t \leq 3\}, +, \times\}$ be a neutrosophic finite real quaternion vector space (linear algebra) over the field Z_{11}.

We see $(iI)^2 = 10I$, $I^2 = I$, $(kI)^2 = 10I$ and $(jI)^2 = 10I$.
Clearly S is finite dimensional over Z_{11}.

Example 1.35: Let $W = \{a \in P_N = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle Z_{19} \cup I\rangle;\ 0 \leq t \leq 3\}, +, \times\}$ be the neutrosophic vector space of finite real quaternions over the field Z_{19}.

We can as in case of usual spaces define neutrosophic finite real quaternion matrix vector spaces over the field Z_p.

This is illustrated by some examples.

Example 1.36: Let S = $\{(a_1, a_2, a_3) \mid a_i \in P_N = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in \langle Z_{43} \cup I\rangle\ 0 \leq t \leq 3\}, 1 \leq i \leq 3\}$ be the neutrosophic finite real quaternion row matrix vector space over the field Z_{43}. S is finite dimensional over Z_{43}.

Example 1.37: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle|\ a_i \in P_N = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in \langle Z_{23} \cup I\rangle; 0 \leq t \leq 3\}, 1 \leq i \leq 9, \times_n \right\}$$

be the neutrosophic finite real quaternion column vector space over the field Z_{23}.

V is finite dimensional over the field Z_{23} both as a vector space as well as the linear algebra over Z_{23}.

Example 1.38: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix} \middle|\ a_i \in P_N = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in \langle Z_5 \cup I\rangle; 0 \leq t \leq 3\}, 1 \leq i \leq 40, +, \times_n \right\}$$

be the neutrosophic finite real quaternion vector space (linear algebra) of matrices over the field Z_5.

M is finite dimensional over the field Z_5. M is also finite dimensional as a linear algebra over the field Z_5.

Now super matrices of neutrosophic real finite quaternions can also be built. This is considered as a matter of routine and left as an exercise to the reader.

Next we proceed onto introduce the notion of finite neutrosophic complex modulo integer real quaternion vector spaces and linear algebras over the field Z_p.

This will be only illustrated by examples.

Example 1.39: Let V = $\{x \mid x \in P_{NC} = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle Z_{23} \cup I\rangle);\ 0 \leq t \leq 3\}, +\}$ be the neutrosophic complex modulo integer finite real quaternion vector space over the field Z_{23}.

V is finite dimensional over Z_{23}.

Example 1.40: Let V = $\{x \mid x \in P_{NC} = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle Z_3 \cup I\rangle);\ 0 \leq t \leq 3;\ +\}$ be the finite neutrosophic complex modulo real quaternion vector space over the field Z_3.

M is finite dimensional over Z_3. M is also a linear algebra which is non commutative over Z_3.

We can have matrix of finite real complex modulo integer neutrosophic quaternion vector space (linear algebra) over the field Z_p.

This will be exhibited by some examples.

Example 1.41: Let V = $\{(x_1, x_2, x_3, x_4, x_5, x_6) \mid x_i \in P_{NC} = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle Z_{29} \cup I\rangle\_;\ 0 \leq t \leq 3\}, 1 \leq i \leq 6, +\}$ be the finite complex neutrosophic modulo real quaternion vector space of row matrices over the field Z_{29}.

Clearly dimension of V over Z_{29} is finite.

Example 1.42: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{19} & a_{20} & a_{21} \end{bmatrix} \middle| \; a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle Z_{13} \cup I \rangle); 0 \leq t \leq 3\}, 1 \leq i \leq 21, +, \times_n \right\}$$

be the real finite neutrosophic complex modulo integer real quaternions vector space of matrices over the field Z_{13}.

Dimension of M over Z_{13} is finite. Clearly under the natural product $\times_n$. M is a linear algebra which is non commutative and is of finite dimension over Z_{13}.

Example 1.43: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle Z_{17} \cup I \rangle); 0 \leq t \leq 3\}, 1 \leq i \leq 16, +, \times_n \right\}$$

be the finite complex modulo neutrosophic real quaternion vector space (linear algebra) over the field Z_{17}.

V is finite dimensional over Z_{17}.

Example 1.44: Let

$$W = \left\{ \left(\begin{array}{c|cc|c} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{array} \right) \middle| \; a_p \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle Z_{13} \cup I \rangle); 0 \leq t \leq 3\}, 1 \leq p \leq 8, +, \times_n \right\}$$

be the finite complex modulo integer neutrosophic real quaternion vector space (linear algebra) over the field Z_{13}.

Next we proceed on to find substructures in these vector spaces (linear algebras).

This will be illustrated by examples.

Example 1.45: Let V = $\{x \mid x \in P_{NC} = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in \langle Z_{13} \cup I\rangle; 0 \leq t \leq 3\}, i^2 = j^2 = k^2 = ijk = 12, ij = 12ji = k, jk = 12kj = i, ki = 12ik = i, I^2 = I, (Ij)^2 = 12I$ and so on $+, \times\}$ be the vector space or linear algebra of finite real quaternions.

We have $W_1 = \{x \mid x \in \langle Z_{13} \cup i\rangle = a + bi; i^2 = 12\} \subseteq M$ is a subspace of M over Z_{13}.

We have many such subspaces.

$W_2 = \{x \mid x \in \langle Z_{13} \cup k\rangle = a + bk; k^2 = 12\} \subseteq M$ is also a subspace of M.

Example 1.46: Let

$$T = \{(a_1, a_2, a_3, a_4) \mid a_m \in P = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in Z_7; \\ 0 \leq t \leq 3\}\ 1 \leq m \leq 4, +, \times\}$$

be the finite real quaternion vector space over the field Z_7.

$P_1 = \{(a_1, 0, 0, 0) \mid a_1 \in P\} \subseteq T$ is a subspace of finite real quaternions.

$P_2 = \{(0, a_2, 0, 0) \mid a_2 \in P\} \subseteq T$ is a subspace of finite real quaternions.

$P_3 = \{(0, 0, a_3, 0) \mid a_3 \in P\} \subseteq T$ is a subspace of finite real quaternions.

$P_4 = \{(0, 0, 0, a_4) \mid a_4 \in P\} \subseteq T$ is a subspace of finite real quaternions.

$T = P_1 + P_2 + P_3 + P_4$ is a direct sum and
$P_i \cap P_j = \{(0, 0, 0, 0)\}$

We also have subspaces of T which cannot be written as a direct sum.

Consider $L_1 = \{(a_1, 0, 0, 0) \mid a_1 \in (Z_{13} \cup k)\} \subseteq T$ is a subspace of T.

$L_2 = \{(0, a_2, 0, 0) \mid a_2 \in \langle Z_{13} \cup i\rangle\} \subseteq T$ is a subspace of T.

$L_3 = \{(0, 0, a_3, 0) \mid a_3 \in \langle Z_{13} \cup j\rangle\} \subseteq T$ is a subspace of T.

$L_4 = \{(0, 0, 0, a_4) \mid a_4 \in Z_{13}\} \subseteq T$ is a subspace of T.

However $L_i \cap L_j = \{(0, 0, 0, 0)\}$, $i \neq j$, $1 \leq i, j \leq 4$ but $W = L_1 + L_2 + L_3 + L_4 \subseteq T$ and hence is not a direct sum.

We can have the concept of direct sum only in some cases.

Further L_i is orthogonal with L_j if $i \neq j$, $1 \leq i, j \leq 4$.

However L_i is not the orthogonal complement of L_j. We cannot have the complement of L_i to be such that $L_i \cup L_i^c = T$ this is impossible $1 \leq i \leq j$. $L_i^c = \{x \in T \mid (x.y) = (0, 0, 0, 0)$ for all $y \in L_i\}$.

Now $L_i^c = \{(0, a, b, c) \mid a, b, c \in P\} \subseteq T$ and $L_i \cap L_i^c = (0, 0, 0, 0)$ and $L_i + L_i^c \neq T$ is only a proper subset of T.

We can derive all the properties of vector spaces without any difficulty. This work is considered as a matter of routine.

We can define linear transformation provided both the real quaternion spaces are defined over the same field Z_p.

Now we can find (subalgebra) subspaces in case of vector spaces of complex modulo integer real quaternions, neutrosophic vector spaces of real quaternions and complex modulo integer neutrosophic vector space of real quaternions.

All these will be described by an example or two before we develop the concept of linear transformation and linear operators.

Example 1.47: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_s \in C(Z_{31}); \right.$$

$$\left. 0 \leq s \leq 3\}, 1 \leq i \leq 6, +, \times_n \right\}$$

be the finite complex modulo integer real quaternion matrix vector space (linear algebra) defined over the field Z_{31}.

We see V has subspaces which are as follows:

$$W_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in C(Z_{31}) \right\} \subseteq V$$ is a subspace of V over Z_{31}.

$$W_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_2 \in C(Z_{31}) \right\} \subseteq V$$ is again a subspace of V over Z_{31}.

$$W_3 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ a_3 \\ \vdots \\ 0 \end{bmatrix} \right| a_2 \in C(Z_{31}) \right\} \subseteq V$$ is also a subspace of V over $F = Z_{31}$.

$$W_4 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \end{bmatrix} \right| a_4 \in C(Z_{31}) \right\} \subseteq V$$ is a subspace of V over Z_{31}.

$$W_5 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \end{bmatrix} \right| a_5 \in C(Z_{31}) \right\} \subseteq V$$ is a subspace of V over Z_{31}.

and $$W_6 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \end{bmatrix} \right| a_6 \in C(Z_{31}) \right\} \subseteq V$$ is a subspace of V over Z_{31}.

Clearly $W_i \cap W_j = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$ if $i \neq j$; $1 \leq i, j \leq 6$.

Further $W_1 + W_2 + \ldots + W_6 \subseteq V$ and is not V so is not a direct sum.

Infact all these six subspaces $W_1, W_2, \ldots, W_6$ are only real subspaces of finite complex modulo integer subspaces.

None of them is the subspaces of real quaternions.

Now we see $S_1 = \left\{\begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| a_1 \in \langle Z_{31} \cup i\rangle\right\} \subseteq V$;

$$S_2 = \left\{\begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| a_1 \in \langle Z_{31} \cup j\rangle\right\} \subseteq V;$$

$$S_3 = \left\{\begin{bmatrix} 0 \\ 0 \\ a_3 \\ \vdots \\ 0 \end{bmatrix} \middle| a_3 \in \langle Z_{31} \cup k\rangle\right\} \subseteq V;$$

$$S_4 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \end{bmatrix}\right| a_4 \in Z_{31}\right\} \subseteq V;$$

$$S_5 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \end{bmatrix}\right| a_5 \in C(Z_{31})\right\} \subseteq V \text{ and}$$

$$S_6 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \end{bmatrix}\right| a_6 \in C(Z_{31})\right\} \subseteq V \text{ are all subspaces of V and}$$

none of them real quaternion subspaces.

We see $S_i \cap S_j = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$ if $i \neq j$; $1 \leq i, j \leq 6$.

We see $S_1 + S_2 + \ldots + S_6 \subseteq V$ and is not a direct sum.

Let $B_1 = \left\{\left.\begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_1 \in P_C\right\} \subseteq V$

$B_2 = \left\{\left.\begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_1 \in P_C\right\} \subseteq V;$

$B_3 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ a_3 \\ \vdots \\ 0 \end{bmatrix}\right| a_3 \in P_C\right\} \subseteq V;$

$B_4 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \end{bmatrix}\right| a_4 \in P_C\right\} \subseteq V;$

$B_5 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \end{bmatrix}\right| a_5 \in P_C\right\} \subseteq V$ and

$$B_6 = \left\{ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \end{array}\right] \middle| \; a_6 \in P_C \right\} \subseteq V \text{ are all subspaces of V.}$$

All of them are quaternion subspaces and

$$B_i \cap B_j = \left\{ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \right\} \text{ if } i \neq j;\ 1 \leq i, j \leq 6 \text{ and}$$

$V = B_1 + B_2 + \ldots + B_6$ is a direct sum.

Example 1.48: Let

$$S = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{array}\right] \middle| \; a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid \right.$$

$$\left. b_s \in \langle Z_7 \cup I\rangle;\ 0 \leq s \leq 3\},\ 1 \leq i \leq 12, +, \times_n \right\}$$

be the finite neutrosophic real quaternion vector space (linear algebra) over the field Z_7.

Let $T_1 = \left\{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in \langle Z7 \cup I \rangle\right\} \subseteq S$,

$$T_2 = \left\{\begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in \langle Z_7 \cup I \rangle\right\} \subseteq S,$$

$$T_3 = \left\{\begin{bmatrix} 0 & 0 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_3 \in \langle Z_7 \cup I \rangle\right\} \subseteq S,$$

$$T_4 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ a_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_4 \in \langle Z_7 \cup I \rangle\right\} \subseteq S,$$

$$T_5 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & a_5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_5 \in \langle Z_7 \cup I \rangle\right\} \subseteq S,$$

$T_6 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_5 \in \langle Z_7 \cup I \rangle\right\} \subseteq S$ and so on.

$$T_{10} = \left\{\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{10} & 0 & 0 \end{array}\right] \middle| \; a_{10} \in \langle Z_7 \cup I\rangle\right\} \subseteq S$$

$$T_{11} = \left\{\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{11} & 0 \end{array}\right] \middle| \; a_{11} \in \langle Z_7 \cup I\rangle\right\} \subseteq S \text{ and}$$

$$T_{12} = \left\{\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{12} \end{array}\right] \middle| \; a_{11} \in \langle Z_7 \cup I\rangle\right\} \subseteq S$$

are all neutrosophic subspaces of S and $T_i \cap T_j = \left\{\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]\right\}$

if $i \neq j$; $1 \leq i, j \leq 6$ and $V = T_1 + T_2 + \ldots + T_{12} \subseteq S$ is not a direct sum.

None of the subspaces of S are real quaternions vector subspaces of S.

$$\text{Let } D_1 = \left\{\left[\begin{array}{ccc} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \middle| \; a_1, a_2 \in P_N\right\} \subseteq S,$$

$$D_2 = \left\{\left.\begin{bmatrix} 0 & 0 & a_3 \\ a_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_3, a_4 \in P_N\right\} \subseteq S,$$

$$D_3 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_5, a_6 \in P_N\right\} \subseteq S,$$

$$D_4 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_7 & a_8 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_7, a_8 \in P_N\right\} \subseteq S,$$

$$D_5 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_9 \\ a_{10} & 0 & 0 \end{bmatrix}\right| a_9, a_{10} \in P_N\right\} \subseteq S \text{ and}$$

$$D_6 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{11} & a_{12} \end{bmatrix}\right| a_{11}, a_{12} \in P_N\right\} \subseteq S$$

be six subspaces of S over the field Z_7.

Clearly $D_i \cap D_j = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$ if $i \neq j$; $1 \leq i, j \leq 6$ and

$S = D_1 + D_2 + D_3 + D_4 + D_5 + D_6$ is a direct sum.

Thus we can have finite number of subspaces some collection will lead to direct sum and some will never lead to a direct sum.

Such study is a matter of routine and is left for the reader.

Example 1.49: Let

$$M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_9 \end{bmatrix} \middle| \; a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_s \in C(\langle Z_{19} \cup I\rangle);\right.$$

$$\left. 0 \leq p \leq 3\},\ 1 \leq t \leq 9, +, \times_n\right\}$$

be the real finite neutrosophic complex modulo integer real quaternions of finite order.

We see several subspaces exist in case of M.

We will denote them by

$$P_1 = \left\{\begin{bmatrix} a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in P_{NC}\right\};$$

$$P_2 = \left\{\left.\begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_2 \in P_{NC}\right\},$$

$$P_3 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ a_3 \\ \vdots \\ 0 \end{bmatrix}\right| a_3 \in P_{NC}\right\} \text{ and}$$

$$P_9 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ a_9 \end{bmatrix}\right| a_9 \in P_{NC}\right\}$$

are all subspaces of M and

$$P_i \cap P_j = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\}; 1 \leq i, j \leq 9,$$

further $M = P_1 + P_2 + \ldots + P_9$.

Thus M is a direct sum of subspaces.

We have several ways of representing M as a direct sum.

$$\text{Consider } B_{1,2} = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_1, a_2 \in P_{NC}\right\} \subseteq M,$$

$$B_{3,4} = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ a_3 \\ a_4 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_3, a_4 \in P_{NC}\right\} \subseteq M,$$

$$B_{5,6} = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ a_6 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right| a_5, a_6 \in P_{NC}\right\} \subseteq M,$$

$$B_{7,8} = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_7 \\ a_8 \\ 0 \end{bmatrix}\right| a_7, a_8 \in P_{NC}\right\} \subseteq M \text{ and}$$

$$B_9 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_9 \end{bmatrix}\right| a_9 \in P_{NC}\right\} \subseteq M$$

are all subspaces of M.

$$B_i \cap B_j = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\}; \text{ if } i \neq j,$$

$M = B_{1,2} + B_{3,4} + B_{5,6} + B_{7,8} + B_9$ is a direct sum.

Consider $D_1 = \left\{\left.\begin{bmatrix} a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_1 \in P_C \subseteq P_{NC}\right\} \subseteq M$;

$$D_2 = \left\{\left.\begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_2 \in P_C \subseteq P_{NC}\right\} \subseteq M;$$

$$D_3 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ a_3 \\ \vdots \\ 0 \end{bmatrix}\right| a_3 \in P_C \subseteq P_{NC}\right\} \subseteq M,$$

$$D_4 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_4 \in P_C \subseteq P_{NC}\right\} \subseteq M;$$

$$D_5 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_5 \in P_C \subseteq P_{NC} \right\} \subseteq M \text{ and so on.}$$

$$D_9 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ a_9 \end{bmatrix} \middle| \; a_9 \in P_C \subseteq P_{NC} \right\} \subseteq M$$

are all subspaces of M. It is noted

$$D_i \cap D_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}; \; 1 \leq i, j \leq 9, \text{ and}$$

$L = D_1 + D_2 + \ldots + D_9 \subseteq M$, so is not a direct sum. So we have subspaces in M which can never be made into a direct sum.

None of $D_1, D_2, \ldots, D_8$ can be completed to get a direct sum.

However using D_9 we can get the direct sum or complete it to a direct sum.

Example 1.50: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in P_{NC} = \{b_0 + b_1i + b_2j + b_3k \mid \right.$$

$$\left. b_s \in C(\langle Z_{59} \cup I\rangle);\ 0 \leq s \leq 3\},\ 1 \leq i \leq 9, +, \times_n \right\}$$

be the finite neutrosophic complex modulo integer real quaternion vector space of finite dimension over Z_{59}.

B has several subspaces and B can be written as a direct sum of 2 spaces 3 spaces and soon say upto 9 spaces.

For instance

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_i \in P_{NC};\ 1 \leq i \leq 3 \right\} \subseteq B \text{ and}$$

$$P_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \; a_i \in P_{NC};\ 1 \leq i \leq 6 \right\} \subseteq B$$

are subspaces of B such that

$$P_1 \cap P_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

We can write using P_i's as direct sum $P_1 + P_2 = B$.

Consider

$$T_1 = \left\{\begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \end{bmatrix} \middle| \; a_i \in P_{NC};\ 1 \leq i \leq 3\right\} \subseteq B,$$

$$T_2 = \left\{\begin{bmatrix} 0 & a_1 & 0 \\ 0 & a_2 & 0 \\ 0 & a_3 & 0 \end{bmatrix} \middle| \; a_i \in P_{NC};\ 1 \leq i \leq 3\right\} \subseteq B \text{ and}$$

$$T_3 = \left\{\begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & a_3 \end{bmatrix} \middle| \; a_i \in P_{NC};\ 1 \leq i \leq 3\right\} \subseteq B$$

are subspaces of B.

$$\text{Further } T_i \cap T_j = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\};$$

$1 \leq i, j \leq 3$. $T_1 + T_2 + T_3 = B$ is the direct sum of subspaces of B.

Example 1.51: Let $M = \{(a_1 | \; a_2 \; a_3 \; | \; a_4 \; a_5 \; a_6) \mid a_t \in P_{NC} = \{b_0 + b_1i + b_2j + b_3k \mid b_j \in C(\langle Z_{11} \cup I\rangle);\ 0 \leq j \leq 3\},\ 1 \leq t \leq 9, +\}$ be the neutrosophic finite complex modulo integer real quaternion vector space over the field Z_{11}.

$P_1 = \{(a_1 | \; 0 \; 0 \; | \; 0 \; 0 \; 0) \mid a_1 \in P_{NC}\} \subseteq M,$

$P_2 = \{(0 \; | \; a_2 \; 0 \; | \; 0 \; 0 \; 0) \mid a_2 \in P_{NC}\} \subseteq M,$

$P_3 = \{(0 \; | \; 0 \; a_3 \; | \; 0 \; 0 \; 0) \mid a_3 \in P_{NC}\} \subseteq M,$

$$P_4 = \{(0 \mid 0\ 0 \mid a_4\ 0\ 0) \mid a_4 \in P_{NC}\} \subseteq M,$$

$$P_5 = \{(0 \mid 0\ 0 \mid 0\ a_5\ 0) \mid a_5 \in P_{NC}\} \subseteq M,$$

and $P_6 = \{(0 \mid 0\ 0 \mid 0\ 0\ a_6) \mid a_6 \in P_{NC}\} \subseteq M$
be real quaternion neutrosophic complex modulo integer vector subspaces of M.

Clearly $P_i \cap P_j = \{(0 \mid 0\ 0 \mid 0\ 0\ 0)\}$ for $1 \leq i, j \leq 6$ and
$P_1 + P_2 + \ldots + P_6 = M$ is the direct sum.

We can have several subspaces.

Next we proceed onto give examples the notion of linear transformation and linear operators.

Example 1.52: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle|\ a_i \in P_{NC} = \{b_0 + b_1i + b_2j + b_3k \mid b_s \in Z_{43};\ 0 \leq s \leq 3\},\ 1 \leq i \leq 9\ \right\}$$

and

$$N = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle|\ a_i \in P = \{b_0 + b_1i + b_2j + b_3k \mid b_s \in Z_{43};\ 0 \leq s \leq 3\},\ 1 \leq i \leq 12\right\}$$

be finite real quaternion vector space over the field Z_{43}.

Define $T : M \to N$ by

$$T\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ a_3 & a_4 \\ 0 & 0 \\ a_5 & a_6 \\ 0 & 0 \end{bmatrix},$$

$$\text{for } \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \in M;$$

it is easily verified T is a linear transformation from M to N.

Define $S : N \to M$ by

$$S\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 & a_3 + a_4 & a_5 + a_6 \\ a_7 + a_8 & a_9 + a_{10} & a_{11} + a_{12} \end{bmatrix},$$

$$\text{for } \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \in N.$$

S is a linear transformation from N to M.

Thus we can define linear transformation provided the spaces are built on the same field Z_p.

Example 1.53: Let

$$M = \left\{ \left[\begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{matrix}\right] \middle| \; a_i \in P_C = \{b_0 + b_1i + b_2j + b_3k \mid b_s \in C(Z_{17}); \right.$$

$$\left. 0 \leq s \leq 3\}, 1 \leq i \leq 9 \right\}$$

and

$$R = \left\{ \left[\begin{matrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{matrix}\right] \middle| \; a_i \in P = \{b_0 + b_1i + b_2j + b_3k \mid b_s \in C(Z_{17}); \right.$$

$$\left. 0 \leq s \leq 3\}, 1 \leq i \leq 9 \right\}$$

be two finite complex modulo integer real quaternion vector spaces defined over the field Z_{17}.

A map $T : S \to R$ defined by

$$T\left\{\left[\begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{matrix}\right]\right\} = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

is a linear transformation of S to R.

We can have several such linear transformation.

Example 1.54: Let S = {(a_1, a_2, a_3, a_4) | $a_t \in P_C = \{b_0 + b_1i + b_2j + b_3k \mid b_j \in C(Z_3); 0 \leq j \leq 3\}, 1 \leq t \leq 4$} and

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| \; a_t \in P_{NC} = \{b_0 + b_1i + b_2j + b_3k \mid b_p \in C(\langle Z_3 \cup I \rangle); 0 \leq p \leq 3\}, 1 \leq t \leq 8 \right\}$$

be the two finite real quaternion vector spaces over the field Z_3.

Define a map T : S → M by

$$T(a_1, a_2, a_3, a_4) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \\ a_3 & 0 \\ 0 & a_4 \end{bmatrix}.$$

It is easily verified T is a linear transformation.

This type of linear transformation can be defined and developed as a matter of routine.

We now proceed onto give a examples of linear operators.

Example 1.55: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_t \in P = \{b_0 + b_1i + b_2j + b_3k \mid b_p \in Z_{23}; 0 \leq p \leq 3\}, 1 \leq t \leq 9 \right\}$$

be the finite real quaternion space over the field Z_{23}.

Define $T : S \to S$ by

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}.$$

T is a linear operator on S.

Define $R : S \to S$ by

$$R\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 & a_2 \\ 0 & a_3 & 0 \\ a_4 & 0 & a_5 \end{bmatrix}.$$

R is a linear operator on S. We can have several such linear operators on S.

Example 1.56: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_t \in P_C = \{b_0 + b_1i + b_2j + b_3k \mid b_p \in C(Z_7);\ 0 \le p \le 3\}, 1 \le t \le 15\right\}$$

be the finite complex modulo integer real quaternion vector space over the field Z_7.

$T : V \to V$ defined by

$$T\begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \\ \vdots & \vdots & \vdots \\ a_{13} & 0 & 0 \end{bmatrix}$$

is a linear operator on V.

We can have several linear operators on V.

Now study of linear operators is also a matter of routine and hence left as an exercise to the reader.

However if we try to build inner product of these linear algebras of finite real quaternions we face with a lot of problems.

In the first place we see in case of vector space of finite real quaternions the product does not belong to the field.

That is we are not in a position to define a map from a pair of vectors in V to the field Z_p of scalars over which V is defined.

For if i, k $\in$ V, $\langle i, k\rangle = (n-1)j \notin Z_p$.

So in the first place we have to over come this problem.

Secondly we may have $\langle x, x\rangle = 0$ even if $x \neq 0$.

For if $x = i + j + k \in P = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_t \in Z_3\}$ then $\langle x, x\rangle = 0$ but $x \neq 0$.

For
$$\begin{aligned}\langle x, x\rangle &= \langle i + j + k, i + j + k\rangle \\ &= i^2 + j^2 + k^2 \text{ (as } i^2 = 2 = j^2 = k^2) \\ &= 2 + 2 + 2 \\ &= 0 \text{ (mod 3)}.\end{aligned}$$

Thus to over come these we define Smarandache vector space of real quaternions defined over the ring P of finite quaternions where

$$P = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in Z_p ; 0 \leq t \leq 3\}.$$

We see if this way it is define we can have a special type of inner product called the pseudo inner product. So we just define the notion of S-vector space of real quaternions over the S-finite real quaternions.

DEFINITION 1.6: *Let V be a vector space of finite real quaternions defined over the Smarandache ring P of finite real quaternions. We then define V to be a Smarandache vector space of finite real quaternions defined over P.*

We will illustrate this situation by some examples.

Example 1.57: Let

$$S = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \right| a_t \in P = \{b_0 + b_1i + b_2j + b_3k \mid b_p \in Z_{11}; \right.$$

$$0 \leq p \leq 3\}, 1 \leq t \leq 5\}$$

be the S-finite real quaternion vector space (linear algebra) over the S-ring P.

Example 1.58: Let W = $\{(a_1, a_2, \ldots, a_8) \mid a_t \in P_C = \{b_0 + b_1i + b_2j + b_3k \mid b_p \in C(Z_p); 0 \leq p \leq 3\}, 1 \leq t \leq 8, 0 \leq p \leq 3, +, \times\}$ be the finite real quaternion S-vector space over the S-ring $C(Z_{19})$ (or over the S-ring P_C).

W has inner product defined over it only when W is defined over P_C and not over $C(Z_{19})$.

Example 1.59: Let

$$S = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{19} & a_{20} \end{bmatrix} \right| a_i \in P_N = \{b_0 + b_1i + b_2j + b_3k \mid \right.$$

$$b_p \in \langle Z_{11} \cup I\rangle;\ 0 \leq p \leq 3\},\ 1 \leq i \leq 20\}$$

be the S-finite real quaternion vector space over the S-ring P_N. We can define on S an inner product which is as follows:

$$\text{If } X = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{19} & a_{20} \end{bmatrix} \text{ and } Y = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ \vdots & \vdots \\ b_{19} & b_{20} \end{bmatrix} \in S$$

$$\langle X, Y\rangle = \sum_{i=1}^{20} a_i b_i \quad \langle\ ,\ \rangle : S \to P_N \text{ is an inner product on S.}$$

Example 1.60: Let $M = \{(x_1, x_2, x_3) \mid x_i \in P_{NC} = \{a_0 + a_1i + a_2j + a_3k \mid a_p \in C(\langle Z_5 \cup I\rangle;\ 0 \leq p \leq 3\},\ 1 \leq i \leq 3, +, \times\}$ be the Smarandache finite real quaternion finite neutrosophic complex modulo integer vector space over the S-ring P_{NC}.

On M we can define an inner product as follows:

Let $x = (3iI + 2i_Fk, 0, 2kIi_F)$ and $y = (2i_FIj, 3i_F + 4iI, 2 + 3i_F) \in M$.

$$\langle x, y\rangle = \langle(3iI + 2i_Fk, 0, 2kIi_F), (2i_FIj, 3i_F + 4iI, 2 + 3i_F)\rangle$$

$= (3iI + 2i_Fk)(2i_FIj) + 0 + 2Ii_Fk \times (2 + 3i_F)$
$= 6kIi_F + 4 \times 4 \times 4iI + 4Ii_Fk + 6I \times 4 \times k$
$= kIi_F + 4iI + 4Ii_Fk + 4kI$
$= 4iI + 4Ik \in P_{NC}$.

This is the way inner product is defined on M.

That is if $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in M$ then

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3.$$

In general $\langle x, y\rangle \neq \langle y, x\rangle$ for $x, y \in M$.

This is also a difference between usual inner product and special pseudo inner product.

Consider $\langle y, x\rangle = \langle(2i_FIj\ 4iI + 3i_F, 2 + 3i_F), (3iI + 2i_Fk, 0, 2kIi_F)\rangle$

$= 2i_FIj \times 3iI + 2i_Fk + 0 + (2 + 3i_F)(2kIi_F)$
$= 6i_FI4k + 4 \times 4i \times I + 4kIi_F + 6 \times 4kI$
$= 4i_FIk + Ii + 4Ii_Fk + 4kI$
$= 3i_FIk + Ii + 4kI \in P_{NC}$.

Clearly for this $x, y \in M$ we see $\langle x, y\rangle \neq \langle y, x\rangle$.

Example 1.61: Let $P = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_{43}, 0 \leq i \leq 3, i^2 = j^2 = k^2 = ijk = 42; ij = 42ji = k, jk = 42kj = i, ki = 42ik = j, +, \times\}$ be the finite ring of real quaternions.

$$V = \left\{\begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \end{bmatrix} \middle| d_j \in P; 1 \leq j \leq 3\right\}$$

be a S-real quaternion vector space defined over P.

We define $\langle A, B\rangle$ for $A, B \in V$ as follows:

If A = $\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$ and B = $\begin{bmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{bmatrix}$ $\in$ V

then $\langle A, B\rangle = x_1y_1 + x_2y_2 + x_3y_3 + \ldots + x_9y_9 \in P$.

Thus $\langle\ ,\ \rangle : V \to P$ is an inner product on V. We can define inner products.

Now having introduced special pseudo inner product on V we call a S-vector space of real quaternions over a S-real finite quaternion ring to be a pseudo special inner product space provided we have a pseudo special inner product defined on that S-space.

All properties related to inner product spaces are derivable in case of these special pseudo inner product spaces also. Such study is considered as matter of routine and hence left as an exercise to the reader.

Now we can define linear functionals on these pseudo inner product spaces. Let V be a S-finite real quaternion space on which special inner produce is defined on the S-finite real quaternion ring P_C.

We see map $f : V \to P_C$ which is such that f is a linear transformation. For $\alpha \in V$; $f(\alpha) \in P_C$.

We will illustrate this situation by some examples.

Example 1.62: Let $V = \{(x_1, x_2, x_3) \mid x_i \in \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(Z_{17}),\ 0 \le t \le 3\}, 1 \le i \le 3, +, \times\}$ be the S-finite complex modulo integer real quaternion vector space over the S-complex finite modulo integer ring P_C.

Define $f : V \to C(Z_{17})$ by $f(x_1, x_2, x_3) = x_1 + x_2 + x_3$; f is a linear functional on V.

Example 1.63: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle Z_{19} \cup I \rangle); \right.$$

$$\left. 0 \leq t \leq 3\}, 1 \leq i \leq 6, +, \times_n \right\}$$

be the S-finite neutrosophic complex modulo integer vector space of finite real quaternions over the S-ring P_{NC}.

Define $f : V \to P_{NC}$ by

$$f\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right) = a_1 + a_2 + \ldots + a_6 \in P_{NC}.$$

$$\text{If } A = \begin{bmatrix} 3i + 8ji_F + 9i_F k \\ 4i_F I + 3i_F k \\ 7I + 2 \\ 6Ii_F i + 6i_F jI \\ 0 \\ 2I + 3i_F i + 5i_F kI \end{bmatrix} \in V.$$

$$f(A) = 3i + 8ji_F + 9i_F k + 4i_F I + 3i_F k + 7I + 2 + 6Ii_F i + 6i_F Ij + 0 + 2I + 3i_F i + 5i_F kI$$

$= (2 + 9I + 4i_FI) + (3 + 6Ii_F + 3i_F)i + (12i_F + 5i_FI)k + (8i_F + 6Ii_F)j \in P_{NC}$.

f is a linear functional on V.

Example 1.64: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in P_C = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in C(Z_5); \right.$$

$$0 \leq t \leq 3,\ 1 \leq i \leq 9\}$$

be the S-finite complex modulo integer real quaternion vector space over the S-ring P_C.

Define a map $f : M \to P_C$ as follows:

Let $x = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \in M$ then $f(x) = x_1 + x_5 + x_9 \in P_C$.

In particular if $x = \begin{bmatrix} 3i_F + i & 2i_F + 3i + k & 3ik \\ 0 & 4ki_F + 2i_Fi & 0 \\ 2i_F & 3i_Fi + k & 2i_Fj + k + i \end{bmatrix} \in M$

then

$$f(x) = 3i_F + I + 4ki_F + 2i_Fi + 2i_Fj + k + i$$

$$= 3i_F + (2 + 2i_F)i + 2i_Fj + (4i_F + 1)k \in P_C.$$

This is the way the linear functional on V is defined.

Example 1.65: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in Z_{19}; 0 \leq t \leq 3\}, +, \times \right\}$$

be the S-finite real quaternion vector space over the S-ring P.

Define for $X \in S$

$$f(X) = \sum_{i=1}^{12} a_i \text{ then } f : S \to P$$

is a linear functional on S.

Interested reader can study the notion of dual S-space and other related properties of S. Since it is considered as a matter of routine it is left as an exercise to the reader.

Now we can define polynomial ring of finite real quaternions.

Let $P = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_i \in Z_n, \; 0 \leq i \leq 3, i^2 = j^2 = k^2 = ijk = (n-1), ij = (n-1)ji = k; jk = (n-1)kj = i, ki = (n-1)ik = j, +, \times\}$ be the finite ring of real quaternions.

$$\text{Define } P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P, +, \times \right\};$$

x an indeterminate is a ring defined as a polynomial ring of finite real quaternions in the variable x.

P[x] is only a non commutative ring as P is a non commutative ring. P[x] is of infinite order. P[x] can have zero divisors.

For any n as P[x] can have zero divisors as P has zero divisors. Clearly P $\subseteq$ P[x].

We will first illustrate this situation by some examples.

Example 1.66: Let

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in Z_6;\ 0 \leq t \leq k\}, +, \times \right\}$$

be the finite real quaternion polynomial ring.

P[x] has zero divisors.

For if $p(x) = (3i + 3j + 3k)x^3 + 3kx + (3 + 3j)$ and

$q(x) = (4i + 2j)x + 2kx + 4 \in P[x]$.

It is easily verified p(x). q(x) = 0.

Thus p(x) is a zero divisor.

P[x] is a non commutative ring.

For let $t = 3ix + j$ and

$$\begin{aligned} s &= jx^3 + k \in P[x] \\ ts &= (3ix + j)(jx^3 + k) \\ &= 3ijx^4 + j^2x^3 + 3ikx + jk \\ &= 3kx^4 + 5x^3 + 3 \times 5jx + i \\ &= 3kx^4 + 5x^3 + 3jx + i \qquad \text{--- I} \end{aligned}$$

$$\begin{aligned} \text{Consider } st &= (jx^3 + k)(3ix + j) \\ &= 3jix^4 + 3kix + j^2x^3 + kj \\ &= 3 \times 5kx^4 + 3jx + 5x^3 + 5i \\ &= 3kx^4 + 5x^3 + 3jx + 5i \qquad \text{--- II} \end{aligned}$$

Clearly I and II are different hence P[x] is non commutative that is st ≠ ts for the given t, s ∈ P[x].

Thus P[x] is a non commutative ring of infinite order and has zero divisors.

Example 1.67: Let

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in Z_{11};\right.$$

$0 \leq t \leq 3$; $i^2 = j^2 = k^2 = ijk = 10$; $ij = 10ji = k$, $jk = 10kj = i$, $ki = 10ik = j\} = P, +, \times\}$ be the finite real quaternion polynomial ring. P[x] is Smarandache ring as $Z_{11} \subseteq P[x]$.

We see $Z_{11} \subseteq P \subseteq P[x]$.

THEOREM 1.6: *Let*

$$P_N = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in Z_n;\right.$$

$$0 \leq t \leq 3\}, +, \times\}$$

be the polynomial ring of finite real quaternions.

(i) P[x] has zero divisors.
(ii) P[x] has finite number of units.
(iii) P[x] is a non commutative ring and is of infinite order.
(iv) P[x] is a S-ring only if Z_n is a S-ring or Z_n is a field.

The proof is direct and hence left as an exercise to the reader.

Now we see if p(x) ∈ P[x] then p(x) can be differentiated but integration as in case of usual polynomial rings is little difficult.

We will illustrate this situation by some examples.

Example 1.68: Let

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in Z_{12};\right.$$

$$\left. 0 \leq t \leq 3; +, \times\} \right\}$$

be the polynomial ring of real finite quaternions.

Let $p(x) = 3ix^4 + 6jx^2 + 3 \in P[x]$

$$\frac{dp(x)}{dx} = 12ix^3 + 6j \times 2x + 0 = 0.$$

Thus in case of polynomials in p(x) non constant polynomials can have their derivatives to be zero.

Now suppose $q(x) = ix^{11} + 2ix + 3 \in P[x]$.

Now $\int q(x)\, dx$

$$= \int ix^{11} + 2ix + 3dx$$

$$= \frac{ix^{12}}{12} + \frac{2ix^2}{2} + 3x + c.$$

Clearly $12 \equiv 0 \pmod{12}$ so this integral of this polynomial $q(x) \in P[x]$ remains undefined.

This is the type of problems we face while integrating.

However the differential exist but the conclusion the constant polynomial alone when differentiated gives zero is false.

Now we say $p(x) \in P[x]$ has a root α if $p(\alpha) = 0$ where $\alpha \in P$.

We can solve for roots for $p(x) \in P[x]$, however we are not sure whether every $p(x) \in P[x]$ is solvable.

Let $p(x) = x + i$ is such that when $x = 11i$;
$p(11i) = 11i + I = 0$.

So $x = –i = 11i$ is the root. So we can not say all linear equations in x are solvable.

$P(x) = 3x + 4j \in P[x]$

Hence $3x = –4j$, $3x= 8j$.

But $3x = 8j$ cannot give the value for x as 3 is a zero divisor in Z_{12}. So this type of linear equations are not solvable in general. We declare this equation has no solution.

Consider $4x + 5k =0$.

We see $4x = 7k$ but x has no solution as 4 is an idempotent in Z_{12}. So in P[x] sometimes we may not be in position to solve even the linear equations.

Thus $(ai + bj + cj) x + t = 0$ where $t, a, b, c \in Z_n$ and if $a^2 + b^2 + c^2 = 0$ then also this linear equation is not solvable.

Thus the major difference between the usual polynomials and polynomials is finite real quaternions rings. So we will reach a stage even a linear (first degree) polynomial has no solution.

Consider the polynomial $8x^2 + (4i + 2k)x + 11j = 0$ then it is factored as $8x^2 + 4xi + 2xk + ik = 0$ ($ik = 11j$).

$$4x (2x+ i) +(2x + i)k = 0$$
$$(2x + i) (4x + k) = 0.$$

Coefficients are from Z_{12} so this second degree eqution can be factored into linear terms but linear terms do not have a solution as the coefficient of x is a zero divisors in this case and in another case it is an idempotent as well as a zero divisor.

This sort of study leads to difficult situations to define the notion of solvability of linear equations.

Thus if $p(x) \in P[x]$ is a polynomial of degree n with coefficients form the finite neutrosophic ring of real quaternions P.

We say $p(x) \in P[x]$ is completely solvable if

$$p(x) = (a_1x + b_1) \ldots (a_nx + b_n) \qquad \mathrm{I}$$

such that each $a_ix + b_i$ is a solvable linear equation and each a_i is not an idempotent or a zero divisor but is invertible in P.

We say p(x) is linearly decomposable but not completely solvable even if one of the linear equations $a_ix + b_i$ is not solvable.

We say p(x) is not linearly decomposable that is representation of the form I is not possible if p(x) cannot be represented as linear product.

p(x) is totally irreducible if p(x) cannot be written as q(x) r(x) where $r(x) \neq p(x)$ or 1 and $q(x) \neq r(x)$ or 1.

All linear polynomials are totally irreducible.

Consider $p(x) = 4x^7 + 1 \in P[x]$ is a totally irreducible. $(4, 3 \in Z_{12})$. Study in this direction is also interesting.

For solving such equations is not easy even if P(x) is built over Z_p; p a prime.

Now we can define right ideal, left ideal and ideal in P[x] which is considered as a matter of routine for they are just like finding ideals in non commutative ring R[x] where R is non commutative and is finite order.

Now we can give one of two examples of them before we proceed to define vector spaces or linear algebras using these P[x].

Example 1.69: Let

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in Z_3; \; 0 \leq t \leq 3; +, \times\} \right.$$

be the polynomial ring with coefficients from the ring of real quaternions.

Consider $p(x) = (i + j + k)\, x^3 + 1 \in P[x]$ we can generate the ideal I by p(x); I = {All polynomials of degree greater than or equal to three with coefficients from P}.

Now we see one has to stick to the property of multiplying the values given to x as ax for xa will give a different solution. Though we may say ax = xa but once the x takes a value in P we see $ax \neq xa$ in general.

This problem must also be kept in mind while working with this special type of non commutative ring.

Thus if $p(x) = a_0 + a_1x + \ldots + a_nx^n$ and if $\alpha = x$, $\alpha \in P$ then $p(\alpha) = a_0 + a_1\alpha + \ldots + a_n\alpha^n$.

Clearly $\alpha a_i \neq a_i \alpha$ for all $a_i \in P$.

Example 1.70: Let

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in Z_{40}; \right.$$

$0 \leq t \leq 3$; $i^2 = j^2 = k^2 = ijk = 39$, $ij = 39ji = k$, $jk = 39kj = i$, $ki = 39ik = j\}$, $+, \times\}$ be the polynomial ring of real quaternions.

$$M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \{0, 10, 20, 30\} \subseteq Z_{40} \right\} \subseteq P\,[x]$$

is an ideal of P[x].

P[x] has atleast five ideals.

Next we proceed onto define the notion of $P_C[x]$ polynomial ring of complex modulo integer finite real quaternions.

Let $P_C[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \right.$ $C(Z_n)_;\ 0 \leq t \leq 3\}$ be the finite comple modulo integer real quaternion polynomial ring. $P_C[x]$ is of infinite order. $P_C[x]$ is non commutative.

Clearly $P[x] \subseteq P_C[x]$ and P[x] is only a subring of $P_C[x]$ and is not an ideal.

We will illustrate this situation by some examples.

Example 1.71: Let

$$P_C[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(Z_{12}); \right.$$
$$\left. 0 \leq t \leq 3; +, \times \right\}$$

be the finite complex modulo integer real quaternion polynomial ring.

$P_C \subseteq P_C[x]$ is a subring $P[x] \subseteq P_C[x]$ is also a subring and none of them are ideals.

Example 1.72: Let

$$P_C[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(Z_{17}); \right.$$
$$\left. 0 \leq t \leq 3; +, \times \right\}$$

be the finite complex modulo integer real quaternion polynomial ring. $P_C[x]$ has linear equations which are not solvable.

Take p(x) = (2i + 2j + 3k) x + 5j $\in P_C[x]$, this is a linear equation but (2i + 2j + 3k)x = 12j but we cannot find the value of x as (2i + 2j+ k) is a zero divisor in $P_C[x]$.

Example 1.73: Let

$$P_C[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(Z_{15}); \right.$$
$$\left. 0 \leq t \leq 3\}; +, \times \right\}$$

be the polynomial ring of finite complex modulo integer. $P_C[x]$ has ideals and subrings of finite order.

Take $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{15} \right\}$ is only a subring of $P_C[x]$.

Next we define neutrosophic finite real quaternion polynomial ring.

$$P_N[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid \right.$$
$$\left. b_t \in \langle Z_n \cup I \rangle; 0 \leq t \leq 3; +, \times \right\}$$

is defined as the polynomial ring of neutrosophic finite real quaternion ring.

$P_N[x]$ also has subrings and ideals. Infact all equations of the form $p(x) = Ix^n + a_{n-1} x^{n-1} + \ldots + a_0$ are not solvable as $I^2 = I$ is an idempotent.

Thus $Ix + a = p(x)$ has no solution for x.

We will illustrate this by some examples.

Example 1.74: Let

$$P_N[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle Z_{42} \cup I\rangle;\ 0 \leq t \leq 3;\ +, \times\} \right.$$

be the ring of neutrosophic polynomial quaternions. $P_N[x]$ is a non commutative ring of infinite order. $P_N[x]$ has subrings and ideals.

Example 1.75: Let

$$P_N[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle Z_{17} \cup I\rangle;\ 0 \leq t \leq 3;\ +, \times\} \right.$$

be the neutrosophic finite real quatertions polynomial ring.

All polynomial whose highest degree coefficient as I will be not solvable.

Next we define the notion of finite complex modulo integer neutrosophic real quaternion polynomial ring.

Let

$$P_{NC}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C\langle Z_n \cup I\rangle;\ 0 \leq t \leq 3\};\ +, \times \right\}.$$

($P_{NC}[x]$) is infinite and is not commutative $P_{NC}[x]$ also has zero divisors, ideals, subrings. Several linear polynomials have no solution in $P_{NC}[x]$.

Example 1.76: Let

$$P_{NC}[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle Z_7 \cup I\rangle); 0 \leq t \leq 3; +, \times\}\right.$$

be the polynomial complex neutrosophic finite real quaternion ring. $P_{NC}[x]$ has ideals, subrings, zero divisors and units.

Example 1.77: Let

$$P_{NC}[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle Z_{12} \cup I\rangle); 0 \leq t \leq 3; +, \times\}\right.$$

be the polynomial complex neutrosophic finite real quaternion ring. $P_{NC}[x]$ has ideals and subrings;

$p(x) = (4i + 6j)x + (3i + 4j + 2k + 2) \in P_{NC}[x]$ is not solvable for x as the coefficient of x is a zero divisors.

So we see even linear equation cannot be solved in case of $P_{NC}[x]$, polynomial rings with neutrosophic complex finite real quaternion coefficients.

For $p(x) = 4i\, Ix + 3j = 0$ has no solution.

$q(x) = 6Ii_F x + 2Ij$ has no solution and so on.

Thus these rings display a very different properties.

Hence solving equations in them is a difficult job however if we have to consider polynomials for finding eigen values and eigen vectors.

To find eigen values and eigen vectors we have some advantages as well as some hurdles.

The advantage being that all characteristic equations will have the highest coefficient to be always one. This will help us solve the linear equations without any difficulty.

Secondly the main disadvantage is that we cannot define the vector space or linear algebra over the field but only over the S-rings which are real quaternion rings or finite real complex modulo integer quaternion rings or finite neutrosophic complex real modulo integer quaternion rings.

Thus the very concept of eigen values exist we need the basic vector space or linear algebra to be a Smarandache vector space or Smarandache linear algebra over the real quaternion S-rings P or P_C or P_N or P_{NC}.

With this in mind we now proceed onto define S-vector spaces (S-linear algebras) or vector spaces or linear algebras using P[x] or $P_C[x]$ or $P_N[x]$ or $P_{NC}[x]$.

Let P[x] be the real finite quaternion vector space over the field Z_p.

P[x] is also a linear algebra over Z_p.

On similar line we have $P_C[x]$, $P_N[x]$ and $P_{NC}[x]$ built over P_C or P_N or P_{NC} respectively.

We will illustrate this situation by some examples.

Example 1.78: Let

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in Z_{11};\right.$$

$$\left. 0 \leq t \leq 3; +\} \right.$$

be a vector space over the field Z_{11}. P[x] is also a linear algebra of finite real quaterion over the field Z_{11}.

Example 1.79: Let

$$P_C[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(Z_5); \; 0 \leq t \leq 3\}; +\right\}$$

be the vector space over Z_5 or a S-vector space over $C(Z_5)$.

In both cases we see $P_C[x]$ has subspaces and (S-subspaces).

Study in this direction is also a matter of routine.

Likewise we can have using polynomials $P_N[x]$ and $P_{NC}[x]$ also which will be described by examples.

Example 1.80: Let

$$P_N[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle Z_{19} \cup I\rangle; \; 0 \leq t \leq 3\right\}$$

be the vector space of neutrosophic real quaternions over the field Z_{19} or S-vector space defined over $\langle Z_{19} \cup I\rangle$ the neutrosophic S-ring.

$$W = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle Z_{19} \cup I\rangle\right\}$$

is a S-subspace of $P_N[x]$ over $\langle Z_{19} \cup I\rangle$; however W is also a subspace over Z_{19}.

Example 1.81: Let

$$P_{NC}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle Z_{13} \cup I\rangle);\ 0 \leq t \leq 3\}, +, \times \right\}$$

be a vector space of Z_{13} or S-vector space over $\langle Z_{13} \cup I\rangle$ or a S-vector space over $C(Z_{13})$ or a S-vector space over $C(\langle Z_{13} \cup I\rangle)$ or a S-vector space over P or a S-vector space over P_C or S-vector space over P_N or S-vector space over P_{NC}.

Thus we can have 7 types of S-vector spaces using the same $P_{NC}[x]$; however only $P_{NC}[x]$ defined over P_{NC} alone can give the solution to eigen values / eigen vectors or for defining the concept of linear functionals and inner product spaces.

Now finally we just give one or two examples regarding eigen values or eigen vectors.

Already to this end the concept of matrices with entries from P_{NC} (or P_C or P_N or P) have been defined.

Let $A = (a_{ij})_{n \times n}$ matrix with entries from P or P_C or P_N or P_{NC}. We see as in case of usual matrices we can define the notion of eigen values and eigen vectors however for the solution to exist we need to have the spaces defined over P or P_N or P_C or P_{NC}, then only we can have solution for the characteristic equation.

This will be represented by an example or two.

$$\text{Let } A = \begin{bmatrix} 5i + j & 3i + j + k \\ 0 & 2 + 3i + 4j + k \end{bmatrix}$$

$5i + j,\ 3i + j + k,\ 2 + 3i + 4j + k \in P = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_t \in Z_7;\ 0 \leq t \leq 3\}$

$$|A - \lambda| = \begin{bmatrix} 5i + j - \lambda & 2i + k \\ 1 + j & 2 + 3i + 4j + k - \lambda \end{bmatrix}$$

$$= (5i + j - \lambda)\ (2 + 3i + 4j + k - \lambda) = 0$$

implies $\lambda = 6j + 2i$ or $\lambda = 5 + 4i + 3j + 6k$.

This is the way it is solved.

Hence if the vector space is defined over Z_7 this characteristic values will have no relevance.

Let $A = \begin{bmatrix} 3 + i & 2j + k \\ 1 + j & 4k + j + 2 \end{bmatrix}$ be 2×2 matrix with entries from P.

$$|A - \lambda| = \begin{bmatrix} 3 + i - \lambda & 2i + k \\ 1 + j & 4k + j + 2 - \lambda \end{bmatrix}$$

$= \quad (3 + i - \lambda)\ (4k + j + 2 - \lambda) - (1 + j)\ (2i + k)$

$= \quad (3 + i)\ (4k + j + 2) - \lambda\ (4k + j + 2) - \lambda\ (3 + i) + \lambda^2 - (1 + j)\ (2j + k)$

$= \quad 12k + 3j + 6 + 4 \times 6j + k + 2i - \lambda(5 + i + j + 4k) + \lambda^2 - (2j + 2 \times 6 + k + i).$

Thus $\lambda^2 + \lambda\ (2 + 6i + 6j + 3k) + (4k + 6j + 2i + 6) = 0$.

Solving even a quadratic equation cannot be done.

This is left as an open conjecture.

For we can solve second degree equation in field of characteristic zero but solving equations with coefficients from P or P_C or P_N or P_{NC} happens to be one of the challenging problems of the present day.

$$\text{Let } A = \begin{bmatrix} 2i_F i + j & 3i + 4i_F k + j \\ 0 & 3i_F k + 4i_F j + i \end{bmatrix}$$

where $2i_F i+j$, $3i + 4i_F k + j$, $3i_F k + 4i_F j + i \in C(Z_5)$

$$\text{Now } |A - \lambda| = \begin{vmatrix} 2i_F i + j - \lambda & 3i_F + 4i_F k + j \\ 0 & 3i_F k + 4i_F j + i - \lambda \end{vmatrix}$$

$= (2i_F i + j - \lambda)\,(3i_F k + 4i_F j + i - \lambda) = 0.$

Thus $(2i_F i + j)\,(3i_F k + 4i_F j + i) - (2i_F i + j)\,\lambda -$
$(3i_F k + 4i_F j + i)\,\lambda + \lambda^2 = 0$

$\lambda^2 + \lambda(4j + 3i_F i + 4i + i_F j + 2i_F k) + (2i_F i + j)\,(4i_F j + i + 3i_F k)$
$= 0$

$\lambda^2 - ((3i_F + 4)I + (4 + i_F)\,j + 2i_F k) + 8 \times 4k + 4i_F \times 4 +$
$2i_F \times 4 + 4k + 6 \times 4 \times 4j + 3i_F i = 0$

that is

$\lambda^2 + ((3i_F + 4)i + (4 + i_F)\,j + 2i_F k) + 4i_F + k + j + 3i_F i = 0.$

Solving this equation is difficult.

Now we work with entries in P_N.

$$\text{Let } A = \begin{bmatrix} 3I + 2jI & 0 \\ 4I + 4jI + k & 2i + 3jI \end{bmatrix}$$

where $3I + 2jI$, $4I + 4jI + k$, $2i + 3jI \in P_N = \{a_0 + a_1 i + a_2 j + a_3 k \mid$
$a_t \in \langle Z_5 \cup I\rangle;\ 0 \le t \le 3\}$.

$$|A - \lambda| = \begin{vmatrix} 3I + 2jI - \lambda & 0 \\ 4I + 4jI + k & 2i + 3jI - \lambda \end{vmatrix}$$

$= (3I + 2jI - \lambda)\ (2i + 3jI - \lambda) = 0$

$(3I + 2jI)\ (2i + 3jI) - \lambda\ (2i + 3jI) - \lambda\ (3I + 2jI) + \lambda^2 = 0$

$\lambda^2 + \lambda\ (2I + 3i) + (3I + 2jI)\ (2i + 3jI) = 0$

$\lambda^2 + \lambda\ (2I + 3i) + (6Ii + kI + 9jI + 6 \times 4 \times I) = 0$

$\lambda^2 + \lambda\ (2I + 3i) +(Ii + kI + 4jI + 4I) = 0.$

Solving even this equation is difficult.

Thus it is left as an open conjecture to solve even equations of second degree in case of finite quaternion rings P or P_C or P_N or P_{NC}.

Let

$$A = \begin{bmatrix} 3Ii_F i + 2i_F j + Ik & 0 \\ 4i_F I + 2Ij + 3Ii_F & 2 + 2I + 3i_F I + 4i_F Ik \end{bmatrix}$$

where $3Ii_Fi + 2i_FI + Ik$, $4i_FI + 2Ij + 3Ii_F$, $2 + 2I + 3i_FI + 4i_FIk \in P_{NC} = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle Z_5 \cup I\rangle)_;\ 0 \leq t \leq 3\}$.

$$|A - \lambda| = \begin{vmatrix} 3Ii_F i + 2i_F j + ik - \lambda & 0 \\ 4i_F I + 2Ij + 3Ii_F & 2 + 2I + 3i_F I + 4i_F Ik - \lambda \end{vmatrix}$$

$= (3Ii_Fi + 2i_Fj + I - \lambda)\ (2 + 2I + 3i_FI + 4i_FIK - \lambda)$

$= \quad (3Ii_Fi + 2i_Fj + Ik)\ (2 + 2I + 3i_F + 4i_FIk + \lambda^2 -$
$\lambda\ (2 + 2I + 3i_FI + 4i_FIk + 3Ii_Fi + 2i_Fj + Ik)$

$= \lambda^2 - \lambda\ (2+ 2I + 3i_FI + 3Ii_Fi + 2i_Fj + (4i_FI + I)k) + (6Ii_Fi + 4i_Fj + 2Ik + 6i_FIi + 4Ii_Fj + 2Ik + 9I \times 4i + 6 \times 4jI + 3i_FIk + 12I \times 4 \times 4j + 8 \times 4 \times iI + 4i_FI \times 4)$

$= \lambda^2 + \lambda\,(3 + 3I + 2i_FI + 2Ii_Fi + 3i_Fj + (i_FI + 4I)k) + (Ii_F + (3I + 2Ii_F)I + (4i_F + 4Ii_F + 3I)j + (4I + 3Ii_F)k)$ is the quadratic in λ^2 and finding a solution for this is extremely difficult.

Thus finding eigen values and eigen vectors of the S-vector spaces of finite real quaternions, S-vector spaces of finite complex modulo integer real quaternions, S-vector spaces of finite neutrosophic real quaternions and S-vector spaces of finite complex modulo integer neutrosophic real quaternions defined over the S-rings P, P_C, P_N and P_{NC} respectively.

This study is innovative and interesting. However finding solution to the related characteristic polynomials happens to be a difficult task even if it is a quadratic equation.

However if we have only diagonal values for the matrices and if the nth degree polynomial can be linearly factorized then as the coefficient of every α is one we can always have a solution.

$$\text{Let } A = \begin{bmatrix} 3i_F & 0 & 6 \\ 0 & 4i_F + 3iI & 0 \\ 2i_F & 0 & 3i_FIi + ji_F \end{bmatrix};$$

to find the eigen values of A.

$$|A - \lambda| = \begin{bmatrix} 3i_F - \lambda & 0 & 6 \\ 0 & 4i_F + 3iI - \lambda & 0 \\ 0 & 0 & 3i_FIi + i_Fj - \lambda \end{bmatrix}$$

$= (3i_F - \lambda)(4i_F + 3i_F - \lambda)\,(3i_FIi + i_Fj - \lambda)$
$= 0$ implies

$$3i_F - \lambda = 0$$
$$4i_F + 3Ii - \lambda = 0$$
$$3i_FIi + i_Fj - \lambda = 0$$

$\lambda - 3i_F = 0$; $\lambda = 3i_F$, $\lambda = 4i_F + 3Ii$ and $\lambda = 3i_F\,Ii + i_Fj$.

Hence to find eigen values for a square matrix with values from P or P_C or P_N or P_{NC}.

We suggest one can make use row echelon method and make as many zeros as possible and then solve the equation.

In many cases it may result in the linear form of the representation of the characteristic polynomial.

To the best of the authors knowledge this happens to be the workable method for solving equations in P or P_C or P_N or P_{NC}.

$$\text{Let } A = \begin{bmatrix} 3i_F j & 2i_F + 2k & 4i + 3j \\ 2i_F j + 2 & 4k & 2 + j \\ 0 & 2 + 2j & 3i_F k \end{bmatrix}$$

be the given matrix with the entries from

$P = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in C(Z_5), 1 \leq i \leq 3\}$.

We try to make as many zeros as possible by simple row reduction method.

Add row (1) with row (2) in A and place it in the first row of A.

$$A \sim \begin{bmatrix} 2 & 2i_F + k & 4i + 4j + 2 \\ 2i_F j + 2 & 4k & 2 + j \\ 0 & 2 + 2j & 3i_F k \end{bmatrix} = A_1$$

Adding (1) and (2) rows of A_1 and placing it in the 2^{nd} row gives

$$\begin{bmatrix} 2 & 2i_F + k & 4i + 4j + 2 \\ 2i_F j + 4 & 2i_F & 4i + 4 \\ 0 & 2 + 2j & 3i_F k \end{bmatrix} = A_2.$$

Multiply the 2nd row by 3 (mod 5).

$$A_2 \sim \begin{bmatrix} 2 & 2i_F + k & 4i + 4j + 2 \\ i_F j + 2 & i_F & 2i + 2 \\ 0 & 2 + 2j & 3i_F k \end{bmatrix} = A_3$$

Multiply 2nd row of A_3 by 3 and add with first row we get the following.

$$A_3 \sim \begin{bmatrix} 3i_F j + 3 & k & 4j + 3 \\ i_F j + 2 & i_F & 2i + 2 \\ 0 & 2 + 2j & 3i_F k \end{bmatrix} = A_4$$

In A_4 add row (1) and row (2) and place it in 2nd row

$$A_4 \sim \begin{bmatrix} 3i_F j + 3 & k & 4j + 3 \\ 4i_F j & i_F + k & 2i + 4j \\ 0 & 2 + 2j & 3i_F k \end{bmatrix} = A_5.$$

Multiply first row of A_5 by 4 we get

$$A_5 \sim \begin{bmatrix} 2i_F j + 2 & 4k & j + 2 \\ 4i_F j & i_F + k & 2i + 4j \\ 0 & 2 + 2j & 3i_F k \end{bmatrix} = A_6$$

Add row (1) with (2) and place it in row (2) in A_6.

$$A_6 \sim \begin{bmatrix} 2i_F j + 2 & 4k & j + 2 \\ i_F j + 3 & i_F & 2i + 2 \\ 0 & 2 + 2j & 3i_F k \end{bmatrix} = A_7$$

Multiply row 3 of A_7 by i we get

$$A_7 \sim \begin{bmatrix} 2i_F j + 3 & 4k & j + 2 \\ i_F j + 3 & i_F & 2i + 2 \\ 0 & 2i + 3k & 3i_F j \end{bmatrix} = A_8.$$

Like this one can row reduce the matrices and obtain a matrix with maximum number of zero.

It is kept on record that making the diagonal elements to be non zero or big does not matter or intervene the simplification we need to make only non diagonal elements zero.

Multiply the 3rd row of A_8 by 4 we get

$$A_8 \sim \begin{bmatrix} 2i_F j + 3 & 4k & j + 2 \\ i_F j + 3 & i_F & 2i + 2 \\ 0 & 3 + 4k & 2i_F j \end{bmatrix} = A_9$$

and so on.

It is pertinent to keep on record that one need to know that such reduction is also a difficult job.

The next difficult task is finding eigen vectors using these eigen values.

Now having defined these S-vector spaces we shall call the eigen values only as S-eigen values.

Next we sugget a few problems. Some of the problems are very difficult and some of them can be taken as open conjectures.

However only by solving these problems one will under stand the depth involved in arriving at a solution.

Further only when the study is based on the respective S-rings P or P_C or P_N or P_{NC} the S-space has some nice features otherwise that when defined over Z_p it does not enjoy several of the properties.

Problems

1. Find the special features enjoyed by the finite ring of real quaternions.

$$P = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_p; 0 \leq i \leq 3\}.$$

   (i) Can P be a S-ring?
   (ii) Can P have S-subrings which are not S-ideals?
   (iii) Can P have S-ideals?
   (iv) Can P have subrings which are not S-ideals?
   (v) Can P have subrings which are not S-subrings?
   (vi) Can P have S-units?
   (vii) Can P have S-zero divisors?
   (viii) Can P have zero divisors which are not S-zero divisors?
   (ix) Can P have ideals which are not S-ideals?
   (x) Can P have S-idempotents?
   (xi) Can P have idempotents which are not S-idempotents?
   (xii) Describe or develop any other property associated with P.

2. Let R = $\{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_{11}; 0 \leq i \leq 3, i^2 = j^2 = k^2 = 10 = ijk, ij = 10(ji) = k, jk = 10(kj) = i, ki = 10(ik) = j, +, \times\}$ be the ring of finite real quaternions.

   Study questions (i) to (xii) of problem 1 for this S.

3. Let S = $\{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_{11}; 0 \leq i \leq 3, i^2 = j^2 = k^2 = 16 = ijk, ij = 16(ji) = k, jk = 16(kj) = i, ki = 16(ik) = j, +, \times\}$ be the ring of finite complex modulo integer real quaternions.

   (i) Study questions (i) to (xii) of problem 1 for this R.

(ii) Find o(S).

4. Let B = $\{a_0 + a_1i + a_2j + a_3k \mid a_i \in C(\langle Z_7 \cup I\rangle); 0 \leq i \leq 3, i_F^2 = 6, (i_F1)^2 = 1, i^2 = j^2 = k^2 = 6 = ijk, ij = 6(ji) = k, jk = 6(kj) = i, ki = 6(ik) = j; +, \times\}$ be the finite neutrosophic complex modulo integer real ring of quaternions.

   (i) Study questions (i) to (xii) of problem 1 for this B.
   (ii) Find o(B).

5. Let P_{NC} = $\{a_0 + a_1i + a_2j + a_3k \mid a_i \in C(\langle Z_{24} \cup I\rangle); 0 \leq i \leq 3, +, \times\}$ be the ring of real finite quaternions.

   Study questions (i) to (xii) of problem 1 for this P_{NC}.

6. Let P_C = $\{a_0 + a_1i + a_2j + a_3k \mid a_i \in C(Z_{14}); 0 \leq i \leq 3, i^2 = j^2 = k^2 = 13 = ijk, ij = 13(ji) = k, jk = 13(kj) = i, ki = 13(ik) = j\}, i_F^2 = 13, (i_Fi)^2 = (i_Fj)^2 = (i_Fk)^2 = 1; +, \times\}$ be the ring of finite modulo integer real quaternions.

   (i) Find o(P_C).
   (ii) Study questions (i) to (xii) of problem 1 for this P_C.

7. Prove $P \subseteq P_C \subseteq P_{NC}$ and $P \subseteq P_N \subseteq P_{NC}$.
   Prove P_C, P_{NC} and P_N are all S-vector spaces over P, provided P is a S-ring of finite real quaternions.

8. Compare the four real quaternion rings P, P_C, P_N and P_{NC} for a fixed Z_n.

9. Under what conditions P or P_N or P_C or P_{NC} will have commutative subring of finite order; that is for what values of n of Z_n.

10. Describe and develop some interesting features about the rings P, P_C, P_N or P_{NC}.

11. Describe the semi idempotents in P, P_C, P_N or P_{NC}.

12. Obtain some special features enjoyed by finite complex modulo integer real quaternion rings.

13. Let V = $\{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_{43}, 0 \leq i \leq 3, i^2 = j^2 = k^2 = 42 = ijk, ij = 42(ji) = k, jk = 42(kj) = i, ki = 42(ik) = j, +\}$ be the vector space of finite real quaternions over the field Z_{43}.

    (i) Find a basis of V over Z_{43}.
    (ii) What is the dimension of V over Z_{43}?
    (iii) How many subspaces of V exist?
    (iv) Find $\mathrm{Hom}_{Z_{43}}(V, V) = S$.
    (v) Is S a vector space over Z_{43}?
    (vi) If S is a vector space over Z_{43}; what is the dimension of S over Z_{43}?

14. Let V = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in P = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in Z_{13}; 0 \leq t \leq 3\}\ 1 \leq i \leq 5; +\}$ be the vector space of finite real quaternions over the field Z_{13}.

    Study questions (i) to (vi) of problem 13 for this V.

15. Let P = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle|\ a_i \in P_C = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in Z_{17};\right.$

    $0 \leq t \leq 3; i^2 = j^2 = k^2 = 16 = ijk, ij = 16(ji) = k, jk = 16(kj) = i, ki = 16(ik) = j, +\}$ be the vector space of real quaternions over the field Z_{17}.

    Study questions (i) to (vi) of problem 13 for this P.

16. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \end{bmatrix} \middle| \right.$ $a_i \in P_N = \{b_0 + b_1i$ $+ b_2j + b_3k \mid b_t \in C(\langle Z_{11} \cup I\rangle); 0 \leq t \leq 3; 1 \leq i \leq 20, +\}$ be the vector space of real quaternions over the field Z_{11}.

    Study questions (i) to (vi) of problem 13 for this M.

17. Let B = $\left\{\begin{pmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \end{pmatrix} \middle| \right.$ $a_i \in P_C = \{b_0 + b_1i + b_2j$ $+ b_3k \mid b_t \in C(Z_7); 0 \leq t \leq 3\}$ $1 \leq i \leq 20, +\}$ be the vector space of real quaternions over the field Z_7.

    Study questions (i) to (vi) of problem 13 for this B.

18. Let $P_{NC}[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \middle| \right.$ $a_i \in P_{NC} = \{b_0 + b_1i + b_2j + b_3k \mid$ $b_t \in C(\langle Z_{19} \cup I\rangle); 0 \leq t \leq 3\}$ be the vector space over Z_{19}.

    (i) Is $P_{NC}[x]$ infinite dimension over Z_{19}?
    (ii) Find subspaces of $P_{NC}[x]$ over Z_{19}.
    (iii) Can $P_{NC}[x]$ have any subspace of finite dimension over Z_{19}?

19. Find some special features enjoyed by $P_{NC}[x]$ as a ring of finite real quaternions.

20. How will one solve polynomials in $P_{NC}[x]$?

21. Construct a method of solving polynomials in $P_C[x]$.

22. Find the method of constructing polynomials of nth degree in $P_{NC}[x]$.

23. Let $P_N[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle Z_{19} \cup I\rangle;\ 0 \leq t \leq 3, +, \times\}\right.$ be the polynomial ring of real quaternions.

    (i) Find right ideals which are not left ideals.
    (ii) Find two sided ideals of $P_N[x]$.
    (iii) Give some subrings which are not ideals.
    (iv) Find S-ideals if any.
    (v) Find a S-subring which is not an S-ideal.
    (vi) Can $P_N[x]$ have ideals which not S-ideals?
    (vii) Can $P_N[x]$ have ideals which are not S-ideals?
    (viii) Can $P_N[x]$ have S-zero divisors?
    (ix) Can $P_N[x]$ have S-idempotents?

24. Let $P_{NC}[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle Z_7 \cup I\rangle);\ 0 \leq t \leq 3, +, \times\}\right.$ be the ring of finite complex neutrosophic real quaternion polynomials.

    Study questions (i) to (ix) of problem 23 for this $P_{NC}[x]$.

25. Find the special features enjoyed by S-vector spaces.

26. Let $V = \{(a_1, a_2, \ldots, a_{10}) \mid a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(Z_{11});\ 0 \leq t \leq 3,\ 1 \leq i \leq 10, +\}$ be a S-vector space over the S-ring P_C.

    (i) Find the S-basis of V over P_C.
    (ii) What is the dimension of V over P_C?
    (iii) Find $\mathrm{Hom}_{P_C}(V, V) = R$.
    (iv) Find $\mathrm{Hom}_{P_C}(V, P_C) = S$.
    (v) What is dimension of R as a S-vector space?

(vi) What is dimension of S over P_C?
(vii) Write V as a direct sum.

27. Let M = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in\right.$ $C(\langle Z_3 \cup I\rangle)$; $0 \le t \le 3\}$ $1 \le i \le 10$, +, $\times_n\}$ be a S-linear algebra finite complex neutrosophic real quaternions over the S-ring P_{NC}.

Study questions (i) to (vii) of problem 26 for this M.

28. Let W = $\left\{ \begin{bmatrix} a_1 & a_2 & \cdots & a_7 \\ a_8 & a_9 & \cdots & a_{14} \end{bmatrix} \middle| a_i \in P_C = \{b_0 + b_1 i + b_2 j\right.$ $+ b_3 k \mid b_t \in C(Z_{41})$; $0 \le t \le 3\}$ $1 \le i \le 14$, +, $\times_n\}$ be the S-linear algebra finite complex modulo integer real quaternions over the S-ring P_C.

Study questions (i) to (vii) of problem 26 for this W.

29. Let M = $\left\{ \begin{bmatrix} a_1 & a_2 & \cdots & a_{10} \\ a_{11} & a_{12} & \cdots & a_{20} \end{bmatrix} \middle| a_i \in P_N = \{b_0 + b_1 i + b_2 j\right.$ $+ b_3 k \mid b_t \in \langle Z_7 \cup I\rangle$; $0 \le t \le 3\}$ $1 \le i \le 20\}$ be the S-vector space of neutrosophic finite real quaternions over P_N.

N = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & \cdots & \cdots & a_8 \\ a_9 & \cdots & \cdots & a_{12} \\ a_{13} & \cdots & \cdots & a_{16} \end{bmatrix} \middle| a_i \in P_N = \{b_0 + b_1 i + b_2 j +\right.$

$b_3k \mid b_t \in \langle Z_7 \cup I\rangle;\ 0 \leq t \leq 3\}\ 1 \leq i \leq 16, +\}$ be the S-vector space of neutrosophic finite real quaternions over P_N.

(i) Find $\text{Hom}_{P_N}(M, N) = S$.

(ii) What is the algebraic structure enjoyed by S?

(iii) Find $R_1 = \text{Hom}_{P_N}(M, M)$ and $R_2 = \text{Hom}_{P_N}(N, N)$.

(iv) What is the algebraic structure enjoyed by R_1 and R_2?

(v) Find $B_1 = \text{Hom}_{P_N}(M, P_N)$ and $B_2 = \text{Hom}_{P_N}(N, P_N)$.

(vi) What is the algebraic structure enjoyed by B_1 and B_2?

(vii) Can there exist a $T \in \text{Hom}(M, N)$ such that T has T^{-1}?

(viii) Find a linear transformation $L \in \text{Hom}(N, M)$ so that $\ker L \neq (0)$.

30. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix} \right|$ $a_i \in P_{NC} = \{b_0 + b_1i +$

$b_2j + b_3k \mid b_t \in C(\langle Z_5 \cup I\rangle);\ 0 \leq t \leq 3\}\ 1 \leq i \leq 30\}$ and

$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \right|$ $a_i \in P_{NC} = \{b_0 + b_1i + b_2j + b_3k$

$\mid b_t \in C(\langle Z_5 \cup I\rangle);\ 0 \leq t \leq 3\}\ 1 \leq i \leq 30\}$be the S-vector space of finite neutrosophic complex modulo integer real quaternion over the S-ring P_{NC}.

Study questions (i) to (viii) of problem 29 for this V and W.

31. Let S = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \cdots & \cdots & \cdots & a_{10} \\ a_{11} & \cdots & \cdots & \cdots & a_{15} \\ a_{16} & \cdots & \cdots & \cdots & a_{20} \\ a_{21} & \cdots & \cdots & \cdots & a_{25} \end{bmatrix}\right|$ $a_i \in P_C = \{b_0 + b_1i +$

$b_2j + b_3k \mid b_t \in C(Z_{19}); 0 \leq t \leq 3\}$ $1 \leq i \leq 25, +, \times_n\}$ and

R = $\left\{\begin{bmatrix} a_1 & a_2 & \cdots & a_7 \\ a_8 & a_9 & \cdots & a_{14} \\ a_{15} & a_{16} & \cdots & a_{21} \\ a_{22} & a_{23} & \cdots & a_{28} \end{bmatrix}\right|$ $a_i \in P_C = \{b_0 + b_1i + b_2j +$

$b_3k \mid b_t \in C(Z_{19}); 0 \leq t \leq 3\}$ $1 \leq i \leq 28, +, \times_n\}$ be two S-linear algebra of finite complex modulo integer real quaternions over the S-ring P_C.

Study questions (i) to (viii) of problem 29 for this S and R.

32. Obtain some special features associated with P[x]. (P the finite ring of real quaternions over Z_n).

33. Prove linear equations can not be solved in general in P[x].

34. Find a new method of solving polynomial equations in P[x].

35. Show differentiation and integration in general do not follow the usual law in case of P[x] or P_C[x] or P_N[x] or P_{NC}[x].

36. Derive a method of solving second degree polynomial equation in P[x] or P_C[x] or P_N[x] or P_{NC}[x].

37. Let $P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in Z_{40};\ 0 \leq t \leq 3\} \right.$ be the polynomial ring of real quaternions.

    (i) Find ideals of P[x]
    (ii) Can P[x] have S-ideals?
    (iii) Find S-subrings which are not ideals of P[x].
    (iv) Find ideals which are not S-ideals.
    (v) Show P[x] has infinite number of zero divisors.
    (vi) Find a method of solving p(x) = 0 in P[x].
    (vii) Can P[x] have S-zero divisors?
    (viii) Can P[x] have idempotents which are not S-idempotents?
    (ix) Can P[x] have S-units?
    (x) Define the concept of semi idempotents in P[x].
    (xi) Can P[x] have S-semi idempotents?

38. If in P[x], Z_{40} in problem 37 is replaced by Z_{43} study questions (1) to (xi) of problem 37 for that P[x].

39. Let $P_C[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P_C = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in C(Z_{12});\ 0 \leq t \leq 3\}, +, \times \right\}$ be the finite complex modulo integer real quaternion polynomial ring.

    Study questions (i) to (xi) of problem 37 for this $P_C[x]$.

40. If in problem 39, Z_{12} is replaced by Z_{19}.

    Study questions (i) to (xi) of problem 37 for this $P_C[x]$.

41. Let $P_N[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in P_N = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in \langle Z_{15} \cup I\rangle;\ 0 \leq t \leq 3\}, +, \times \right\}$ be the finite neutrosophic real quaternion polynomial ring.

Study questions (i) to (xi) of problem 37 for this $P_N[x]$.

42. If in problem (41) $\langle Z_{15} \cup I\rangle$ is replaced by $\langle Z_{23} \cup I\rangle$ study questions (i) to (xii) of problem 37 for this $P_N[x]$.

43. Let $P_{NC}[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle Z_6 \cup I\rangle);\ 0 \leq t \leq 3\}, +, \times\right\}$ be the polynomial ring of finite complex neutrosophic modulo integer real quaternions.

    Study questions (i) to (xi) of problem 37 for this $P_{NC}[x]$.

44. If in problem (43) $C(\langle Z_6 \cup I\rangle)$ is replaced by $C(\langle Z_5 \cup I\rangle)$ Study questions (i) to (xi) of problem 37 for this $P_{NC}[x]$.

45. Let $P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in Z_{13};\ 0 \leq t \leq 3\}, +\right\}$ be a vector space over the field Z_{13}.

    (i) Find a basis of $P[x]$ over Z_{13}.
    (ii) What is the dimension of $P[x]$ over Z_{13}?
    (iii) Find subspaces of $P[x]$.
    (iv) Find Hom $(P[x], P[x]) = M$.
    (v) What is the algebraic structure enjoyed by M?
    (vi) If $P[x]$ is a S-vector space over P study question (i) to (v) for that S-vector space.
    (vii) Let $N = \{\text{Hom}(P[x], P)\}$; what is the algebraic structure enjoyed by N?

46. Let $P_N[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle Z_{19} \cup I\rangle;\ 0 \leq t \leq k\}, +\right\}$ be the S-vector space defined over P_N.

(i) Study questions (i) to (v) of problem 45 for this $P_N[x]$.
(ii) If P_N is replaced by Z_{19} study questions (i) to (v) problem 45 for that vector space over Z_{19}.

47. Let $P_C[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_C = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in C(Z_{43});\ 0 \leq t \leq 3\}, +\right\}$ be a vector space of finite complex real quaternions over the field Z_{43}.

(i) Study questions (i) to (v) of problem 45 for this $P_C[x]$.
(ii) If Z_{43} is replaced by $C(Z_{43})$ study questions (1) to (v) for that S-vector space $P_C[x]$ over $C(Z_{43})$.

48. Let $P_{NC}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_{NC} = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in C(\langle Z_7 \cup I\rangle);\ 0 \leq t \leq 3\}, +\right\}$ be the vector space of finite complex neutrosophic real quaternions over the field Z_7.

(i) Study problems (i) to (v) of question 45 for this $P_{NC}[x]$.
(ii) If Z_7 is replaced by $C(\langle Z_7 \cup I\rangle)$ study question (i) to (v) for that S-vector space $P_{NC}[x]$ over P_{NC}.

49. Obtain any special features enjoyed by the S-vector spaces $P_{NC}[x]$, $P_N[x]$, $P_C[x]$ and $P[x]$ over the S-ring P_{NC}, P_N, P_C and P respectively.

50. Find the eigen values and eigen vectors associated with the matrix;

$$A = \begin{bmatrix} 3i + 4j + 2k & 2 + i \\ 4j + 8k + i & 1 + 7i + 10j + 3k \end{bmatrix}$$

3i + 4j + 2k, 2 + i, 4j + 8k + i and 1 + 7i + 10j + 3k $\in$ P = $\{a_0 + a_1 i + a_2 j + a_3 k \mid a_t \in Z_{11};\ 0 \le t \le 3\}$.

51. Find the eigen values of the matrix

$$S = \begin{bmatrix} 3i+4j+3Ik+I & 0 & 3i+4kI \\ 2i+3jI+Ik & 7i+8Ij & 0 \\ 0 & 4i+9Ij+11kI & 8iI+k \end{bmatrix}$$

where the entries are from

$P_N = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_t \in \langle Z_{13} \cup I\rangle;\ 0 \le t \le 3\}$.

52. Can any special means be invented to find eigen values when the entries of the matrix are from P or P_C or P_N or P_{NC}?

53. Let $A = \begin{bmatrix} 3Ii_F + ki_F & 2Ij + 3i_F Ik & 0 \\ 4Ii_F j + 4Ik & 2i_F Ik + iI & Ij \\ i_F I + jIi_F & ki_F + ji_F + I & I + i_F + j \end{bmatrix}$

where the entries of A are from

$P_{NC} = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_t \in C(\langle Z_5 \cup I\rangle);\ 0 \le t \le 3\}$.

Find the eigen values and eigen vectors associated with the matrix A.

54. Can the theorem of diagonalization be obtained for S-vector spaces over P_{NC}?

55. Study question (54) if P_{NC} is replaced by P_C.

56. Is Cayley's theorem true in case of S-vector spaces over P_N?

57. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \, a_i \in P_C = \{b_0 + b_1 i + b_2 j \right.$

$+ b_3 k \mid b_t \in C(Z_{43});\ 0 \leq t \leq 3\}\ 1 \leq i \leq 12\}$ be the S-vector space over P_C.

(i) Study all properties associated with M.
(ii) Define the inner product of M.
(iii) Can Hom $(M, P_C) = S$ be isomorphic with M as a S-vector space?
(iv) What is the dimension of Hom $(M, M) = V$ over P_C?
(v) Find a $T \in V$ so that T is not invertible.
(vi) Find a T in V so that $\ker T \neq (0)$.
(vii) Find a T in V that $\ker T = (0)$.

58. Let $W = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \end{bmatrix} \middle| \, a_i \in P_{NC} = \{b_0 + b_1 i + \right.$

$b_2 j + b_3 k \mid b_t \in C(\langle Z_7 \cup I\rangle;\ 0 \leq t \leq 3\};\ 1 \leq i \leq 18, +, \times_n\}$

and $V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \, a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + \right.$

$b_3 k \mid b_t \in C(\langle Z_7 \cup I\rangle;\ 0 \leq t \leq 3\}\ 1 \leq i \leq 18, +, \times_n\}$ be a S-vector spaces defined over P_{NC}.

(i) Find a basis of V and W
(ii) Find Hom $(V, W) = S$.
(iii) What is the dimension of S over P_{NC}?

(iv) Find Hom (V, V) and Hom(W, W).
(v) Find Hom (V, P_{NC}) = L_1, and Hom (W, P_{NC}) = L_2. Is $L_1 \cong L_2$?
(vi) Define two different types of inner products on V and W.
(vii) Show these inner product spaces are of same dimension over P_{NC}.
(viii) Study (i) to (iv) if P_{NC} is a replaced by P_N.
(ix) Study (i) to (iv) if P_{NC} is a replaced by P_C.
(ix) Study (i) to (iv) if P_{NC} is a replaced by P.

59. Study the special features enjoyed by S-linear functionals using S-vector spaces over P or P_C or P_N or P_{NC}.

60. Prove only if the linear algebras /vector spaces are defined over P or P_C or P_N or P_{NC} we can have the concept of pseudo special inner product on them.

61. Can we define bilinear forms in S-vectors spaces defined over P or P_C or P_N or P_{NC}?

Chapter Two

INFINITE QUATERNION PSEUDO RINGS USING [0, n)

In this chapter we for the first time introduce the notion of pseudo quaternion rings of infinite order using [0, n), n ≥ 2. For n=1 we cannot define this notion.

These infinite quaternion rings does not satisfy several of the usual properties of the real ring of quaternions.

In this chapter several interesting features about them are described and developed.

DEFINITION 2.1: *Let* $Q = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, n)$ $n \geq 2$, $0 \leq t \leq 3$, $i^2 = j^2 = k^2 = ijk = n–1$, $ij = (n–1)ji = k$, $jk = (n–1)kj = i$, $ki = (n–1)ik = j$, $+\}$ *be a group which is commutative and of infinite order under +. This Q will be known as the group of infinite interval real quaternions.*

Q is commutative and has subgroups of finite order also.

We will first illustrate this by some examples.

Example 2.1: Let Q = $\{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 5), i^2 = j^2 = k^2 = ijk = 4, ij = 4ji = k, jk = 4kj = i, ki = 4ik = j, 0 \leq t \leq 3, +\}$ be the interval group of real quaternions $|Q| = \infty$.

P = $\{a_0 + a_1i + a_2j + a_3k \mid a_t \in Z_5, 0 \leq t \leq 3\} \subseteq Q$ is a finite subgroup of Q.

Example 2.2: Let S = $\{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 12), 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 11, ij = 12ji = k, jk = 12kj = i, ki = 4ik = j, +\}$ be the group under +.

S has several subgroups of finite order.

Finding subgroups of infinite order is an interesting work.

We can have several such groups of interval finite quaternions of infinite order.

We see the interval real quaternions under × is a semigroup.

DEFINITION 2.2: *Let S = $\{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, n)$ $0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = n-1, ij = (n-1)ji = k, jk = (n-1)kj = i, ki = (n-1)ik = j, \times\}$ be a semigroup under product known as the interval real quaternions semigroup, is of infinite order which is non commutative and is a monoid. Infact these semigroups have zero divisors.*

They are S-semigroups provided [0, n) is a S-semigroup.

We will illustrate this situation by some examples.

Example 2.3: Let S = $\{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 7), 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 6, ij = 6ji = k, jk = 6kj = i, ki = 6ik = j, \times\}$ be a interval real quaternion semigroup under ×.

x = $(3i + 2j + k) \in S$ is a zero divisor for $x^2 = (3i + 2j + k)^2$ $= 9i^2 + 4j^2 + k^2 + 6ij + 6ji + 2jk + 2kj + 3ik + 3ki$

$$= \quad 9 \times 6 + 4 \times 6 + 6 + 6k + 6 \times 6 \times k + 2i + 2 \times 6i + 3j + 3 \times 6j$$
$$= \quad 54 + 24 + 6 + 42k + 14i + 3j + 18j$$
$$= \quad 0.$$

Thus x is a zero divisor in S and
$3^2 + 2^2 + 1^2 = 0 \pmod 7$. Also if $x = 3.5i$ and $y = 4k \in S$

$$x \times y = 3.5i \times 4k = 14.0 \times 6j = 0 \pmod 7.$$

We have several zero divisors in S.

This semigroup also have units $3.5 = 1 \pmod 7$. We see how product operation is performed in S.

$x= (1 + i) \in S$

$$x^2 = (1 + i)^2 = 1 + 2i + i^2 = 2i.$$

$$x_1^2 = (1 + 2i)^2 = 1 + (2i)^2 + 4i = 1 + 4 \times 6 + 4i = 4 + 4i.$$

$$x_2^2 = (1 + 3i)^2 = 1 + (3i)^2 + 6i = 1 + 9 \times 6 + 6i = 6 + 6i.$$

$$x_3^2 = (1 + 4i)^2 = 1 + 16 \times 6 + 8i = 97 + 8i = i+6 \text{ and so on.}$$

Let $y = i + j \in S$, $y^2 = 6 + 6 + k + 6k$
$y^2 = 5$.

Example 2.4: Let $S = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 12), i^2 = j^2 = k^2 = ijk = 11, ij = 11ji = k, jk = 11kj = i, ki = 11ik = j, 0 \leq t \leq 3, \times\}$ be the interval real quaternion semigroup which is non commutative and of infinite order.

Consider $2i + 2j + 2k = x \in S$;
$x^2 = (2i + 2j + 2k)^2$

$= \quad 4i^2 + 4j^2 + 4k^2 + 4ij + 4ji + 4jk + 4kj + 4ki + 4ik$
$= \quad 4 \times 11 + 4 \times 11 + 4 \times 11 + 4k + 4 \times 11 \times k + 4i + 4 \times 11i + 4j + 4 \times 11 \times j$
$= \quad 0 \pmod{12}$ is a zero divisor in S.

Example 2.5: Let $S = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 17), i^2 = j^2 = k^2 = ijk = 16, ij = 16ji = k, jk = 16kj = i, ki = 16ik = j, 0 \leq t \leq 3, \times\}$ be the interval semigroup of real quaternions of infinite order.

S is non commutative and has zero divisors, idempotents and units.

THEOREM 2.1: *Let $S = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, n)\ 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = n-1, ij = (n-1)ji = k, jk = (n-1)kj = i, ki = (n-1)ik = j, \times\}$ be the interval semigroup of real quaternions;*

(i) S is non commutative and is of infinite order.
(ii) S has zero divisors, units and idempotents.
(iii) S has subsemigroups of finite order.
(iv) S has subsemigroups of infinite order.

The proof is direct and hence left as an exercise to the reader.

Example 2.6: Let $S = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 42), i^2 = j^2 = k^2 = ijk = 41, ij = 41ji = k, jk = 41kj = i, ki = 41ik = j, 0 \leq t \leq 3, \times\}$ be the real quaternion interval semigroup. S has units, zero divisors and idempotents.

S has both finite and infinite subsemigroup.

S is a Smarandache subsemigroup. For consider the table given by $\{i, j, k, (n-1)i, (n-1)k, (n-1)j, (n-1), 1\}$.

×	1	n–1	(n–1)i	(n–1)j	(n–1)k	i	j	k
1	1	n–1	(n–1)i	(n–1)j	(n–1)k	i	j	k
n–1	n–1	1	i	j	k	(n–1)i	(n–1)j	(n–1)k
(n–1)i	(n–1)i	i	(n–1)	k	(n–1)j	1	(n–1)k	j
(n–1)j	(n–1)j	j	(n–1)k	(n–1)	i	k	1	(n–1)i
(n–1)k	(n–1)k	k	j	(n–1)i	(n–1)	(n–1)j	i	1
i	i	(n–1)i	1	(n–1)k	j	(n–1)	k	(n–1)j
j	j	(n–1)j	k	1	(n–1)i	(n–1)k	(n–1)	i
k	k	(n–1)k	(n–1)j	i	1	j	(n–1)i	(n–1)

Clearly this is a non commutative group of order 8.

Inview of this we have the following theorem.

THEOREM 2.2: *Let $S = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, n)$ $0 \leq t \leq 3$, $i^2 = j^2 = k^2 = ijk = n–1$, $ij = (n–1)ji = k$, $jk = (n–1)kj = i$, $ki = (n–1)ik = j$, $\times\}$ be the interval real quaternion semigroup. S is a Smaradache semigroup and $G = \{i, j, k, 1, (n–1), (n–1)i, (n–1)k, (n–1)j\} \subseteq S$ is a group of order 8 in S.*

Proof is direct and hence left as an exercise to the reader.

Example 2.7: Let $M = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 19)$, $i^2 = j^2 = k^2 = ijk = 18$, $ij = 18ji = k$, $jk = 18kj = i$, $ki = 18ik = j$, $0 \leq t \leq 3$, $\times\}$ be the interval real quaternions semigroup of infinite order.

Now we can build infinitely many group of interval real quaternions and semigroup of interval real quaternions. This situation described by the following examples.

Example 2.8: Let M = $\{(a_1, a_2, \ldots, a_5)\ a_t \in Q = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 26),\ i^2 = j^2 = k^2 = ijk = 25, ij = 25ji = k, jk = 25kj = i, ki = 25ik = j, 0 \leq t \leq 3, +\}$ be the group of interval real quaternions.

M has subgroups of both finite and infinite order.

Let $N_1 = \{(a_1, 0, 0, 0, 0) \mid a_1 \in Q = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 26),\ i^2 = j^2 = k^2 = ijk = 25, ij = 25ji = k, jk = 25kj = i, ki = 25ik = j, +\} \subseteq M$,

$N_2 = \{(0, a_2, 0, 0, 0) \mid a_2 \in Q\} \subseteq M$,

$N_3 = \{(0, 0, a_3, 0, 0) \mid a_3 \in Q, +\} \subseteq M$,

$N_4 = \{(0, 0, 0, a_4, 0) \mid a_4 \in Q, +\} \subseteq M$ and

$N_5 = \{(0, 0, 0, 0, a_5) \mid a_5 \in Q, +\} \subseteq M$ are all interval quaternion subgroups of M and all of them are also normal subgroups of M.

We see $W = N_1 + N_2 + N_3 + N_4 + N_5$.

Consider $P_1 = \{(a_1, 0, 0, 0, 0) \mid a_1 \in [0, 26), +\} \subseteq W$,

$P_2 = \{(0, a_2, 0, 0, 0) \mid a_2 \in [0, 26), +\} \subseteq W$,

$P_3 = \{(0, 0, a_3, 0, 0) \mid a_3 \in [0, 26), +\} \subseteq W$,

$P_4 = \{(0, 0, 0, a_4, 0,) \mid a_4 \in [0, 26), +\} \subseteq W$ and

$P_5 = \{(0, 0, 0, 0, a_5) \mid a_5 \in [0, 26), +\} \subseteq M$ are all interval subgroups of M and all of them are a normal subgroups of M as M is always commutative under +.

Clearly $M \neq N_1 + N_2 + N_3 + N_4 + N_5$.

Infact $N_1 + N_2 + \ldots + N_5$ is only an interval subgroup of W and that subgroup is also a normal subgroup of W.

Thus this group has several subgroups.

Example 2.9: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| \; a_i \in Q = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, 24), \right.$$

$$i^2 = j^2 = k^2 = ijk = 23, \; ij = 23ji = k, \; jk = 23kj = i,$$

$ki = 23ik = j$, $0 \leq t \leq 3$, $0 \leq i \leq 9$, $+\}$ be the interval quaternion group of column matrices.

M has subgroups all which are normal subgroups of M.

$$\text{Let } L_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in Q, +\right\} \subseteq M,$$

$$L_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_2 \in Q, +\right\} \subseteq M,$$

$$L_3 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_3 \in Q, +\right\} \text{ and so on and}$$

$$L_9 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_9 \end{bmatrix}\right| a_9 \in Q, +\right\} \text{ be the subgroups of M.}$$

$L_1 + L_2 + \ldots + L_9 = M$ is a direct sum.

Example 2.10: Let

$$M = \left\{\left.\begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ \vdots & \vdots & & \vdots \\ a_{64} & a_{65} & \ldots & a_{70} \end{bmatrix}\right| a_i \in \{b_0 + b_1 i + b_2 j + b_3 k \mid\right.$$

$b_t \in [0, 19)$, $i^2 = j^2 = k^2 = ijk = 18$, $ij = 18ji = k$, $jk = 18kj = i$, $ki = 18ik = j\}$, $0 \leq i \leq 70$, $+\}$ be the interval real quaternion group of infinite order.

M has several subgroups which are normal.

$$P_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & \ldots & 0 \\ a_{10} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ a_{64} & 0 & \ldots & 0 \end{bmatrix}\right| a_1, a_{10}, a_{19}, a_{28}, a_{37}, a_{46}, a_{53},\right.$$

$$\left. a_{64} \in [0, 19)\right\} \subseteq M,$$

$$P_2 = \left\{\left.\begin{bmatrix} 0 & a_2 & 0 & \ldots & 0 \\ 0 & a_{11} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_{65} & 0 & \ldots & 0 \end{bmatrix}\right| a_2, a_{11}, a_{20}, a_{29}, a_{38}, a_{47},\right.$$

$$\left. a_{56}, a_{65} \in [0, 19)\right\} \subseteq M,$$

$$P_3 = \left\{\left.\begin{bmatrix} 0 & 0 & a_3 & 0 & \ldots & 0 \\ 0 & 0 & a_{12} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{66} & 0 & \ldots & 0 \end{bmatrix}\right| a_3, a_{12}, a_{21}, a_{30}, a_{39},\right.$$

$$\left. a_{48}, a_{57}, a_{66} \in [0, 19)\right\} \subseteq M,$$

$$P_4 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & a_4 & 0 & \ldots & 0 \\ 0 & 0 & 0 & a_{13} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & a_{67} & 0 & \ldots & 0 \end{bmatrix}\right| a_4, a_{13}, a_{22}, a_{31}, a_{40},\right.$$

$a_{49}, a_{58}, a_{67} \in [0, 19)\} \subseteq M$ and so on;

$$P_9 = \left\{\left.\begin{bmatrix} 0 & 0 & \dots & a_9 \\ 0 & 0 & \dots & a_{18} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & a_{70} \end{bmatrix}\right| a_2, a_{11}, a_{20}, a_{29}, a_{38}, a_{47},\right.$$

$$\left. a_{56}, a_{65} \in [0, 19)\right\} \subseteq M$$

be the collection of subgroups of M.

P_i's are only subgroups of M.

$$T_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}\right| a_1 \in \{\{b_0 + b_1i + b_2j + b_3k\},\right.$$

$$\left. b_t \in [0, 19), 0 \leq t \leq 3\right\} \subseteq M,$$

$$T_2 = \left\{\left.\begin{bmatrix} 0 & a_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}\right| a_2 \in \{\{b_0 + b_1i + b_2j + b_3k \mid\right.$$

$$\left. b_t \in [0, 19), 0 \leq t \leq 3\right\} \subseteq M, \ldots,$$

$$T_{70} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{70} \end{bmatrix}\right| a_{70} \in [0, 24)\right\} \subseteq M$$

are all subgroups of M and all of them are normal subgroups of M.

We see $T_1 + T_2 + \ldots + T_{70} = V \subseteq M$ is only a subgroup of M.

Such study is interesting and involves lot of properties.

Example 2.11: Let

$$R = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| a_i \in \{b_0 + b_1i + b_2j + b_3k \mid$$

$$b_t \in [0, 16), 0 \leq t \leq 3\}, 1 \leq i \leq 9, +\}$$

be the interval group of real quaternions.

R has subgroups and normal subgroups.

$$T_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right| a_1 \in Z_{16}\} \subseteq R$$

is only subgroup which is also a normal in R.

Several such finite and infinite subgroups of R exist all of which are normal in R.

Example 2.12: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \right| a_i \in \{b_0 + b_1i + b_2j + b_3k \mid$$

$b_t \in [0, 40)$, $0 \leq t \leq 30$, $i^2 = j^2 = k^2 = ijk = 39$, $ij = 39ji = k$, $jk = 39kj = i$, $ki = 39ik = j$, $+\}$ be the interval group of real quaternions.

$$\text{Let } P_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| a_1, a_2, a_3 \in [0, 40), +\right\} \subseteq V,$$

P_1 is a subgroup which is normal.

$$\text{Let } B_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| a_1, a_2, a_3 \in \{b_0 + b_1 i + \right.$$

$$b_2 j + b_3 k \mid b_t \in [0, 40), 0 \leq t \leq 30, + \subseteq V,$$

B_1 is a subgroup of V and B_1 is a normal subgroup of V.

Thus V has normal subgroups. All normal subgroup of V are of infinite order or infinite order.

All the finite subgroups of V are normal.

Inview of this we have the following theorem.

THEOREM 2.3: *Let $V = \{(a_{ij})$ be a $m \times n$ matrix $1 \leq i \leq m$ and $1 \leq j \leq n$, $a_{ij} \in \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, s), 0 \leq t \leq 3, +\}$ be the interval group of real quaternions.*

(i) *Every subgroup of V is normal.*

(ii) Every normal subgroup of V is of finite or of infinite order.

Proof is direct hence left as an exercise to the reader.

Example 2.13: Let T = {(a_1 | a_2 a_3 | a_4 a_5) | $a_i \in$ {$b_0 + b_1i + b_2j + b_3k$ | $b_t \in [0, 10)$, $0 \leq t \leq 3$, $1 \leq i \leq 5$, +} be the interval group of real quaternions.

T has atleast ${}_5C_1 + {}_5C_2 + \ldots + {}_5C_5$ number of infinite subgroups which are normal.

We have atleast ${}_5C_1$ + + ${}_5C_2$ + + ${}_5C_3$ + + ${}_5C_4$ number of subgroups of finite order which are normal. All subgroups are normal.

Example 2.14: Let

$$P = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ \hline a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \end{array}\right] \middle| \; a_i \in \{b_0 + b_1i + b_2j + b_3k \mid \right.$$

$$b_t \in [0, 40), 0 \leq t \leq 3, 1 \leq i \leq 7, +\}$$

be the interval real group of real quaternions of super column matrices.

P has subgroups of finite order say

$$M_1 = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \hline a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \end{bmatrix} \right| a_i \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in Z_{43}, \right.$$

$$0 \leq t \leq 3, 1 \leq i \leq 7\}$$

$$M_2 = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \hline a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \end{bmatrix} \right| a_i \in \{b_0 + b_1i \right.$$

$$b_0, b_1 \in Z_{43}, 1 \leq i \leq 7\}$$

$$M_3 = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \hline a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \end{bmatrix} \right| a_i = \{b_0 + b_1j,\ b_0, b_1 \in Z_{43}, 1 \leq i \leq 7\} \right.$$

$$M_4 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ \hline a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \end{array} \right] \middle| \; a_i = \{b_0 + b_1k \,,\, b_0, b_1 \in Z_{43}, 1 \leq i \leq 7\} \right.$$

$$M_5 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ \hline a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \end{array} \right] \middle| \; a_i = \{b_0\, i + b_1j \,,\, b_0, b_1 \in Z_{43}, 1 \leq i \leq 7\} \right.$$

$$M_6 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ \hline a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \end{array} \right] \middle| \; a_i = \{b_0i + b_1j, b_0, b_1 \in Z_{43}, 1 \leq i \leq 7\} \right.$$

$$M_7 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \hline a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \end{bmatrix} \middle| \; a_i = \{b_0 i + b_1 j, b_0, b_1 \in Z_{43}, 1 \leq i \leq 7\} \right.$$

are seven finite subgroups of P and are of finite order.

Now we illustrate some semigroups of interval real quaternions and the properties associated with them.

Example 2.15: Let S = {(a_1, …, a_8) | $a_i \in \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, 12), 0 \leq t \leq 3, 1 \leq i \leq 8, \times\}$ be the semigroup of interval real quaternions.

S is non commutative and has subsemigroups of both finite and infinite order.

Example 2.16: Let S = {($a_0 + a_1 i + a_2 j + a_3 k \mid a_t \in [0, 15), 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 14, ij = 14ji = k, jk = 14kj = i, ki = 14ik = j, \times\}$ be the interval real quaternion semigroup of infinite order.

This has subsemigroups of both infinite and finite order which are not ideals only subsemigroups.

Let $P_1 = \{a \mid a \in [0, 15) \subseteq S$ be a interval subsemigroup of infinite order and is not an ideal of S.

$P_2 = \{a + bi \mid a, b \in [0, 15), i^2 = 14\} \subseteq S$ is again an interval subsemigroup of infiite order and is not an ideal of S.

$P_3 = \{a + bj \mid a, b \in [0, 15), \times, j^2 = 14\}$ is an interval subsemigroup of infinite order which is not an ideal.

$P_4 = \{a + bk \mid a, b \in [0, 15)\} \subseteq S$ is a subsemigroup of infinite order which is not an ideal.

$P_5 = \{a \mid a \in Z_{15}\} \subseteq S$ is a subsemigroup of finite order.

$P_6 = \{a + bi \mid a, b \in Z_{15}\} \subseteq S$ is also a subsemigroup of finite order.

$P_7 = \{a + bj \mid a, b \in Z_{15}, j^2 = 14\} \subseteq S$ is also a subsemigroup of finite order.

$P_8 = \{a + bk \mid a, b \in Z_{15}\} \subseteq S$ is a subsemigroup of finite order.

$P_9 = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in Z_{15}, 0 \leq t \leq 3\} \subseteq S$ is a subsemigroup of finite order. None of these finite order subsemigroups or ideals of S.

Example 2.17: Let $S = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 9), 0 \leq t \leq 3, 1 \leq i \leq 6, \times\}$ be the semigroup of infinite order.

S is a interval row matrix real quaternion semigroup.

$P_1 = \{(a_1, 0, 0, 0, 0, 0) \mid a_1 \in \{b_0 + b_1i + b_2j + b_3k\}$ with $b_t \in [0, 9); 0 \leq t \leq 3\}, \times\} \subseteq S$,

$P_2 = \{(0, a_2, 0, 0, 0, 0) \mid a_2 \in \{b_0 + b_1i + b_2j + b_3k\}$ with $b_t \in [0, 9); 0 \leq t \leq 3\}, \times\} \subseteq S, \ldots,$

$P_6 = \{(0, 0, \ldots, a_6) \mid a_6 \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 9); 0 \leq t \leq 3\}, \times\} \subseteq S$ are all subsemigroups of infinite order and all of them are also ideals of S.

$P_{12} = \{(a_1, a_2, 0, \ldots, 0) \mid a_1, a_2 \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 9); 0 \leq t \leq 3\}, \times\} \subseteq S$ and so on.

$P_{16} = \{(a_1, 0, 0, \ldots, a_6) \mid a_1, a_6 \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 9)$; and

$P_{56} = \{(0, 0, 0, 0, a_5, a_6) \mid a_5, a_6 \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 9); 0 \leq t \leq 3\}, \times\} \subseteq S$ are all subsemigroups of S which are also ideals of S.

Thus S has atleast ${}_6C_1 + {}_6C_2 + \ldots + {}_6C_5$ number of subsemigroups which are also ideals of S.

S has atleast $4({}_6C_1 + {}_6C_2 + \ldots + {}_6C_6)$ number of subsemigroups of finite order which are not ideals of S.

Example 2.18: Let

$$M = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix}\right| a_i \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 12),\right.$$

$$\left. 0 \leq t \leq 3\}, 1 \leq i \leq 15, \times_n\right\}$$

be the interval semigroup of real quaternions.

$$T_1 = \left\{\left.\begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_1 \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in\right.$$

$$\left.[0, 12), 0 \leq t \leq 3, \times_n\right\} \subseteq M,$$

$$T_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ \vdots \\ 0 \end{bmatrix} \middle| \, a_2 \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 12), 0 \le t \le 3, \times_n\} \right\} \subseteq M, \ldots,$$

$$T_{15} = \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ a_{15} \end{bmatrix} \middle| \, a_{15} \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 12), 0 \le t \le 3, \times_n\} \right\} \subseteq M$$

are all such semigroups of M which are also ideals of M.

Likewise

$$B_{1,3,10,15} = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_3 \\ 0 \\ \vdots \\ a_{10} \\ 0 \\ \vdots \\ 0 \\ a_{15} \end{bmatrix} \middle| \, a_1, a_3, a_{10}, a_{15} \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in Z_{15}, 0 \le t \le 3, \times_n\} \right\}$$

is a subsemigroup of M which is not an ideal of M.

We have atleast ${}_{15}C_1 + {}_{15}C_2 + \ldots + {}_{15}C_{14}$ number of subsemigroups which are ideals of M.

We have atleast ${}_{15}C_1 + {}_{15}C_2 + \ldots + {}_{15}C_{15}$ number of finite subsemigroups which are not ideals.

Example 2.19: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{25} \end{bmatrix} \middle| a_i \in \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, 25), 0 \leq t \leq 3\}, 1 \leq i \leq 25, \times_n \right\}$$

be the interval semigroup of real quaternions semigroup of infinite order.

S has atleast ${}_{25}C_1 + {}_{25}C_2 + {}_{25}C_3 + \ldots + {}_{25}C_{24}$ number of infinite order subsemigroups of S which are ideals.

S has atleast ${}_{25}C_1 + {}_{25}C_2 + {}_{25}C_3 + \ldots + {}_{25}C_{24} + {}_{25}C_{25}$ number of finite order subsemigroup none of which are ideals of S.

S has infinite number of zero divisors. S has only finite number of units and idempotents.

Example 2.20: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{45} & a_{46} & a_{47} & a_{48} \end{bmatrix} \middle| a_i \in \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, 11), 0 \leq t \leq 3\}, 1 \leq i \leq 48, \times_n \right\}$$

be the interval semigroup of real quaternions. N is non commutative and is of infinite order. N has ideals and subsemigroups which are not ideals.

$$P_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1 \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in Z_{11}, 0 \leq t \leq 3, \times_n\}\right.$$

is a subsemigroup of finite order which is not an ideal of N.

We see N has atleast ${}_{48}C_1 + {}_{48}C_2 + {}_{48}C_3 + \ldots + {}_{48}C_{48}$ number of finite subsemigroup which are not ideals.

N also has atleast ${}_{48}C_1 + {}_{48}C_2 + \ldots + {}_{48}C_{47}$ number of infinite subsemigroup which are ideals.

Let

$$B = \left\{\left.\begin{bmatrix} a_1 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1 \in Z_{11} \text{ and } a_2 \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 11), 0 \leq t \leq 3\}, \times_n\right\} \subseteq N$$

be a subsemigroup of infinite order which is not an ideal.

Thus N has several subsemigroups of infinite order which are not ideals of N.

However N has infinite number of zero divisors but only finite number of units and idempotents.

Now having seen examples of semigroups and groups of interval real quaternions we now proceed onto define develop and describe the concept of pseudo interval ring of real quaternion.

DEFINITION 2.3: *Let* $R = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, n), 0 \leq t \leq 3, +, \times\}$ *be defined as the pseudo real quaternion interval ring.*

We know R under the operation "+" is an abelian group.

R under the operation × is a semigroup.

Clearly $a \times (b + c) \neq a \times b + a \times c$ *for all a, b, c* $\in R$*; hence we call R as a pseudo real quaternion interval ring.*

R is of infinite order.

Further R is non commutative. R has zero divisors and idempotents.

We will illustrate this situation by some examples.

Example 2.21: Let $R = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 6), 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 5, ij = 5ji = k, jk = 5kj = i, ki = 5ik = j, +, \times\}$ be the pseudo real quaternion interval ring. $o(R) = \infty$.

Consider $x = 2i$ and $y = 3j \in R$.

$$x \times y = 2i \times 3j = 0 \pmod 6$$

and $y \times x = 3j \times 2i = 0 \pmod 6$; hence R has zero divisors.

Consider $3 \in R$, $3 \times 3 = 3 \pmod 6$ and $4 \in R$ is such that $4 \times 4 = 4 \pmod 6$ thus R has idempotents.

Now let $(1.3i + 4.2j + 2.1k) = x$ and
$0.6i + 0.8k = y \in R$.

$$\begin{aligned}
x \times y &= (1.3i + 4.2j + 2.1k)\,(0.6i + 0.8k) \\
&= 0.78 \times i^2 + 2.52ji + 1.26\,ki + 1.04ik + 3.36jk + 1.68k^2 \\
&= 0.78 \times 5 + 2.52 \times 5k + 1.26 \times j + 1.04 \times 5j + 3.36 \times i + 1.68 \times 5 \\
&= 3.90 + 12.60k + 1.26j + 5.20j + 3.36i + 8.40 \\
&= 3.90 + 2.60k + 1.26j + 5.20j + 3.36i + 2.4 \\
&= 0.3 + 2.60k + 0.46j + 3.36i \quad \ldots \quad \text{I}
\end{aligned}$$

Consider $y \times x =$
$(0.6i + 0.8k) \times (1.3i + 4.2j + 2.1k)$

$$\begin{aligned}
&= 0.78 \times 5 + 1.04j + 2.52k + 3.36 \times 5i + 1.26 \times 5j + 1.68 \times 5 \\
&= 3.90 + 1.04j + 2.52k + 16.80i + 6.30j + 8.40 \\
&= 1.34j + 2.52k + 4.80i + 0.30 \quad \ldots \quad \text{II}
\end{aligned}$$

Clearly I and II are distinct hence R is a non commutative pseudo ring.

Let $x = 0.3i$, $y = 1.2j$ and $z = 0.4j \in R$

$$\begin{aligned}
x \times (y + z) &= 0.3i \times (1.2j + 0.4j) \\
&= 0.3i \times 1.6j \\
&= 0.48\,k \quad \ldots \quad \text{I}
\end{aligned}$$

$$\begin{aligned}
\text{Now } & x \times y + x \times z \\
&= 0.3i \times 1.2j + 0.3i \times 4j \\
&= 0.36ij + 0.12ij \\
&= 0.\,48k \quad \ldots \quad \text{II}
\end{aligned}$$

Hence I and II are identical so in this case the triple x, y, $z \in R$ satisfies the distributive law.

A natural question is can the distributive law be true in case of every triple $x, y, z \in R$.

Consider $x = 5.1i$; $y = 4.2k$ and $z = 3.8k \in R$.

$$\begin{aligned} \text{Now } x \times (y + z) &= 5.1i \times (4.2k + 3.8k) \\ &= 5.1i \times 2k \\ &= 10.2\, ik = 10.2 \times 5j \\ &= 3j \qquad \ldots \quad \text{I} \end{aligned}$$

We now find

$$\begin{aligned} x \times y + x \times z &= 5.1i \times 4.2k + 5.1\, i \times 3.8k \\ &= 5 \times 3.42\, j + 1.38 \times 5j \\ &= 17.10j + 6.90j \\ &= 24j \\ &= 0 \qquad \ldots \quad \text{II} \end{aligned}$$

I and II are distinct.

Hence the distributive law is not true in case of R, thus R is only a pseudo ring.

Example 2.22: Let
$R = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 7), 0 \leq t \leq 3, +, \times\}$ be the pseudo real quaternion interval ring. Clearly $o(R) = \infty$ and R is non commutative.

R has zero divisors for if $x = 3.5$ and $y = 2 \in R$ then $x \times y = 0 \pmod 7$.

Let $x = 3.5i$ and $y = (2k + 4j) \in R$.

We see $x \times y = 0 \pmod 7$.

Consider $x = 3.1i$, $y = 0.9j$ and $z = 6.1j \in R$.

$$\begin{aligned} x \times (y + z) &= 3.1i \times (0.9j + 6.1j) \\ &= 3.1 \times 7j \\ &= 0 \pmod 7 \qquad \ldots \quad \text{I} \end{aligned}$$

Consider $x \times y + x \times z = 3.1i \times 0.9j + 3.1i \times 6.1j$
$= 2.79ij + 4.91ij$
$= 2.79\ k + 4.91\ k$
$= 0.7 \pmod{k}$ … II

I and II are distinct hence R is only a pseudo real quaternion interval ring.

We see $P_1 = \{a \mid a \in [0, 7), +, \times\}$ is a pseudo interval subring of R.

Clearly P_1 is not an ideal of R.

$P_2 = \{a_0 + a_1i \mid a_0, a_1 \in [0, 7), +, \times\} \subseteq R$ is again a pseudo interval subring of R which is not an ideal of R. Both P_1 and P_2 are commutative subrings of R.

Let $P_3 = \{a_0 + a_1j \mid a_0, a_1 \in [0, 7), +, \times\} \subseteq R$ be again a interval pseudo subring of R and P_3 is also a commutative pseudo subring of R.

Let $P_4 = \{b_0 + b_1k \mid b_0, b_1 \in [0, 7)\} \subseteq R$ be a pseudo interval subring of R which is commutative.

Now

$B_1 = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in Z_7, 0 \leq t \leq 3k, +, \times\} \subseteq R$ is a subring of R which is only a real quarternion finite ring which is not pseudo.

$B_2 = \{a_0 + a_1i \mid a_0 , a_1 \in Z_7, +, \times\}$is a subring of R of finite order which is not pseudo.

$B_3 = \{a_0 + a_1j \mid a_0, a_1 \in Z_7, +, \times\}$be a subring of R of finite order which is not pseudo.

$B_4 = \{a_0 + a_2k \mid a_0, a_2 \in Z_7, +, \times\}$be a subring of R of finite order which is not pseudo.

Now B_1, B_2, B_3 and B_4 are not ideals of R only subrings.

We give more examples.

Example 2.23: Let $R = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 12), i^2 = j^2 = k^2 = ijk = 11, ij = 11ji = k, jk = 11kj = i, ki = 11ik = j, 0 \leq t \leq 3, +, \times\}$ be the pseudo interval real quaternion ring of infinite order. R is non commutative. R has zero divisors, units and idempotents.

It is left as an open conjecture whether pseudo interval real quaternions rings contains proper ideals?

Can R have right ideals which are not left ideals?

Study in this direction is innovative and interesting.

We can build matrix pseudo interval real quaternion rings which will be described by examples.

Example 2.24: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{18} \end{bmatrix} \middle| \, a_i \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 11), \right.$$

$$0 \leq t \leq 3\}, i^2 = j^2 = k^2 = ijk = 10, ij = 10ji = k, jk = 10kj = i, \\ ki = 10ik = j, +, \times_n\}$$

be the pseudo column matrix and real quaternion ring of infinite order.

M is non commutative M has pseudo subrings of infinite order which are not ideals.

$$\text{For } P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ a_{18} \end{bmatrix} \middle| \; a_1, a_2 \in [0, 11) \text{ and } a_{18} \in \{b_0 + b_1 i \mid \right.$$

$$\left. b_0, b_1 \in [0, 11)\}, +, \times_n \right\}$$

is only a pseudo subring which is not an ideal of M. Clearly P is of infinite order.

Example 2.25: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| \; a_j \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid \right.$$

$$\left. b_t \in C(\langle [0, 7) \cup I \rangle), 0 \leq t \leq 3, +, \times\}; 1 \leq j \leq 30, +, \times_n \right\}$$

be the pseudo interval neutrosophic complex finite modulo integer real quaternion ring. $|M| = \infty$.

M is non commutative.

M has atleast $3({}_{30}C_1 + {}_{30}C_2 + \ldots + {}_{30}C_{29} + 1)$ number of finite order subring of real quaternions which are not pseudo.

M has atleast ${}_{30}C_1 + {}_{30}C_2 + \ldots + {}_{30}C_{29}$ number of subrings of infinite order which are pseudo.

Apart from this M has pseudo subrings of infinite order which are not ideals the above said ${}_{30}C_1 + {}_{30}C_2 + \ldots + {}_{30}C_{29}$ number of pseudo rings are also ideals of M.

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in P_{NC} \right\} \subseteq M$$

is a pseudo subrings which is also a pseudo ideal of M. M has infinite number of zero divisors and only finite number of idempotents and units.

Example 2.26: Let

$$W = \left\{ \left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \\ a_8 \\ a_9 \\ a_{10} \end{array} \right] \middle| a_p \in P_N = \{b_0 + b_1i + b_2j + b_3k \mid \right.$$

$$\left. b_t \in \langle [0, 7) \cup I \rangle, 0 \leq t \leq 3, 1 \leq p \leq 10, +, \times\}; +, \times_n \right\}$$

be the pseudo interval neutrosophic finite real quaternion ring of super column matrices.

Clearly W is non commutative and is of infinite order. W has both commutative rings of finite and infinite order.

W has atleast ${}_{10}C_1 + {}_{10}C_2 + \ldots + {}_{10}C_9$ number of pseudo ideals.

Further W has atleast
$5({}_{10}C_1 + {}_{10}C_2 + \ldots + {}_{10}C_9 + {}_{10}C_{10})$ number of finite subrings which are not pseudo and are not ideals of W.

Example 2.27: Let $M = \{(a_1 \mid a_2\ a_3 \mid a_4 \mid a_5\ a_6\ a_7 \mid a_8) \mid a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C([0, 12)); 0 \leq t \leq 3, +, \times\}$ be the pseudo interval complex modulo finite integer real quaternion super row matrix ring. $o(M) = \infty$.

M is non commutative but has commutative subrings of both finite and infinite order.

Now we can use the pseudo interval quaternion ring to build group pseudo interval rings.

Let R be the pseudo interval ring $R = \{[0, n), +, \times\}$. G be any group

$$RG = \left\{\sum_{i=1}^{n} a_i g_i \,;\, n < \infty, a_i \in [0, n), g_i \in G; +, \times\right\} \text{ under usual}$$

+ and × is a pseudo interval ring defined as the pseudo interval group ring.

We will first illustrate this situation by some examples.

Example 2.28: Let $R = [[0, 5), +, \times\}$ be the pseudo interval ring. $G = S_3$ the permutation group. RG be the pseudo interval group ring. RG is of infinite order.

Let $x = 0.35p_1 + 2.1\, p_2 + 4.5$
and $y = 2p_3 + 4p_1 + 1 \in RG$.

$$\text{where } p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3.$$

$x \times y = (0.35p_1 + 2.1p_2 + 4.5) \times (2p_3 + 4p_1 + 1)$

$= \quad 1.4\,p_1^2 + 8.4p_2p_1 + 18p_1 + 0.7p_1p_3 + 4.2p_2p_3 + 9p_3 + 0.35p_1 + 2.1p_2 + 4.5$

$= \quad 1.4 + 3p_4 + 3p_1 + 0.35\,p_1 + 2.1p_2 + 4p_3 + 4.5 + 0.7p_4 + 4.2p_5$

$= \quad 0.9 + 3.35p_1 + 3.7p_4 + 2.1p_2 + 4p_3 + 4.2p_5.$

It is easily verified RS_3 = RG is only a pseudo interval ring. RG has zero divisors, units and idempotents. RG has pseudo ideals and finite subrings. RG is non commutative.

Example 2.29: Let R = {[0, 6), +, ×} be the pseudo interval ring G = $\{g \mid g^5 = 1\}$ be the cyclic group of order 5. RG be the pseudo interval group ring.

Clearly G $\subseteq$ RG and R $\subseteq$ RG.

RG is a commutative pseudo ring. RG has units, zero divisors and idempotents.

Let $x = 3.7g^2 + 4.8g + 1.3$ and
$y = 4g^4 + 3g^2 + 2g + 5 \in RG$.

$x + y \quad = (3.7g^2 + 4.8g + 1.3) + 4g^4 + 3g^2 + 2g + 5$

$= 4g^4 + 3g^2 + 2g + 5$

$= 4g^4 + 0.7g^2 + 0.8g + 0.3 \in RG.$

$$\begin{aligned} x \times y &= (3.7g^2 + 4.8g + 1.3)\,(4g^4 + 3g^2 + 2g + 5) \\ &= 14.8g^6 + 15.2g^5 + 5.2g^4 + 11.1g^4 + 14.4g^3 \\ &\quad + 3.9g^2 + 7.4g^3 + 5.6g^2 + 2.6g + 18.5g^2 + 24.0g \\ &\quad + 6.5 \pmod 6). \end{aligned}$$

(using the fact $g^5 = 1$ we get)

$$\begin{aligned} &= 4.8g + 3.2g^5 + 5.2g^4 + 5.1g^4 + 2.4g^3 + 3.9g^2 \\ &\quad + 1.4g^3 + 5.6g^2 + 2.6g + 0.5g^2 + 0 + 0.5 \\ &= 3.2g^5 + 4.3g^4 + 3.8g^3 + 4g^2 + 1.4g + 0.5 \in RG. \end{aligned}$$

This is the way product is defined. RG is an infinite commutative pseudo interval ring which has zero divisors and units.

This has subrings and subrings which are not ideals are also in RG. $R \subseteq RG$ and $G \subseteq RG$.

Example 2.30: Let $B = \{\langle [0, 9) \cup I\rangle, +, \times\}$ be the pseudo interval neutrosophic ring. $G = S_4$. BG is the pseudo interval neutrosophic group ring. BG is non commutative and has units and zero divisors.

$$B \subseteq BG \text{ and } G \subseteq BG. \quad |BG| = \infty.$$

Example 2.31: Let $B = \{\langle [0, 28) \cup I\rangle, +, \times\}$ be the pseudo interval neutrosophic ring. $G = D_{2,11}$ be the dihedral group. MG be the group ring of G over M.

M has zero divisors and is non commutative and is of infinite order.

Example 2.32: Let

$$B = \left\{ \sum_{i=1}^{n} a_i g^i \;\middle|\; g^i \in D_{2,7} \text{ and } a_i \in C(\langle [0, 7) \cup I\rangle) \right.$$

be the pseudo group ring of the group G over the pseudo interval ring C⟨[0, 7) ∪ I⟩} be the pseudo group ring of the group G over the pseudo interval ring {C⟨[0, 7) ∪ I⟩), ×, +}.

Example 2.33: Let B = {C([0, 12)), +, ×} be the pseudo interval complex modulo integer ring.

G = S_7 be the symmetric group of degree 7. BS_7 be the interval pseudo group ring.

Example 2.34: Let M = C(⟨[0, 17) ∪ I⟩), +, ×} be the pseudo interval ring of finite complex neutrosophic integer number. G = $D_{2,7} \times S_5$ be the group.

MG be the pseudo interval group ring. MG has zero divisors, units and idempotents.

Now we proceed onto define the notion group ring using pseudo interval real quaternion rings.

Let P denote the pseudo interval ring of real quaternions, P_C the pseudo interval complex modulo integer ring of real quaternions.

P_N the pseudo interval ring of neutrosophic real quaternions.

P_{NC} the pseudo interval complex modulo integer neutrosophic ring of real quaternions.

Now using P or P_C or P_N or P_{NC} we can build group rings using pseudo interval rings of real quaternions.

This will be illustrated by some examples.

Example 2.35: Let PG be the pseudo interval group ring of finite real quaternions, where

$P = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 3), 0 \leq t \leq 3, +, \times\}$ and $G = \langle g \mid g^7 = 1\rangle$ be the group.

Example 2.36: Let

$P = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 9), 0 \leq t \leq 3, +, \times\}$ be the pseudo interval ring. $G = S_5$.

$$PS_5 = \left\{\sum_{i=1}^{n} d_i g_i \,\middle|\, g_i \in S_5 \text{ and } d_i \in P\right\}$$

be the pseudo interval group ring of real quaternions. PS_5 is non commutative and has zero divisors, units and idempotents.

Example 2.37: Let

$P_C = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 15), 0 \leq t \leq 3, +, \times\}$

be the pseudo interval complex modulo integer real quaternion ring. $G = \{g \mid g^{12} = 1\}$ be the cyclic group of order 12.

P_CG be the pseudo group rig of interval modulo integer complex real quaternios of the group G over P_C.

Example 2.38: Let

$P_N = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in \langle[0, 23) \cup I\rangle, +, \times\}$

be the interval pseudo neutrosophic ring of real quaternions.

$G = D_{2,7}$ be the dihedral group.

P_NG be the group ring of G over P_N. P_NG has zero divisors, units and idemponents.

Study in this direction is routine but interesting and innovative.

Example 2.39: Let

$$P_{NC} = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in \langle[0, 21) \cup I\rangle, +, \times\}$$

be the neutrosophic complex modulo integer interval pseudo ring of real quaternions. $G = D_{2,7}$ be the dihedral group.

$P_{NC}G$ be the pseudo interval complex modulo integer neutrosophic real quaternion group ring of the group G over the pseudo interval ring P_{NC} of real modulo finite complex neutrosophic quaternions.

Next we proceed onto give examples of pseudo interval semigroup ring.

Example 2.40: Let $R = \{[0, 42), +, \times\}$ be the pseudo interval ring. $S = \{d \mid d \in Z_{42}, \times\}$ be the semigroup.

$$RS = \left\{\sum_{i=1}^{n} a_i d_i \;\middle|\; n < 8, d_i \in Z_{42}, d_0 = 0, a_i \in R; +, \times\right\}$$

be the pseudo interval semigroup ring of the semigroup S over the pseudo ring R.

Example 2.41: Let $R = \{C[0, 31), +, \times\}$ be the pseudo interval ring complex modulo integer ring. $S = S(4)$ be the symmetric semigroup of (1 2 3 4).

$$RS = \left\{\sum_{i=1}^{n} a_i s_i \;\middle|\; s_i \in S \text{ and } a_i \in R, +, \times\right\}$$

be the pseudo interval complex modulo integer semigroup ring.

RS has zero divisors units and idempotents.

$$R \subseteq RS \text{ and } S \subseteq RS.$$

Example 2.42: Let B = $\{\langle[0, 92) \cup I\rangle, +, \times\}$ be the interval pseudo neutrosophic integer ring. G = $\{Z_{192}, \times\}$ be the semigroup BG the interval pseudo neutrosophic semigroup ring BG has zero divisors, units and idempotents.

G = $\{Z_{192}, \times\}$ be the semigroup BG the interval pseudo neutrosophic semigroup ring BG has zero divisors, units and idempotents.

Example 2.43: Let L = $\{C(\langle[0, 7) \cup I\rangle, +, \times\}$ be the pseudo interval integer ring of complex finite modulo integer neutrosophic numbers. S = S(4) be the symmetric semigroup of degree four. LS be the semigroup ring of interval complex modulo integer neutrosophic numbers.

LS is a Smarandache ring.

Example 2.44: Let
P = $\{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 15), 0 \leq t \leq 3, +, \times\}$ be the pseudo interval finite real quaternion ring. S = S(7) be the symmetric semigroup. PS be the pseudo interval finite real quaternion semigroup ring of the semigroup S over the pseudo ring P. PS has zero divisors and units.

Example 2.45: Let
$P_C = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C([0, 18)), 0 \leq t \leq 3, +, \times\}$ be the pseudo interval complex modulo integer finite real quaternion ring.

S = $\{Z_{15}, \times\}$ be the semigroup. P_CS be the pseudo interval semigroup ring of real quaternions of complex modulo integers.

P_CS has zero divisors, units and idempotents.

Example 2.46: Let
$P_N = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in \langle[0, 24) \cup I\rangle$ and $0 \leq t \leq 3, +, \times\}$ be the pseudo interval neutrosophic real quaternion ring. Let S = S(6) be the symmetric semigroup on (1 2 3 4 5 6).

P_NS be the pseudo interval neutrosophic real quaternion semigroup ring of S over P_N.

Example 2.47: Let

$P_C = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C([0, 24)) \cup I\rangle, 0 \leq i \leq 3, i_F^2 = 23, +, \times\}$ be the pseudo interval finite complex modulo integer ring of real quaternions.

S = $\{Z_{42}, \times\}$ is be the semigroup under product.

$P_CS = \left\{\sum_{i=0}^{41} a_i g_i \,\middle|\, g_0 = 0,\ g_1 = 1,\ g_2 = 2 \text{ and so on } g_{41} = 41;\right.$ and $a_i \in P_C$; $0 \leq i \leq 41\}$ be the semigroup S over the interval complex modulo integer quaternion pseudo interval ring P_C, the semigroup S.

P_CS is non commutative semigroup pseudo ring with zero divisors, units and idempotents. P_CS has subrings which are not ideals as well as subrings which are ideals.

Let $x = 13.7i_Fg_{10} + 10.3g_4 + 20.4g_2 + 12.5j + 10k$

and $y = (2i + 4j)\, g_5 + (5i + 10j + 20k)g_6 + 4j + 10i + 5k + 20 \in P_CS$.

$$\begin{aligned} x \times y &= (13.7i_Fg_{10} + 10.3g_4 + 20.4g_2 + 12.5j + 10k)) \\ &\quad \times (2i + 4j)g_5 + (5i + 10j + 20k)\, g_6 + 4j + 10i \\ &\quad + 5k + 20 \\ &= (27.4i_Fi + 16.8i_Fj)g_8 + (20.6i + 3j)\, g_{20} + \\ &\quad (16.8i + 9.6j)\, g_{10} + (9.5k + 20j + 22 + 8i)\, g_5 + \\ &\quad (20.5ii_F + 17ji_F + 10i_Fk)\, g_{18} + \\ &\quad (3.5i + 7j + 14k)g_0 + \ldots + 8k + 22 \in P_CS. \end{aligned}$$

This is the way product is performed on P_CS.

Infact P_CS has zero divisors.

For take $x = (12i + 6ji_F)\, g_{10}$ and
$y = (4ji_F + 8i_Fk)\, g_{12} + 12g_5 \in P_CS$.

We see $x \times y = 0$

Infact we have several zero divisors in P_CS.

Clearly $S \subseteq P_CS$ and $P_C \subseteq P_CS$. Product is a matter of routine as in case of usual semigroup rings. Only distributive law is not true leading us to call them as pseudo ring.

Example 2.48: Let $P_N = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in \langle[0, 19) \cup I\rangle, 0 \leq t \leq 3, +, \times\}$ be the pseudo interval neutrosophic finite real quaternion ring.

$S = S(3)$ be the symmetric semigroup. $P_NS(3)$ be the pseudo interval neutrosophic finite real quaternion semigroup ring. $P_NS(3)$ is no commutative has zero divisors and units.

Example 2.49: Let $P_{NC} = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in C(\langle[0, 4) \cup I\rangle), 0 \leq t \leq 3, +, \times\}$ be the pseudo interval finite complex modulo integer neutrosophic real quaternion ring. $S = \{Z_{12}, \times\}$ be the semigroup. $P_{NC}S$ be the semigroup ring which is a pseudo interval neutrosophic semigroup ring.

Example 2.50: Let Z be the ring of integers. $G = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 5), 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 4, ij = 4ji = k, jk = 4kj = i, ki = 4ik = j, \times\}$ be the semigroup. ZG be the semigroup ring of the semigroup G over the ring Z.

Clearly this is the semigroup of real finite quaternion.

Let $x = 2\,(0.3 + 4.2i + 3.4j + 2.4k) + 3$ and

$y = 4 + 2\,(2i + 3j) \in ZG$.

$$
\begin{aligned}
x \times y &= 8\,(0.3 + 4.2i + 3.4j + 2.4k) + 12 \\
&\quad + 14\,(0.6i + 8.4 \times 4 + 6.8ji + 4.8ki + 0.9j \\
&\quad + 12.6ij + 10.2j^2 + 7.2kj)\; 3(0.3 + 4.2i + 3.4j \\
&\quad + 2.4k) + 2 + 4\,(0.6i + 3.6 + 2.2k + 4.8j + 0.9j \\
&\quad + 2.6k + 0.8 + 3.8i) \\
&= (0.9 + 2 + 4.4 + 0.8) + (2.6 + 2.4 + 3.8)i \\
&\quad + (0.2 + 4.8 + 0.9)j + (2.2 + 2.2 + 2.6)k \\
&= 3.1 + 3.8i + 0.9j + 2k \in ZG.
\end{aligned}
$$

This is the way product is performed.

Example 2.51: Let $R = Z_{12}$ and

$$S = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in \langle[0, 3) \cup I\rangle, \times\}$$

be the ring of modulo integers and interval neutrosophic real finite quaternion semigroup respectively.

RS be the semigroup ring. RS is of infinite order.

By this way we can built real quaternion semigroup rings of complex finite modulo integers

$P_C = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C([0, 8)), 0 \leq t \leq 3, \times\}$ be the interval semigroup of complex real quaternions.

$P_{NC} = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle[0, 10) \cup I\rangle); 0 \leq t \leq 3, \times\}$ be the interval semigroup of complex neutrosophic real quaternions.

Using them we can build semigroup rings.

We proceed onto suggest some problems for the reader.

Problems

1. Study the special features associated with interval group; G = $\{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, m);$ $0 \leq t \leq 3$, $i^2 = j^2 = k^2 = ijk = (m-1)$, $ij = (m-1)ji = k$, $jk = (m-1)kj = i$, $ki = (m-1)ik = j, +\}$

2. Find all finite subgroups of G in problem 1 where $m = p_1^{t_1} p_2^{t_2} ... p_r^{t_r}$; $r \leq m$, $t_j \geq 1$, p_i's are distinct prime $1 \leq i \leq r$.

3. Does G have subgroups of infinite order?

4. What is the subgroup in S_n (S_n the infinite symmetric group) which is isomorphic to G in problem (1) when m in G takes the values m = 2, m = 3, …?

5. Let $G = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 8); 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 7, ij = 7ji = k, jk = 7kj = i, ki = 7ik = j, +\}$ be the group.

   Find a S(n) in which G is a subgroup.

6. Prove an interval group of real quaternions has subgroups which are of finite order and characterize them.

7. Find the special features enjoyed by interval complex modulo integer real quaternion groups $B = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle[0, n)\rangle); 0 \leq t \leq 3, +\}$.

   (i) Characterize subgroups of B.

   (ii) Characterize all subgroups of B finite order.

   (iii) Find any other special features enjoyed by B.

8. Let $M = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle[0, n) \cup I\rangle;$ $0 \leq t \leq 3, +\}$ be the interval group of neutrosophic complex modulo integer real quaternions.

   Study questions (i) to (iii) of problem 7 for this M.

9. Let $N = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle[0, n) \cup I\rangle;$ $0 \leq t \leq 3, +\}$ be the interval group of complex neutrosophic real quaternion group.

   Study questions (i) to (iii) of problem 7 for this N.

10. Let $T = \{(a_1, a_2, \ldots, a_{10}) \mid a_t \in \{b_0 + b_1i + b_2j + b_3k$ where $b_r \in [0, 15), 0 \leq r \leq 3\}; 1 \leq t \leq 10, +\}$ be the interval group of real quaternion.

    Study questions (i) to (iii) of problem 7 for this T.

11. Let $S = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix}\right| a_t \in \{b_0 + b_1i + b_2j + b_3k \mid b_m \in\right.$

    $\langle[0, 23) \cup I\rangle, 0 \leq m \leq 3; 0 \leq t \leq 12, +\}$ be the interval group of neutrosophic real quaternions.

    Study questions (i) to (iii) of problem 7 for this S.

12. Let V = $\left\{\begin{bmatrix} a_1 & a_2 & ... & a_{10} \\ a_{11} & a_{12} & ... & a_{20} \\ \vdots & \vdots & & \vdots \\ a_{61} & a_{62} & ... & a_{70} \end{bmatrix} \right|$ $a_t \in \{b_0 + b_1i + b_2j +$

$b_3k \mid b_m \in C(\langle[0, 5) \cup I\rangle)$; $0 \le m \le 3$; $0 \le t \le 70$, +} be the interval complex neutrosophic real quaternion group under +.

Study questions (i) to (iii) of problem 7 for this V.

13. Let S = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right|$ $a_t \in \{b_0 + b_1i + b_2j +$

$b_3k \mid b_m \in C([0, 27))$; $0 \le m \le 3$; $1 \le t \le 16$, +} be the interval group of complex real quaternions.

Study questions (i) to (iii) of problem 7 for this S.

14. Let T = $\left\{\left(\begin{array}{cc|ccc|c|ccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ a_{10} & ... & ... & ... & ... & ... & ... & ... & ... \end{array}\right) \right|$ a_t

$\in \{b_0 + b_1i + b_2j + b_3k \mid b_m \in C([0, 29))$; $0 \le m \le 3$; $1 \le t \le 18$, +} be the interval group of complex real quaternions.

Study questions (i) to (iii) of problem 7 for this T.

15. Let $N = \left\{ \left[\begin{array}{cc} a_1 & a_2 \\ \hline a_3 & a_4 \\ \hline a_5 & a_6 \\ a_7 & a_8 \\ \hline a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \\ \hline a_{15} & a_{16} \end{array} \right] \right| a_t \in \{b_0 + b_1 i + b_2 j + b_3 k \mid b_m$

    $\in C\langle[0, 124) \cup I\rangle;\ 0 \leq m \leq 3\}\ 1 \leq t \leq 16, +\}$
    be the interval group of complex real quaternions.

    Study questions (i) to (iii) of problem 7 for this N.

    Obtain any other special features enjoyed by them.

16. Can the semigroup $S_K = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_j \in [0, 5),\ 0 \leq j \leq 3,\ i^2 = j^2 = k^2 = ijk = 4,\ ij = 4ji = k,\ jk = 4kj = i,\ ki = 4ik = j, \times\}$ be embedded in an infinite symmetric group S(n). (n any appropriate value may be infinite)?

17. Let
    $M = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_j \in [0, 12),\ 0 \leq t \leq 3, \times\}$
    be the interval semigroup of complex modulo integer finite real quaternions.

    (i) Prove M is of infinite order.
    (ii) Prove M is non commutative.
    (iii) Can M have ideals of finite order?
    (iv) Can M have S-subsemigroups?
    (v) Can M have ideals which are not S-ideals?
    (vi) Can M have S-zero divisors?
    (vii) Can M have S-units?
    (viii) Can M have S-idempotents?
    (ix) Can M have units which are not S-units?

18. Let M = $\{a_0 + a_1i + a_2j + a_3k \mid a_j \in C(\langle[0, 23) \cup I\rangle), 0 \leq t \leq 3, \times\}$ be the special interval semigroup of complex neutrosophic real quaternions.

    Study questions (i) to (ix) of problem 17 for this M.

19. Let M = $\{a_0 + a_1i + a_2j + a_3k \mid a_j \in C([0, 45)), 0 \leq t \leq 3, \times\}$ be the special interval semigroup of complex modulo real quaternions.

    Study questions (i) to (ix) of problem 17 for this S.

20. Let T = $\{(a_1, a_2, \ldots, a_{12}) \mid a_p \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in [0, 140); 0 \leq t \leq 3; \times, 1 \leq p \leq 12\}$ be the interval semigroup of row matrix of real quaternions.

    Study questions (i) to (ix) of problem 17 for this T.

21. Let A = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix} \middle| a_i \in \{b_0 + b_1i + b_2j + b_3k \mid b_m \in\right.$

    $C(\langle[0, 15) \cup I\rangle); 0 \leq m \leq 3; \times\} \times_n, 1 \leq i \leq 15\}$

    be the interval semigroup of column matrix of complex neutrosophic real quaternions.

    Study questions (i) to (xi) of problem 17 for this A.

22. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \end{bmatrix} \right| a_i \in \{b_0 + b_1i + b_2j + b_3k$

    $| b_t \in ([0, 17)); 0 \leq t \leq 3; \times\} \times, 1 \leq i \leq 21\}$ be the interval real quaternions of complex modulo integers.

    Study questions (i) to (xi) of problem 17 for this M.

23. Let M = $\{a_0 + a_1i + a_2j + a_3k \mid a_j \in ([0, 24)), 0 \leq t \leq 3, \times\}$ be the special interval semigroup of complex modulo real quaternions.

    Study questions (i) to (ix) of problem 17 for this M.

    If M is considered as a pseudo complex modulo integer real quaternion pseudo ring study the following questions.

    (i) Prove o(M) = ∞.
    (ii) Show M is non commutative.
    (iii) Can M have S-zero divisor?
    (iv) Is every zero divisor of M a S-zero divisor?
    (v) Can M have idempotents which are not S-idempotents?
    (vi) Find S units in any in M.
    (vii) Find ideals which are not S-ideals in M.
    (viii) Find S-subrings in M.
    (ix) Can a finite subring of M be a S-subring justify?
    (x) Obtain any other special property associated with M.

24. Let W = $\{a_0 + a_1i + a_2j + a_3k \mid a_j \in ([0, 43)), 0 \leq t \leq 3, +, \times\}$ be the pseudo interval real quaternion ring.

    Study questions (i) to (x) of problem 23 for this W.

25. Let $S = \{a_0 + a_1i + a_2j + a_3k \mid a_j \in C([0, 28)), 0 \leq t \leq 3, +, \times\}$ be the pseudo interval complex finite modulo integer ring of real quaternions.

    Study questions (i) to (x) of problem 23 for this S.

26. Let $A = \{a_0 + a_1i + a_2j + a_3k \mid a_j \in ([0, 43)), 0 \leq t \leq 3, +, \times\}$ be the pseudo interval real quaternion ring.

    Study questions (i) to (x) of problem 23 for this A.

27. Let $W = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in C(\langle[0, 41) \cup I), 0 \leq t \leq 3, +, \times\}$ be the pseudo interval real quaternion ring.

    Study questions (i) to (x) of problem 23 for this W.

28. Let $M = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in (\langle[0, 23) \cup I\rangle), 0 \leq t \leq 3, +, \times\}$ be the pseudo interval neutrosophic real quaternion ring.

    Study questions (i) to (x) of problem 23 for this M.

29. Let $S = \{a_0 + a_1i + a_2j + a_3k \mid a_j \in (\langle[0, 124) \cup I\rangle), 0 \leq t \leq 3, +, \times\}$ be the pseudo interval neutrosophic real quaternion ring.

    Study questions (i) to (x) of problem 23 for this S.

30. Let $M = \{(a_1, a_2, \ldots, a_{12}) \mid a_i \in P = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in ([0, 42)); 0 \leq t \leq 3; \times\} \times, 1 \leq i \leq 12\}$ be the interval row matrix real quaternions ring.

Study questions (i) to (x) of problem 23 for this M.

31. Let W = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \right| a_i \in \{b_0 + b_1i + b_2j + b_3k \mid b_t \in$ C([0, 23)); $0 \leq t \leq 3$; +, ×} +, $\times_n$, $1 \leq i \leq 12\}$ be the pseudo interval complex modulo integer real quaternion column matrix ring.

Study questions (i) to (x) of problem 23 for this W.

32. Let S = $\left\{\begin{pmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \end{pmatrix} \right| a_i \in \{b_0 + b_1i + b_2j +$ $b_3k \mid b_t \in (\langle[0, 43) \cup I\rangle)$; $0 \leq t \leq 3$; +, ×} +, $\times_n$, $1 \leq i \leq 30\}$ be the pseudo interval matrix ring of neutrosophic real quaternions.

Study questions (i) to (x) of problem 23 for this S.

33. Let B = $\left\{\begin{bmatrix} a_1 & a_2 & \dots & a_7 \\ a_8 & a_9 & \dots & a_{14} \\ \vdots & \vdots & & \vdots \\ a_{43} & a_{44} & \dots & a_{49} \end{bmatrix} \right| a_i \in \{b_0 + b_1i + b_2j +$

$b_3k \mid b_t \in C(\langle[0, 21) \cup I\rangle); 0 \leq t \leq 3; +, \times\} +, \times_n, 1 \leq i \leq 49\}$ be the pseudo interval complex modulo integer neutrosophic real quaternion ring.

Study questions (i) to (x) of problem 23 for this B.

34. Let PG be the pseudo groupring of interval real quaternion where $P = \{a_0 + a_1i + a_2j + a_3k \mid a_t \in [0, 23), 0 \leq t \leq 3, +, \times\}$ be the pseudo interval real quaternion ring over the group $G = S_4$.

    Study questions (i) to (x) of problem 23 for this PG.

35. Let $P_CG = \left\{\sum_{i=1}^{n} a_i g_i \,\middle|\, n < 8, a_i \in P_C = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in C([0, 23)); 0 \leq t \leq 3; +, \times\}, g_i \in D_{2,7}, +, \times\right\}$ be the pseudo interval group ring of finite complex modulo integer real quaternion.

    Study questions (i) to (x) of problem 23 for this P_CG.

36. Let $D = \left\{\sum_{i=1}^{n} a_i g_i \,\middle|\, a_i \in P_N = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in (\langle[0, 7) \cup I\rangle), 0 \leq t \leq 3; +, \times\}, g_i \in S_3\right\} = P_N S_3$ be the pseudo interval group ring of neutrosophic real quaternions.

    Study questions (i) to (x) of problem 23 for this $P_NS_3 = D$.

37. Let B = $\left\{ \sum_{i=1}^{n} a_i g_i \right|$ $a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid$

$b_t \in C(\langle[0, 12) \cup I\rangle)$, $0 \leq t \leq 3$; $+, \times\}$, $1 \leq i \leq t$; $g_i \in G = \{g \mid g^m = 1\}$ be a cyclic group of order m, $+, \times\}$ be the pseudo interval group ring of finite complex neutrosophic integer of real quaternions.

Study questions (i) to (x) of problem 23 for this $P_{NC}G = B$.

38. Let M = $\left\{ \sum_{i=1}^{n} a_i s_i \right|$ $a_i \in [0, 7)$, $s_i \in S(4)$,

S(4) the symmetric semigroup of mappings of the set (1 2 3 4), $+, \times\}$ be the pseudo interval semigroup ring.

Study questions (i) to (x) of problem 23 for this M.

39. Let TG = $\left\{ \sum_{i=0}^{13} a_i g_i \right|$ $a_i \in C([0, 23)$; $g_i \in \{Z_{14}, \times\}$

where $g_0 = 1$, $g_1 = 1$, …, $g_{13} = 13$, $1 \leq i \leq 13$, $+, \times\}$ be the pseudo interval finite complex modulo integer semigroup ring.

Study questions (i) to (x) of problem 23 for this TG.

40. Let PG = $\left\{ \sum_{i=0}^{19} a_i g_i \right|$ $a_i \in \langle[0, 25) \cup I\rangle$; $g_i \in (Z_{20}, \times)$ $= G$, $g_0 = 1$, $g_1 = 1$, …, $g_{19} = 19$, $0 \leq i \leq 19$, $+, \times\}$ be the pseudo interval neutrosophic semigroup ring of the semigroup G over pseudo interval neutrosophic ring P.

    Study questions (i) to (x) of problem 23 for this PG.

41. Let S = $\left\{ \sum_{i=1}^{n} a_i s_i \right|$ $a_i \in C(\langle[0, 4) \cup I\rangle)$; $s_i \in S(5)$; $1 \leq i \leq n$, $n = |S(5)|$, $+, \times\}$ be the pseudo interval complex modulo integer neutrosophic semigroup ring.

    Study questions (i) to (x) of problem 23 for this S.

42. Obtain some special properties enjoyed by pseudo interval semigroup rings over pseudo interval ring P (or P_C or P_N or P_{NC})

43. Let S = $\left\{ \sum_{i=1}^{n} a_i x_i^i \right|$ $a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid$ $b_t \in [0, 5)$, $0 \leq t \leq 3$; and $x_i \in (Z_{40}, \times)$ $x_0 = 0$, $x_1 = 1$, …, $x_{39} = 39$; $0 \leq i \leq 39$; $+, \times\}$ be the pseudo interval finite and quaternion semigroup ring.

    Study questions (i) to (x) of problem 23 for this S.

44. Let $M = \{P_N G \mid P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle [0, 19) \cup I \rangle, 0 \leq t \leq 3; +, \times\}$ and $G = S(7)$ is the symmetric semigroup$\}$ be the pseudo interval neutrosophic real quaternion semigroup ring of the semigroup G over P_N.

    Study questions (i) to (x) of problem 23 for this $P_N G = M$.

45. Let $S = \{P_C G \mid P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle [0, 24)), 0 \leq t \leq 3; +, \times\}$ be the pseudo interval finite complex modulo integer ring of real quaternions.

    $G = \{Z_{12}, \times\}$ be the semigroup $S = P_C G$ the pseudo interval semigroup ring.

    Study questions (i) to (x) of problem 23 for this $S = P_C G$.

46. Let $S = \{P_{NC} S(20) \mid P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle [0, 26) \cup I \rangle), 0 \leq t \leq 3; +, \times\}$ be the pseudo interval ring of finite complex modulo integer neutrosophic real quaternion ring and S(20) the symmetric of degree 20.

    $B = P_{NC} S(20)$ be the pseudo interval semigroup ring.

    Study questions (i) to (x) of problem 23 for this $B = P_N S(20)$.

    Does this pseudo interval ring enjoy any other special properties?

Chapter Three

PSEUDO INTERVAL POLYNOMIAL RINGS AND PSEUDO INTERVAL FINITE REAL QUATERNION POLYNOMIAL RINGS

In this chapter we authors introduce the notion of pseudo interval polynomial rings with coefficients from $[0, n)$, pseudo interval polynomial neutrosophic ring with coefficients from $\langle [0, n) \cup I \rangle$, pseudo interval polynomial finite complex modulo integer ring coefficient from $C([0, n))$ and pseudo interval polynomials with the complex neutrosophic modulo integer ring coefficients from $C(\langle [0, n) \cup I \rangle)$.

Further all the four types of pseudo interval quaternion polynomial rings can be constructed. These rings are special for even the linear polynomial equations may have more than one root and some do not have roots. Study of these properties happen to be a great task and several open conjectures are given.

We now proceed onto define, develop and describe them.

DEFINITION 3.1: *Let R[x] =* $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, n)\right\}$*, (R[x], +) is an abelian group. (R[x], ×) is a commutative semigroup. Thus (R[x], ×, +) be defined as the pseudo polynomial interval ring as the distributive law is not true. Further R[x] is an infinite pseudo polynomial interval ring.*

We will give examples of them and show how solving of polynomials in them take place.

Example 3.1: Let

$$R[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 12), +, \times\right\}$$

be the pseudo polynomial interval ring.

Let $p(x) = 6x^3 + 3x^2 + 6x + 6$
and $q(x) = 4x^2 + 8 \in R[x]$.

We see $p(x) \times q(x) = 0$.

Thus R[x] has zero divisors.

Some units are 11, 5 and 7 $\in$ R[x] are such that $11^2 = 1$, $5^2 = 1$ and $7^2 = 1$.

R[x] also has idempotents.

For 4 and 9 $\in$ R[x] are such that $4^2 = 4 \pmod{12}$ and $9^2 = 9 \pmod{12}$.

However number of units and idempotents are only finite in number, but the number of zero divisors is infinite.

Example 3.2: Let

$$R[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 7), +, \times\right\}$$

be a pseudo polynomial interval ring. R[x] has zero divisors, finite number of units and no idempotents.

Example 3.3: Let $R[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 10), +, \times\right\}$ be a pseudo polynomial interval ring.

R[x] has infinite number of zero divisors. Only finite number of idempotents and units.

Let $p(x) = 3.7x^3 + 4.1x + 2.5$
$q(x) = 0.5x^2$ and $r(x) = 0.8x^2 \in R[x]$.

$$\begin{aligned}
&p(x) \times (q(x) + r(x)) \\
&= (3.7x^3 + 4.1x + 2.5) \times (0.5x^2 + 0.8x^2) \\
&= 3.7x^3 + 4.1x + 2.5 \times 1.3x^2 \\
&= 4.81x^5 + 5.33x^3 + 3.25x^2 \qquad \ldots \quad \text{I}
\end{aligned}$$

Consider $p(x) \times q(x) + p(x) \times r(x)$

$$= 3.7x^3 + 4.1x + 2.5 \times 0.5x^2 + 3.7x^3 + 4.1x + 2.5 \times 0.8x^2$$

$$= 1.85x^5 + 2.05x^3 + 1.25x^2 + 2.96x^5 + 3.28x^3 + 2.00x^2$$

$$= 4.81x^5 + 5.33x^3 + 3.25x^2 \qquad \ldots \quad \text{II}$$

I and II are identical so for the triple the distributive law is true.

We have some triples in R[x] where the distributive law is true.

Now we give a few properties about these pseudo rings.

THEOREM 3.1: *Let*

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, p), p \text{ a prime}, +, \times \right\}$$

be the pseudo polynomial interval ring.

(i) R[x] has only finite number of units.
(ii) R[x] has no non trivial idempotents.
(iii) R[x] has infinite number of zero divisors.

The proof is direct and hence left as an exercise to the reader.

THEOREM 3.2: *Let*

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, n), n \text{ a composite number}, +, \times \right\}$$

be the pseudo interval polynomial ring.

(i) R[x] has infinite number of zero divisors.
(ii) R[x] has only finite number of units.
(iii) R[x] has finite number of idempotents if and only if $Z_n \subseteq [0, n)$ has non trivial idempotents.

Proof is direct and hence left as an exercise to the reader.

Now we show a pseudo polynomial interval ring is a Smarandache pseudo polynomial interval ring if and only if R[x] has a proper subset S such that S is a subring of R and S is not a pseudo ring.

Example 3.4: Let

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 25), +, \times \right\}$$

be the pseudo interval polynomial ring. We see $Z_{25}[x] \subseteq R[x]$ is a subring of R[x].

Clearly in $Z_{25}[x]$ the distributive law is true. Hence R[x] is a Smarandache pseudo interval polynomial ring.

Example 3.5: Let

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 23), +, \times \right\}$$

be the pseudo polynomial interval ring. R[x] is a Smarandache pseudo polynomial interval ring as $Z_{23}[x]$ and Z_{23} are subrings which satisfy the distributive law.

Example 3.6: Let

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 48), +, \times \right\}$$

be the pseudo interval polynomial ring. R[x] is a S-pseudo interval polynomial ring for it has several subrings which satisfy the distributive laws.

THEOREM 3.3: *Let*

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, n), +, \times \right\}$$

be the pseudo interval polynomial ring. R[x] is a S-pseudo interval polynomial ring.

Proof is direct hence left as an exercise to the reader.

Example 3.7: Let

$$R[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 15), +, \times\right\}$$

be the pseudo interval polynomial ring.

Let $p(x) = 3x^5 + 5x^3 + 7 \in R[x]$.

The derivative of p(x) is $\dfrac{dp(x)}{dx} = \dfrac{d(3x^5 + 5x^3 + 7)}{dx}$

$$= 15x^4 + 15x^2 + 0$$
$$= 0 \pmod{15}.$$

Thus contrary to usual polynomial rings we see in this case we have the derivative of polynomials is also zero.

This is the marked difference between the usual polynomial ring over reals and pseudo polynomial rings over the interval [0, n).

Let $p(x) = 3.1107x^{15} + 3x^5 + 7.2 \in R[x]$;

$\dfrac{dp(x)}{dx} = 0$; however $p(x) \in R[x]$ is not a constant polynomial.

Now we see in several case integration of $p(x) \in R[x]$ is not defined.

We see $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$ if $n + 1 \in Z_m$ and $n + 1$

is not a unit in Z_m then we see the integral is not defined.

Consider $\int p(x)\ dx = x^{14} + x^4 + 2 \in R[x]$

$$\int p(x)\,dx \quad = \int (x^{14} + x^4 + 2)\,dx$$

$$= \frac{x^{15}}{15} + \frac{x^5}{5} + 2x + c\,.$$

We see $15 \equiv 0 \pmod{15}$ and $5 \in [0, 15)$ is a zero divisor so the first two terms have no meaning.

Hence we cannot in general integrate all the polynomials in R[x].

So finding differential or integral in case of polynomial in R[x] happens to be a difficult task for in many cases it may not be defined.

Here we can define only polynomials as an abstract concept.

However it is pertinent to record at this juncture we have used these polynomials in $Z_p[x]$ in the construction of codes and so much so in coding theory (p a prime or a power of a prime).

So when polynomials are used with coefficients from Z_p, we are not in a position to give it a geometrical interpretation. However we have been using them as polynomial with real coefficients.

We will say a polynomial p(x) is differentiable provided $\frac{dp(x)}{dx} \neq 0$ if p(x) is not a constant polynomial.

Next we encounter with the problem of solving these polynomials $p(x) \in R[x]$.

We see if $p(x) \in R[x]$ is such that the highest degree of x's coefficient is a zero divisor or an idempotent then certainly $p(x) \in R[x]$ will not satisfy all the fundamental properties of polynomials.

We will describe these situations by some examples.

Example 3.8: Let

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 6), +, \times \right\}$$

be the pseudo interval polynomial ring.

Let $p(x) = 3x + 4 \in R[x]$. We see this has no solution.

So for the first time we encounter with the problem of not able to solve linear equations in R[x].

Let $q(x) = 2x + 5 \in R[x]$ we see q(x) has no root for 2 is a zero divisor in [0, 6).

Further $p(x) = 4x + 2 \in R[x]$ has no solution.

However $q(x) = 5x + 3 \in R[x]$ is solvable as 5 has inverse or 5 is a unit in [0, 6).

So $5x + 3 = 0$ gives $5x \times 5 + 15 = 0$, $x + 3 = 0$, $x = 3$ is a root of q(x) for $q(3) = 5 \times 3 + 3 = 0 \pmod 6$.

Example 3.9: Let

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 5), +, \times \right\}$$

be the polynomial pseudo interval ring.

We see if p(x) = 1.7x + 2 $\in$ R[x] then p(X) is not solvable as 1.7 has no inverse or 1.7 is not a unit in [0, 5).

So ax + b = p(x) $\in$ R[x] is solvable if and only if a is a unit in [0, n).

Example 3.10: Let

$$R[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 24), +, \times\right\}$$

be the pseudo interval polynomial ring.

Let p(x) = 4x + 3 $\in$ R[x] is not solvable in R[x].

q(x) = 23x + 1 $\in$ R[x] is solvable in R[x] as 23 is a unit in [0, 24).

So 23x + 1 = 0 gives 23.23x + 23 = 0 which gives x + 23 = 0, so x = 1. Thus q(1) = 23.1 + 1 = 0 (mod 24). Hence is solvable.

But p(x) = 8x + 3 $\in$ R[x] is not solvable.

Inview of all these examples and practical problems faced by us we put forth the following theorem.

THEOREM 3.4: *Let R[x] =* $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, n), +, \times\right\}$ *be the polynomial pseudo interval ring.*

A linear polynomial p(x) = ax + b (a, b $\in$ [0, n)) is solvable if a is a unit in [0, n).

Proof follows from the fact if p(x) = ax + b, a, b $\in$ [0, n) is such that a is a unit then

ax + b = 0 implies
acx + bc = 0 where c in [0, n) is such that ac = 1 so that
x + bc = 0 giving x = –bc; hence the claim.

Conversely if ax + b is solvable in R[x], we see we have x = t such that at + b = 0 which implies t is a root.

But we may have many such t's so the solution for a linear equation is not unique.

Inview of this we first give some examples.

Example 3.11: Let

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 15), +, \times \right\}$$

be the pseudo polynomial interval ring.

Consider p(x) = 5x + 5 ∈ R[x] we see x = 2 is a root for p(2) = 5 × 2 + 5 ≡ 0 mod 15, x = 5 is a root for
p(5) = 5 × 5 + 5 ≡ 0 (mod 15).

x = 8 is a root for
p(8) = 5 × 8 + 5 ≡ 0 (mod 15).
x = 11 is a root for p(11) = 5 × 11 + 5 = 0 (mod 15)
x = 14 is a root for p(14) = 5 × 14 + 5 = 0 (mod 15)

Hence the equation p(x) = 5x + 5 ∈ R[x] has 5 roots.

This is not possible in case of usual polynomials.

R[x] behaves in a odd way.

Infact it flouts the basic theorem which states that a nth degree polynomial in the variable x has n and only n roots.

A linear polynomial has more than one root.

So we are at this stage not able to comprehend the behaviour of pseudo polynomial interval rings.

Consider $p(x) = 3x + 3 \in R[x]$. This linear equation $p(x) = 3x + 3$ we will find the number of roots in $[0, 15)$.

Consider $x = 4$, $p(4) = 3.4 + 3 = 0 \pmod{15}$.
$x = 9$ is a root for $p(9) = 3 \times 9 + 3 \equiv 0 \pmod{15}$.

$x = 14$ is also a root for $p(14) = 3 \times 14 + 3 = 45 = 0 \pmod{15}$ is a root. Thus 4, 9 and 14 are roots for a linear equation $3x + 3$.

Consider $r(x) = 6x +6 \in R[x]$, $x = 4$ is a unit for

$$r(4) = 6 \times 4 + 6 \equiv 0 \pmod{15}.$$

$x = 9$ is again a root for $r(9) = 6 \times 9 + 6 \equiv 0 \pmod{15}$.

Consider $x = 14$ is a root for
$r(14) = 6 \times 14 + 6 = 84 + 6 \equiv 0 \pmod{15}$.

Thus $6x + 6 = r(x)$ has 3 roots 4, 9 and 14.

Hence we see in general a linear equation can have more than one root.

Consider $s(x) = 7x + 7 \in R[x]$.

$7x + 7 = 0$; $7x \times 13 + 7 \times 13 = x + 1$;

$x + 1 = 0$; $x = 14$ is the only root of this linear equation. The fact follows from the information $7 \times 13 \equiv 1 \pmod{15}$ that is 7 is unit in $[0, 15)$.

$x + a \in R[x]$ has a unique root. $2x + 5 = p(x) \in R[x]$ has the following roots.

$x = 5$ is a root for $2 \times 5 + 5 = 0 \pmod{15}$.

2 is a unit as $8 \times 2 \equiv 1 \pmod{15}$.

$4x + 7 = p(x) \in R[x]$.

This equation has a unique root as 4 is a unit in [0, 15].

$4 \times 4 = 1 \pmod{15}$. $x = 2$ is the unique root of $q(x)$.
$8x + 3 = r(x) \in R[x]$.

This equation has a unique root as $8 \times 2 \equiv 1 \pmod{15}$ $x = 9$ is the unique root as $8 \times 9 + 3 \equiv 0 \pmod{15}$.

$9x + 9 = t(x) \in R[x]$.

$x = 4$ is a root for $t(4)$
$= 9 \times 4 + 9 = 45 \equiv 0 \pmod{15}$.

$x = 9$ is also a root
$t(x) = 9 \times 9 + 9 = 90 \equiv 0 \pmod{15}$

$x = 14$ is also a root as
$t(14) = 9 \times 14 + 9 \equiv 0 \pmod{15}$.

4, 9 and 14 are the roots of $t(x)$.

Consider $9x + 6 = a(x) \in R[x]$, what are the roots of $a(x)$?
$x = 1$ is a root for $a(1) = 9 + 6 \equiv 0 \pmod{15}$.
$x = 6$ is a root for $a(6) = 9 \times 6 + 6 = 60 \equiv 0 \pmod{15}$.

$x = 11$ is also a root for
$a(11) = 9 \times 11 + 6 = 99 + 5 = 105 \equiv 0 \pmod{15}$.

Hence $a(x)$ has 1, 6 and 11 to be roots.

Let $m(x) = 9x + 2 \in R[x]$. This has no solution.

Inview of all these we have the following theorem.

THEOREM 3.5: *Let*

$$R[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, n), +, \times\right\}$$

be the pseudo interval polynomial ring.

(1) A linear polynomial p(x) = ax + b (a, b $\in$ [0, n)) can have more than one root if a is not a unit in [0, n).
(2) A linear polynomial p(x) = ax + b has a unique root if a is a unit in [0, n).

The proof is direct hence left as an exercise to the reader.

It is left as an open conjecture to find the number of roots of ax + b = p(x) $\in$ R[x] where [0, n) is such that n is a composite number.

Example 3.12: Let

$$R[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 24), +, \times\right\}$$

be the pseudo interval polynomial ring. To find roots of a second degree polynomial in R[x].

Let $3x^2 + 4x + 17 = p(x) \in R[x]$.

To find the roots of p(x)
$p(1) = 3.1^2 + 4.1 + 17 = 0$

So 1 is a root of p(x).

It is difficult to find the other root.

Example 3.13: Let

$$R[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 8), +, \times\right\}$$

be a pseudo polynomial interval ring.

Consider the polynomial $p(x) = 4x^2 + 7x + 7 \in R[x]$.

We see $x = 3$ is a root for

$$\begin{aligned} p(3) &= 4 \times 9 + 7 \times 3 + 7 \\ &= 0 \pmod 8. \end{aligned}$$

Now factorize p(x), we get

$p(x) = (x + 5)(4x + 3)$ since 4 is a zero divisor in [0, 8) we see $4x = 5$ is the other root and x value cannot be got. For $(4x + 3)(x + 5) = 4x^2 + 3x + 20x + 15$

$$= 4x^2 + x + 7 \pmod 8.$$

The polynomial p(x) is reducible but we cannot get all the roots of p(x).

So we cannot claim a second degree equation has two roots. One root may exist another may not exist or may exist but not uniquely.

Consider $p(x) = 6x^2 + 3x + 3 \in R[x]$.

$$p(x) = (2x + 3)(3x + 1).$$

Thus $2x = 5$, $3x = 7$ which shows
$6x = 7$ and $6x = 6$; and $6x = 6$.

Now $p(x) = 7x^2 + 4x + 5 \in R[x]$.

$p(x) = (x+5)(7x + 1)$; the roots are
$x = 3$ and $x = 1$

$$\begin{aligned} p(1) &= 7 + 4 + 9 = 0 \pmod 8) \\ p(3) &= 7 \times 9 + 4 \times 3 + 5 \\ &= 63 + 12 + 5 \\ &= 0 \ (\text{mod } 8). \end{aligned}$$

Since the coefficient of highest degree is 7 and it is a unit hence p(x) has two distinct roots.

Let $p(x) = x^2 + 4x + 3 \in R[x]$.

Clearly $p(1) = 1 + 4 + 3 \equiv 0 \ (\text{mod } 8)$ and $p(3) = 9 + 12 + 3 \equiv 0 \ (\text{mod } 8)$.

Consider the polynomial $p(x) = 7x^2 + 2x \in R[x]$.

The four roots of p(x) are 0, 2, 4 and 6.

$$\begin{aligned} &\text{Now } p(0) = 0 \\ p(2) &= 7 \times 4 + 2 \times 2 = 28 + 4 \\ &= 0 \ (\text{mod } 8) \\[1em] p(4) &= 7 \times 16 + 8 \\ &= 0 \ (\text{mod } 8) \ \text{ and} \\[1em] p(6) &= 7 \times 36 + 12 \\ &= 28 + 12 = 0 \ (\text{mod } 8). \end{aligned}$$

Thus equations of degree two has four roots in this case. Hence we cannot say a second degree equation has only two roots.

Further we can say a nth degree polynomial in general can have more than n-roots or even less than n roots.

Now we proceed onto study by examples pseudo polynomials interval complex modulo integer rings.

Let C([0, n)) = {a + bi_F | a, b $\in$ [0,n); i_F^2 = n–1} be the complex interval modulo integers.

P[x] = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ a_i $\in$ C([0, n)), +, ×} is defined as the pseudo interval complex finite modulo integer polynomial ring. Here also we have several properties just like pseudo interval polynomial ring R[x].

In the first place R[x] $\subseteq$ P[x] as a proper subring, however it is not an ideal.

We will first illustrate this situation by some examples.

Example 3:14: Let

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in C([0, 5)),\ i_F^2 = 4, +, \times \}$$

be the pseudo interval complex modulo integer polynomial ring.

Let p(x) = $3i_Fx^3 + (2+i_F)\,x + 1$ and
q(x) = $(4+2i_F)x^2 + (1+i_F)\,x + 3 \in$ P[x].

p(x) + q (x) = $3i_Fx^3 + (2 + i_F)\,x + 1 + (4 + 2i_F)\,x^2 +$
$(1 + i_F)x + 3$

= $3i_Fx^3 + (4 + 2i_F)x^2 + (3 + 2i_F)\,x + 4$.

It is easily verified p(x) + q(x)

= q (x) + p(x) for all p(x), q(x) $\in$ p(x).

p(x) × q(x) = $(3i_Fx^3 + (2 + i_F)\,x + 1) \times ((4 + 2i_F)x^2 +$
$(1 + i_F)\,x + 3)$

$$= 3i_F\,(4 + 2i_F)x^5 + (2+i_F)\,(4 + 2i_F)x^3 + (4 + 2i_F)x^2$$
$$+ 3i_F\,(1 + i_F)\,x^4 + (2 + i_F)\,(1 + i_F)\,x^2 + (1 + i_F)x$$
$$+ 9i_Fx^3 + (2 + i_F)\,3x + 3$$

$$= (12i_F + 6 \times 4)x^5 + (8 + 4i_F + 4i_F + 2 \times 4)x^3$$
$$+ (4 + 2i_F)x^2 + (3i_F + 3 \times 4)x^4 + (2 + 2i_F + i_F$$
$$+ 4)x^2 + (1 + i_F)x + 9i_F\,x^3 + (6 + 3i_F)x + 3$$

$$= (2i_F + 4)x^5 + (2i_F + 1)\,x^3 + (4 + 2i_F)x^2 + (3i_F + 2)x^4$$
$$+ (1 + 3i_F)x^2 + (2 + 4i_F)x + (4i_F + 3)$$

$$= (4 + 2i_F)x^5 + (2 + 2i_F)x^4 + (1 + 3i_F)x^3 + (0)x^2$$
$$+ (2 + 4i_F)x + 3.$$

This is the way $\times$ operation is performed on P[x].

Consider $p(x) = 3i_Fx + 2 \in P[x]$.

$$3\,i_F^2\,x + 2i_F = 0$$
$$3 \times 4x + 2i_F = 0$$
$$2x + 2i_F = 0$$
$$x + i_F = 0 \qquad x = 4i_F \text{ is the root of } p(x).$$

$p(4i_F) = 3i_F \times 4i_F + 2$
$= 2 \times 4 + 2 = 0 \pmod 5$.

This equation has a unique solution.

Consider $1.231x + 0.7 = q(x)$.

We see 1.231 has no inverse that is $1.231 \times t = 1$ is not possible in [0, 5).

Hence we cannot find a value for x.

However $1.231x + 0.7 = 0$ implies $1.231x = 4.3$.

Example 3.15: Let

$$P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in C([0, 6)),\ i_F^2 = 5, +, \times\right\}$$

be the pseudo interval complex modulo integer polynomial ring.

Let $3x + 2.1 = p(x) \in P[x]$.

To solve p(x).

$3x + 2.1 = 0$ implies $3x = 3.9$.

We cannot further solve for x as 3 is an idempotent $2x + 3 = q(x) \in P[x]$.

$2x = 3$ and we cannot find any other root as 2 is a zero divisor in [0, 6).

Now $2.5x + 4 = r(x) \in P[x]$
$2.5x + 4 = 0$ that is $5x + 2 = 0$
$25x + 10 = 0$ $\quad\quad x + 2 = 0$ or $x = 4$;
So $2.5x + 4 = r(x)$.

Now $r(4) = 2.5 \times 4 + 4 = 14 + 4 = 0 \pmod 6$.

So the above form of simplification gives the value of x to be 4.

So $x = 4$ is a root of r(x).
Let $1.5x + 2 = s(x) \in P[x]$ to solve for x.

We see 1.5 in [0, 6) is a zero divisor as
$1.5 \times 4 = 6.0 = 0 \pmod 6$.

So we cannot solve for x.
Let $(1 + i_F)x + (2+3i_F) = m(x) \in P[x]$.

Solving m(x) is a tricky task.

We see we may not have a root for m(x). Thus solving even linear equations in P[x] happens to be challenging.

Example 3.16: Let

$$P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in C(Z_{18});\ i_F^2 = 17, +, \times\right\}$$

be the pseudo interval complex modulo integer polynomial ring. P[x] has zero divisors.

$p(x) = 2x^3 + 4$; $q(x) = 9x^7 + 9i_F \in P[x]$ is such that
$p(x)\, q(x) \equiv 0 \pmod{18}$.

Finding ideals and subrings are a matter of routine.

We proceed onto give examples of pseudo interval neutrosophic polynomial rings.

Example 3.17: Let

$$B[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle [0, 9) \cup I\rangle;\ I^2 = I, , +, \times\right\}$$

be the pseudo interval polynomial neutrosophic ring.

Let $p(x) = 0.9x^7 + 3.2x^3 + 6.01$ and
$q(x) = 5.2x^6 + 2.1 \in B[x]$

$p(x) \times q(x) = (0.9Ix^7 + 3.2x^3 + 6.01) \times (5.2x^6 + 2.1)$

$= 4.68Ix^{13} + 7.64x^9 + 4.252x^6 + 1.89Ix^7 + 6.72x^3 + 3.621 \in B[x]$

$p(x) + q(x) = 0.9Ix^7 + 3.2x^3 + 6.01 + 5.2x^6 + 2.1$

$= 0.9Ix^7 + 5.2x^6 + 3.2x^3 + 8.11 \in B[x].$

$p(x) = 5Ix + 2.1 \in B[x]$ to solve for the root of p(x)

$5xI + 2.1 = 0$
$10xI + 4.2 = 0$ this gives $xI = 4.8$.

We see we cannot find the value of x as I is an idempotent.

$q(x) = 4x + 3.2 \in B[x].$

$4x + 3.2 = 0$ implies $28x + 22.4$
$x + 4.4 = 0$ so that $x = 4.6$.

Thus the solution is unique as the coefficient of x in this linear equation is a unit in $\langle[0, 9) \cup I\rangle$.

Consider
$p(x) = (3xI + 1)(2x + I) \times (6x + 2.1I)((2I+3)x + 4.2) \in B[x]$ a polynomial of degree four p(x) is linearly reducible still p(x) does not contain unique four roots.

We see $p(x) = 0$ implies
$(3xI + 1)(2x + I)(6x + 2.1I)((2I + 3)x + 4.2) = 0$ which is turn implies

$3xI + 1 = 0$
$2x + I = 0$
$6x + 2.1I = 0$ and
$(2I + 3)x + 4.2 = 0.$

Now all these 4 linear equations are not solvable for some have unique roots and some lack it.

Now consider $3xI + 1 = 0$
We see $3xI = 9$ is the solution as 3 is a zero divisor in [0, 9).

Consider $2x + I = 0$ this implies
$(2x + I) \times 5 = 0$

$10x + 5I = 0$ that is
$x + 5I = 0$ which implies $x = 4I$.

Hence $4I + 5I \equiv 0 \pmod 9$.

So this equation $2x + I$ has a unique solution, the main reason being 2 to the coefficient of x is a unit in [0, 9)

Next consider $6x + 2.1I = 0$; we see 6 is a zero divisor in [0, 9) so

$6x = 6.9I$ is the only value x can take.

Finally $(2I + 3)x + 4.2 = 0$ gives
$(2I + 3)^2 x + 4.2 (2I + 3) = 0$.

Thus $(4I + 9 + 2I)x + 8.4I + 126 = 0$

$5Ix + (8.4I + 3.6) = 0$.

Since $I^2 = I$ is an idempotent we cannot uniquely solve even this equation.

Further even a linearly reduced polynomial p(x) in B[x] need not be solvable uniquely.

Example 3.18: Let

$$B[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle[0, 12) \cup I\rangle;\ I^2 = I, +, \times\right\}$$

be the pseudo interval neutrosophic polynomial ring.

$N = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_{12}, +, \times\right\}$ is a subring of B[x] of infinite order and is not an ideal B[x].

$$M = \left\{ \sum a_i x^i \,\middle|\, a_i \in \langle Z_{12} \cup I\rangle, +, \times \right\}$$

is again only a subring and not an ideal.

All these rings are all of infinite order.

Let $p(x) = (3 + 4I)x^7 + (7I + 4)x^2 + 5I \in B[x]$

We can differentiate and integrate p(x) which is given in the following.

$$\frac{dp(x)}{dx} = \frac{d}{dx}\ ((3 + 4I)x^7 + (7I+4)x^2 + 5I)$$

$$7\,(3 + 4I)x^6 + 2(7I + 4)x + 0$$
$$= (9 + 4I)\, x^6 + (2I + 8)x \in B[x].$$

Now $\int p(x)\, dx = \int ((3 + 4I)x^7 + (7I + 4)x^2 + 5I)\, dx$

$$= \frac{(3+4I)x^8}{8} + \frac{(7I+4)x^3}{3} + 5Ix + C\,.$$

We see $\frac{1}{8}, \frac{1}{3}$ has no meaning as in [0, 12) they lead to zero divisors.

So this p(x) cannot be integrated.

We can integrate only a few of them.

Example 3.19: Let

$$B[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle [0, 13) \cup I\rangle; +, \times \right\}$$

be the pseudo interval neutrosophic polynomial ring.

We can integrate all polynomials of degree less than 12 and all polynomials of degree (13 – 1)n; n any integer.

For if $p(x) = 2Ix^{12} + C \in B[x]$

then $\int p(x)\, dx = \int 2Ix^{12}\, dx + C\, dx$

$$= \frac{2Ix^{13}}{13} + cx + d;$$

d a constant, the first term is undefined as $\frac{1}{13}$ is not defined in this case.

Consider $p(x) = (4I + 7)x^{25} + x + 3 \in B[x]$

$\int p(x)\, dx = \int ((4x + 7)x^{25} + x + 3)\, dx$

$$= \frac{(4x+7)x^{26}}{26} + \frac{x^2}{2} + 3x + C;$$

The first term is not defined as $\frac{1}{26}$ is undefined.

Thus integration cannot be carried out for all $p(x) \in B[x]$.

Example 3.20: Let

$$T[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in C(\langle [0, 8) \cup I \rangle);\ i_F^2 = 7; \right.$$
$$\left. I^2 = I\ (i_F I)^2 = 7I, +, \times \right\}$$

be the pseudo interval neutrosophic finite complex modulo integer polynomial ring. T[x] has subrings of infinite order which are not ideals. Further T[x] has linear equations which cannot be solved.

Let $p(x) = 2Ix + (2i_F + 3I + 1) \in T[x]$.

p(x) does not contain unique root as 2I is a zero divisor in $\langle [0, 8) \cup I \rangle$.

Consider $5x + 3.2 = q(x) \in T[x]$; we see 5 is a unit in [0, 8).

So $25x + 16.0 = 0$ gives $x = 0$; but $5.0 + 3.2 = q(0)$ is not a root.

Consider $5x + 3.7 = t(x) \in T[x]$.

$x + 3.7 \times 5 = 0$; this gives $x + 18.5 = 0$ so that $x = 5.5$.

Now $t(5.5) = 5 \times 5.5 + 3.7 = 2.75 + 3.7 = 7.2$.

So 5.5 is not a root of t(x). Even if 5 is a unit in [0, 8) this linear equation is not solvable.

Consider $5x + 3 = m(x) \in T[x]$.

$$5x + 3 = 0$$
$$25x + 15 = 0.$$

This gives $x + 7 = 0$ so $x = 1$.

Thus $m(1) = 5.1 + 3 = 0 \pmod 8$.
This root is unique.

Thus some linear equations may have a root and some of them may not have a root so this study is both innovative ad interesting.

Example 3.21: Let

$$M[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in C(\langle[0, 12) \cup I\rangle);\ i_F^2 = 11; \right.$$

$$\left. I^2 = I\ (i_F I)^2 = 11I, +, \times \right\}$$

be the complex modulo integer finite pseudo neutrosophic polynomial ring.

M[x] has several subrings of infinite order.

Integration and differentiation can be performed on polynomial in M[x] some may be well defined and some may not be defined.

Study in this direction is also interesting. However all linear polynomials cannot be solved. For solutions may or may not exist.

Further even if a polynomial p(x) in M[x] is linearly reducible still the solution may or may not exist.

Even if ax + b = 0 with a unit in [0, 12) still the solution may or may not exist.

This study is open to researchers.

Now we can as in case of usual rings define polynomials pseudo rings in more than one variable.

Let $S[x_1, x_2] = \left\{ \sum_{i \leq i, j = \infty} a_{ij} x_1^i x_2^j \,\middle|\, a_{ij} \in [0, n) \right.$ with x_1 and x_2 two indeterminate such that $x_1x_2 = x_2x_1\}$ then we define $S[x_1, x_2]$ to be the pseudo interval polynomial ring in the variables x_1 and x_2.

We can have all properties. Just we describe how the operations of + and × are carried out on $S[x_1, x_2]$ by a few examples.

Example 3.22: Let

$$S[x_1, x_2] = \left\{ \sum_{i \leq i, j = \infty} a_{ij} x_1^i x_2^j \,\middle|\, a_{ij} \in [0, 5), x_1x_2 = x_2x_1, +, \times \right\}$$

be the pseudo interval polynomial ring in the two variables x_1 and x_2.

Let $p(x_1, x_2) = 3x_1^3 x_2^2 + 4.1x_1 + 2.3x_2 + 1.5$
and $q(x_1, x_2) = 2x_1^2 + 3x_2^2 + 4x_1 x_2 + 1 \in S[x_1, x_2]$.

We find $p(x_1, x_2) + q(x_1, x_2)$

$= (3x_1^3 x_2^2 + 4.1x_1 + 2.3x_2 + 1.5) + (2x_1^2 + 3x_2^2 + 4x_1 x_2 + 1)$

$= 3\ x_1^3 x_2^2 + 4.1x_1 + 2.3x_2 + 2x_1^2 + 3x_2^2 + 4x_1x_2 + 2.5 \in S[x_1, x_2]$.

Now we find $p(x_1, x_2) \times q(x_1, x_2) = (3x_1^3 x_2^2 + 4.1x_1 + 2.3x_2 + 1.5) (2x_1^2 + 3x_2^2 + 4x_1x_2 + 1)$

$= x_1^5 x_2^2 + 3.1x_1^3 + 4.6\ x_2\ x_1^2 + 3x_1^2 + 4x_1^3x_2^4 + 2.3x_1 x_2^2 + 1.9x_2^3 + 4.5x_2^2 + 2x_1^4x_2^3 + 1.4x_1^2 x_2 + 4.2x_1 x_2^2 + x_1x_2 + 3x_1^3 x_2^2 + 4.1x_1 + 2.3x_2 + 1.5$

$= x_1^5 x_2^2 + 2x_1^4x_2^3 + 4x_1^3 x_2^2 + 3.1x_1^3 + x_1^2 x_2 + 3x_1^2 + 3.5x_1 x_2^2 + 1.9x_2^3 + x_1x_2 + 3x_1^3 x_2^2 + 4.1x_1 + 2.3x_2 + 1.5$.

This is the way product operation is performed.

Infact $S[x_1, x_2]$ is a commutative pseudo polynomial ring and has zero divisors.

Example 3.23: Let

$$S[x_1, x_2] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 12), x_1 x_2 = x_2 x_1, +, \times\right\}$$

be the pseudo interval polynomial ring in the two variables x_1 and x_2.

$S[x_1, x_2]$ has infinite subrings and has zero divisors.

$$\text{Take } 4x_1^3 x_2^2 + 8x_1^2 x_2 + 4 = p(x)$$

and $q(x) = 6x_1^3 x_2 + 3x_1^2 x_2 + 3 \in S[x_1, x_2]$.

Clearly $p(x)\, q(x) = 0$, hence the claim.

Solving polynomial equations happens to be a difficult task in case of $S[x_1, x_2]$.

Example 3.24: Let

$$S[x_1, x_2] = \left\{\sum_{0 \le i, j \le \infty} a_{ij} x_1^i x_2^j \,\middle|\, a_{ij} \in C([0, 15)), x_1x_2 = x_2x_1, +, \times\right\}$$

be the pseudo interval complex modulo integer interval polynomial ring in the variables x_1 and x_2.

$S[x_1, x_2]$ has zero divisors, infinite order subrings and a few units and idempotents.

Solving equations happens to be a difficult problem.

We can differentiate with respect to x_1 and x_2 and also integrate with respect to both the variables.

Let $p(x_1, x_2) = 5x_1^3 x_2^2 + 3x_1 x_2 + 5x_2^2 \in S[x_1, x_2]$.

$$\frac{dp(x_1x_2)}{dx_1} = 15x_1^2\ x_2^2 + 3x_2$$

$$\frac{d^2p(x_1x_2)}{dx_1dx_2} = 3$$

$$\frac{dp(x_1x_2)}{dx_2} = 10x_1^3 + 3x_1 + 10x_2$$

$$= \frac{dp(x_1x_2)}{dx_2dx_1} = 30x_1^2 + 3 = 3.$$

This is the way differentiation is done.

Integration also can be done as the matter of routine for may or may not be defined for all polynomial.

Example 3.25: Let

$$S[x_1, x_2] = \left\{ \sum_{0 \le i, j \le \infty} a_{ij} x_1^i x_2^j \,\middle|\, a_{ij} \in C(\langle [0, 24) \cup I \rangle); \right.$$
$$\left. x_1x_2 = x_2x_1, +, \times \right\}$$

be the pseudo interval neutrosophic polynomial ring. S has subrings of infinite order.

Study of zero divisors, units and idempotents is an interesting feature but it is considered as a matter of routine.

Example 3.26: Let

$$S[x_1, x_2] = \left\{ \sum_{0 \leq i, j \leq \infty} a_{ij} x_1^i x_2^j \,\middle|\, a_{ij} \in C(\langle [0, 12) \cup I \rangle);\ i_F^2 = 11; \right.$$
$$\left. I^2 = I,\ (i_F I)^2 = 11I, +, \times \right\}$$

be the pseudo interval complex modulo integer neutrosophic polynomial ring of infinite order. $S[x_1, x_2]$ has finite number of units and idempotents but has infinite number of zero divisors.

4, 9, I, 4I, 9I are some of the idempotents of $S[x_1, x_2]$. Some of the units are 7, 5 and 11 are some of the units of $S[x_1, x_2]$.

Example 3.27: Let

$$S[x_1, x_2, x_3] = \left\{ \sum_{0 \leq i, j, k \leq \infty} a_{ij} x_1^i x_2^j x_3^k \,\middle|\, a_{ijk} \in C(\langle [0, 7) \cup I \rangle); \right.$$
$$\left. x_i\, x_j = x_j\, x_i;\ 1 \leq i, j \leq 3, +, \times \right\}$$

be a pseudo interval polynomial finite complex modulo integer neutrosophic ring in three variables x_1, x_2 and x_3.

$$P[x_1] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in C(\langle 0, 7) \cup I \rangle), +, \times \right\} \subseteq S[x_1, x_2, x_3]$$

is only a subring and not an ideal.

$$P[x_1, x_2] = \left\{ \sum_{i=0}^{\infty} a_{ij} x_1^i\, x_2^j \,\middle|\, a_{ij} \in [0, 7), +, \times \right\} \subseteq S\,[x_1, x_2, x_3]$$

is only a subring and not an ideal of $S[x_1, x_2, x_3]$

This ring has only finite number of units but has infinite number of zero divisors.

Example 3.28: Let

$$S[x_1, x_2, x_3, x_4] = \left\{ \sum_{0 \le i,j,k,l \le \infty} a_{i,j,k,l}\, x_1^i x_2^j x_3^k x_4^l \,\middle|\, a_{i,j,k,l} \in [0, 20), \right.$$
$$\left. x_i x_j = x_j x_i;\ 1 \le i, j \le 4,\ +, \times \right\}$$

be the pseudo interval polynomial ring in the variables x_1, x_2, x_3 and x_4. S has zero divisors, units and idempotents.

Now we can have pseudo interval polynomial rings in n variable ($n \ge 2$).

Let $F[x_1, \ldots, x_n] = \{\sum a_1, \ldots, a_n,\ x_1^i, \ldots, x_n^{i_n} \mid a_i, \ldots, a_n \in [0, m)$ (or $C(([0, m)$ or $\langle [0, m) \cup I\rangle$ or $C(\langle[0, m) \cup I\rangle))$, $x_{i_j} x_{i_k} = x_{i_k} x_{i_j}$; $1 \le k, j \le m, \times, +\}$ be defined as the pseudo interval polynomial ring (or pseudo interval finite complex modulo integer polynomial ring or pseudo interval neutrosophic polynomial ring or pseudo interval neutrosophic complex modulo integer polynomial ring respectively).

Study in this direction is new and lots of new methods can be discovered from this study.

Even solving for roots in case of polynomial equations needs new and innovative techniques.

Next we proceed onto describe finite real quaternion interval polynomial ring by an example or two.

Example 3.29: Let

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, 18), \right.$$

$0 \leq t \leq 3$, $i^2 = j^2 = k^2 = ijk = 17$, $ij = 17ji = k$, $jk = 17kj = i$, $ki = 17ik = j$, $+$, $\times\}$ be the pseudo interval real quaternion polynomial ring. $o(P[x]) = \infty$.

P[x] is non commutative. P[x] has zero divisors, units and idempotents. P[x] has subrings. Differentiation and integration can be performed on polynomials in P[x].

Some of these properties will be described.

$$\text{Let } p(x) = (5i + 3j)\, x^2 + 2kx + (8i + j + 4)$$
$$\text{and } q(x) = 3kx^5 + (2i + j + k) \in P[x].$$

We find $p(x) + q(x)$ and $p(x) \times q(x)$.

$$\begin{aligned}
p(x) + q(x) = {} & (5i + 3j)x^2 + 2kx + (8i + j + 4) + 3kx^5 + (2i + j + k) \\
= {} & (5i + 3j)x^2 + 3kx^5 + 2kx + (10i + 2j + 4 + k) \in P[x].
\end{aligned}$$

$$\begin{aligned}
p(x) \times q(x) = {} & [(5i + 3j)x^2 + 2kx + (8i + j + 4)] \times [3kx^5 + (2i + j + k)] \\
= {} & (5i + 3j)\, 3k\, x^7 + 6k^2\, x^6 + (8i + j + 4)\, 3kx^5 + (5i + 3j)\,(2i + j + k)\, x^2 + 2k\,(2i + j + k)\, x + (8i + j + 4)\,(2i + j + k) \\
= {} & (15 \times 17j + 9i)\, x^7 + 6 \times 17x^6 + (24 \times 17j + 3i + 12k)x^5 + (10 \times 17 + 6ji + 5ij + 3j^2 + 5ik + 3jk)\, x^2 + (4ki + 2kj + 2 \times 17)x + (16 \times 17 + 2ji + 8i + 8ij + j^2 + (16 \times 17 + 2ji + 8i + 8ij + j2 + 4j + 8ik + jk + 4k) \\
= {} & (j + 9i)x^7 + 12x^6 + (3i + 12k + 12j)x^5 + (8 + 12k + 5k + 15 + 13j + 3i)x^2 + (4j + 16i + 16)x + (4k + i + 4j + 17 + 10j + 8i + 8k + 16k + 2)
\end{aligned}$$

$$= \quad (9i + j)x^7 + 12x^6 + (3i + 12k + 12j)\,x^5 + (17k + 5 + 13j + 3i)\,x^2 + (4j + 16i + 16)x + (10k + 1 + 14j + 9i) \in P[x].$$

This is the way the sum and product are carried out in P[x].

Consider the derivative of p(x)

$$\frac{dp(x)}{dx} = \frac{d}{dx}\ (5i + 3j)x^2 + 2kx + (8i + j + 4)$$

$= (10i + 6j)\,x + 2k \in P[x]$.

This is the way derivations are carried out.

Only the property derivative of a constant polynomial is zero is not true in case of P[x]. For in P[x] we can have polynomials such that which are not constant yet their derivative is zero.

For consider $s(x) = 9kx^4 + (6i + 6k)x^3 + 12 \in P[x]$

$$\frac{ds(x)}{dx} = \frac{d}{dx}[9x^4 + (6i+6k)x^3 + 12]$$

$$= 36x^3 + 3\,(6i + 6k)x^2 + 0.$$
$$= 0 + 0 \pmod{18}.$$

Hence the claim.

Finally we cannot integrate in general polynomials in P[x].

For consider $p(x) = (3i + 2j + 4k)x^8 + (2i + 4j + 5k + 1)x^5 + 8i \in P[x]$.

We find the integral of $p(x) \in P[x]$

$$\int p(x)\,dx = \int [(3i + 2j + 4k)\,x^8 + (2i + 4j + 5k + 1)\,x^5 + 8i]\,dx$$

$$= \frac{(3i+2j+4k)x^9}{9} + \frac{(2i+4j+5k+1)x^6}{6} + 8ix + c$$

Since $\frac{1}{9}, \frac{1}{6}$ are not defined in [0, 18) the integral of p(x) is not defined.

Thus some of the polynomials can be integrated.

Example 3.30: Let

$$P[x] = \left\{ \sum_{s=0}^{\infty} a_s x^s \,\middle|\, a_s \in P = \{ b_0^s + b_1^s i + b_2^s j + b_3^s k \mid b_t^s \in [0,43), \right.$$

$0 \le t \le 3$, $i^2 = j^2 = k^2 = 42 = ijk$, $ij = 42ji = k$, $jk = 42kj = i$, $ki = 42ik = j\}, +, \times\}$ be the pseudo interval finite real quaternion polynomial ring we see all polynomials of degree less than 42 can be integrated.

Further if $p(x) \in P[x]$. p(x) should not have non zero coefficient for x^{42}, x^{85}, x^{128}, …, $x^{(n\times 43-1)}$, n = 1, 2, …, a finite natural integer then p(x) can be integrated.

Here all polynomials in P[x] whose power of x is 43, 86, 129, … n43 = 43n ($n \in N$, a natural integer) are such that their derivative is always zero.

Thus if $p(x) = ix^{86} + (3j + 2i + k)x^{43} + (8i + 4) \in P[x]$ then

$$\frac{dp(x)}{dx} = 0.$$

In view of this we have the following theorem.

THEOREM 3.6: *Let*

$$P[x] = \left\{ \sum_{s=0}^{\infty} a_s x^s \;\middle|\; a_s \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0,p), \right.$$

$0 \leq t \leq 3$, p a prime, $i^2 = j^2 = k^2 = ijk = p–1$, $ij = (p–1)ji = k$, $jk = (p–1)kj = i$, $ki = (p–1)ik = j\}, +, \times\}$ be the pseudo interval finite real quaternion polynomial ring.

(i) *All polynomials are integrable except those with degree 2p–1, …, (np–1).*

(ii) *The derivative of all powers of x with degree p_1, p_2, …, np are such that it is zero.*

The proof is direct and hence left as an exercise to the reader.

Next we provide examples of pseudo interval finite complex modulo integer real quaternion polynomial rings and some of the related properties of them.

Example 3.31: Let

$$P_C[x] = \left\{ \sum_{s=0}^{\infty} a_s x^s \;\middle|\; a_s \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \right.$$

$C([0, 12))$, $0 \leq t \leq 3$, $i^2 = j^2 = k^2 = ijk = 11$, $ij = 11ji = k$, $jk = 11kj = i$, $ki = 11ik = j$, $i_F^2 = (ki_F)^2 = 1\} +, \times\}$ be the pseudo interval finite complex modulo integer real quaternion polynomial ring.

We see all linear equation in $P_C[x]$ are not uniquely solvable in general. Let $p(x) = 5x + 2 \in P[x]$ we see p(x) is uniquely solvable for x = 2 is a root as $p(2) = 5 \times 2 + 2 \equiv 0 \pmod{12}$.

$p(x) = 5x + 3.1 \in P[x]$, has no solution. Thus even though the coefficient of x is a unit in [0, 12) we see this polynomial $p(x) = 5x + 3.1$ has no solution in [0, 12).

Thus we see the problem of even solving a linear equation in $P_C[x]$ happens to be an open conjecture.

However product and sum of the elements in $P_C[x]$ can be done without any difficulty.

Let $p(x) = [(3ii_F + 4j + 2i_F + k)x^3 + (4i_Fk + 2i_Fj + 3)]$ and $q(x) = [(3i_F + 4k + (2 + i_F)j)\ x + (2 + 3i_F)k] \in P_C[x]$.

$p(x) + q(x) = [(3ii_F + 4j + 2i_F + k)x^3 + (4i_Fk + 2i_Fj + 3)] + [(3i_F + 4k + (2 + i_F)j)\ x + (2 + 3i_F)k]$

$= \quad (3ii_F + 4j + 2i_F + k)x^3 + (3i_F + 4k + (2 + i_F)j)\ x + (2 + 7i_F)\ k\ (2i_Fj+3)] \in P_C[x]$.

Consider $p(x) \times q(x) = [(3ii_F + 4j + 2i_F + k)x^3 + (4i_Fk + 2i_Fj + 3)] \times [(3i_F + 4k + (2 + i_F)j)\ x + (2 + 3i_F)k]$

$= (2i_F + k + 4j + 3ii_F)\ (3i_F + 4k + (2 + i_F)j)\ x^4\ (4i_Fk + 2i_Fj + 3)\ (3i_F + 4k + (2 + i_F)j\ )\ x + (2i_F + k + 4j + 3ii_F)\ (2 + 3i_F)\ kx^3 + (4i_Fk + 2i_Fj + 3)\ (2 + 3i_F)k$

$= (6 \times 11 + 3i_Fk + 12i_Fj + 9i \times 11 + 8i_Fk + 4 \times 11 + 16i + 12iki_F + 2(2 + i_F)i_F + 4\ (2 + i_F) \times 11 + 3\ (2 + i_F)i_Fij)\ x^4 + (12 \times 11k + 6 \times 11j + 9i_F + 16i_F \times 11 + 8i_Fjk + 12k + 4i_F\ (2 + i_F)kj + 2i_F\ (2 + i_F) \times 11 + 3\ (2 + i_F)j)x + (4i_Fk + 2k_2 + 8jk + 6ii_Fk + 6 \times 11k + 31i_Fk^2 + 12ji_Fk + 9i \times 11k)\ x^3 + (8i_Fk^2 + 4 \times 11jk + 6k + 4i_Fk^2 + 6 \times 11jk + 9i_Fk)$

$= (6 + 3i_Fk + 0 + 3i + 8i_Fk + 8 + 4i + 0 + 4i_Fj + 2 \times 11j + 88 + 44i_F + 61i_Fk + 3 \times 11k]\ x^4 +$

$(0 + 6j + 9i_F + 8i_F + 8i_Fi + 0 + 8i_F \times 11i + 4 \times 11 \times 11i + 8i_F + 2 \times 11 \times 11 + 6j + 3i_Fj)\ x + (4i_Fk + 10 + 8 \times I + 6i_Fj + 6k + 9i_F+0 + 9j)\ x^3 + (4i_F + 8i + 6k + 8i_F + 6i + 9i_Fk)$

$= (6 + 8i_F + (9 + 5i_F)k + 7i + (10 + 4i_F)j)\ x^4 + (2 + i_F) + (4 + 0)I + 3i_Fj)\ x + (10 + 9i_F + 8i + (9 + 6i_F)j + 6 + 4i_F)\ k)\ x^3 + (2i + (6 + 9i_F)k) \in P_C[x].$

This is the way product is performed in $P_C[x]$.

We see $P_C[x]$ is of infinite order non commutative but has infinite number of zero divisors only finite number of units and idempotents.

Example 3.32: Let $P_C[x] = \left\{ \sum_{s=0}^{\infty} a_s x^s \,\middle|\, a_s \in P_C = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in C([0, 11)), 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 10, ij = 10ji = k, jk = 10kj = i, ki = 10ik = j\} +, \times \right\}$ be the pseudo interval finite complex modulo integer real quaternion polynomial ring.

We see this ring also has zero divisors and finite number of units.

Integration and differentiation can be performed with appropriate modifications.

We see product and sum of any two polynomials can be obtained.

Example 3.33: Let $P_N[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_s \in P_N = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in C([0, 24)), 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 23, ij = 23ji = k, jk = 23kj = i, ki = 23ik = j\} +, \times \right\}$ be a pseudo neutrosophic real quaternion polynomial ring.

Let $p(x) = (2I + 3)\ x^4 + (7I + 8)\ x^2 + 9I$ and
$q(x) = (8I + 7) + (3I + 2)x + (7I + 19)\ x^2 \in P[x]$.

$p(x) + q(x) = [(2I + 3)\, x^4 + (7I + 8)\, x^2 + 9I] + [(8I + 7) + (3I + 2)x + (7I + 19)\, x^2]$

$= (2I + 3)\, x^4 + (14I = 2)\, x^2 + (3I + 2)x + (17I + 7) \in P_N[x]$.

$p(x) \times q(x) = [(2I + 3)x^4 + (7I + 8)x^2 + 9I]\, [(7I + 19)x^2 + 9I]\, [(7I + 19)x^2 + (3I + 2)x + (8I + 7)]$

$= (2I + 3)\,(7I + 19)\, x^6 + (7I + 8)\,(7I + 19)x^4 + 9I\,(7I + 8)x^2 + (2I + 3)\,(3I + 2)x^5 + (7I + 8)\,(3I + 2)x^3 + 9I\,(3I + 2)x + (2I + 3)\,(8I + 7)\, x^4 + (7I + 8)\,(8I + 7)\, x^2 + 9I\,(8I + 7)$

$= (14I + 21I + 38I + 57)\, x^6 + (49I + 56I + 133I + 152)x^4 + (72I + 63I)x^2 + (6I + 9I + 4I + 6) + (21I + 24I + 14I + 16)\, x^3 + (27I + 18I)\, x + 16I + 24I + 14I + 21)x^4 + (56I + 56 + 49I + 64I)\, x^2 + (72I + 63I)$

$= (I + 19)x^6 + (22I + 8)\, x^4 + 15Ix^2 + (6 + 19I)\, x + (11I + 16)\, x^3 + 21I\, x + (6I + 2I)\, x^4 + (11I + 18)x^2 + 15I$

$= (I + 19)\, x^6 + (5 + 4I)x^4 + (11I + 16)x^3 + (6 + 16I)\, x + (2I + 18)x^2 + 15I$.

This is the way product is performed in $P_N[x]$.

Example 3.34: Let $P_N[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_s \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle [0, 11) \cup I \rangle),\ 0 \le t \le 3,\ i^2 = j^2 = k^2 = ijk = 10,\ ij = 10ji = k,\ jk = 10kj = i,\ ki = 10ik = j\}\ +, \times \right\}$ be the pseudo interval neutrosophic real quaternion polynomial ring.

$P_N[x]$ is non commutative and has zero divisors units and idempotents. $P_N[x]$ has all other properties appropriately.

Now we proceed on to give examples of pseudo interval pseudo interval finite complex modulo integer neutrosophic real quaternion polynomial ring.

Example 3.35: Let $P_{NC}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_s \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle[0, 7) \cup I\rangle), 0 \leq t \leq 3, i^2 = j^2 = k^2 = ijk = 6, ij = 6ji = k, jk = 6kj = i, ki = 6ik = j\} +, \times \right\}$ be the pseudo interval finite complex modulo integer neutrosophic real quaternion polynomial ring.

This polynomial ring has zero divisors, units and idempotents.

We just show how sum and product are performed on $P_{NC}[x]$.

Let $p(x) = (6I + 2.4i\ i_F) + (4 + 2i_F + I + i_F I)\ k)\ x^2 + 0.4 + 0.2I + 5i_F I)\ x + 3i_F Ik$ and

$q(x) = (2I + 3i)\ x^2 + (4 + 2i_F I + 3i + 4k) \in P_{NC}[x]$.

We find $p(x) + q(x) = [(6I + 2.4i\ i_F) + (4 + 2i_F + I + i_F I)\ k)\ x^2 + (0.4 + 0.2I + 5i_F I)x + 3i_F Ik] + [(2I + 3i)\ x^2 + (4 + 2i_F I + 3i + 4k)]$ $(8I + 2.4ii_F + 3i + (4 + 2i_F + I + Ii_F)k)\ x^2\ (0.4 + 0.2I + 5iFI)x + (4 + 2i_F I + 3i + 4k + 3i_F Ik) \in P_{NC}\ [x]$.

We find $p(x) \times q(x) = [6I + 2.4ii_F + (4 + 2i_F + I + Ii_F)k] \times (2I + 3i)\ x^4 + (0.4 + 0.2I + 5i_F I)\ (4 + 2i_F I + 3i + 4k)\ x + 3i_F k\ (4 + 2i_F I + 3i + 4k)$

$= (5I + 4.8i_F I + (3I + 6Ii_F)k + (5 + 6i_F + 3I + 3Ii_F)j\ ]\ x^4\ (0.8I + 0.4I + 10i_F I + 1.2i + 0.6iI + 15i_F I)\ x^3 + (6i_F Ik + 9 \times ji_F i)\ x^2 + (6I \times 4 + 9.6ii_F + (16 + 8i_F + 4I + 4i_F I)k + 12i_F I + 4.8i \times 6(8i_F I + 4 \times 6I + 2i_F I + 2I \times 6)k + + (18Ii + 7.2 \times 6i_F + 12 \times j + 6 \times i_F j + 3Ij + 3i_F Ij + 24Ik + 9.6 \times 6j\ i_F + 16 \times 6 + 8i_F \times 6 + 4I \times 6 + 4i_F I \times 6]\ x^2 + (1.6 + 0.8I + 20i_F I + 1.2i + 0.6Ii + 15i_F Ii + 1.6k + 0.8Ik + 20i_F\ Ik)x + (12i_F k + 6 \times 6kI + 9 \times 6ji_F + 12 \times 6 \times i_F)\ (\text{mod } 7)$.

This is the way the product is defined.

Of course the reader is assigned the simple task of simplifying the equations.

We give some more examples.

Example 3.36: Let $P_{NC}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle[0, 24) \cup I\rangle), 0 \le t \le 3, +, \times\} \right.$ be the pseudo interval real quaternion polynomial ring.

All subrings in $P_{NC}[x]$ are of infinite order. Some of the subrings of infinite order are not ideals of $P_{NC}[x]$.

Infact $P_{NC}[x]$ has infinite number of zero divisors, only finite number of units and idempotents.

Further $P[x] \subseteq P_N[x] \subseteq P_{NC}[x]$ as subrings and
$P[x] \subseteq P_C[x] \subseteq P_{NC}[x]$.

We see these subrings are pseudo and behave differently.

Now we proceed onto define pseudo interval linear algebras and vector spaces using pseudo interval polynomial rings. Further we see only using strong Smarandache polynomial rings which are pseudo we can build pseudo inner product and linear functionals.

DEFINITION 3.2: *Let V ={P[x] / P[x] is the group under + with coefficients from the interval [0, n); n a prime}. V is a vector space over Z_n defined as the interval polynomials vector space.*

Clearly V is of infinite dimension over Z_n.

V has both finite and infinite subspaces.

All these concepts will be illustrated by some examples.

Example 3.37: Let $V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 23), + \right\}$ be the polynomial interval vector space over the field Z_{23}.

$M = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_{23}\right\} \subseteq V$ is a finite dimensional vector subspace of V over Z_{23}.

However $N = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_{23}, +\right\} \subseteq V$ is a subspace of V over Z_{23} but is of infinite dimension.

Example 3.38: Let $P = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 43), +\right\}$ be the interval polynomial ring over the field Z_{43}. P is an infinite dimensional polynomial interval vector space over Z_{43}.

Now if we build polynomial interval vector spaces over the S-ring Z_n; we call such polynomial vector spaces as S-polynomial interval vector spaces.

We will illustrate this situation by some examples.

Example 3.39: Let $V = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 23), +\right\}$ be the S-interval polynomial vector space over the S-ring Z_{12}.

Example 3.40: Let $V = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 45), +\right\}$ be the S-interval polynomial vector space over the S-ring Z_{45}.

Now we define an interval polynomial vector space over Z_p to be a pseudo interval polynomial linear algebra over the field Z_p. Since P[x] under product and + is only a pseudo interval ring as the distributive law of product over + is not true.

If we define using S-vector spaces of interval polynomial we define them as S-interval polynomial linear algebra.

This will be illustrated by the following examples.

Example 3.41: Let

$$V = \{P[x] \mid P\,[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 11), +, \times\right\}$$

be the pseudo interval polynomial linear algebra over the field Z_{11}.

We have sublinear algebras of finite dimension over Z_{11}.

$S = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_{11}, +, \times\right\} \subseteq V$ is a sublinear algebra of finite dimension and the subalgebra is not pseudo.

For {1, x} is a basis of S.

Example 3.42: Let $V = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 5), +, \times\right\}$ be the pseudo interval polynomial linear algebra over the field Z_5.

The dimension of V over Z_5 is infinite.

However V has sublinear algebras of finite dimension also.

Example 3.43: Let $V = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 15), +, \times\right\}$ be the S-pseudo interval polynomial linear algebra over the S-ring Z_{15}. V is infinite dimensional over Z_{15}.

Example 3.44: Let $M = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 46), +, \times\right\}$ be a S-pseudo interval polynomial polynomial linear algebra over the S-ring Z_{46}.

We have the following theorem the proof of which is direct.

THEOREM 3.7: *Let V =* $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, n), +, \times\right\}$ *(n is such that either n is a prime or n is such that* Z_n *is a S-ring) be the pseudo interval polynomial linear algebra (or S-linear algebra) over* Z_n*. V is a interval polynomial vector space or a interval polynomial S-vector space over* Z_n*.*

However a interval polynomial vector space over Z_n (or [0, n) interval polynomial interval vector space over S-ring Z_n) in general need not be a pseudo interval polynomial linear algebra or a S-pseudo polynomial interval linear algebra over Z_n (or the S-ring Z_n).

Proof: One way is direct. To prove the other part we give an example.

Consider V = $\left\{\sum_{i=0}^{20} a_i x^i \,\middle|\, a_i \in [0, 7), 0 \le i \le 20, +\right\}$; V is only a interval vector space of polynomial and is not a pseudo interval polynomial linear algebra over the field Z_7.

For if $p(x) = 5.2x^{19} + 3x + 0.8$ and
$q(x) = 2x^5 + 3x + 0.6 \in V$;

$$p(x) \times q(x) = (5.2x^{19} + 3x + 0.8) \times (2x^5 + 0.3x + 0.6)$$

$$= 10.4x^{24} + 6x^6 + 1.6x^5 + 1.56x^{20} + 0.9x^2 + 0.24x + 3.12x^{19} + 1.8x + 0.24 \notin V.$$

Hence the claim.

Now we will illustrate some more properties of these spaces.

In the first place as in case of usual spaces linear transformation can be defined only when the pseudo interval spaces are defined over the same field or the same S-ring.

Further for this we need to built other types of algebraic structures using the interval polynomial pseudo linear algebra as we would only land in linear operators in that case.

Example 3.45: Let V = $\left\{ \sum_{i=0}^{20} a_i x^i \,\middle|\, a_i \in [0, 19), 0 \le i \le 20, +\right\}$ and W = $\left\{ \sum_{i=0}^{10} a_i x^i \,\middle|\, a_i \in [0, 19), 0 \le i \le 10, +\right\}$ be any two interval polynomial vector spaces defined over the field Z_{19}.

Define T : V $\to$ W by

$$T\left(\sum_{i=0}^{20} a_i x^i \right) = \sum_{i=0}^{10} a_i x^i \text{ (and } a_{11}x^{11} = 0 = a_{12}x^{12} = \ldots = a_{20}x^{20}).$$

It is easily verified T is a linear transformation from V to W.

Now we give examples of matrices built using interval polynomial groups and interval pseudo polynomial rings in the following.

Example 3.46: Let W = $\{(a_1, a_2, a_3) \mid a_i \in P[x] = \left\{ \sum_{i=0}^{\infty} b_i x^i \,\middle|\, b_j \in [0, 29), +\right\}; 1 \le i \le 3, +\}$ be a interval matrix polynomial vector space over the field Z_{29}.

Example 3.47: Let V = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \,\middle|\, a_i \in P[x] = \left\{ \sum_{i=0}^{\infty} b_i x^i \,\middle|\, b_j \in [0, 33), +\right\}; 1 \le i \le 4, +\right\}$ be the polynomial interval column matrix vector space over the S-ring Z_{33}.

$$\text{Let } A = \begin{bmatrix} 0.4x^2 + 3x + 7.1 \\ 0 \\ 5.1x^7 + 2x^3 + 1.7 \\ 3.9x^{12} + 4.271 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 10.5x^7 + 3.6x^2 + 6.001 \\ 7.51x^{11} + 20.7x^{10} + 3.1106 \\ 7.81x^{20} + 11.3x^3 + 8.13x \\ 14.37x^{16} + 10.3x^2 + 9.9 \end{bmatrix} \in V.$$

We find A + B as follows:

$$A + B = \begin{bmatrix} 0.4x^7 + 3x + 7.1 + 10.5x^7 + 3.6x^2 + 6.001 \\ 0 + 7.51x^{11} + 20.7x^{10} + 3.1106 \\ 5.1x^7 + 2x^3 + 1.7 + 7.81x^{20} + 11.3x^3 + 8.13x \\ 3.9x^{12} + 4.271 + 14.37x^{16} + 10.3x^2 + 9.9 \end{bmatrix}$$

$$= \begin{bmatrix} 4x^2 + 10.5x^7 + 3x + 13.101 \\ 7.51x^{11} + 20.7x^{10} + 3.1106 \\ 7.81x^{20} + 5.1x^7 + 1.7 + 13.3x^3 + 8.13x \\ 14.37x^{16} + 3.9x^{12} + 10.3x^2 + 14.171 \end{bmatrix} \in V.$$

This is the way the operation + is performed on V.

Example 3.48: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in P[x] \right.$$

$$= \left\{ \sum_{i=0}^{\infty} b_i x^i \,\middle|\, b_j \in [0, 17), +\}; 1 \leq i \leq 9\right\}$$

be the polynomial interval matrix vector space over the field Z_{17}.

W has atleast ${}_9C_1 + {}_9C_2 + \ldots + {}_9C_8$ number of subspaces of infinite dimension over Z_{17}.

Example 3.49: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle|\, a_j \in P[x] = \left\{ \sum_{i=0}^{\infty} b_i x^i \,\middle|\, b_i \in [0, 48), +\right\}; \right.$$

$1 \leq j \leq 18, +\}$ be the interval polynomial matrix S-vector space over the S-ring Z_{48}.

S has atleast ${}_{18}C_1 + {}_{18}C_2 + \ldots + {}_{18}C_{17}$ number of S-subspaces of infinite dimension over Z_{48}.

Example 3.50: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{27} \end{bmatrix} \middle|\, a_j \in P[x] \right.$$

$= \left\{ \sum_{i=0}^{\infty} b_i x^i \,\middle|\, b_i \in [0, 12), +\}; 1 \leq i \leq 27, +\right\}$ be the S-polynomial interval matrix vector space over the S-ring Z_{12}.

Several interesting properties can be derived as a matter of routine and hence left as an exercise to the reader.

Example 3.51: Let V = {(a_1, a_2, …, a_8) | $a_i \in P[x] = \left\{ \sum_{j=0}^{\infty} b_j x^i \,\middle|\, b_j \right.$ $\in [0, 5)$, $1 \le i \le 8$, +} and

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| a_i \in P[x] = \left\{ \sum_{i=0}^{\infty} b_j x^j \,\middle|\, b_j \in [0, 5), + \right\}; \right.$$

$1 \le i \le 16$, +} be interval polynomial matrix vector spaces over the field Z_5.

We can define linear operators from V to S.

Define T : V → S by

$$T\{(a_1, a_2, \ldots, a_8) = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

T is a linear transformation from V to S.

We can build several such linear transformation from V to S.

Example 3.52: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \, a_i \in P[x] = \left\{ \sum_{j=0}^{\infty} b_j x^j \middle| \, b_j \in [0, 21), + \right\}; 1 \leq i \leq 6, + \right\}$$

and

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \, a_i \in P[x] = \left\{ \sum_{j=0}^{\infty} b_j x^j \middle| \, b_j \in [0, 21), + \right\}; \right.$$

$1 \leq i \leq 12, +\}$ be the interval matrix polynomial vector spaces over the S-ring Z_{21}.

Define a linear transformation

$$T : M \to S \text{ by}$$

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & a_4 \\ a_5 & 0 & a_6 & 0 \end{bmatrix};$$

it is easily verified T is a linear transformation from M to S.

Interested reader can study the algebraic structure of $\text{Hom}_{Z_{21}}$ (M, S) and this work is considered as a matter of routine.

Now consider two pseudo interval matrix polynomial linear algebras over a field.

We show by examples how to define linear transformation using them.

Example 3.53: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| a_i \in P[x] = \left\{ \sum_{j=0}^{\infty} b_j x^j \middle| b_j \in [0, 13), + \right\}; \right.$$

$$1 \leq i \leq 18, +, \times_n \}$$

and

$$V = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \end{pmatrix} \middle| a_i \in P[x] = \left\{ \sum_{j=0}^{\infty} b_j x^j \middle| b_j \in [0, 13), \right. \right.$$

$$+, \times\}; 1 \leq i \leq 18, +, \times_n \}$$

be any two pseudo interval polynomial linear algebras of matrices under natural product over the field Z_{13}.

Define a linear transformation T from V to W in the following way $T : V \to W$;

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{16} & a_{17} & a_{18} \end{bmatrix}\right) = \begin{pmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \end{pmatrix}.$$

Clearly T is a pseudo linear transformation from V to W as the basic structures V and W are only pseudo polynomial interval linear algebras over Z_{13}.

This study is also considered as a matter of routine so interested reader can study.

Now the basic problem we encounter with these spaces and linear algebras is that we are not in a position to define inner products or linear functionals if they are defined on Z_n, so we make a modification in our definition.

We define the notion of Smarandache strong interval polynomial vector spaces (linear algebras) define over the pseudo interval Smarandache ring [0, n).

Let $V = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, n), +\right\}$ be defined as the Smarandache strong interval polynomial vector space over the Smarandache pseudo interval ring [0, n).

Only now we can define pseudo inner product from

$V \times V \to [0, n)$ as follows.

If $p(x) = p_0 + \ldots + p_n x^n$ and

$q(x) = q_0 + \ldots + q_n x^n \in V$ with $p_i\, q_j \in [0, n)$.

Define $\langle p(x), q(x)\rangle = \sum_{\substack{i=0\\j=0}}^{n^m} p_i q^j \in [0, n)$ then $\langle\ ,\ \rangle$ is defined as the pseudo inner product.

We call it pseudo for $\langle p(x), p(x)\rangle = 0$ is possible even if $p(x) \neq 0$.

That is why in the first place we call them as pseudo inner product space.

We may have other properties to be true or false depending on the structure. The definition of pseudo linear functionals is a matter of routine.

We can build this only when the space is defined over [0, n).

We illustrate these situations by some examples.

Example 3.54: Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 15), +, \times \right\}$$

be the pseudo interval polynomial Smarandache strong linear algebra over the S-pseudo interval ring [0, 15) define inner product $\langle\ ,\ \rangle\ V \times V \to [0, 15)$ as follows:

$$\text{If } p(x) = 0.7x^3 + 2x + 7$$
$$\text{and } q(x) = 5x^2 + 0.5x + 1.2 \in V.$$

$\langle p(x), q(x)\rangle = 0.7 \times 5 + 2 \times 5 + 7 \times 5 + 0.7 \times 0.5 + 2 \times 0.5 + 2 \times 0.5 + 7 \times 0.5 + 0.7 \times 1.2 + 2 \times 1.2 + 7 \times 1.2 \pmod{15}$

$= 3.5 + 10 + 35 + 0.35 + 1 + 3.5 + 0.84 = 2.4 + 8.4 \pmod{15}$

$= \quad 4.99.$

This is the way inner product is defined.

We can define inner product in other ways also say by finding sum of product even powers of x (or odd powers of x) and so on.

We will illustrate them by an example or two.

Example 3.55: Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 12), +, \times \right\}$$

be the S-strong interval pseudo linear algebra over the S-ring [0, 12).

Define $\langle\ \ \rangle : V \times V \to [0, 12)$ as follows:

If p(x), q(x) $\in$ V then $\langle p(x), q(x)\rangle$ = sum of the product of the even powers of x only.

For if $p(x) = 0.8x^6 + 9.37x^5 + 2x^2 + 3$
and $q(x) = 3.89x^7 + 4.2x^6 + 9.37x^3 + 4.578x + 2 \in S$.

$\langle p(x), q(x)\rangle = 0.8 \times 4.2 + 2 \times 4.2 + 0.8 \times 2 + 2 \times 2 + 3 \times 4.2 + 3 \times 2 \pmod{12}$

$$= 3.36 + 8.4 + 1.6 + 4 + 12.6 + 6$$
$$= 11.36 \pmod{12} \in [0, 12).$$

This is yet another way of defining the pseudo inner product on V.

We see $\langle p(x), q(x)\rangle = 0$ is also possible even without $p(x) = 0$ or $q(x) = 0$ also $\langle p(x), q(x)\rangle = 0$ is also possible.

For take $p(x) = 4x^4 + 3.7x^3 + 8x^2$ and
$q(x) = 3x^4 + 6x^2 + 7.1x^3 + 9.32x \in V$.

We see $\langle p(x), q(x)\rangle = 0 \pmod{12}$.

However $p(x) \neq 0$. This study can be carried out with appropriate modification.

Next we can define linear functionals from V to [0, n) and this will pave way for the concept of dual spaces.

If $p(x) = p_0 + p_1x + \ldots + p_nx^n$ one simple way of defining linear functional, $f : V \to [0, n)$ for $p(x) \in V$.

$$f(p(x)) = \sum_{i=0}^{n} p_i \pmod{n}.$$

It is pertinent to keep on record that we can define in other ways also as f(p(x)) = sum of odd powers or = sum of even power and so on.

It is flexible and is in the hands of the researcher to define in any way to suit the purpose of the study.

The reader can study the pseudo dual space and it is a matter of routine with simple and appropriate modifications.

Now having seen that the concept of linear functional and pseudo inner product can be defined in a natural way only when the space is a Smarandache proceed on to define these concepts in case of finite real quaternions.

DEFINITION 3.3: *Let* $P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, n), +, n \text{ a prime}\}\right.$ *be defined as the interval polynomial finite real quaternion vector space over* Z_n, *n a prime.*

We will illustrate this by some examples.

Example 3.56: Let $P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, 19), 0 \leq i \leq 3, +\} \right.$ be the interval polynomial finite real quaternion vector space over Z_{19}.

We see if $p(x) = 0.7x^3 + 2x^2 + 8.31 \in P[x]$.

If $a = 9.5 \in [0, 19)$, $9.5 \times p(x) = 9.5\,(0.7x^3 + 2x^2 + 8.31)$

$= 6.65x^3 + 0 + 4.115 \in P[x]$.

Let $p(x) = 0.3x^4 + 7x^2 + 2$ and
$q(x) = 6.4x^2 + 15.2 \in P[x]$;

$p(x) \times q(x) = (0.3x^4 + 7x^2 + 2) \times (6.4x^2 + 15.2)$
$= 1.92x^6 + 4.56x^4 + 44.8x^4 + 106.4x^2 + 12.8x^2 + 30.4$
$= 0.2x^6 + 4.56x^6 + 6.8x^4 + 5.2x^2 + 11.4.$

We cannot define inner product or linear functional for this space.

Example 3.57: Let $P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_j \in [0, 12), 0 \leq j \leq 3, +\} \right.$ be the interval vector space of real quaternions over the S-pseudo interval quaternion ring.

Example 3.58: Let $P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_j \in [0, 143), 0 \leq j \leq 3, +\} \right.$ be the interval vector space of real quaternions over the S-pseudo interval quaternion ring P.

Example 3.59: Let $P_C[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, 43), 0 \leq t \leq 3, +\}, +\right\}$ be the interval vector space of real quaternions over the field Z_{43}. $P_C[x]$ is infinite dimensional over Z_{43}.

Example 3.60: Let $P_C[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, 28), 0 \leq t \leq 3, +\}, +\right\}$ be the interval S-vector space of complex real quaternions over the S-ring Z_{28}. This has S-subspaces of finite and infinite order.

Example 3.61: Let $P_C[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, 19), 0 \leq t \leq 3, +\}, +\right\}$ be a Smarandache strong interval vector space of real quaternions over the interval pseudo Smarandache ring of real quaternions of complex finite modulo integers P_C.

On $P_C[x]$ we can define pseudo inner product and linear functionals.

Example 3.62: Let $P_N[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle Z_{11} \cup I\rangle; 0 \leq t \leq 3, +\}, +\right\}$ be the interval finite real quaternion neutrosophic vector space over the field Z_{11}.

This space is also of infinite dimensional over Z_{11}.

However $P_N[x]$ has finite and finite dimensional subspaces.

For $P = \left\{\sum_{i=0}^{3} a_i x^i \,\middle|\, a_i \in Z_{11}, 0 \leq i \leq 3\right\} \subseteq P_N[x]$ is a finite dimensional subspace.

Take $T = \left\{ \sum_{i=0}^{5} a_i x^i \right| a_i \in \langle Z_{11} \cup I\rangle; 0 \le i \le 5, +\} \subseteq P_N[x]$; T is again finite dimensional neutrosophic vector subspace over Z_{11}.

Example 3.63: Let $P_N[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle [0, 19) \cup I\rangle; 0 \le t \le 3, +\}, +\}$ be the S-interval real quaternion vector space over the neutrosophic S-ring $\langle Z_{19} \cup I\rangle$. This has finite dimensional S-vector subspace.

Example 3.64: Let $P_N[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle [0, 17) \cup I\rangle; 0 \le t \le 3, +\}, +\}$ be the interval vector space real quaternions of polynomials over the S-pseudo neutrosophic real quaternions ring P_N.

In this case we can build inner product on $P_N[x]$. We can also define on $P_N[x]$ a linear functional.

Let $f : P_N[x] \to P_N$.

$$f(p(x)) = \sum_{i=0}^{\infty} p_i \pmod{17} \text{ for any } p(x) \in P[x].$$

Example 3.65: Let $P_{NC}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle [0, 23) \cup I\rangle; 0 \le t \le 3\}$ be the interval finite complex modulo integer neutrosophic real quaternion vector space over the field Z_{23}.

P_{NC} has finite subspaces as well as infinite subspaces.

Example 3.66: Let $P_{NC}[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle [0, 43) \cup I\rangle; 0 \le t \le 3, +\}$ be the S-interval neutrosophic finite complex modulo integer neutrosophic real quaternion vector space over $\langle Z_{43} \cup I\rangle$ (or $C(Z_{43})$ or $C(\langle[Z_{43} \cup I\rangle)$.

On all these spaces we cannot built linear functionals or inner product.

Only when the spaces are built over P_C or P or P_N or P_{NC} we can define provided they are pseudo interval real quaternion linear algebra.

We will illustrate this by an example or two.

Example 3.67: Let $P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, 48); 0 \leq t \leq 3\}, +, \times\right\}$ be the Smarandache pseudo interval linear algebra of real quaternion polynomials over the S-pseudo ring P.

On P[x] we can define inner product as well as linear functionals.

Example 3.68: Let $P_C[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, 23); 0 \leq t \leq 3, +\}, +, \times\right\}$ be the pseudo interval Smarandache strong linear algebra of finite complex modulo integer real quaternions over the S-pseudo interval complex finite modulo integer ring.

We see $P_C[x]$ has several subspaces.

Example 3.69: Let $P_N[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle[0, 29) \cup I\rangle; 0 \leq t \leq 3\}, +, \times\right\}$ be the Smarandache strong pseudo interval neutrosophic real quaternion polynomial vector space over the S-neutrosophic pseudo interval ring P_N.

Example 3.70: Let $P_{NC}[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C(\langle[0, 28) \cup I\rangle);\ 0 \leq t \leq 3\}, +, \times\right\}$ be the Smarandache strong pseudo interval special finite complex modulo integer real quaternion neutrosophic polynomial linear algebra over the S-pseudo finite neutrosophic complex modulo integer real quaternion ring.

$P_{NC}[x]$ has several S-sublinear algebras.

We see $P_{NC}[x]$ is made into pseudo inner product linear algebra.

Further we can define linear functionals from $P_{NC}[x] \to P_{NC}$.

Thus we see one can built matrix Smarandache strong pseudo linear algebras using $P_{NC}[x]$ or $P_C[x]$ or $P_N[x]$ or $P[x]$.

This is considered as a matter of routine and hence left as an exercise to the reader. However we give examples of them.

Example 3.71: Let $V = \{(a_1, a_2, \ldots, a_{10}) \mid a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in [0, 7);\ 0 \leq t \leq 3\}, +\}, +, \times\}$ be the Smarandache strong pseudo interval polynomial ring with coefficients from the pseudo interval real quaternion ring P.

V can be made into an inner product space.

We can define linear functionals from V to P.

We see V has atleast ${}_{10}C_1 + {}_{10}C_2 + \ldots + {}_{10}C_9$ number of S-subalgebras.

Example 3.72: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C([0, 29)); \right.$$

$$\left. 0 \leq t \leq 3\}, +\}, +, \times_n \right\}$$

be the Smarandache strong pseudo interval column matrix finite real quaternion linear algebra over the finite real quaternion pseudo S-ring P_C.

T has also ${}_9C_1 + {}_9C_2 + \ldots + {}_9C_8$ number of S-sublinear algebras. T has zero divisors which are infinite in number.

Example 3.73: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in P_C = \left\{ \sum_{j=0}^{\infty} m_j x^j \middle| m_j \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid \right.\right.$$

$$\left. b_t \in C([0, 17)); 0 \leq t \leq 3, +\}, +, \times_n \right\}$$

be the Smarandache strong special interval polynomial pseudo linear algebra with complex finite modulo integer of real quaternions coefficients over P_C.

We can find substructures in them.

Infact S is infinite dimension over P_C. S has infinite number of zero divisors and a few idempotents.

Example 3.74: Let S = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \middle| a_i \in P_N[x] = \right.$ $\left\{\sum_{i=0}^{\infty} b_i x^i \middle| b_i \in P_N = \{c_0 + c_1 i + c_2 j + c_3 k \mid c_t \in \langle [0, 21) \cup I\rangle);\right.$ $0 \leq t \leq 3, +\}, +, \times_n\}$ be the Smarandache strong pseudo interval neutrosophic finite real quaternions polynomials column matrix linear algebra over the S-pseudo interval ring.

Example 3.75: Let S = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| a_i \in P_{NC}[x] = \right.$ $\left\{\sum_{i=0}^{\infty} m_i x^i \middle| m_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C\langle [0, 42) \cup I\rangle);\right.$ $0 \leq t \leq 3, +\}, +, \times_n, 1 \leq i \leq 3\}$ be the Smarandache strong pseudo linear algebra of matrix with coefficients from S-interval polynomial ring of real neutrosophic complex modulo integer quaternions.

S has units, zero divisors and idempotents.

Study of these is considered as a matter of routine.

We suggest the problems some of which are at research level.

Problems

1. Obtain some special features enjoyed by pseudo interval polynomial rings with coefficients from [0, n).

2. Let R[x] = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 7), +, \times\right\}$ be the pseudo interval polynomial ring.

   (i) Find subrings which are not ideals.
   (ii) Find ideals in R[x].
   (iii) Can R[x] have idempotents?
   (iv) Can R[x] have S-units?
   (v) Can R[x] have ideals?
   (vi) Does R[x] contain S-subrings which are not S-ideals?
   (vii) Does R[x] contain ideals which are not S-ideals?
   (viii) Can we have the concept of S-idempotents in R[x]?
   (ix) Can R[x] have S-zero divisors?

3. Let R[x] = $\{\sum a_i\, x_i \mid a_i \in [0, 48), +, \times\}$ be the pseudo interval polynomial ring.

   Study questions (1) to (ix) of problem 2 for this R[x].

4. Let B[x] = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle [0, 11) \cup I\rangle, +, \times\right\}$ be the special interval neutrosophic polynomial ring.

   (i) Study questions (i) to (ix) of problem two for this B[x].
   (ii) Distinguish between R[x] ad B[x].

5. Let B[x] = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle [0, 24) \cup I\rangle, +, \times\right\}$ be the special interval neutrosophic polynomial ring.

   (i) Study questions (i) to (ix) of problem two for this B[x].
   (ii) Distinguish between the usual interval polynomial ring and interval neutrosophic polynomial ring.

6. Let $T[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in C([0, 23)), +, \times \right\}$ be the interval complex modulo integer polynomial ring.

   (i) Study questions (i) to (ix) of problem 2 for this T[x].
   (ii) Compare T[x] with other two types of rings.

7. Let $T[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in C([0, 56)), +, \times \right\}$ be the interval complex modulo integer polynomial ring.

   (i) Study questions (i) to (ix) of problem 2 for this T[x].

8. Let $M[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle [0, 29) \cup I \rangle, +, \times \right\}$ be the interval polynomial finite complex modulo integer neutrosophic ring.

   (i) Study questions (i) to (ix) of problem 2 for this M[x].

9. Let $M[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle [0, 15) \cup I \rangle, +, \times \right\}$ be the interval finite complex modulo neutrosophic integer polynomial ring.

   (i) Study questions (i) to (ix) of problem 2 for this M[x].

10. Let $P[x] = \left\{ \sum_{s=0}^{\infty} a_s x^s \,\middle|\, a_s \in P = \{ b_0^s + b_1^s i + b_2^s j + b_3^s k;\ b_t^s \in [0, 23);\ 0 \le t \le 3, +, \times \right\}$ be the pseudo interval polynomial ring of finite real quaternions.

Study questions (i) to (ix) of problem 2 for this P[x].

11. Let P[x] = $\left\{\sum_{s=0}^{\infty} a_s x^s \,\middle|\, a_s \in P = \{ b_0^s + b_1^s i + b_2^s j + b_3^s k;\ b_t^s \in [0, 48); 0 \leq t \leq 3, +, \times\}\right.$ be the pseudo interval real quaternion polynomial ring.

    Study questions (i) to (ix) of problem 2 for this P[x].

12. Let $P_c[x] = \left\{\sum_{s=0}^{\infty} a_s x^s \,\middle|\, a_s \in P_c = \{ b_0^s + b_1^s i + b_2^s j + b_3^s k;\ b_t^s \in C([0, 19); 0 \leq t \leq 3, +, \times\}\right.$ be the pseudo interval polynomial ring with coefficients from the complex modulo integer of finite real quaternions.

    Study questions (i) to (ix) of problem 2 for this $P_c[x]$.

13. Let $P_c[x] = \left\{\sum_{s=0}^{\infty} a_s x^s \,\middle|\, a_s \in P_c = \{ b_0^s + b_1^s i + b_2^s j + b_3^s k;\ b_t^s \in C([0, 14); 0 \leq t \leq 3, +, \times\}\right.$ be the pseudo polynomial real finite complex modulo integer quaternion ring.

    Study questions (i) to (ix) of problem 2 for this $P_c[x]$.

14. Let $P_N[x] = \left\{\sum_{s=0}^{\infty} a_s x^s \,\middle|\, a_s \in P_N = \{ b_0^s + b_1^s i + b_2^s j + b_3^s k;\ b_t^s \in \langle[0, 59) \cup I\rangle; 0 \leq t \leq 3,\ i^2 = j^2 = k^2 = 58 = ijk,\ ij = 58ji = k,\ jk = 58kj = i,\ ki = 58ik = j\}, +, \times\}\right.$ be the pseudo interval polynomial ring with neutrosophic modulo integer coefficients.

    Study questions (i) to (ix) of problem 2 for this $P_N[x]$.

15. Let $P_N[x] = \left\{ \sum_{s=0}^{\infty} a_s x^s \,\middle|\, a_s \in P_N = \{ b_0^s + b_1^s i + b_2^s j + b_3^s k; \right.$ $b_t^s \in \langle [0, 32) \cup I \rangle; 0 \leq t \leq 3, +, \times\}$ be the pseudo interval polynomial ring with coefficient from the neutrosophic finite modulo integer of real quaternions.

    Study questions (i) to (ix) of problem 2 for this $P_N[x]$.

16. Let $P_{NC}[x] = \left\{ \sum_{s=0}^{\infty} a_s x^s \,\middle|\, a_s \in P_{NC} = \{ b_0^s + b_1^s i + b_2^s j + b_3^s k; \right.$ $b_t^s \in C(\langle [0, 18) \cup I \rangle); 0 \leq t \leq 3, +, \times\}$ be the pseudo interval neutrosophic complex modulo integer real quaternion polynomial ring.

    Study questions (i) to (ix) of problem 2 for this $P_{NC}[x]$.

17. Let $P_{NC}[x] = \left\{ \sum_{s=0}^{\infty} a_s x^s \,\middle|\, a_s \in P_{NC} = \{ b_0^s + b_1^s i + b_2^s j + b_3^s k; \right.$ $b_t^s \in C(\langle [0, 83) \cup I \rangle); 0 \leq t \leq 3, +, \times\}$ be the pseudo interval neutrosophic complex modulo integer real quaternion polynomial ring.

    Study questions (i) to (ix) of problem 2 for this $P_{NC}[x]$.

18. Let $P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \right.$ $[0, 31), 0 \leq t \leq 3, +, \times\}$ be the interval vector space of polynomial with real quaternion coefficients over Z_{31}.

    (i) What is the dimension of P[x] over Z_{31}?
    (ii) Find subspaces of P[x].
    (iii) Find Hom (P[x], P[x]) = S.
    (iv) What is the algebraic structure enjoyed by S?
    (v) Can P[x] have subspaces of finite dimension?
    (vi) Is it possible to define inner product of on P[x]?

(vii) Is it possible to define linear functional in a natural way?

19. Let $P_C[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_C = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C([0, 42)), 0 \le t \le 3, +, \times\}\right.$ be the Smarandache strong pseudo interval real quaternion polynomial linear algebra over the S-pseudo interval real ring of complex modulo integer quaternions.

    Study questions (i) to (vii) of problem 18 for this $P_C[x]$.

20. Find some special features enjoyed by Smarandache strong pseudo interval linear algebras over the S-pseudo ring P (or P_C or P_N or P_{NC}).

21. Let $P_N[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_N = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in \langle [0, 97) \cup I\rangle, 0 \le t \le 3, +, \times\}\right.$ be the Smarandache strong pseudo interval real quaternion polynomial linear algebra over the S-pseudo interval real ring of complex modulo integer quaternions.

    Study questions (i) to (vii) of problem 18 for this $P_N[x]$.

22. Let $P_{NC}[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t \in C\langle [0, 42) \cup I\rangle), 0 \le t \le 3, +, \times\}\right.$ be the Smarandache strong pseudo interval complex finite modulo integer neutrosophic real quaternion polynomial linear algebra over the S-interval pseudo complex modulo integer real quaternion ring.

    Study questions (i) to (vii) of problem 18 for this $P_{NC}[x]$.

23. Let S = $\{(a_1, a_2,\ldots, a_7) \mid a_j \in P_{NC}[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in P_{NC}\right.$ $= \{b_0 + b_1i + b_2j + b_3k \mid b_t \in C(\langle[0, 128) \cup I\rangle), 0 \leq t \leq 3,$ $+, \times\}, 1 \leq j \leq 7, +, \times\}$ be the Smarandache strong pseudo interval complex finite modulo integer neutrosophic real quaternion polynomial linear algebra over the S-interval pseudo complex modulo integer real quaternion ring.

    Study questions (i) to (vii) of problem 18 for this S.

24. Let M = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \,\middle|\, a_i \in P_C[x] = \left\{\sum_{i=0}^{\infty} m_i x^i \,\middle|\, m_i \in P_C = \{b_0 + \right.\right.$ $b_1i + b_2j + b_3k \mid b_t \in C([0, 23)), 0 \leq t \leq 3, +, \times\}, 1 \leq i \leq 9, +, \times_n\}$ be the Smarandache strong pseudo interval column matrix real quaternion complex modulo integers over the S-pseudo interval ring P_C.

    Study questions (i) to (vii) of problem 18 for this M.

25. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{25} \end{bmatrix} \,\middle|\, a_i \in P_N[x] = \right.$

    $\left\{\sum_{i=0}^{\infty} m_i x^i \,\middle|\, m_i \in P_N = \{b_0 + b_1i + b_2j + b_3k \mid b_t \in \right.$ $\langle[0, 27) \cup I\rangle, 0 \leq t \leq 3, +, \times\}, 1 \leq j \leq 5, +, \times_n\}$ be the

Smarandache strong real quaternion neutrosophic polynomial interval pseudo linear algebra over the S-neutrosophic interval real quaternion ring.
Study questions (i) to (vii) of problem 18 for this M.

26. Let N = $\left\{\left[\begin{array}{c|c|cccc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & \dots & \dots & \dots & \dots & \dots & a_{14} \\ a_{15} & \dots & \dots & \dots & \dots & \dots & a_{21} \\ a_{22} & \dots & \dots & \dots & \dots & \dots & a_{28} \end{array}\right] \middle| a_i \in\right.$
$P_{NC}[x] = \left\{\sum_{i=0}^{\infty} m_i x^i \middle| m_i \in P_{NC} = \{b_0 + b_1 i + b_2 j + b_3 k \mid b_t\right.$
$\in C(\langle [0, 3) \cup I \rangle)$, $0 \leq t \leq 3, +, \times\}$, $1 \leq j \leq 28, +, \times_n\}$ be the Smarandache strong interval neutrosophic finite complex modulo integer real quaternion polynomial super row matrix pseudo linear algebra defined over the S-neutrosophic complex modulo integer real quaternion pseudo ring.

Study questions (i) to (vii) of problem 18 for this N.

27. Obtain some special features enjoyed by these pseudo inner product spaces.

28. In case of N in problem 26 find Hom (N, P_{NC}) = V.

29. What is the algebraic structure enjoyed by V in problem 28?

30. Find W = Hom (N, N), N given in problem 26.

31. What is the algebraic structure enjoyed by W in problem 30?

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INDEX

N

P

R

S

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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Here finite real quaternion rings, and finite complex modulo integer quaternion rings, neutrosophic finite quaternion rings, complex neutrosophic quaternion rings for the first time are introduced and analysed. All these rings behave in a very unique way.

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