“Rekenen-informatica”: Informatics for Primary School Mathematics

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Abstract

A number of issues is listed which arise within primary school mathematics and where a perspective of informatics may shed some new light on the matter. Together these points prove that there are many different possible connections between informatics and primary school mathematics, each of which merit further investigation and clarification. A rationale for further investigation of these issues is given.

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1 Introduction

Teaching elementary arithmetic (in Dutch "rekenen") as a part of teaching school mathematics (in Dutch "rekenen-wiskunde") at a primary school level takes place throughout the world. From a scientific point of view that part of arithmetic is rather uninteresting, its mathematics is nearly trivial while its logic is remarkably unexplored, and the main research efforts about it concern the following aspects:

1. What goals to set in relation to the age group and other characteristics of the students. (I will speak of students irrespective of their age, maturity, level, or ambitions).

2. How to measure ability and competence in the area of arithmetic and how to assess progress,

3. How to determine the quality of learning outcomes for an entire population for a region or for a country.

4. How to organise the teaching of arithmetic at a large institutional scale, how to allocate and distribute responsibilities, which responsibilities to leave to schools and teachers, parents, and of course students, and which responsibilities to centralise at higher levels of aggregation.

5. Assessment of the effectiveness of teaching methods and materials.

6. Cognitive psychology including the cognitive neuroscience of arithmetic.

7. Investigating the practical value of arithmetical skills for later stages of school development.

I will use PSM to denote primary school mathematics with a focus on its elementary kernel involving numbers, ordering, counting, classification, arithmetic, some sets, graphs, and other containers, and some algebra.

1.1 Focus on PSM content matter

Missing entirely from the listing of research efforts above is the subject itself: what is it that one intends to convey and to what extent is that subject itself a source for further reflection. This topic may in principle be approached along the following well-known lines:

1. Finding connections with more sophisticated mathematics such as: combinatorics, logic, proof theory, geometry, number theory, linear algebra, rings, and fields. It seems fair to say that none of these connections provides anything beyond confronting the students with simplified versions of expositions of very well-known parts of more advanced mathematics.

2. Taking logic and reasoning very seriously and dealing with elementary arithmetic in those terms. However, it is somewhat unhelpful that mathematical logic presumes the mastery of elementary arithmetic rather than that it supports the acquisition of that mastery. Providing connections between philosophical logics and elementary arithmetic is possible in principle but no significant tradition has yet emerged as the purpose of the introduction of philosophical logic is usually more grands style so to say than contemplating $2 + 3 = 5$.

3. Looking at applications of elementary arithmetic in realistic or even real settings. Here it seems impossible to go beyond simplistic introductions to aspects of physics, chemistry, and economics.

4. As a fourth and most promising option I see having an informatics perspective on elementary arithmetic. This seems to lead immediately to a wealth of novel considerations which are entirely insensitive to the elementary character of the underlying arithmetic.

My focus is entirely on the last option. I claim that by viewing elementary arithmetic, or say primary school mathematics (PSM), as a subject embedded within informatics it instantaneously becomes highly interesting and non-trivial at the same time. This perspective has been labeled (in Dutch) rekenen-informatica in [4] where an attempt is made to survey its characteristics. I will use the phrase informatics for primary school mathematics (I4PSM) as an English counterpart of *rekenen-informatica*. I4PSM includes the teaching of similar content to higher age groups, that is school is not meant as an indication of student age but as an indication of student maturity relative to the subject.

1.2 Informatics informed viewpoints on PSM

Thinking about PSM from the perspective of informatics has changed my view on PSM entirely. Even the simplest aspects of it become at once intriguing and challenging from an informatics perspective. I hope to illustrate that viewpoint with the listing of cases below. Each case involves an assertion or cognition which can be found at the initial level of PSM teaching. I then try to indicate how notions from informatics may come into play in connection with that assertion or cognition.

However, apart from providing pleasant clues to contemplate snippets of PSM, the informatics perspective changed my view on PSM in a more radical, and perhaps indefensible, way. My views on PSM and the potential relevance of I4PSM change in time and don't easily stabilise, but that will probably not go away, although subjectively speaking there seems to be some sort of convergence. My current view on the matter is summarised in the following listing of beliefs (viewpoints, opinions).

1. I4PSM is the preferred home for PSM, thus replacing mathematics and logic at the same time as candidates for providing that hosting.

2. PSM at the earliest level provide much room for content development.

3. The most relevant learning outcomes of I4PSM are not anymore the conventional arithmetical skills. The latter will become a side effect of "deeper" insights emerging from an I4PSM perspective on arithmetic.

4. Central to all of this is that I4PSM cannot ignore the following concepts: syntax, semantics, model, model refinement, validation, and reasoning strategy.

5. Central for I4PSM too is the idea that at its lowest level many building blocks of PSM are inconsistent to such an extent that a careful reasoning strategy based on accepting a paraconsistent foundation is needed. The chunck and permeate strategy of [13] seems to be very promising in this respect, see also [3].

6. And most importantly: reasonings strategies for PSM must deal with views (held by student and teacher alike) in specific content matters (say the question "what is a fraction") that fluctuate between different portions that are jointly inconsistent. This fluctuation is widespread and touches many aspects of PSM at the same time. Some more specific remarks must be made:

   (a) Communication between individuals about aspects of PSM presupposes temporary congruence of the fluctuation states of these individuals. It seems that human agents people are able to achieve that congruence almost instantaneously.

   (b) These fluctuations cannot simply be removed by "doing mathematics properly". The widespread hypothesis that eventually and not later than at a university level all foundational problems concerning PSM disappear is mistaken.

   (c) The fluctuations just mentioned can be compared to the fluctuating interpretations of the Necker cube. By viewing PSM (or parts of it) as a subject which is amenable for cognitive understanding in terms of, concepts with appropriate definitions and reasoning patterns, one moves from an entirely mechanical understanding (comparable with the 2D interpretation of the Necker cube) to a deeper understanding (comparable to a choice of one of the consistent 3D interpretations of the Necker cube). But just as in the case of the Necker cube: the deeper view has different mutually inconsistent forms which unavoidably appear to an individual in an alternating manner. (This observation was made in [3] in the context of fractions but it seems to apply much more widely.

7. The viewpoint that conventional two-valued (classical) predicate logic explains PSM (or reasoning within PSM) is mistaken. It comes nowhere near. Working with a three valued logic over one or more datatypes (or alternatively over a many sorted datatype) containing an error element for each sort in combination with the use of sequential logical connectives, seems to be needed at least.

1.3 Languageless PSM

A remarkable aspect of PSM is that part of it is presented as if it can be acquired without any use of explanatory language. This is comparable with walking and speaking, both of which are learned at young age without a complementary theoretical account of it being simultaneously part of the leaning outcomes and inputs.

It is a common belief that initially PSM can be taught with the use of very limited linguistic support, like a baby is taught to drink and to eat. Thus some languageless, and almost physical, operational skill (or skill set) is meant with “rekenen” (elementary arithmetic). This viewpoint has several important consequences:

1. Moving to an informatics informed setting is likely to require language support in order to explain a large variety of designs.

2. A languageless mathematics cannot afford ambiguity or choice. Because any explanation of division by zero involves non-trivial choices and design decisions, the bottom line is that conventional mathematics ceases providing language based explanations long before division by zero enters the picture. Asking about the meaning of $1/0$ is problematic because it refutes the alleged languageless character of the mathematical operation. One can’t ask about the meaning of an expresion without using words.

   In a languageless world Skinner box like training may bring a person to the point that (s)he will assume that an error has been made whenever decision by zero appears in whatsoever form. This conditioning does not require a language based understanding of what it is about, on the contrary. Neither does it require an explicit statement that dividing by zero is forbidden. All that is needed is some languageless negative feedback upon any attempt to divide by zero.

3. Summing up: the question about existence and value of $1/0$ arises by necessity once the expression $1/0$ is considered to be rational syntax. But even a weaker assertion leads to that question, namely the assertion that one should (or even that one is is permitted) to find a language based understanding of fractions.

   Thus where the wish to have a 3D understanding of the Necker cube induces a fluctuating perspective on its 3D structure, a wish to .have a language based interpretation of arithmetic introduces a paraconsistent setting which can be properly dealt with along the lines of Chunk and Permeate from [13]. Introducing a distinction between syntax and semantics requires a language based understanding and will also unavoidably lead to paraconsistency issues.

1.4 Some speculative conclusions followed by a cautious disclaimer

From the assumption that PSM is conventionally taught in a languageless context focusing on mechanical abilities rather than on reflective understanding I have drawn some rather drastic conclusions:

1. I believe that the absence of reflection in a verbal form and the absence of attempts to cast in words what is done has a negative impact on student’s understanding of PSM. This is a potentially controversial assertion which by itself requires justification of a kind that is beyond the scope of this paper.

2. In fact much PSM teaching is reflection averse. That is, it expects students not to reflect on the content at hand, as if that were a risk for their future. Many questions are supposed not to not be asked by the students and many answers given by teachers to such questions don't (or rather would not when the test was made) stand scrutiny.

3. In many cases PSM negatively contributes to the intellectual maturity of students, because it systematically frustrates any possibility of asking critical questions, and even presents itself as self-evident which is quite unconvincing.

4. The idea that conventional PSM teaching promotes critical thinking is plainly wrong. The claim that PSM must be understood, and for that reason that contributes to the development of a capability for understanding complex states of affairs is problematic.

5. By moving towards I4PSM the following aspects are introduced:

   (a) systematic embedding in in formal natural language,

   (b) tolerance for critical questions and eagerness to reflect about those,

   (c) awareness of a distinction between syntax and semantics,

   (d) awareness of the logic (reasoning strategy) that one actually needs to argue in the setting of PSM.

1.4.1 A cautious disclaimer

These drastic conclusions, however, are far from being conclusive about PSM. If I4PSM can be made operational and if it proves helpful for many students, I expect it to be advantageous in a more far reaching manner, because the worries mentioned above in 1, 2, and 3 may be remedied as a consequence.

If, however, practical teaching based on I4PSM merely provides additional conceptual complications, the instrumental value of arithmetical skills is probably still so high that it pays off for many students to go ahead as usual and not to worry about the mentioned worries (in 1, 2, and 3) in the usual manner. This balance may change in time, however, with the relevance of reflection on the way up, and of calculation on the way down.

Much work needs to be done to develop I4PSM to the level of maturity that it potentially can productively impact the practice of PSM teaching in a relevant manner. But I consider this to be a challenge and an opportunity at the same time. Working along these lines PSM teaching can perhaps be turned into science teaching, and in particular into information science teaching.

1.5 Incremental learning versus incremental consistency

The presence, if only temporarily, in a student's mind, of an inconsistent picture of an initial part of PSM may be unavoidable. I don't know to what extent fast leaning of PSM is bound to move through stages of knowledge and comprehension that are logically inconsistent, and to what extent it is possible at all to arrive at a consistent picture via an incrementally growing series of consistent approximations of that picture. It seems that this matter requires explicit attention before a commitment to a paraconsistent reasoning strategy is engaged.

2 On the identity $2 + 3 = 5$

The identity $2+3 = 5$ will not raise much attention when shown in the context of PSM. It may serve as an example in an explanation, or it may arise from an exercise that reads like "$2 + 3 = ...$". An intriguing aspect of the identity $2 + 3 = 5$ (an of all similar one's), however, is that it provides a connection with informatics in several disparate ways. Below I will survey some different perspectives on that equation, each being of relevance for I4PSM.

1. It is common that a text fragment like "$2 + 3 = ...$" is presented without any further commenting text. This absence suggests that the student considered as an agent, is supposed to infer from the context that something needs to be done (for instance signalled by the dots, and by the occurrence of the fragment in a usually vertical listing of between 2 and 10 similar fragments), in addition the student is supposed to know or to guess that an answer (the information to be provided as a textual extension of the given fragment) consists of a sequence of decimals without redundant leading zeroes, and the student is supposed to know that the answer must be such that the equality sign will be justified afterwards, that is although "$2 + 3 = ...$" may not be true, "$2 + 3 = 5$" is true. Indeed a fragment without truth value must be changed into a true fragment.

   Doing the exercise may be viewed as a mechanical task for which a verbal explanation in terms of the underlying semantics is irrelevant. This can be learned by example and that may be the primary manner for a machine learning approach to model the learning of arithmetical capabilities of a human learner. And it may be close to how humans perform that same task. In other words, just as a machine learning approach to natural language translation can be performed on the basis of statistics while almost ignoring linguistics, "natural arithmetic" may be machine learnable without any explicit modelling of cognitive processes.

   This view supports the identification of a special place or role of "natural arithmetic" next to (perhaps in advance of but not included in) mathematics, mathematics being an activity which like linguistics presupposes explicit awareness of meaning as well as reasoning supported by that awareness.

2. Suppose a student produces an answer as follows: $2+3 = 3+2$ and then is told that this answer is wrong. Can this wrongness be explained otherwise than by asserting that somehow $2 + 3 \neq 5$? Are we in a situation where

the insight that an answer is valid is incompatible with asserting that competing answers are invalid, because we lack the language to express what is wrong? Or does $3+2$ not qualify for being a potentially competing answer whereas 6 qualifies as such. And if the latter were true would we agree or disagree that $20^{53-49} = \ldots$ is correctly answered by 1.6×10^5. Is "answer" what is written on the RHS (righthand side) of an equation, or is it primarily a rather complex and context dependent category of syntax, perhaps occurring in a setting where we prefer or even insist not to speak about syntax.

These considerations reveal a link with informatics because instances of text types of major importance such as computer programs must be first and for all understood in syntactic terms. Indeed the semantic view on the meaning of programs is usually lagging behind the syntactic view that is acquired first. A semantic view either presupposed a mathematica model of program effectuation, or it involves assumptions (which will vary from case to case) about machines used for putting a program into effect.

3. In some cases $2 + 3$ is referred to as "an addition".[1] Now $2 + 3 = 5$ but 5 is not an addition. Clearly the equality of $2 + 3$ and 5 does not extend to "being an addition". Lefthand side (LHS) and righthand side (RHS) of the equation don't correspond in that respect. There are various ways to look at the matter.

   (a) One way to look at the matter is too view $- + -$ as a polymorphic operator with two types:

      i. $- +_e -$ constructs an expression from two expressions, and
      ii. $- +_n -$ constructs a number from two numbers.

   Digit sequences are polymorphic as well and can be typed as numbers and as expressions. Type inference works as follows: the preferred typing of $- + -$ is $- +_e -$ but in a context with an equality operator coercion enforces typing of an occurrence of $- + -$ as $- +_n -$.

   (b) Another way to look at the ambiguous typing of $-+-$ is to follow the line of thought of [3] where a similar issue is discussed in connection with fractions. In that work there are two algebras involved and by overlaying these an inconsistency arises. Paraconsistent reasoning in chunk and permeate style allows one to move back and forth between both interpretations of the division operator and more generally of the entire arithmetical syntax.

   These alternate views, as well as the occurrence of their alternation in the mind of a human agent, can be compared to the alternation of different 3 dimensional interpretations of the Necker cube. The 2 dimensional interpretation of the Necker cube may be compared to the mechanical understanding of arithmetic as suggested

[1] For instance in TAL [17].

above. Once this understanding is explained in terms of cognitive modelling based on some concept of number and reasoning about numbers a diffraction into different and mutually inconsistent viewpoints emerges. Paraconsistency arises and some careful reasoning method is need to avoid problematic inferences and making indefensible mistakes.

(c) Another line of thought is to view $-+-$ exclusively as a construction for expressions and to insist that the equality sign expresses equality of an appropriate abstraction say ν. Thus when reading $2+3=5$ one thinks of $\nu(2+3)=\nu(5)$, but when referring to the LHS (left hand side) of the equation $2+3=5$ one refers to $2+3$ rather than to to $\nu(2+3)$.

(d) Viewing different typing of $-+-$ as features of the formalism, one may try to analyse the difficulties in terms of so-called feature interaction (see [15]).

4. A most plausible connection with informatics is to view $2+3=5$, and in particular the way in which 5 is found from $2+3$ as an instance of of term rewriting. But when working out the details of that interpretation the story turns out to be remarkably complex.

5. Yet another perspective is the idea that there is an algorithm which can be carried out by hand and which produces 5 from inputs 2 and 3. The intriguing aspect of this explanation is that the question "what is an algorithm" has no obvious answer. In particular it is unclear to what extent "algorithm" is a mathematical notion or a notion that can be defined in the context of informatics, with recent work pointing in the negative direction. For a recent attempt to proceed along the path towards a definition of the concept of an algorithm see [8]. That paper also provides some historic information and key references about the subject.

3 Multiplication

Deviant publishing (see [16]) recently produced a preview for a new method for teaching arithmetical skills from first principles to certain fractions of the population of students. Below I will refer to that text as DPII (Deviant-Preview Instap/Instroom). From a few pages of DPII one may extract a wealth of incentives for thinking about connections with informatics. This wealth is astonishing to the extent that by itself it calls for an explanation, which I have not yet found. My remarks are not at all meant as criticism on DPII, on the contrary, my remarks rather indicate that DPII provides valuable "raw material" from which one may proceed in many directions.

Below I have modified text fragments slightly from corresponding occurrences in DPII in order not to copy text fragments form the mentioned source, but I wish to be very clear about DPII as the source of ideas.

1. "2×6 is a multiplication."

   This leads to just the same issues as mentioned above with $2 + 3$ is an addition. In the context of multiplication an approach with term rewriting and thinking in terms of algorithms is probably even more rewarding than with addition.

2. "$3 + 3 + 3 + 3 = 12$".

   This seemingly trivial assertion leads to the question what operator we are looking at. Is this a 4-place infix addition operator? And must we assume the presence of similar infix operators for all finite arities.[2]

   One might consider having infinitely many operations for repeated addition as overly complex. And one might prefer to think in terms of an associative (and in this case also commutative) operation which allows the deletion of brackets and so on. But calculating modulo associativity is a complicated story which one may not be able to communicate at the stage to teaching for which DPII is meant.

   I now believe that repeated addition gives rise to an infinite number of operation. From these only finitely many are actually used in any specific didactic method and simplifications of the picture in terms of brackets and associativity comes much later and will be based on an understanding of the combined working of a plurality of addition operations.

   A somewhat speculative option is to view $3+3+3+3$ as (an instantiation of) a 4-place partial operation $(- + - + - + - + -)$ that requires that all of its arguments are the same. This operation may be named "adding a number a number of times", and then to insist that $2 + 2$ is the result of "adding 2 once".

3. "Multiplication is adding a number a number of times."

   This kind of explanation of multiplication can be found in DPII. So consider $2 + 2$, what is this "number of times"? And is it at all possible "to add a number"? The question can be raised how to make sense of such explanations. One way is to understand the sentence as an abbreviation of a longer sentence which is valid, e.g. "Multiplication (of two numbers) is adding a (first) number a (second) number (minus one) of times (to itself)."

   Another way to look at the matter is to assume that a mechanical competence for multiplication is taught and that the competence to express in words what multiplication achieves is not part of the intended learning outcomes. Then the explanation is merely an attempt to link some words to the process of multiplication without any pretense to provide a definition from which the formal mechanics of multiplication may be derived.

[2] I recall that the arity of an operation is its number of arguments.

4. Following DPII style one may write: "you see $5 + 5 + 5$ walnuts, it total your see 15 walnuts." The use of "in total you see X Y's" seems to be restricted to the case that X is "an answer" and Y is not a number (can one see numbers?) Is there any ground for such restrictions?

4 Concurrent containers and nested containers

In DPII we find pictures and photographs of real life scenes that provide an incentive for counting and for arithmetic. In particular some scenes concern a plurality of transparant containers each containing and identical number of some kind of item. This leads to many questions. I will discuss some of these:

1. "in figure X (*a photograph*) you find two baskets containing 4 apples each; together you see 8 apples."

   This leads to the question: do we see apples or merely pictures of apples? And does it matter if we make that sort of distinction. Are we counting apples or pictures of apples. In the latter case, is counting apples a case of applying the ability to count pictures of apples (in a realistic context) to the counting of certain kinds of entities in a real context. Can it be the case, if only in principle, that this application reveals some kind of mistake in the theoretical work?

   These questions seem futile but are related to the far from trivial question what we know about a program once its correctness has been formally established? What form of validation comes after verification? Must a provably correct program still be tested?

2. "...so we have 4 + 4 = 8 apples."

   Clearly bracketing can't be as follows: "...so we have 4 + 4 = (8 apples)." That is apples don't serve as a dimension, in which case one would expect: "...so we have 4 apples + 4 apples = 8 apples." Is there a calculus of dimensions next to this particular form of language?

3. When dealing with boxes and apples in the descriptions of DPII the number of apples is made explicit while the number of boxes seems to disappear from the formalisation of scenes. What explains this asymmetry? Is there a form of abstraction going on which is left implicit, is there a notion of system behind the story which might be made more explicit. If we think in terms of meters $2m + 3m = 5m$ is a plausible way of calculating with dimensions, making m as a dimension. Is apple a dimension as well, and box. Should we write $2A + 3A = 5A$, with A a unit of apple and and $B + B = 2B$, with B representing a (unit of) box?

4. In the context of apples, however, it is plausible to think in terms of systems with $- \mid\mid -$ representing the parallel (concurrent, simultaneous) composition of two systems. Then $A \mid\mid A$ represents the concurrent presence of two apples. However unusual this notation may seem at first sight,

using $- \mid\mid -$ for the concurrent composition of systems has a long tradition in informatics (for my own experience with the use of that notation I refer to [10], and [7]).

5. Being liberal about what constitutes a system is quite common in informatics as well. Having some freedom in the design of expressions for systems is nowadays the use rather than the exception. If we denote a box with 4 apples with $B(4A)$ then we find the following system expression for two as system consisting of two of such boxes $B(4A) \mid\mid B(4A)$. If we introduce $\#_A(-)$ as a function that counts numbers of apples we find $\#_A(B(4A)) = 4$ and $\#_A(P \mid\mid Q) = \#_A(P) + \#_A(Q)$ and thus $\#_A(B(4A) \mid\mid B(4A)) = 4 + 4$. This formalisation of depicted scenes may well precede becoming aware that $4 + 4 = 8$. It also adds to a much more liberal and creative understanding of expressions and formulae, and finally: this is how a dimensional calculus works with apples, looking at the example this way removes a bias from natural sciences which suggests that the apples are merely helpful for teaching while meters would be a meaningful "dimension". That is not true, apples are constants of type system, the most important type of the engineering part of informatics, and systems are primarily composed in parallel, which is only rarely denoted with $- + -$.

6. In very few pages DPII makes use of some 10 different kinds of containers. Similar questions seem not to make use of the same syntax. Fore instance suppose that Alice has bought two packages each counting three bars of chocolate. Now the question is: how may bars of chocolate has Alice bought. Here are some answers, which one's are correct?

   (a) 6,

   (b) 6 bars,

   (c) 6 chocolate bars,

   (d) 6 bars of chocolate.

   One may dispute that this has to do with arithmetic and claim that these questions are about language only.

7. $2€ + 3€ = 5€$.

   Here € serves as a unit of a particular money, the Euro. This unit is to be seen as unit of account and not as a suggestion that one-Euro coins must be used or are meant. In this sense Euro's are a dimension like meters. We assume that the number of Euro's is written in front of the Euro sign.[3] Thinking in terms of a money of account writing $2€ + 3€ = 5€$ is plausible but writing $2€ \mid\mid 3€ = 5€$ is not.

[3]This convention has an alternative. as one might prefer writing 3€ over € 3.

The latter is plausible, however, if we want to speak of a money of exchange. We may write $€2||€2||€2 = 3€2$ where $€2$ is meant as a dimension corresponding to a physical unit of money (in this case 2 Euro coins). A general amount of cash may be written as say: $5€2||3€5||70€10$. A function v (value) turns cash amounts into quantities of account; its defining equations are: $v(k€u) = k \times u€$, and $v(X \mid\mid Y) = v(X) + v(Y)$.

Assuming a universal system of bank accounts indexed by natural numbers, then $k€\#17886$ indicates or represents k Euro on bank account 17886. If the bank account contains less than 17886 Euro, say 549 Euro, then at that very moment, $1250€\#17886$ represents 549 Euro only. A typical mode of payment is handing over an amount say $723€\#17886||5€2 ||3€5||70€10$.

5 Division

In TAL [17] we find the proposal that $1/0 = 1$. The final pages of [18] are also devoted to this suggestion for division by zero. This particular suggestion can be dealt with in a setting of algebraic specifications which was done in detail in [9]. My conclusion, however, is that working with $1/0 = 0$ (see e.g. [12]) would be an option preferable to having $1/0 = 1$. I currently think that in school division by zero ought to be dealt with along the lines of the common meadows of [11].

In [14] the suggestion is made to have a reference level on arithmetic with does without division. That is a very sure manner to keep division by zero issues on a distance at least at the lowest competence levels. This raises the question whether or not it is practical to have a reference level on initial arithmetic without division.

6 Bounded arithmetic

In DPII we find assertions of the following kind "for level *Instroom* we will cover numbers with arithmetic up to 1000 only, for level *Instap* we work with numbers up to 100 only." A part of the course works with numbers up to 10 only.

The relevance of these questions to informatics is immediate. A finite arithmetic either uses partial functions, or cyclical addition or works with explicit errors. All of this requires meticulous specification and comparison to find out what form of presentation produces the best fit with educational practice.

Technical questions are:

1. Working in level *Instap* what is 95+12?
2. Working in level *Instroom* what is 950 + 120?
3. What is the reason that answers to such questions need not be discussed within the course material?

4. How to define a reference level in a bounded arithmetic, are students supposed to be aware of the bound (and if so how to deal with overflow) or is the bound implicit as a limiting constraint on exercises only?

5. Is the existence of a sequence of increasing bounds relevant, will it lead to a systematic design of progressively more inclusive finite datatypes for bounded arithmetics?

6. The reference level F1 (from the Dutch framework of reference levels for arithmetical competence) seems not to make use of a finite bound. However it seems to be an option to view that level as based on a bounded arithmetic just as well, for instance with a bound of say 10^{15}. Will this perspective create a more coherent story?

7 Further remarks

Many more incentives for contemplating the reaction between informatics and PSM can be found in DPII. Below I will list four such theme's leaving the working out of any details to further work.

7.1 Closed world assumptions

Many questions and exercises in DPII have the following form: last week John has worked 5 hours on monday and 7 hours on Tuesday. How many hours has John worked last week? In order to arrive at a reliable answer the additional information is needed that John has not worked on any other day of last week. This is a consequence of a closed world assumption, a classical notion in artificial intelligence. It seems that in quiet some cases DPII invites an implicit closed world assumption.

7.2 Instruction sequencing

DPII contains a program which might be phased as an instruction sequence with backward jumps. It is supposed to be effectuated by hand while the student evaluates tests with the help of a pocket calculator (for determining where to jump with a goto instruction).

The primitives of the instruction sequence notation are interesting. There is a set of instructions which can be turned into a sequence. Using the notations of program algebra ([6, 1, 2]) the program has the form $H_1; ...; H_n$, where H_i has one of two possible forms: either $H_i = \pounds k_i; \#\#\pounds n_i \times m_i$ or $H_i = \pounds k_i; !@p_i$ with p_i representing a picture of a reward that the person effectuating the inseq has won by terminating at that (tagged) termination instruction. It seems plausible to tell students in detail how this kind of instruction sequencing works.

7.3 Deontic logic

One of the exercises of DPII involves deontic logic. Deontic logic is a topic from philosophical logic which has been used extensively in informatics in the context of agent theories.

We consider Peter who is supposed to plant 7 rows of trees (so he must plant 7 rows of trees), and Peter plants 20 trees in each row. How many trees must Peter plant? Probably the expected answer is 140 trees but the validity of that answer may be questioned.

We may simplify the situation by assuming that Peter must plant one row only and can plan at least one p_t^1 and at most two trees p_t^2 in that row. With $O(p)$ we express that Peter must (is obliged) to do p. So we know that $O(p_t^1 \vee p_t^2)$. It is a well-known observation in deontic logic that one may not infer $O(p_t^1) \vee O(p_t^2)$ from this assumption.

Now we assume that Peter plants one tree, that is p_t^1. The additional fact that p_t^1 does not change much. Even if we assume that $O(p_t^1) \to p_t^1$ and $O(p_t^2) \to p_t^2$ these assumptions suffice to demonstrate that $\neg O(p_t^2)$ but the same assumptions don't not suffice to infer $O(p_t^1)$.

7.4 Pie charts

A well-known representation is to have pictures of circles that have been cut in pieces of equal size using straight lines from the center to the perimeter. Then a number of these parts, usually chosen adjacent is coloured grey and students are asked to choose a fraction from a listing of fractions which is said to belong to the picture. Typically if the pie has been cut in two pieces and one piece is grey the corresponding fraction is supposed to be $\frac{1}{2}$. Now there is no such thing as a fraction belonging to a pie chart of this kind. In any case there is no obvious correspondence. Another way of looking at the matter is as follows. First we introduce a collection of expressions which can model pie char pictures of the mentioned form. This collection is one of may options for instance with pies that have been decomposed in 5 pieces the expressions all have the following form:

$$\frac{b_1 + b_2 + b_3 + b_4 + b_5}{1 + 1 + 1 + 1 + 1}$$

Here the following conventions are in order:

1. $b_i \in \{0, 1\}$ for $i \in \{1, ..., 5\}$,

2. we use the language of fractions and distinguish a numerator and a denominator,

3. the number of summands in the denominator represents the number of parts of the pie,

4. this number always exceeds 0,

5. a 1 in the numerator represents a grey part,

6. the summands in the numerator enumerate the parts clockwise starting from the part that contains the top element from the circle and extends a non-zero amount in the clockwise direction,

7. the formalisation of a circle can represent non adjacent grey parts just as well,

8. there is a 1-1 correspondence between pie charts and the expressions under the following assumptions: (i) at least one cut from the center runs upwards in the vertical direction (provided the pie has been split in more than one part), (ii) the radius of the circle is known, and (iii) the colors are known.

Now first of all one exercises with formalisation of pies (finding an expression E_p given a pie p), and with the implementation (drawing) of a pie given an expression E of the above kind for it. Only once this has been mastered one proceeds with a semantic step: E_p is understood as a fraction and fractions are identified if they represent the same rational number. Now the following questions and observations must and can be studied:

1. what have p and q in common if $E_p = E_q$?

2. a partition refinement splits every part once more in the same number of parts, while inheriting the colors. If q is a partition refinement of q then $E_p = E_q$. Is the converse true?

3. Now suppose that $E_p = E_q$ and that all grey parts of p are adjacent and such that all grey parts of q are adjacent, then there must be r_p and r_q where (i) p is a refinement of r_p, (ii) q is a refinement of r_q such that r_p and r_q have the same number of parts, and (iii) r_p results from r_q via a rotation of the pie.

The merit of this view on pies and charts is that the modelling phase, which is the essential phase from the perspective of informatics, is not taken for granted. In particular the quality of the model can be studied in detail.

8 A disclaimer and a claim

I do not claim that working out the details of the issues raised in the preceding sections will lead to a change or even to an improvement of the mentioned course material. However, it is necessary to contemplate these questions in order to find out precisely what it is that one may be teaching in PSM under the heading of arithmetic (in Dutch *rekenen*[4]).

[4]In The Netherlands it is nowadays common not to include *rekenen* in *wiskunde* (mathematics) but to speak of *rekenen-wiskunde* if some mix of *rekenen* and *wiskunde* is meant.

8.1 A claim

What I do claim instead is that working further on the basis of the observations made above will allow to introduce aspects from informatics to PSM. To be more precise the following aspects from informatics may be linked with the mentioned issues respectively.

1. Distinction between syntax and semantics, typing regimes with coercion, design of signatures for abstract datatypes, term rewriting and expression evaluation, rewriting with surjective pairing and unpairing, paraconsistent reasoning and default logic.

2. Parsing, overloading, rewriting modulo associativity and commutativity, infinite signatures, two-level grammars.

3. Formal verification versus requirements validation; reasoning about systems on the bias of knowledge about programs and machines.

4. Formal architectures, architecture extraction, abstraction operations, module algebra, abstract datatypes.

5. Design of finite datatypes, use of error values, logics with error values, three valued logic, term rewriting, priority rewriting, datatypes with partial functions.

In addition I claim that these considerations may contribute to the insights that classical formalisation of elementary arithmetic in first order two valued and many-sorted predicate logic may provide.

8.2 A conclusion

Further elaboration of the details of issues that were touched only superficially in this paper seems to be a precondition for subsequent work towards the development of novel course material and towards a realisation or implementation of I4PSM including the definition of so-called reference levels, and the development of appropriate tests.

References

[1] J.A. Bergstra. Putting instruction sequences into effect. `arXiv:1110.1866 [cs.PL]`, (2011).

[2] J.A. Bergstra. A terminology for instruction sequencing. `viXra:1502.0228`, (2015).

[3] J.A. Bergstra and I. Bethke. Note on paraconsistency and the logic of fractions. `arXiv:1410.8692v2[cs.LO]`, (2015).

[4] J.A. Bergstra, I. Bethke and A. Ponse. Rekenen-Informatica. *UvA ReportTCS14-12*,`https://ivi.fnwi.uva.nl/tcs/pub/tcsreports/TCS1412.pdf`, (2014).

[5] J.A. Bergstra, G.P.A.J. Delen, and S.F.M. van Vlijmen. Stratified outsourcing theory. `http://arxiv.org/abs/1109.6536`,(2010).

[6] J.A. Bergstra and M.E. Loots. Program algebra for sequential code. *Journal of Logic and Algebraic Programming*, 51 (2): 125-156, (2002).

[7] J.A. Bergstra and C.A. Middelburg. Thread algebra for strategic interleaving. *Formal Aspects of Computing*, 19 (4) pp. 445–474, (2007).

[8] J.A. Bergstra and C.A. Middelburg. On algorithmic equivalence of instruction sequences for computing bit string functions. `arXiv:1402.4950[cs.LO]`, (2014).

[9] J.A. Bergstra and C.A. Middelburg. Division by zero in non-involutive meadows. *Journal of Applied Logic*, 13(1): 1–12 (2015).

[10] J.A. Bergstra and A. Ponse. Register machine based processes. *J. ACM*, 48 (6): 1207–1241, (2001).

[11] J.A. Bergstra and A. Ponse. Division by zero in common meadows. `arXiv:1406.6878 [math.RA]`, (2014).

[12] J.A. Bergstra and J.V. Tucker. The rational numbers as an abstract data type. *Journal of the ACM*, 54 (2), Article 7 (2007).

[13] B. Brown and G. Priest. Chunk and Permeate, a paraconsistent inference strategy. Part I: the infinitesimal calculus. *Journal of Philosophical Logic*, 33:379–388 (2004).

[14] J. Jeene. Towards Testing at Intermediate Levels and the Definition of Intermediate Competences in the Learning Path for Arithmetic Between the Basis Level and the Reference Level 2F. `http://vixra.org/abs/1503.0036`, (2015).

[15] K. Kimbler and L.G. Bouma (Eds.) Feature interactions in telecommunications and software systems V. IOS Press, (1998).

[16] Irene Lugten, Sari Wolters, Sarah Brusell, Manon Keuenhof, Kim Klappe, Maartje van Middelaar, Martine Knijnenberg, Rob Lagendijk, Marloes Kramer, Jelte Folkertsma, Cyriel Kluiters, Jasper van Abswoude, and Rieke Wynia. Proefkatern. Startrekenen Vooraf, Op weg naar 1F/Startrekenen Instap, Rekenen tot 100. *Uitgeverij Deviant, Amersfoort*, (2015).

[17] J. Nelissen and A. Treffers. Onderwijskader. *In:van den Heuvel-Panhuizen,M.,Buijs, K., Treffers, A. (eds.) Kinderen Leren Rekenen (in Dutch)*, pp. 147–162. Noordhoff Uitgevers, Groningen (2000).

[18] A. Treffers. Weg van het cijferen, rekenmethodes vanaf 1800 tot heden. *ISBN 978909028793, Uitgeverij Reni Casoli*, (2015).

A Properties of this particular paper

The first Appendix contains information which is specific for this paper, the subsequent Appendices provide the necessary explanation.

A.1 Licencing

A.2 Minstroom Research Nopreprint Series Number

This is #7 from the Minstroom Research Nopreprint Series (in brief Minstroom Research NPP#7). The other papers in the series are listed below. In these texts Minstroom Research has been abbreviated to MRbv. I have changed the abbreviation to make it independent of the legal form and to avoid the introduction of an acronym MRbv that is in use for several other meanings already.

1. Minstroom Research NPP#1: "Decision taking avoiding agency", `http://vixra.org/abs/1501.0088` (2015); this paper is not explicitly labeled as a nopreprint but it sufficiently meets the criteria as listed below, though it lacks a defensive novelty analysis which admittedly is a deficiency,

2. Minstroom Research NPP#2: "A nopreprint on algebraic algorithmics: paraconsistency as a afterthought", `http://vixra.org/abs/1501.0203` (2015),

3. Minstroom Research NPP#3: "A nopreprint on the pragmatic logic of fractions", `http://vixra.org/abs/1501.0231` (2015),

4. Minstroom Research NPP#4: "Personal multithreading, account snippet proposals and missing account indications", `http://vixra.org/abs/1502.0204` (2015).

5. Minstroom Research NPP#5: "Terminology for instruction sequencing", `http://vixra.org/abs/1502.0204` (2015).

6. Minstroom Research NPP#6: "A SWOT analysis for Instruction Sequence Theory", `http://vixra.org/abs/1502.0231` (2015).

A.2.1 NPP Subseries on I4PSM

"Informatics for primary school mathematics (I4PSM)" is coined as the name for a theme within Minstroom Research. For that theme a subseries of the NPP series is maintained. The paper is the first entry in that subseries, which is reflected in its extended code: Minstroom Research NPP#7 I4PSM#1.

A.2.2 Rationale for I4PSM as a theme within Minstroom Research

I4PSM as a theme requires a rationale on which a rationale for having a subseries of the NPP series devoted to that theme can be based. This rationale is specified in a number of items below.[5]

1. I4PSM is a new area which I expect to provide business opportunities in the near future. I hope that such opportunities may be exploited from the platform of Minstroom Research.

2. I consider it a personal ambition to explore I4PST. I intend to use Minstroom Research as a vehicle for the expression of that ambition.

3. In spite of the small scale, in terms of sourcing the situation is somewhat involved:[6] as a person interested in I4PST I make use of a service provided by Minstroom Research to host a project on a specific topic. Instantiating that service to I4PST leads to Minstroom Research maintaining a project on I4PST. This perspective splits my role as a director of Minstroom Research, now a purely managerial role seeing to it that certain services are provided, from my role as a researcher. In the latter capacity there is a variety of possible interactions between Minstroom Research and myself:

    (a) *technically not the current situation, but undeniably a practical way to insulate Minstroom Research from the negative impact of a prob-*

[5] I must apologise for the remarkable complexity of these considerations. But all of it simply arises when applying stratified sourcing theory (e.g. see [5]) to this particular case. In fact the setting up of Minstroom Research may be considered a cases study on sourcing theory as it has been put forward in [5] and papers cited there.

If further themes are developed within Minstroom Research it is plausible that the rationale will be modularised in a generic part about the purpose, mechanism, and scope of themes within Minstroom Research and a specific part containing aspects specific for a certain theme. At some stage this need for further modularisation will induce refactoring of the structure of the appendices.

One may ask why such considerations merit open publication. To that question my response is that given the fact that Minstroom Research creates a setting where the meaning of an assertion like $2 + 3 = 5$ is not taken for granted, the structure theory of Minstroom Research must not be taken for granted either. The difficulties of developing that structure theory must be faced with an open mind and without fear of being considered pedantic or ignorant of existing best practices. I have chosen to be as explicit as possible about the learning curve that goes with the setting up of Minstroom Research, irrespective of the significant risk that Minstroom Research may fail to deliver its hoped for functionality.

[6] I now consider the design of Minstroom Research, including its positioning in the context of other entities and organisations, to constitute a fruitful case study for the application of the sourcing theory that I have been developing in cooperation with Guus Delen and Bas van Vlijmen in the context of software product management (see [5]).

lematic business case concerning I4PSM: me (as a person outside Minstroom Research) intending to realise an ambition, and for that reason paying Minstroom Research for its services in as far as these are helpful to this objective,

I may in addition require Minstroom Research to provide decision taking as a service in order to simplify my task for achieving the mentioned ambition (such decision are then taken by myself in the capacity of a Minstroom Research project manager, which certainly introduces a different perspective)

(b) *the current situation (sourcement with shared sourcing):* an informal deal has been established for an interaction with closed pockets between me (as a person outside Minstroom Research) having an interest in advancing an objective and Minstroom Research (represented by me as its director) which has an interest in maintaining a project portfolio of projects related to its mission statement and with some further development perspective (assuming that in my capacity as being responsible for Minstroom Research I have agreed that I4PSM meets these requirements).

My interest in PSM (in my role as a person outside Minstroom Research) requires further explanation and to some extent justification:

i. I am a volunteer in an organization called PRAGO situated in Utrecht, The Netherlands. Website: `www.prago.nl`; PRAGO stands for *"Praktijkgericht onderwijs"* which may be translated as "practice oriented education"; PRAGO is active in 7 locations in and around Utrecht, with students ranging from 20 to 60 years of age, often without any formal former school qualification. My own role in PRAGO is serving as the chairperson of the Board; in that capacity I also take a special interest in supporting the development within PRAGO of teaching methods and materials for arithmetic.

ii. The objective of PRAGO is to teach individuals, who for some reason failed to acquire these skills in school, elements of Dutch language as well as elements of arithmetic, including initial informatics skills. I take an interest in specifying as clearly as possible what it is that PRAGO hopes to convey about school mathematics. I work under the assumption that by incorporating informatics aspects to the "classical" PSM content the story will become more meaningful and simpler to communicate.

iii. In my role an academic researcher in informatics I have contributed to work (e.g. [3, 11]) for which the best options for valorisation may lie in the area of PSM rather than in computing, while at the same time the academic setting seems less amenable to the realisation of such particular valorisation options. Bridging the seemingly unbridgeable gap between this line of academic research and the context of PRAGO is a remarkable challenge which I have taken on board.

(c) *not the current situation, unlikely to apply in the near future:* me being paid by Minstroom Research for having provided a useful idea with market potential which Minstroom Research is exploiting as a business opportunity.

(d) *the preferred situation, not yet achieved, requiring an insourcing transition from the current sourcement as specified in 3b above:* me, as a Minstroom Research employee being paid in part (by Minstroom Research) for doing work for the realisation of objectives that (accidentally) I happened to have formulated myself (as an external person, and to which I agreed to use Minstroom Research as a platform in my role as being responsible for Minstroom Research).

I notice that the micro-institutional scale introduces conceptual problems with overlapping roles for the same person which will be absent in a larger scale operation.

4. Pursuing that objective many tasks may be involved that neither constitute research nor the valorisation of research. A time horizon of 10 years starting with the publication of this document is in order. Both these features point in the direction of the use of a micoinstitution (a notion used in Minstroom Research NPP#6).

5. Constructing a body of knowledge on how informatics may impact the lowest level of mathematical teaching leads to a volume of elementary observations that deviates from the expected outcome of academic research, by being less systematic and more explorative; failures must be reported and kept for future references just as well as positive outcomes.

6. Much work on I4PST precedes the formulation of teaching methods for elementary mathematics that is informed by informatics.

7. Developing informatics based content matter that can be used in the initial case of primary school requires desk research rather than fundamental research, that work can be done from Minstroom Research.

A.2.3 Subseries rationale

A subseries needs a definite rationale allowing readers to understand why papers in the subsection belong together, and why the work is done at all. A subseries rationale is based on a rationale for a theme to which the subseries is devoted. The rationale is be expressed at a higher level of abstraction than that of a particular instance of the subseries. The reasons listed below are rather generic for that reason.

1. The subseries contains work only for which the relating with existing or proposed teaching material is evident.

2. Transferability (proven or merely conceivable) for an educational setting is not a prerequisite for a paper to qualify for the subseries.

3. The subseries mechanism makes it simpler for a reader to find out which of the Minstroom Research NPPs relate to primary school mathematics.

4. Open development (as guaranteed by the publication of intermediate results and considerations in the form of an NPP) of the material reflects the position that in relation to this particular theme (I4PSM) the business aspects of Minstroom Research are merely a means to an and (promoting I4PSM) rather than an end in itself.[7]

A.3 Minstroom Research Document Class

This paper has document class B in the Minstroom Research Document classification scheme. This scheme is detailed in Appendix C. This classification refers to the body of the paper with the exclusion of the Appendices.

A.3.1 Justification of this particular Minstroom Research document classification

In this particular case the classification in class B has the following motivation:

1. The nopreprint status is intentional, submission to a (selectively) peer reviewed publication outlet is not intended. (This indicates Minstroom Research as an appropriate affiliation bringing with the need for classification in A B, C, or D). Forthcoming greement of any peer review system with the design decisions in the paper is not sought.

2. Subsequent academic research on the basis of the content of this work is not foreseen by the author. Subsequent non-academic research, however, is expected and intended. Working within Minstroom Research towards material that can be used in practice is also intended.

A.4 Defensive novelty analysis

A nopreprint ought to be equipped with a so-called defensive novelty analysis. An explanation of this this notion as well as an explanation of why it is needed in the case of a nopreprint is given in Appendix B below). For this paper I put forward the following arguments:

- The work only formulates directions for further work.

[7] That same consideration need not necessarily apply to other themes where Minstroom Research is active or will be active. Obviously in a theme where the business objectives of Minstroom Research outweigh the thematic development objectives per se, the development of cumulative knowledge cannot simply be archived in an open NPP subseries. In such cases a combination of secrecy, that is documents which are not made public, or paid IP protection by means of patents, or an effective use of copyrights must be applied. At this stage the only theme "personal multi-threading" is of this second kind within Minstroom Research.

- The claim that particular aspects from primary school mathematics can indeed be illustrated or redeveloped meaningfully from an informatics perspective is not made.
- There is no technical content that might be wrong.
- Achieving completeness in the listing of topics from primary school mathematics amenable for treatment from a perspective of informatics is not an objective (and for that reason need not be evaluated by an external reviewer). for the issues that are raised in the paper.

B Formalities and policy statements I: about nopreprints

This Appendix begins with brief historical remarks concerning the possibly novel ideas that are put forward in this Appendix as well as and in the following Appendix. The remaining part of this Appendix spells out the details an rational of nopreprints as a novel class of papers and publications.

B.1 Remarks on micro-history

The development of the concept of a nopreprint document category as well as of of the Minstroom Research document classification scheme including the form of presentation of such matters in appendices of MRbv nopreprints steadily evolves.

This Appendix and the following Appendix constitute (after importing the reference texts) a minor adaptation of essentially the same content that was included (assuming that MRbv in renamed into Minstroom Research) in the Appendices B and C of Minstroom Research NPP#4 (http://vixra.org/abs/1502.0204), which in turn derives from the Appendices of two earlier nopreprints (Minstroom Research NPP#2 and Minstroom Research NPP#3) that were posted as `http://vixra.org/abs/1501.0231` and `http://vixra.org/abs/1501.0203` respectively, which in turn have been derived from the final Section of Minstroom Research NPP#1 (which is `http://vixra.org/abs/1501.0088`).

I apologise for the length of these considerations. I will include similar texts in further documents (either having nopreprint status or written from my MRbv affiliation) expecting that some gradual evolution to a mature, stable and compact form wil result in due time.

B.2 Nopreprints and micro-institutions

This Paragraph and subsequent Paragraphs with are identical (modulo the renaming of MRbv into Minstroom Research) to the corresponding Paragraphs of [2], and are not repeated here for that reason.

C Formalities and policy statements 2: using a private micro-institution as an affiliation

This Appendix is identical to Appendix C (modulo the renaming of MRbv into Minstroom Research) of [2], it will not be repeated here for that reason.