VARIATIONAL THEORY: VARIABLE-INDEPENDENCE AND CONSISTENCY

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Abstract

Variational theory of elasticity is surveyed in the context of mathematical logic in the present paper. The problem of variable-independence of variational principles raised by Chien is discussed. We find that Chien's "High-order Lagrange Multiplier Theory", which deals with the problem of variable-independence and constraint of variational principles, is inconsistent; Luo's system, which is involved in the problem of variable-independence, is involved in contradictions; the conventional understanding of independence of variables of variational principles connotes contradiction. In the context of mathematical logic, variational theory must be established as a mathematical system of logic, excluding vagueness and misunderstanding. By consideration of logic, variable-independence is understood as identity of variables and then formalization of variational theory is a solution to the problem of variable-independence. Two consistent systems for elasticity, the Axiomatic System of Variation and the Formal System of Variation, are suggested in this paper.

Keywords variational theory, variable-independence, consistency , formalization, Axiomatic System of Variation, Formal System of Variation

1. Introduction

Variational calculus, used widely in various areas of mathematics, physics and engineering, is one of the fundamental and important methods of mathematical physics. The Minimum Potential Energy Principle (MPEP hereinafter) is a typical variational principle in elasticity[1,2]. Hu and Washizu suggested individually Hu-Washizu Principle (H-W Principle hereinafter) of three kinds of variables (argument functions)[3-6]. In 1964, Chien derived H-W Principle by using Lagrange Multiplier Method. According to He , "from then on generalized variational principles can be arrived at from a scientific way, not a blind way." [7,8] During the period of 1983-1985, Chien argued that H-W Principle was subject to one kind of constraint conditions, and so one kind of variables in the principle was not independent and the principle was equivalent to Hellinger-Reissner Principle (H-R Principle hereinafter) of two kinds of variables [9-13]. In order to eliminate the "constraint" of H-W Principle, Chien suggested the High-order Lagrange Multiplier Method and then established $G\lambda$ Principle [9-11]. He made the comment, in the Chinese version of his paper, on Chien's work as: the finding of dependence of variables in H-W Principle by Chien led to the birth of High-order Lagrange Multiplier Method, which is the important landmark in the history of development of variational theory [7].However, as pointed in this paper, there exist some contradictions in Chien's theory.

Luo's work [14] is also involved in the problem of variable-independence of variational principles, taking independence of variables in the variational principles for granted. But the problem is not solved because of contradictions involved in the system of the work.

Furthermore, we find that the conventional understanding of independence of variables of variational principles, stated or implied in the works of Washizu, Chien and Luo connotes contradiction [6, 9-11, 14].

In the present paper, we raise and discuss the problem of consistency of variational theory, arguing that consistency is a fundamental requirement of any mathematical theory and variational theory should not be an exception. We assert that variational theory must be established as a mathematical system of logic, excluding vagueness and misunderstanding, if it is a rigorous mathematical theory. We realize that independence of variables should be understood logically as identity of variables, if variational theory is required to be consistent, and formalization of variational theory is a solution to the problem of variable-independence. Then we suggest the Axiomatic System of Variation and the Formal System of Variation for the variational theory of elasticity, which are proved consistent.

2. Fundamental Equations in Elasticity and the Principle of Consistency

2.1. Governing Equations in Elasticity

(a)Equilibrium equations:

$$\sigma_{ij,j} + \bar{F}_i = 0 \qquad (i = 1,2,3) \qquad (in \quad \tau) \quad ; \qquad (2.1)$$

(b)Strain-displacement relations:

$$e_{ij} - (1/2)u_{i,j} - (1/2)u_{j,i} = 0 \qquad (i, j = 1,2,3) \qquad (in \quad \tau) \quad ; \qquad (2.2)$$

(c)Stress-strain relations:

$$\frac{\partial A(e)}{\partial e_{ij}} - \sigma_{ij} = 0 \qquad (i, j = 1,2,3) \qquad (in \quad \tau) \qquad (2.3a)$$

or $$\frac{\partial B(\sigma)}{\partial \sigma_{ij}} - e_{ij} = 0 \qquad (i, j = 1,2,3) \qquad (in \quad \tau) \quad ; \qquad (2.3b)$$

(d)Traction boundary conditions:

$$\sigma_{ij} n_j - \bar{P}_i = 0 \qquad (i = 1,2,3) \qquad (on \quad s_p) \quad ; \qquad (2.4)$$

(e)Displacement boundary conditions:

$$u_i - \bar{u}_i = 0 \qquad (i = 1,2,3) \qquad (on \quad s_u) \quad . \qquad (2.5)$$

In (2.1)-(2.5) Einstein's notations, $\sigma_{ij,j} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}$ for example, are used; τ is the volume domain of the body; s_p, a part of the piecewise smooth boundary of the body, is loaded with components of surface force per unit area $\bar{P}_i$; s_u, the other part of the boundary, is given displacements $\bar{u}_i$; $\bar{F}_i$ are components of body force per unit volume; σ_{ij}, e_{ij} and u_i are stress, strain and displacement components respectively ; $A(e)$ and $B(\sigma)$ are potential and complementary energy density of the body respectively; n_j are direction cosines of the outward normal to the boundary.

The boundary of the elastic body, s, is divided as

$$s = s_p + s_u \qquad (2.6)$$

2.2. Main Variational Principles in Elasticity Discussed in This Paper (functionals are given)

$$\Pi_p = \iiint_\tau \{A(e) - \overline{F}_i u_i\} d\tau - \iiint_{S_p} \overline{P}_i u_i ds \quad \text{(for MPEP)} \quad , \qquad (2.7)$$

$$\Pi_{HW} = \iiint_\tau \{A(e) - \sigma_{ij}[e_{ij} - (1/2)u_{i,j} - (1/2)u_{j,i}] - \overline{F}_i u_i\} d\tau$$
$$- \iint_{S_p} \overline{P}_i u_i ds - \iint_{S_u} \sigma_{ij} n_j (u_i - \overline{u}_i) ds \qquad (2.8)$$

$$\Pi_{HR} = \iiint_\tau \{ B(\sigma) + (\sigma_{ij,j} + \overline{F}_i) u_i \} d\tau$$
$$- \iint_{S_u} \sigma_{ij} n_j \overline{u}_i ds - \iint_{S_p} (\sigma_{ij} n_j - \overline{P}_i) u_i ds \quad , \qquad (2.9)$$

$$\Pi_{G\lambda'} = \iiint_\tau \{A(e) - \sigma_{ij}[e_{ij} - (1/2)u_{i,j} - (1/2)u_{j,i}] - \overline{F}_i u_i + \lambda'[A(e) + B(\sigma) - \sigma_{ij} e_{ij}]\} d\tau$$
$$- \iint_{S_p} \overline{P}_i u_i ds - \iint_{S_u} \sigma_{ij} n_j (u_i - \overline{u}_i) ds \quad , \qquad (2.10)$$

$$\Pi_{G\lambda} = \iiint_\tau \{ B(\sigma) + (\sigma_{ij,j} + \overline{F}_i) u_i + \lambda[A(e) + B(\sigma) - \sigma_{ij} e_{ij}] \} d\tau$$
$$- \iint_{S_u} \sigma_{ij} n_j \overline{u}_i ds - \iint_{S_p} (\sigma_{ij} n_j - \overline{P}_i) u_i ds \quad . \qquad (2.11)$$

2.3. Principle of Consistency Suggested in This Paper

A variational theory is consistent iff (if and only if) there exists no logical contradiction in its system of logic.

3. Chien's Theory and Its Inconsistency

3.1. Chien's Theory of Variational Principles

Chien uses "variable", which is σ_{ij}, e_{ij} or u_i in elasticity, for the term "argument function" [1] or "quantity" [6] of variational principles. It is not difficult to know, by logical consideration, that Chien's theory is actually based on the logical system below, though it is not expounded in his papers[9-11].

3.1.1. Postulates Stated or Implied in Chien's " High-order Lagrange Multiplier Theory"

P3.1. The variational principles of elasticity, whose functionals are formulated by (2.7)-(2.11).

P3.2. Equations (2.1), (2.2), (2.3a) or (2.3b), (2.4) and (2.5) ((2.1-2.5) in Sec.3).

P3.3. Uniqueness Theorem [10]:

For a problem of physics formulated in terms of a complete set of variables, under given constraint conditions the formulation of the functional for the constraint variational principle of the problem is unique. If there exists no constraint condition, the formulation of the functional for the generalized variational principle of the problem is unique.

P3.4. A variable, σ_{ij}, e_{ij} or u_i, can not be both an independent and non-independent variable of a variational principle under discussion. (the law of contradiction of variable-independence)

P3.5. Each of σ_{ij}, e_{ij} and u_i must be either an independent or a non-independent variable of the variational principle under discussion. (the law of excluded middle of variable-independence)

P3.6. An equation from (2.1-2.5) can not be both a constraint and a natural condition of a variational principle under discussion. (the law of contradiction of constraint)

P3.7. Each of (2.1-2.5) must be either a constraint condition of or a natural condition of the variational principle under discussion.(the law of excluded middle of constraint)

P3.8. Any of the variational principle specified by P3.1 can not be both a constraint variational principle and a complete generalized variational principle for the problem of elasticity. (the law of contradiction of identification of the variational principles)

P3.9. Each of the variational principle specified by P3.1 must be either a constraint variational principle or a complete generalized variational principle for the problem of elasticity. (the law of excluded middle of identification of the variational principles)

3.1.2. Definitions Stated or Implied in Chien's " High-order Lagrange Multiplier Theory"

D3.1. A variable is independent for a variational principle iff it is not subject to any constraint condition, which is defined by D3.2 or D3.3, of the variational principle under discussion .

D3.2. A constraint condition for forward inference, of a variational principle, is an algebraic or differential equation which must be substituted into the variational principle or its Euler equations if forward inference process (see D3.5) is exercised.

D3.3. Any two variational principles among those whose functionals are formulated by (2.7)-(2.11) are equivalent to each other iff the sum or the difference of their functionals is equal to zero. The constraint condition for equivalence is the algebraic equation which must be satisfied to make the equivalence true.

D3.4. Natural conditions of a variational principle are algebraic and differential equations which can be deduced from the variational principle if forward inference (see D3.5) is exercised.

D3.5. A forward inference process is a process of deducing natural conditions from a variational principle together with its constraint conditions, if any, following the rules of mathematical inference in Sec.3.1.3.

D3.6. An inverse inference process is a process of deriving a variational principle from (2.1-2.5), following the rules of mathematical inference in Sec.3.1.3. A semi-inverse inference process is a process of transforming a variational equation into a transformed variational equation, following the rules of mathematical inference in Sec.3.1.3.

D3.7. A constraint variational principle is a variational principle with at least one constraint condition.

D3.8. Generalized variational principles are variational principles established by Lagrange Multiplier Method. Complete generalized variational principles are non-constraint variational principles established by Lagrange Multiplier Method and High-order Lagrange Multiplier Method.

D3.9. A proof is a logical argument process under P3.1-P3.9, according to D3.1-D3.9 and following R3.1-R3.9.

3.1.3. Rules of mathematical inference Stated or Implied in Chien's " High-order Lagrange Multiplier Theory"

R3.1. The Fundamental Lemma of the calculus of variations [1] (generalized).

R3.2. The Gauss Theorem in the differential calculus.

R3.3. The shearing stress symmetry ($\sigma_{ij} = \sigma_{ji}$) and the work-and-energy principle in elasticity [2].

R3.4. The rules of operations in the algebraic, differential, integral and variational calculus.

R3.5. Substitution method for forward inference processes (see D3.5): a method to eliminate constraint conditions.

R3.6. Lagrange multiplier method (of the first order or the higher order): a method to eliminate constraint conditions in semi-inverse inference processes (see D3.6).

R3.7. Substitution method for semi- inverse inference processes (see D3.6): a method to introduce constraint conditions into transformed variational equations.

R3.8. Weighted-residual method for inverse inference processes (see D3.6): a method to introduce natural conditions into variational principles.

R3.9. In an equation of constraint condition there exists at least one variable which is subject to the constraint and not independent.

In the following Sec. 3.2-3.5 we will give some proofs within Chien's Theory and show that there exist contradictions in Chien's Theory.

3.2. Contradictions to P3.3 and P3.8: Proofs according to D3.3 and following R3.6, Theorems and Remark

From
$$\Pi_{HR} + \Pi_{HW} = \iiint_{\tau} [A(e) + B(\sigma) - \sigma_{ij} e_{ij}] d\tau = 0 \qquad (3.1)$$

Chien argues [10], according to D3.3, that H-R and H-W Principle are equivalent to each other and each of them is subject to the constraint condition

$$A(e) + B(\sigma) - \sigma_{ij} e_{ij} = 0 \qquad (3.2)$$

That means that two functionals, not unique one, exist for variational principle under a unique constraint.

Now following R3.6, we eliminate the constraint condition of H-R Principle, (3.2), by High-order Lagrange multiplier method and formulate H-W Principle:

$$\Pi_{HW} = -\Pi_{HR} + \lambda_H \iiint_{\tau} [A(e) + B(\sigma) - \sigma_{ij} e_{ij}] d\tau \qquad (3.3)$$

where $\lambda_H = 1$.

And so, Π_{HW} is proved here to be a complete generalized variational principle. We know that Chien established $\Pi_{G\lambda'}$, a complete generalized variational principle, by High-order

Lagrange Multiplier Method[9-11]. So, two functionals, Π_{HW} and $\Pi_{G\lambda'}$, not unique one, exist for functionals of the complete generalized variational principle.

From the proofs above, we have the theorems:

Theorem3.2.1

There exist two functionals, Π_{HR} and Π_{HW} , subject to one constraint (3.2); and, there exist two functionals, Π_{HW} and $\Pi_{G\lambda'}$, of complete generalized variational principles, for elasticity.

Theorem3.2.2

H-W Principle, whose functional is Π_{HW} , is a constraint and also a complete generalized variational principle, of elasticity.

And we have the remark below:

Remark 3.1.

Theorem 3.2.1 is in contradiction to P3.3; Theorem 3.2.2 is in contradiction to P3.8.

3.3. Contradiction of Constraint: Proofs according to D3.6, Theorem and Remark

3.3.1. Proof 3.3.1

The proof is given by Chien [11]. In his proof (2.1) and (2.4) are introduced to formulate the variational equation by weighted-residual method while (2.2), (2.3a) and (2.5) are introduced into the variational equation by substitution method. Therefore, (2.1) and (2.4) become natural conditions of MPEP (see R3.8) while (2.2), (2.3a) and (2.5) become constraint conditions of MPEP (see R3.7).

3.3.2. Proof 3.3.2 Given by us in This Paper

Now (2.3a) is introduced to formulate the variational principle by weighted-residual method while (2.1), (2.2), (2.4) and (2.5) are introduced into the principle by substitution method:

$$\iiint_\tau \left[\frac{\partial A(e)}{\partial e_{ij}} - \sigma_{ij}\right]\delta e_{ij}\, d\tau = 0 \quad . \tag{3.4}$$

Substituting (2.2) into (3.4), then

$$\iiint_\tau \left[\frac{\partial A(e)}{\partial e_{ij}}\delta e_{ij} - \sigma_{ij}\delta u_{i,j}\right] d\tau = 0 \quad , \tag{3.5}$$

then

$$\iiint_\tau [\delta A(e) - (\sigma_{ij}\delta u_i)_{,j} + \sigma_{ij,j}\,\delta u_i]\, d\tau = 0 \quad . \tag{3.6}$$

Substituting (2.1) into (3.6), then

$$\iiint_\tau [\delta A(e) - \overline{F}_i \delta u_i]\, d\tau - \iint_{S_p} \sigma_{ij} n_j \delta u_i\, ds - \iint_{S_u} \sigma_{ij} n_j \delta u_i\, ds = 0 \quad , \tag{3.7}$$

where Gauss Theorem is used.

Substituting (2.4) and (2.5) into (3.7),

$$\iiint_\tau [\delta A(e) - \overline{F}_i \delta u_i]\, d\tau - \iint_{S_p} \overline{P}_i \delta u_i\, ds = 0 \quad , \tag{3.8}$$

then $$\delta\left\{ \iiint_\tau [A(e) - \overline{F}_i u_i]\, d\tau - \iint_{S_p} \overline{P}_i u_i\, ds \right\} = 0 \quad . \tag{3.9}$$

And so, following R3.8, (2.3a) becomes a natural condition of MPEP while, following R3.7, (2.1), (2.2), (2.4) and (2.5) become constraint conditions of MPEP.

3.3.3. Theorem and Remark

From Sec.3.3.1-3.3.2, we have the theorem:

Theorem3.3

Each of (2.1), (2.3a) and (2.4) is a constraint condition and also a natural condition of MPEP.

Then we have the following remark:

Remark 3.2.

Theorem 3.3 is in contradiction to P3.6.

3.4. Contradiction of Constraint Again: Proofs according to D3.5, Theorem and Remark

3.4.1. Proof 3.4.1

The proof is given by Chien [11]:

From (2.7) and following R3.4,

$$\delta\Pi_p = \iiint_\tau \left\{ \frac{\partial A(e)}{\partial e_{ij}} \delta e_{ij} - \overline{F}_i \delta u_i \right\} d\tau - \iint_{s_p} \overline{P}_i \delta u_i ds = 0 \quad . \qquad (3.10)$$

Substituting (2.2) into (3.10),

$$\delta\Pi_p = \iiint_\tau \left\{ \frac{\partial A(e)}{\partial e_{ij}} \delta u_{i,j} - \overline{F}_i \delta u_i \right\} d\tau - \iint_{s_p} \overline{P}_i \delta u_i ds = 0 \qquad (3.11)$$

is obtained. Following R3.2 and R3.4, he arrives at

$$\delta\Pi_p = -\iiint_\tau \left\{ \left(\frac{\partial A(e)}{\partial e_{ij}}\right)_{,j} + \overline{F}_i \right\} \delta u_i d\tau + \iint_{s_p} \left(\frac{\partial A(e)}{\partial e_{ij}} n_j - \overline{P}_i\right) \delta u_i ds = 0 \quad , \quad (3.12)$$

for which (2.5) has been satisfied.

Then, following R3.1, the Euler equations [1] are

$$\left(\frac{\partial A(e)}{\partial e_{ij}}\right)_{,j} + \overline{F}_i = 0 \qquad (in \quad \tau) \qquad (3.13)$$

and

$$\left(\frac{\partial A(e)}{\partial e_{ij}} n_j - \overline{P}_i\right) = 0 \qquad (on \quad s_p) \quad . \qquad (3.14)$$

From P3.7, the stress-strain relationship (2.3a) is required to be substituted into (3.13) and (3.14) to transform them into (2.1) and (2.4), which will become natural conditions of MPEP according to D3.4. Therefore, (2.3a) is a constraint condition according to D3.2.

3.4.2. Proof 3.4.2 by us in This Paper

From (2.7) and following R3.4,

$$\delta\Pi_p = \iiint_\tau \left\{ \frac{\partial A(e)}{\partial e_{ij}} \delta e_{ij} - \sigma_{ij}\delta e_{ij} + \sigma_{ij}\delta e_{ij} + \sigma_{ij,j}\, \delta u_i - \sigma_{ij,j}\, \delta u_i - \overline{F}_i \delta u_i \right\} d\tau$$

$$-\iint_{S_p} \overline{P}_i \delta u_i ds = 0 \quad . \qquad (3.15)$$

We have (see R3.2)

$$\iiint_\tau (\sigma_{ij},_j \, \delta u_i) d\tau = \iiint_\tau [(\sigma_{ij} \delta u_i),_j - \sigma_{ij} \delta u_i,_j \,] d\tau$$

$$= \iint_{S_p} \sigma_{ij} n_j \delta u_i ds - \iiint_\tau \sigma_{ij} \delta [(1/2) u_i,_j + (1/2) u_j,_i \,] d\tau + \iint_{S_u} \sigma_{ij} n_j \delta u_i ds \quad . \qquad (3.16)$$

Substituting (3.16) into (3.15), we get

$$\delta \Pi_p = \iiint_\tau \left\{ (\frac{\partial A(e)}{\partial e_{ij}} - \sigma_{ij}) \delta e_{ij} + \sigma_{ij} \delta [e_{ij} - (1/2) u_i,_j - (1/2) u_j,_i \,] - (\sigma_{ij},_j + \overline{F}_i) \delta u_i \right\} d\tau$$

$$+ \iint_{S_p} (\sigma_{ij} n_j - \overline{P}_i) \delta u_i ds + \iint_{S_u} \sigma_{ij} n_j \delta u_i ds = 0 \quad . \qquad (3.17)$$

From P3.7, (2.2) and (2.5) are required to be substituted into (3.17) and are constraint conditions according to D3.2. Eliminating the constraints by substitution (see R3.5), we obtain

$$\delta \Pi_p = \iiint_\tau \left\{ (\frac{\partial A(e)}{\partial e_{ij}} - \sigma_{ij}) \delta e_{ij} - (\sigma_{ij},_j + \overline{F}_i) \delta u_i \right\} d\tau$$

$$+ \iint_{S_p} (\sigma_{ij} n_j - \overline{P}_i) \delta u_i ds = 0 \quad , \qquad (3.18)$$

where e_{ij} and u_i are independent according to D3.1. And then (2.1) , (2.3a) and (2.4) are natural conditions of MPEP, following R3.1 and according to D3.4.

3.4.3. Theorem and Remark

From Sec. 3.4.1 and Sec.3.4.2, we have the theorem:

Theorem 3.4

Equation (2.3a) is a constraint condition and also a natural condition of MPEP.

And we have the remark:

Remark 3.3

Theorem 3.4 is in contradiction to P3.6.

3.5. Contradiction of Independence of Variable: Proof according to D3.1 and following R3.9, Theorem and Remark

From Theorem 3.3 in Sec. 3.3.3 and Theorem 3.4 in Sec. 3.4.3, following R3.9 and according to D3.1, we have the theorem:

Theorem 3.5

Variable σ_{ij} is both a non-independent and an independent variable of MPEP.

And we have the remark:

Remark 3.4.

Theorem 3.5 is in contradiction to P3.4.

3.6. Inconsistence of Chien's Theory and the Problem of Variable-independence

From Remark 3.1-Remark 3.4, we conclude, according to Principle of Consistency in Sec.2.3, with the remark below:

Remark 3.5.

Chien's theory of variational principles in elasticity, or High-order Lagrange Multiplier Theory, is inconsistent.

Then the problem of variable-independence raised by Chien is not solved by the High-order Lagrange Multiplier Theory because of the inconsistence of the theory. In fact, from Theorem 3.5 in Sec.3.5, a variable can be both independent and non-independent in Chien's theory.

4. Luo's System and Its Problem of Consistency

In his paper, Luo uses "variable", "function" or "field" [14] for the term "argument function" of variational principles. He claims, without proof and explanation, that "the momentum field p_i, the velocity field v_i, the displacement field u_i, the strain field ε_{ij} and the stress field σ_{ij} are five independent variables" in the 5-field variational principle [14]. Then less-field variational principles, including the 3-field Gurtin-type (Hu-Washizu class) generalized variational principle, are inferred by reduction operations. It seems that the problem of variable-independence raised by Chien for H-W Principle has been settled by Luo's work. But if Luo's system (the variational principles, the algebraic equations, the differential equations and the inference in Luo's work) is discussed by logic and if the following prostitute is proposed, some contradictions may exist in Luo's system. The prostitute is:

P4.1. A variable in the system can not be both an independent and a non-independent variable. (The law of contradiction of independence of variables)

The variational principle with five independent fields is inferred from the fundamental equations in linear elastodynamics (Eq.3.1-3.9 in [14]). If the five fields in the fundamental equations are considered to be independent, the conventional understanding of independence of variables will be violated. On the other hand, if the five fields in the fundamental equations are considered to be non-independent, the law of contradiction of independence of variables, P4.1, will be violated because the same five fields in the 5-field variational principle are considered as independent.

Similar contradictions may occur when reduction operations are practiced. The variational principles with less (than five) independent fields are inferred by reduction from the principle with all five independent fields. If all fields in the "when equations" (equations chosen from the fundamental equations Eq. 3.1-3.9 by using "When …" sentences in Sec.V.2-V.5 of [14]) are considered to be independent, the conventional understanding of independence of variables will be violated.. On the other hand, if some fields in "when equations" are considered to be non-independent, the law of contradiction of independence of variables, P4.1, will be violated because they are independent fields in the 5-field variational principle.

The contradiction connoted by the conventional understanding of variable-independence in variational theory will be discussed further in Sec.5.1.

5. Finding a Solution to the Problem of Variable-independence

5.1. Contradiction Connoted by the Conventional Independence of Variables in Variational Theory

5.1.1. Conventional Independence of Variables in Variational Theory

The understanding of " independent variable " is implied in different versions as:

(1) quantity with no subsidiary condition ; [6]

(2) variable not subject to any constraint condition ; [9-11]

(3) variable, or function, or field, without any constraint [14] .

Now we understand the " conventionally independent variable " as whatever version of (1)-(3) above.

5.1.2. Discussion by Means of System 5.1

5.1.2.1. System 5.1

(a)Postulates:

P5.1.1 The variational principle, whose functional is (2.8), where $\overline{F}_i$, $\overline{P}_i$ and $\overline{u_i}$ are different from zero.

P5.1.2 Variables σ_{ij} , e_{ij} and u_i are conventionally independent (see Sec.5.1.1).

(b)Rules of Inference:

R5.1.1 Fundamental Lemma of the calculus of variations [1] (generalized).

R5.1.2 The Gauss Theorem in the differential calculus.

R5.1.3 The shearing stress symmetry ($\sigma_{ij} = \sigma_{ji}$) in elasticity [2].

R5.1.4 The rules of operations in the algebraic, differential and variational calculus.

(c)Definitions

D5.1.1 A proof is a logical argument process starting from P5.1.1-P5.1.2, following R5.1.1-R5.1.4.

5.1.2.2. Contradiction: Proof 5.1

From P5.1.2 we know

$$e_{ij} - (1/2)u_{i,j} - (1/2)u_{j,i} \neq 0 \qquad (i, j = 1,2,3) \qquad (in \quad \tau) \quad , \quad (5.1)$$

otherwise e_{ij} will not be conventionally independent.

On the other hand, it is not difficult to obtain (2.1), (2.2), (2.3a), (2.4) and (2.5) from P5.1.1 and P5.1.2, following R5.1.1-R5.1.4.

Then we find that Eq.(2.2) and (5.1) are in contradiction to each other. And so , the conventional understanding of independence of variables in variational theory connotes contradiction.

5.2. Understanding Variable-independence by Logic

Eliminating constraint by substitution implies that a non-independent variable is not identical to itself in nature, it is a symbol of others. In other words, logically, a non-independent variable "is not" itself in nature, it "is" something else. For example, if e_{ij} is a non-independent variable, it "is not" e_{ij} in nature, it "is" $(1/2)u_{i,j} + (1/2)u_{j,i}$. And so we understand logically that variable-independence is identity of variables, that is, every independent variable is or identical to itself. This understanding by logic of variable-independence is the essence of the solution to the problem of variable-independence of variational theory.

5.3. Characterization of the New Theory to be Established

From Sec.3-4 we know that neither Chien's Theory nor Luo's Theory solves the problem of variable-independence because of the difficulty of inconsistence. To find a solution to the problem, a new theory has to be established, which should be characterized by:

A. It must be a mathematical system of logic excluding vagueness and misunderstanding because vagueness and misunderstanding lead to contradictions.

B. The conventional understanding of independence of variables in variational theory (see 5.1.1) has to be excluded.

C. Variable-independence is understood logically as identity of independent variables (see Sec.5.2), formalization is an approach to formulate identity of independent variables or variable-independence of logic, and so the coming theory will be a formalized theory.

D. Proof of independence of variables will not be included.

Chien raised the variable-independence problem by his "High-order Lagrange Multiplier Theory". Behind Chien's argument is an implied postulate that independence of variables must be proved. But any proof needs independence of at least one kind of variable so that the inference for the proof would be practiced. Then a question arises: why is such a kind of variable so special that its independence needs no proving? No answer exists to this question in Chien's Theory.

E. Concept of constraint will not be included.

When an equation is considered as a conventional constraint condition, it means that it is required to be satisfied in the variation process of the variational principle under discussion, otherwise some contradictions may arise. (A typical example will be formally discussed in Sec.5.4 for MPEP.) That is to say, consistency is a more fundamental issue than constraint. Even in conventional theories, eliminating a conventional constraint is nothing but an inference operation following the inference rules of variational theory. In other words, a constraint condition corresponds to a rule of inference. And so, concept of constraint is unnecessary for variational theories, as long as rules of inference, then consistent systems, are established.

F. Consistence of the logic systems of the coming theory must be proved and can be proved, and consistency of the logic systems will be formalized.

To prove consistence of the Axiomatic System of Variation, the formalized inconsistency equation, or, equivalently, the formalized inconsistency inequality, must be formulated, which will be discussed in Sec. 5.4.

G. It is almost definitely predicted that the logic system of the coming theory will be incomplete.

In Luo's Theory independence of 5 variables is taken for granted, which avoids the contradictions caused by "proof". However, contradictions may arise when complete coverage of variational principles in his paper is pursued by inference of reduction (see Sec.4). Chien's theory is "complete" to cover every variational principle of (2.7)-(2.11) at the expense of consistency. And so we predict that the coming variational theory will be incomplete, if it is required to be consistent.

5.4. Establishment of Formalized Inconsistency

According to Kline , Hilbert argued that the "equation"

$$1 = 0 \tag{5.2}$$

will be formally inferred when a formal logical system is inconsistent [15]. In this sub-section we establish the formalized inconsistency equation of the form of (5.2).

5.4.1. Formalized System 5.4 Suggested for MPEP

(a)Postulates:

P5.4.1. The variational principle, whose functional is (2.7), where $\overline{F}_i$ and $\overline{P}_i$ are different from zero.

P5.4.2. Variables, e_{ij} and u_i, are conventionally independent.

(b)Rules of Inference:

R5.4.1. Fundamental Lemma of the calculus of variations [1] (generalized).

R5.4.2. The rules of operations in the algebraic, differential and variational calculus.

R5.4.3. Formal substitution.

(c)Definitions

D5.4.1. A proof is a logical argument process from P5.4.1-P5.4.2 following R5.4.1-R5.4.3.

5.4.2. Formalized Inconsistency: Proof 5.4

From P5.4.1 and following R5.4.2 we have (3.20).

From (3.20), P5.4.2 and following R5.4.1, we obtain

$$\frac{\partial A(e)}{\partial e_{ij}} = 0 \qquad (in \quad \tau) \qquad , \tag{5.3}$$

$$\overline{F}_i = 0 \qquad (in \quad \tau) \tag{5.4}$$

and

$$\overline{P}_i = 0 \qquad (on \quad s_p) \qquad . \tag{5.5}$$

On the other hand, from P5.4.1,

$$\overline{F}_i \neq 0 \qquad (in \quad \tau) \qquad . \tag{5.6}$$

Equations (5.4) and (5.6) are in contradiction to each other. Formalizing the contradiction and following R5.4.3, we substitute (5.4) formally into (5.6) and have

$$0 \neq 0 \qquad (in \quad \tau) \qquad , \tag{5.7}$$

which is the inconsistency inequality. From (5.7), we have, formally,

$$0 = 1 \qquad (in \quad \tau) \qquad , \tag{5.8}$$

which is the inconsistency equation.

On the other hand, from (5.8) we have (5.7). Then (5.7) and (5.8) are equivalent to each other because they can be deduced from each other.

Similarly, we can establish a formalized inconsistency equation on s_p.

6. A Quasi-formalized Theory of Variational Calculus in Elasticity[16,17]

Based on A-G in Sec.5.3, we establish the Quasi-formalized Theory of Variational Calculus in elasticity.

The Quasi-formalized Theory consists of the general definitions, the Axiomatic System of Variation, proofs and theorems of the Axiomatic System of Variation.

6.1. Definitions

6.1.1. General Definitions

D6.1. (Definition of the Axiomatic System of Variation)

The Axiomatic System of Variation consists of the definitions in the system in Sec.6.1.2, the non-dimensional functions and equations in Sec.6.2, the well-formed functions in Sec.6.3, the equation-building regulations in Sec.6.4, all well-formed equations (wfes), the postulates and their equations in Sec.6.5, the rules of inference in Sec.6.6, proofs and theorems in the system.

D6.2. (Definition of Consistency of an Axiomatic System)

An axiomatic system is consistent iff the formalized inconsistency equation

$$1 = 0 \quad (in \quad \tau \quad and \quad on \quad s) \tag{6.1}$$

is not a theorem in the system.

D6.3. (Definition of Completeness of an Axiomatic System)

An axiomatic system is complete iff every wfe is either a postulate or a theorem in the system.

D6.4. (Definition of Independence of an Axiomatic System)

An axiomatic system is independent iff no wfe of postulate can be deduced from other wfes of postulates in the system.

D6.5. (Definition of Proof of an Axiomatic System)

A proof of an axiomatic system is an inference process concluded by a theorem according to D6.2, D6.3 or D6.4.

6.1.2. Definitions in the Axiomatic System of Variation

D6.6. (Definition of the Formalized Independence of Well-formed Argument Functions)

The formalized independence of well-formed argument functions is that every well-formed argument function is identical to itself.

D6.7. (Definition of Proof in the Axiomatic System of Variation)

A proof in the Axiomatic System of Variation is a finite sequence of wfes deduced from the equations of postulates, following the rules of inference in the system.

D6.8. (Definition of Provability of Wfes)

A wfe is provable to be a theorem iff it can be deduced from the equations of postulates, following the rules of inference in the Axiomatic Syetem.

D6.9. (Definition of the Formalized Consistency Equation)

Identity

$$0 = 0 \quad (in \quad \tau \quad and \quad on \quad s) \tag{6.2}$$

is the formalized equation of consistency.

6.2. Non-dimensional Functions and Equations

N6.1. The non-dimensional Cartesian coordinate system $x_i\,(i = 1,2,3)$ is established for the Axiomatic System of Variation.

N6.2. The bounded and closed 3D-domain τ in the Cartesian coordinate system established by N6.1 is the volume of the elastic body, and s $(s = s_p + s_u)$ is the finite, closed and piecewise-smooth surface of τ .

N6.3. Every function, every equation and every inequality defined in τ and on s in the Axiomatic System of Variation is dimensionless.

6.3. Well-formed Functions

F6.1. A real number is a well-formed number function.

F6.2. The continuous functions, defined in τ and on s, possessing continuous first order partial derivatives with respect to $x_i (i = 1,2,3)$ in τ, that is, stress distribution functions σ_{ij}, strain distribution functions e_{ij} and displacement distribution functions u_i, are the well-formed argument functions.

F6.3. The given, finite and continuous functions, that is, body-force functions $\overline{F}_i$ defined in τ, surface-force functions $\overline{P}_i$ defined on s_p and boundary-displacement functions $\overline{u}_i$ defined on s_u, are the well-formed prescribed functions.

F6.4. The continuous energy density function $A(e)$ in τ, possessing continuous partial derivatives $\frac{\partial A(e)}{\partial e_{ij}}$ in τ, is the well-formed energy density function.

F6.5.A function built from the well-formed functions defined in F6.1-6.4, by means of algebraic and/or differential operations, is a well-formed derived function.

F6.6. A functional built from the well-formed functions defined in F6.1-6.5, by means of integral operation, is a well-formed functional-type function.

F6.7. No function is a well-formed function in the system unless it is compelled to be one by F6.1-F6.6.

6.4. Regulations for Building Well-formed-equations

E6.1. An algebraic equation or algebraic inequality of well-formed functions specified in F6.1-6.5 is a wfe.

E6.2. A differential equation or differential inequality of well-formed functions specified in F6.1-6.5 is a wfe.

E6.3. A variational equation built from a well-formed function specified in F6.6 is a wfe.

E6.4. A variational equation transformed from a variational equation specified in E6.3 by means of operations following the Rules of Inference is a wfe.

E6.5. No equation is a wfe unless it is compelled to be one by E6.1-E6.4.

6.5. Postulates in the Axiomatic System of Variation

P6.1. A well-formed variational equation of elasticity.

P6.2. The well-formed differential equations of elasticity.

P6.3. The formalized independence of well-formed argument functions (see D6.6).

P6.4. The formalized consistency equation (see D6.9).

Therefore, the equations of postulates in the system are (by P6.1-P6.4)

$$\delta\{ \iiint_\tau [A(e) - \sigma_{ij}(e_{ij} - (1/2)u_{i,j} - (1/2)u_{j,i}) - \overline{F}_i u_i] d\tau - \iint_{S_p} \overline{P}_i u_i ds - \iint_{S_u} \sigma_{ij} n_j (u_i - \overline{u}_i) ds \} = 0 \qquad (6.3)$$

where $\overline{F}_i$, $\overline{P}_i$ and $\overline{u_i}$ are different from zero,

$$u_i = u_i \quad (i = 1,2,3) \quad (in \quad \tau \quad and \quad on \quad s), \qquad (6.4)$$

$$e_{ij} = e_{ij} \quad (i, j = 1,2,3) \quad (in \quad \tau \quad and \quad on \quad s), \qquad (6.5)$$

$$\sigma_{ij} = \sigma_{ij} \quad (i, j = 1,2,3) \quad (in \quad \tau \quad and \quad on \quad s), \qquad (6.6)$$

$$0=0 \quad (in \quad \tau \quad and \quad on \quad s), \qquad (6.7)$$

and dimensionless equations (2.1), (2.2), (2.3a), (2.4) and (2.5) ((2.1-2.5A) in Sec.6).

In the equations above the Einstein's notations are applied.

6.6. Rules of Inference

R6.1. $\sigma_{ij}=\sigma_{ji} \quad (i,j=1,2,3) \quad (in \quad \tau \quad and \quad on \quad s) \quad .$ (6.8)

R6.2. $\iiint_\tau (\sigma_{ij}\delta u_i),_j \, d\tau = \iint_{S_p} \sigma_{ij} n_j \delta u_i ds + \iint_{S_u} \sigma_{ij} n_j \delta u_i ds \quad .$ (6.9)

R6.3. Formalized Fundamental Lemma of the calculus of variations in elasticity:

If $\iiint_\tau (F_{ij}\delta\sigma_{ij} + G_{ij}\delta e_{ij} + H_i \delta u_i) d\tau + \iint_{S_p} I_i \delta u_i ds + \iint_{S_u} K_i n_j \delta\sigma_{ij} ds = 0$, (6.10)

$$u_i = u_i \quad (i=1,2,3) \quad (in \quad \tau \quad and \quad on \quad s) \quad , \qquad (6.11)$$

$$e_{ij} = e_{ij} \quad (i,j=1,2,3) \quad (in \quad \tau \quad and \quad on \quad s) \quad , \qquad (6.12)$$

and $\sigma_{ij} = \sigma_{ij} \quad (i,j=1,2,3) \quad (in \quad \tau \quad and \quad on \quad s)$ **,** (6.13)

then $F_{ij}=0 \quad (i,j=1,2,3) \quad (in \quad \tau)$, (6.14)

$$G_{ij}=0 \quad (i,j=1,2,3) \quad (in \quad \tau) \quad , \qquad (6.15)$$

$$H_i=0 \quad (i=1,2,3) \quad (in \quad \tau) \quad , \qquad (6.16)$$

$$I_i=0 \quad (i=1,2,3) \quad (on \quad s_p) \quad , \qquad (6.17)$$

and $K_i=0 \quad (i=1,2,3) \quad (on \quad s_u)$, (6.18)

where F_{ij}, G_{ij}, H_i, I_i, and K_i are continuous functions of $x_i \, (i=1,2,3)$; the Einstein's notations are applied.

R6.4. The rules of algebraic and first-order differential operations for wfes specified in E6.1-6.2. Substitution including formal substitution is allowed except for the forbidden substitution specified in R6.6.

R6.5.The rules of first-order variational and first-order differential operations for wfes specified in E6.3-6.4. Substitution is allowed except for the forbidden substitution specified in R6.6.

R6.6. Rule of Forbidden Substitution:

Substituting any of (2.1-2.5A) into wfes specified in E6.3-6.4 is forbidden.

R6.7.The method of exhaustion.

R6.8. There exists no rule of inference except for R6.1-R6.8.

6.7. Consistency of the Axiomatic System of Variation

Theorem 6.1.

The Axiomatic System of Variation defined by D6.1 is consistent.

Proof 6.1. (by the method of exhaustion, R6.7)(see D6.5):

Sub-process 1:

From (6.4-6.6), (6.1) can not be deduced.

Example 6.1:

From (6.4),

$$u_i - u_i = 0 \quad (i=1,2,3) \quad . \qquad (6.19)$$

Following R6.4, (6.7) is deduced from (6.19).

Sub-process 2:

From (2.1-2.5A), (6.1) can not be deduced.

Example 6.2:

Subtracting each of (2.1-2.5A) from itself, (6.7) is deduced following R6.4.

Sub-process 3:

From (2.1-2.5A) and (6.4)-(6.6), (6.1) can not be deduced.

Example 6.3:

From (6.4),

$$(1/2)u_i,_j = (1/2)u_i,_j \qquad (i = 1,2,3) \qquad (6.20)$$

and

$$(1/2)u_j,_i = (1/2)u_j,_i \qquad (i = 1,2,3) \quad . \qquad (6.21)$$

From (2.2),

$$(e_{ij} - e_{ij}) - [(1/2)u_i,_j - (1/2)u_i,_j] - [(1/2)u_j,_i - (1/2)u_j,_i] = 0 \qquad (i, j = 1,2,3) \quad . \qquad (6.22)$$

Substituting (6.5), (6.20) and (6.21) into (6.22) , (6.7) in τ is deduced.

Sub-process 4:

From (2.1-2.5A) and (6.3)-(6.6), (6.1) can not be deduced.

Example 6.4:

From (6.3) and following R6.1, R6.2 and R6.5,

$$\iiint_\tau [(\frac{\partial A(e)}{\partial e_{ij}} - \sigma_{ij})\delta e_{ij} - (e_{ij} - (1/2)u_i,_j - (1/2)u_j,_i)\delta\sigma_{ij} - (\sigma_{ij},_j + \overline{F}_i)\delta u_i]d\tau$$

$$+ \iint_{s_p} (\sigma_{ij}n_j - \overline{P}_i)\delta u_i ds - \iint_{s_u} (u_i - \overline{u}_i)n_j\delta\sigma_{ij} ds = 0 \quad . \qquad (6.23)$$

From (6.4)-(6.6) and (6.23) and following R6.3,

$$\sigma_{ij},_j + \overline{F}_i = 0 \qquad (i = 1,2,3) \qquad (in \quad \tau) \quad , \qquad (6.24)$$

$$e_{ij} - (1/2)u_i,_j - (1/2)u_j,_i = 0 \qquad (i, j = 1,2,3) \qquad (in \quad \tau) \quad , \qquad (6.25)$$

$$\frac{\partial A(e)}{\partial e_{ij}} - \sigma_{ij} = 0 \qquad (i, j = 1,2,3) \qquad (in \quad \tau) \quad , \qquad (6.26)$$

$$\sigma_{ij}n_j - \overline{P}_i = 0 \qquad (i = 1,2,3) \qquad (on \quad s_p) \quad , \qquad (6.27)$$

$$u_i - \overline{u}_i = 0 \qquad (i = 1,2,3) \qquad (on \quad s_u) \quad . \qquad (6.28)$$

Subtracting (6.24) from (2.1) , (6.25) from (2.2), (6.26) from (2.3a) , (6.27) from (2.4) and (6.28) from (2.5) respectively, 5 equations similar to (6.22) are obtained. And then (6.7) is deduced either by R6.4 or by using (6.4)-(6.6) and R6.4 in the same way as Example 6.3.

Sub-process 5:

From (2.1-2.5A) and (6.3)-(6.7), (6.1) can not be deduced.

Example 6.5:

Adding (6.7) to any algebraic or differential equation within Sub-process1 to Sub-process 4 will not lead to (6.1).

Example 6.6:

Multiplying any algebraic or differential equation within Sub-process 1 to

Sub-process 4 by (6.7) will not lead to (6.1).

Example 6.7:

Substituting (6.7) for any algebraic or differential equation within Sub-process 1 to Sub-process 4, or vice versa, will not lead to (6.1).

From Sub-process 1 to Sub-process 5, (6.1) is not a theorem according to D6.8, then Theorem 6.1. is proved according to D6.2.

6.8. Incompleteness of the Axiomatic System of Variation

Theorem 6.2.

The Axiomatic System of Variation defined by D6.1 is incomplete.

Proof 6.2. (see D6.5, D6.8):

Variational principles other than (6.3), dimensionless MPEP for example, are not theorems in the system. Therefore, the axiomatic system is incomplete according to D6.3.

6.9. On the Independence of the Axiomatic System of Variation

Theorem 6.3.

The Axiomatic System of Variation defined by D6.1 is not independent.

Proof 6.3. (see D6.5)

Equations (6.7) and (2.1-2.5A) are deduced from (6.3)-(6.6) in Sub-process 4 in Sec.6.7, and Theorem 6.3 is proved according to D6.4.

6.10. Minimization of the Axiomatic System of Variation

Equations (6.4)-(6.6) are included in the equations of postulates of the Axiomatic System of Variation because they are explicit formulations of variable-independence. Including (6.4)-(6.7) and (2.1-2.5A) serves the purpose to show compatibility between the formalized variable-independence and the equations of (2.1-2.5A) (see Example 6.3 in Sec. 6.7). Excluding Eq.(6.4)-(6.7) and (2.1-2.5A) (and P6.2 and P6.4) from the equations of postulates will minimize the system and establish the Minimized Axiomatic System of Variation, which is a “pure” (without algebraic and differential equations for postulates) variational formulation of the elastic theory, with variable-independence postulated by P6.3.

7. A Formalized Theory of the Variational Calculus in Elasticity [16,17]

The Formalized Theory consists of the definitions ; the Formal System of Variation; proofs, theorems and interpretation of the Formal System of Variation.

7.1. Definitions

7.1.1. Definitions Concerning the Formal System of Variation

D7.1. (Definition of the Formal System of Variation)

The Formal System of Variation consists of the symbols in Sec.7.2, the rules of formula-building operations in Sec.7.3, the rules of symbol-omitting in Sec.7.4, all well-formed formulas (wffs), the axiom in Sec.7.5 , the rules of inference in Sec.7.6, proofs and theorems in the system.

D7.2. (Definition of Proof of Theorems in the Formal System of Variation)

A proof of theorems in the Formal System of Variation (defined by D7.1) is a finite sequence of separate expressions, in which each expression satisfies one of the following conditions:

(a) It is the axiom of the Formal System of Variation, A7.1;
(b) It is inferred from the expressions previous in the sequence, following the rules of inference, I7.1-I7.12, in the Formal System of Variation;
(c) The end expression(s) of the sequence is/are the theorem(s) proved in the Formal System of Variation.

7.1.2. General Definitions

D7.3. (Definition of Consistency of a Formal System)

A formal system is consistent iff every axiom and every theorem in the system is a wff.

D7.4. (Definition of Completeness of a Formal System)

A formal system is complete iff every wff is an axiom or a theorem in the system.

D7.5. (Definition of Independence of a Formal System)

A formal system is independent iff no axiom can be inferred from other axioms in the system, following the rules of inference in the system.

D7.6. (Definition of Proof of a Formal System)

A proof of a formal system is an inference process concluded by a theorem according to D7.3 , D7.4 or D7.5.

7.2. Symbols

Sym. 7.1. Formula Symbols:

$a, g, f, e, r, c, \sigma, u, t, p, d, v, w, h, q, O$

Sym. 7.2. Prefix Symbols:

i, j, k, δ

Sym. 7.3. Connective Symbols:

$(,), \bullet, +, \rightarrow, =$

Sym. 7.4. Variable Symbols:

(A) Symbol-Variables are s_1 and s_2. Each symbol-variable takes any formula symbol as its value.
(B) Expression-Variables are X and Y. Each expression-variable takes any expression (finite sequence of symbols) as its value.
(C) Wff (well-formed-formula)-Variables are $F_m (m = 1, 2, \cdots, n)$. Each wff-variable takes any wff as its value.
(D) Prefix-Variable is K , which takes any one among i, j *and* k as its value.

7.3. Rules of Formula-Building Operations

B7.1. Every formula symbol is a wff.

B7.2. If X is a wff, then

$$(X) \quad , \tag{7.1}$$

$$K \bullet X \tag{7.2}$$

and

$$\delta \bullet X \tag{7.3}$$

are wffs respectively.

B7.3. If X and Y are wffs respectively, then

$$X + Y \quad , \tag{7.4}$$

$$X \bullet Y \quad , \tag{7.5}$$
$$X \to Y \tag{7.6}$$
and
$$X = Y \tag{7.7}$$
are wffs respectively.

B7.4. No expression is a wff unless it is compelled to be one by B7.1-B7.3.

7.4. Rules of Symbol-Omitting

O7.1. The dot, $\bullet$, need not be explicitly mentioned.

O7.2. The outermost parentheses of an expression need not be explicitly mentioned.

O7.3. The inner parentheses of
$$(X + (Y)) \tag{7.8}$$
need not be explicitly mentioned.

O7.4. The parentheses can be omitted by following the priority-order of grouping and connection (given in the priority-down-order in (7.9)):
$$(\quad), \bullet, +, \to, = \tag{7.9}$$

7.5. Axiom of the Formal System of Variation

A7.1. $$\delta(i(a + (\sigma e) + (\sigma r) + (fu)) + j(pd) + k((hv) + (hw))) = O \tag{7.10}$$

7.6. Rules of Inference

I7.1. Substitution Rule:

If
$$F_1 \to F_2 \ , \tag{7.11}$$
then F_1 can be replaced by F_2, another wff, no matter whether F_1 is a wff or a sub-wff (a part of a wff, which is a wff itself).

I7.2. $$\delta(F_1 + F_2 + \cdots + F_n) \to \delta F_1 + \delta F_2 + \cdots + \delta F_n \tag{7.12}$$

I7.3. $$\delta K F_1 \to K \delta F_1 \tag{7.13}$$

I7.4. $$\delta(s_1 s_2) \to s_1 \delta s_2 + s_2 \delta s_1 \tag{7.14}$$

I7.5. $$K(F_1 + F_2 + \cdots + F_n) \to KF_1 + KF_2 + \cdots + KF_n \tag{7.15}$$

I7.6. $$KF_1 + KF_2 + \cdots + KF_n \to K(F_1 + F_2 + \cdots + F_n) \tag{7.16}$$

I7.7. $$u\delta f \to O \tag{7.17}$$
$$d\delta p \to O \tag{7.18}$$
$$h\delta w \to O \tag{7.19}$$
$$\delta a \to c\delta e \tag{7.20}$$
$$q\delta v + h\delta v \to O \tag{7.21}$$

I7.8. $$F_1 + F_2 \to F_2 + F_1 \tag{7.22}$$

I7.9. $$F_1F_3 + F_2F_3 \to (F_1 + F_2)F_3 \tag{7.23}$$

I7.10. $$F_1 + O \to F_1 \tag{7.24}$$

I7.11. $$i(\sigma\delta r) \to i(g\delta u) + j(t\delta d) + k(q\delta v) \tag{7.25}$$

I7.12. If $$i(F_1\delta u + F_2\delta\sigma + F_3\delta e) + j(F_4\delta d) + k(F_5\delta h) = O \ , \tag{7.26}$$
then $$Fm = O \qquad (m = 1,2,3,4,5) \tag{7.27}$$

I7.13. There exists no rule of inference except for I7.1-I7.13.

7.7. Proof 7.0. in the Formal System of Variation (Given Concisely)

(see D7.2)

The axiom (from (7.10)):

$$\delta(i(a+(\sigma e)+(\sigma r)+(fu))+j(pd)+k((hv)+(hw)))=O \qquad (7.28)$$

By I7.1 and I7.2,

$$\delta i(a+(\sigma e)+(\sigma r)+(fu))+\delta j(pd)+\delta k((hv)+(hw))=O \qquad (7.29)$$

By I7.1 and I7.3,

$$i\delta(a+(\sigma e)+(\sigma r)+(fu))+j\delta(pd)+k\delta((hv)+(hw))=O \qquad (7.30)$$

By I7.1 and I7.2,

$$i(\delta a+\delta(\sigma e)+\delta(\sigma r)+\delta(fu))+j\delta(pd)+k(\delta(hv)+\delta(hw))=O \qquad (7.31)$$

By I7.1, I7.4 and (7.20) of I7.7,

$$i(c\delta e+\sigma\delta e+e\delta\sigma+\sigma\delta r+r\delta\sigma+f\delta u+u\delta f)+j(p\delta d+d\delta p)$$
$$+k(h\delta v+v\delta h+h\delta w+w\delta h)=O \qquad (7.32)$$

By I7.1, I7.5, I7.6, I7.7, I7.8, I7.9, I7.10 and I7.11,

$$i((g+f)\delta u+(e+r)\delta\sigma+(c+\sigma)\delta e)+j((t+p)\delta d)$$
$$+k((v+w)\delta h)=O \qquad (7.33)$$

By I7.12,

$$g+f=O \quad , \qquad (7.34)$$
$$e+r=O \quad , \qquad (7.35)$$
$$c+\sigma=O \quad , \qquad (7.36)$$
$$t+p=O \quad , \qquad (7.37)$$
$$v+w=O \quad . \qquad (7.38)$$

7.8. Consistence of the Formal System of Variation

Theorem 7.1.

The Formal System of Variation defined by D7.1 is consistent.

Proof 7.1. (see D7.6):

(a) The axiom of the system, (7.10), is a wff (see Sec. 7.5).

(b) A wff is built by defining (see B7.1, Sec. 7.3), deriving from a wff (see B7.2, Sec.7.3) or connecting two wffs (see B7.3, Sec. 7.3). Therefore, a wff can only be transformed into another wff following any one of I7.1-I7.11, by which a wff is replaced by another wff (see Sec.7.6).

(c) By I7.12, five wffs are inferred from a wff (see Sec. 7.6).

(d) From (a)-(c) above, every theorem proved in the system is a wff (see D7.2, Sec.7.1).

(e) From (a) and (d), the Formal System of Variation is consistent (see D7.3, Sec. 7.1).

7.9. Incompleteness of the Formal System of Variation

Theorem 7.2.

The Formal System of Variation defined by D7.1 is incomplete.

Proof 7.2. (see D7.6):

A number of wffs,

$$\delta(i(a+(fu))+j(pd))=O \qquad (7.39)$$

for example, are not theorems in the system (see D7.2, Sec. 7.1), and so the Formal System of Variation is incomplete according to D7.4, Sec.7.1.

7.10. Independence of the Formal System of Variation

Theorem 7.3.

The Formal System of Variation defined by D7.1 is independent.

Proof 7.3. (see D7.6):

There is only one axiom (7.10) in the system, so Theorem 7.3 is proved according to D7.5, Sec.7.1.

7.11. Interpretation of the Formal System of Variation

7.11.1. Interpretation: the Dictionary

Symbol	Interpretation
a	$A(e) \quad (in \quad \tau)$
σ	$-\sigma_{ij} \quad (in \quad \tau)$
e	$e_{ij} \quad (in \quad \tau)$
r	$-(1/2)u_{i,j} - (1/2)u_{j,i} \quad (in \quad \tau)$
f	$\overline{F}_i \quad (in \quad \tau)$
u	$-u_i \quad (in \quad \tau)$
p	$-\overline{p}_i \quad (on \quad s_p)$
d	$u_i \quad (on \quad s_p)$
h	$-\sigma_{ij}n_j \quad (on \quad s_u)$
q	$\sigma_{ij}n_j \quad (on \quad s_u)$
v	$u_i \quad (on \quad s_u)$
w	$-\overline{u}_i \quad (on \quad s_u)$
g	$\sigma_{ij,j} \quad (in \quad \tau)$
c	$\dfrac{\partial A}{\partial e_{ij}} \quad (in \quad \tau)$
t	$\sigma_{ij}n_j \quad (on \quad s_p)$
O	0
i	$\iiint_\tau d\tau$
j	$\iint_{s_p} ds$
k	$\iint_{s_u} ds$
δ	δ
(	(
)	)
$\bullet$	$\bullet$
$+$	$+$

$$\rightarrow \qquad \rightarrow (Sign \quad of \quad Substitution)$$

$$= \qquad =$$

$$F_m \qquad f_m(\sigma_{ij}, e_{ij}, u_i) \qquad (m = 1, 2, \cdots, n)$$

7.11.2. Interpretation: Rules of Inference

Rules of Inference Interpretation

I7.11 $$\iiint_\tau \sigma_{ij}\delta[(1/2)u_{i,j} + (1/2)u_{j,i}]d\tau \rightarrow \iiint_\tau (-\sigma_{ij,j}\,\delta u_i)d\tau + \iint_{s_p}(\sigma_{ij}n_j\delta u_i)ds + \iint_{s_u}(\sigma_{ij}n_j\delta u_i)ds \qquad (7.40)$$

I7.12 If $$\iiint_\tau (f_1\delta(-u_i) + f_2\delta(-\sigma_{ij}) + f_3\delta e_{ij})d\tau + \iint_{s_p}(f_4\delta u_i)ds + \iint_{s_u}(f_5\delta(-\sigma_{ij}n_j))ds = 0 \quad , \qquad (7.41)$$

then $$f_m(\sigma_{ij}, e_{ij}, u_i) = 0 \qquad (m = 1,2,3,4,5) \quad . \qquad (7.42)$$

7.11.3. Interpretation: Axiom and Theorems in the Formal System of Variation

The interpretation of the axiom, (7.10), is

$$\delta(\iiint_\tau (A(e) - \sigma_{ij}e_{ij} + \sigma_{ij}((1/2)u_{i,j} + (1/2)u_{j,i}) - \overline{F}_i u_i)d\tau - \iint_{s_p}\overline{p}_i u_i ds - \iint_{s_u}\sigma_{ij}n_j(u_i - \overline{u}_i)ds) = 0 \qquad . \ (7.43)$$

Theorems	Interpretation
(7.34)	(2.1)
(7.35)	(2.2)
(7.36)	(2.3a)
(7.37)	(2.4)
(7.38)	(2.5)

8. Conclusions

(1) Variational theory must be established as a mathematical system of logic, excluding vagueness and misunderstanding.

(2)Of a variational theory, consistency of its system of logic is an essential requirement and a fundamental topic.

(3)Chien's "High-order Lagrange Multiplier Theory" is inconsistent.

(4)There exist contradictions concerning independence of variables in Luo's system.

(5)The conventional understanding of variable-independence connotes contradiction

(6) It is suggested in this paper that variable-independence should be understood logically as identity of variables.

(7) A solution to the problem of variable-independence is formalization of variational theory.

(8) The Axiomatic System of Variation suggested in this paper is consistent, incomplete and not independent.

(9) The Formal System of Variation suggested in this paper is consistent, incomplete and independent.

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