A Mathematical Approach to Physical Realism

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February 25, 2015

Abstract

I propose to ask mathematics itself for the possible behaviour of nature, with the focus on starting with a most simple realistic model, employing a philosophy of investigation rather than invention when looking for a unified theory of physics.

Doing a 'mathematical experiment' of putting a least set of conditions on a general time-dependent manifold results in mathematics itself inducing a not too complex 4-dimensional object similar to our physical spacetime, with candidates for gravitational and electromagnetic fields emerging on the tangent bundle.

This suggests that the same physics might govern spacetime not only on a macroscopic scale, but also on the microscopic scale of elementary particles, with possible junctions to quantum mechanics.

1 Introduction

I want to report of a 'mathematical experiment' and the implications found so far while inventing a somewhat different mathematical framework to represent General Relativity, and spacetime in general, with as few assumptions or axioms as necessary.

But first we shall develop some philosophical prerequisites.

1.1 How Does Mathematics Relate to Physics ?

As physicists, we seek regularities in nature, that is, logical relations. In nature we find logical relations as well as apparently chaotic behaviour, but only the logical relations are accessible to our ability to describe them logically.

Mathematics is our language to describe logical behaviour. So it shall be no miracle that exactly that, which we can describe logically, can be described by mathematics. It is a mere tautology when we take a more humble view on nature, considering that still much of nature's behaviour is inaccessible to our logical understanding.

The 'Unreasonable Effectiveness of Mathematics in the Natural Sciences', as *Wigner* [1] puts it, shall then not seem so unreasonable anymore, since it just reflects our success of reasoning while accessing more and more of nature's logic that is accessible to us in a reasoning way.

Nevertheless, when the logical part of nature can be expressed by mathematics, then we could also ask mathematics, what logic else might be expected from nature, and compare that with our empirical knowledge.

1.2 Ask Mathematics For Reality

Mathematics has often been used to engineer models of physical observations with no regard of realism. Especially quantum mechanics had been formulated with the 'positivistic' philosophy of just deductively building models from empirical results.

Contrary to that, it is proposed to ask mathematics itself to 'reveal her own nature' and to find in an inductive way a mathematical reality which might, at deeper inspection, actually resemble our physical universe.

This way of thinking introduces a philosophy of detection instead of engineering, of investigation rather than invention.

2 An Experimental Spacetime Ansatz

Now follows a brief overview of a previous work [5] by the author, with some focus on philosophical implications regarding the use of mathematics.

The objective of this 'experiment' is to describe and investigate a mathematical model of spacetime, which might at best reproduce a basis of our actual physical reality, aiming at discovering all physical fields on the tangent bundle of the spacetime manifold, instead of ad-hoc introducing them explicitly as separate vector fields.

2.1 The Model

First one should choose a most general conceivable model. The common approach in General Relativity (GR) is to model spacetime as a 4-dimensional $(3 + 1)$ manifold, understood as 3 space dimensions together with a time development.

We will arrive at the same arrangement of dimensions, but at first leave this open and start with a more fundamental approach of only assuming a finite number of $(n + 1)$ dimensions.

2.2 Deconstructing the Perspective

Choosing appropriate viewpoints to investigate the given model is crucial to finding inherent simplifications, which otherwise were hidden in the math. Changes of our perspective alone should not restrict the math in any way, they are 'without loss of generality'.

In GR, the metric tensor, $g_{\mu\nu}$, is regarded as *the* 'fundamental tensor' and the goal is to find metric functions of space and time coordinates. But the metric tensor itself can be constructed from the *Jacobi* matrix, $J^{\alpha}_{\ \mu}$, of an embedding of a space into itself, $g_{\mu\nu} = J^{\alpha}_{\ \mu}\, \eta_{\alpha\beta}\, J^{\beta}_{\ \nu}$, which is a typical textbook exercise. This way some information of the embedding gets lost in forming the metric tensor, so the metric tensor has indeed suffered a loss of generality.

So why not start with the embedding in the first place? The *Jacobi* matrix would then be a more 'fundamental tensor' than the metric.

Now look for a point-wise description of spacetime. In the infinitesimal limit at the local point we would have a flat space, so the absolute value of the embedding matrix shall locally always be the unity matrix, and thus the metric tensor be locally equal to the flat *Minkowski* metric. This view is well-known in differential geometry as a 'local trivialisation of the tangent bundle'.

But the further derivatives of the embedding matrix need not vanish and give us all information about how our vicinity extends into the farth in space and time. With the assumption of holomorphism, even the whole universe should be reconstructible from an infinite power series, if all terms of the series were exactly defined.

Then the matrix logarithm of the local *Jacobi* matrix is 'scale-invariant' in a sense, that at any local point not even an absolute size of the unit vectors is known.

This was an objective of *Weyl*'s 'truly infinitesimal geometry'[1], which he tried to construct in a complicated way, then finally gave up on that. Employing a view on the logarithmic embedding, this scale invariance is already there. We need not invent it, instead just discover it.

2.3 A Minimum Set of Axioms

Up to now, our spacetime model alone has not any rigidity in its structure. We should impose restricting assumptions on it, and the smallest conceivable set of mathematically simple conditions should do, which at the same time restrict the model most effectively. We do not even ask yet for a variational principle.

Surely it is desired to do differential analysis on the model, so a necessary condition is partial differentiability in any spacetime direction, similar to complex differentiability on the space of complex numbers. Spacetime shall be holomorphic, like analytic functions in complex number theory are.

This shall be attained by two axioms:

- The trace of all contractions of all logarithmic derivative tensors shall vanish identically ($\Gamma_{ab}\,\delta^{ab} \equiv 0$, $\Gamma_{abc}\,\delta^{ab} \equiv 0$, $\Gamma_{abc}\,\delta^{bc} \equiv 0$, $\Gamma_{abc}\,\delta^{ac} \equiv 0$, and so on). This is a generalization of the *Laplace* equation to arbitrary tensors, $\Delta T \equiv 0$.
- The chain of successive logarithmic derivative tensors shall start with a single scalar 'master potential' $\Gamma(t, x, y, z)$, thus all derivative tensors are symmetrical in all index pairs (but not necessarily self-adjunct, which would rule out the existence of non-conservative, that is, vortex fields).

This way even the somewhat frightening complexity of GR's tensors gets reduced to a relatively comprehensible system.

[1] 'Wahrhafte Nahegeometrie', after [2]

2.4 Again Adjusting the Perspective

Now assuming holomorphism, then without loss of generality we can express the 'master potential' as a *Taylor* series, which can be expressed from a sum of products of constituting vector forms.

2.5 A (3+1) Spacetime Emerging

A simple $\mathbb{R}^n$, in any number of dimensions, is too stiff to show any phenomenon of gravitation, and does not even allow for rotational transformations in the embedding matrix, since the constituting vectors have to always be perpendicular.

But the *Laplace* condition from above does not rule out the use of imaginary units in the vector forms. When, for example, we employ the components of vectors in $\mathbb{R}^2$ with the complex base vectors $(1, i)$, $i^2 := -1$, this space can be distorted while staying holomorphic. This gives a viable (1+1) spacetime, though which will not support much complexity and obviously does not resemble the reality we live in.

Exactly $\mathbb{R}^4$ with a quaternionic[2] signature $(1, i, j, k)$, $i^2 = j^2 = k^2 = ijk = -1$, $kji = +1$, immediately induces a non-commutative spinor algebra, which allows for rotations and produces the whole of special relativity. It introduces a kind of torsion, but not *Cartan*'s torsion.

Further candidates would be the *Cayley* numbers in $\mathbb{R}^8$, $\mathbb{R}^{16}$ ..., but which are lacking associativity and are therefore rather questionable bases for viable spacetimes.

Thus we arrive at a 'quaternionic spacetime' as a most plausible model of our reality. Defining a quaternionic base vector matrix, its 2nd power gives a *Minkowski* metric,

$$\mathfrak{B} := \left[\begin{array}{c|ccc} 1 & & & \\ \hline & i & & \\ & & j & \\ & & & k \end{array}\right] \quad \Rightarrow \quad \mathfrak{B}\mathfrak{B} = \mathfrak{B}^T\mathfrak{B} = \left[\begin{array}{c|ccc} 1 & & & \\ \hline & -1 & & \\ & & -1 & \\ & & & -1 \end{array}\right] = \eta_{ab} = \eta^{ab},$$

in a natural way without the need to put in the *Minkowski* metric explicitly.
Then the *Laplace* condition of vanishing trace changes to a '*d'Alembert* condition',

$$\frac{1}{{c_0}^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = 0,$$

which introduces a dispersion relation on all logarithmic derivative tensors.

The quaternionic base units imply a fixed pattern of imaginary signatures for each tensor, for example in the 2nd rank,

$$\Gamma_{ab} \sim \left[\begin{array}{c|ccc} +1 & & & \\ \hline & -1 & & \\ & & -1 & \\ & & & -1 \end{array}\right] + \left[\begin{array}{c|ccc} \cdot & i & j & k \\ \hline i & \cdot & & \\ j & & \cdot & \\ k & & & \cdot \end{array}\right] + \left[\begin{array}{c|ccc} \cdot & & & \\ \hline & \cdot & k & j \\ & k & \cdot & i \\ & j & i & \cdot \end{array}\right].$$

[2]Quaternions, or '*Hamilton* numbers', have been discovered as early as in 1843 by *W. R. Hamilton*, who also gave name to the *Hamilton* formalism of classical mechanics and the *Hamilton* operator in quantum mechanics.

2.6 Special Relativity Emerging

For an example of the quaternionic *Lorentz* transform, assume a *Lorentz* boost with rapidity ν_x in direction (x), together with a spatial rotation of angle ϱ_x in the (y,z) plane.

The logarithmic embedding matrix and the corresponding exponentiated *Jacobi* matrix are then

$$\Gamma_{ab} = \left[\begin{array}{c|ccc} \cdot & i\,\nu_x & \cdot & \cdot \\ \hline i\,\nu_x & \cdot & & \\ \cdot & & \cdot & i\,\varrho_x \\ \cdot & & i\,\varrho_x & \cdot \end{array}\right] \xrightarrow{\text{exp}} J^{\mu}{}_{b} = \left[\begin{array}{c|ccc} \cosh(\varphi) & i\,\sinh(\varphi) & \cdot & \cdot \\ \hline \bar{\imath}\,\sinh(\varphi) & \cosh(\varphi) & & \\ \cdot & & \cos(\varrho) & i\,\sin(\varrho) \\ \cdot & & i\,\sin(\varrho) & \cos(\varrho) \end{array}\right],$$

of which at first glance the latter seems to have the wrong symmetries. But notice, that the hyperbolic rotation, though anti-symmetric, is still hermitian and the circular transform is, though pro-symmetric, still anti-hermitian, so the transforms of a (quaternionic) vector come out correctly in a quaternionic context, for instance,

$$\begin{pmatrix} (\cosh(\varphi)\,d - \sinh(\varphi)\,a) \\ i\,(-\sinh(\varphi)\,d + \cosh(\varphi)\,a) \\ j\,(\cos(\varrho)\,b - \sin(\varrho)\,c) \\ k\,(\sin(\varrho)\,b + \cos(\varrho)\,c) \end{pmatrix} = \left[\begin{array}{c|ccc} \cosh(\varphi) & i\,\sinh(\varphi) & \cdot & \cdot \\ \hline \bar{\imath}\,\sinh(\varphi) & \cosh(\varphi) & & \\ \cdot & & \cos(\varrho) & i\,\sin(\varrho) \\ \cdot & & i\,\sin(\varrho) & \cos(\varrho) \end{array}\right] \begin{pmatrix} d \\ i\,a \\ j\,b \\ k\,c \end{pmatrix},$$

similar to the same *Lorentz* transform with purely real signatures,

$$\begin{pmatrix} \cosh(\varphi)\,d - \sinh(\varphi)\,a \\ -\sinh(\varphi)\,d + \cosh(\varphi)\,a \\ \cos(\varrho)\,b - \sin(\varrho)\,c \\ \sin(\varrho)\,b + \cos(\varrho)\,c \end{pmatrix} = \left[\begin{array}{c|ccc} \cosh(\varphi) & -\sinh(\varphi) & \cdot & \cdot \\ \hline -\sinh(\varphi) & \cosh(\varphi) & & \\ \cdot & & \cos(\varrho) & -\sin(\varrho) \\ \cdot & & +\sin(\varrho) & \cos(\varrho) \end{array}\right] \begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix}.$$

2.7 Possible Gravitation

The third-rank derivative tensor, Γ_{abc}, which, following the axioms, happens to be identical to the *Christoffel* symbol of the first kind, contains a logarithmic acceleration together with a spatial stretch, which from the *d'Alembert* condition have to cancel out and give rise to a phenomenon similar to gravitation, even without employing a variational principle.

A proposed gravitational solution is then, not just in the far limit, but exactly, *Newton*'s gravity, with a $1/r$ potential, when measured with the yardstick of an observer walking radially from the gravitational center outwards.

2.8 Possible Electromagnetic Phenomena

Contracting the fourth-rank derivative once, the resulting matrix can be related to an equivalent of the electromagnetic *Faraday* tensor[3],

$$F_{ab} \;:=\; \Gamma^{\boldsymbol{t}}{}_{\boldsymbol{t}ab} \;=\; \left[\begin{array}{c|ccc} \cdot & iE_x & jE_y & kE_z \\ \hline iE_x & \cdot & k\,B_z & j\,B_y \\ jE_y & k\,B_z & \cdot & iB_x \\ kE_z & j\,B_y & iB_x & \cdot \end{array}\right],$$

different from the usual representation, but again with quaternionic units giving it the same anti-hermitian structure.

There is some electromagnetic gauge potential χ together with a *Lorenz* gauge condition,

$$\Phi_{\mathrm{el}} := \partial_t\chi\,, \qquad \vec{A}_{\mathrm{mag}} := \nabla\chi\,, \qquad \partial_t\Phi_{\mathrm{el}} \;\equiv\; \mathrm{div}\,\vec{A}_{\mathrm{mag}}\,.$$

In the fifth-rank derivatives, something similar to *Maxwell*'s equations, up to sign conventions, is reproduced,

$$\mathrm{div}\,\vec{E} =: \rho_{\mathrm{el}}/\varepsilon_0\,, \qquad \mathrm{div}\,\vec{B} \equiv 0\,, \qquad \mathrm{rot}\,\vec{E} \equiv \partial_t\vec{B}\,, \qquad \partial_t\vec{E} \stackrel{?}{=} \mathrm{rot}\,\vec{B} =: \vec{\jmath}/\varepsilon_0\,.$$

Finally in the sixth-rank derivatives a continuity equation shows up, like that of electric charge density, together with some electromagnetic wave equations,

$$\partial_t\,\rho_{\mathrm{el}} \;\equiv\; \mathrm{div}\,\vec{\jmath}, \qquad \Box\vec{E} \;\equiv\; \vec{0}, \qquad \Box\vec{B} \;\equiv\; \vec{0},$$

of which the latter are simply dispersion relations, since the *d'Alembert* condition, per axiom, actually acts on any component of all logarithmic derivative tensors.

Recapitulating those well-known equations with just minor differences and a different representation here would not be a novelty, but novel is them coming out of a simple mathematical model without putting them in to start with.

Indeed, dispersion relations, local gauge potentials and *Lorenz* gauges appear on every rung of the ladder of logarithmic derivatives, with the 'master potential' being the topmost gauge function, but the *Maxwell* equations do not arrive before the fifth rank.

2.9 What Does Mathematics Tell in This Case ?

What smallest set of axioms did we put in?

- Space should have an integer number of dimensions, together with a time dimension,
- The trace of all contractions of all logarithmic derivative tensors must vanish identically,

[3] Factors of $1/c$ are omitted here for simplicity.

- The chain of successive logarithmic derivative tensors starts with a single scalar 'master potential'.

Which adjustments to our viewpoint did we use so far? Observe

- a local 'infinitesimal embedding', that is, the 'local trivialization of the tangent bundle',
- the logarithm of the embedding matrix and all its derivatives as contained in the chain of all derivatives of the 'master potential'.
- the scalar 'master potential' as a *Taylor* series of vector form products.

What do we get out? What does mathematics tell us in response?

- The most reasonable space with maximum flexibility has three similar dimensions, together with a separate time dimension (this is exactly what we see as our physical reality, at least on a macroscopic scale), equipped with a quaternionic algebra (which we didn't see before),
- Flat space has a *Minkowski* metric (in physicality we already found that out, but here we did not explicitly put that condition in from the beginning),
- There is a *Lorentz* transform with boost and rotation as we know it from special relativity,
- Electromagnetic gauge theory is incorporated, together with many more local gauges, for instance between gravitational potential and velocity, or most fundamentally, position in time and space (which are directly gauged by the single topmost 'master potential'),
- Gravitation emerges, though not exactly in accordance with current GR,
- A whole electromagnetic field theory emerges from the tangent bundle of the spacetime manifold alone and needs not be fabricated into the theory as a separate vector bundle.

This points to a possible unification of gravitation with electromagnetism, but through a general relativistic approach instead of trying to quantize gravitation. *Einstein* and others sought for such a realistic unification but failed, possibly due to relying on viewing the metric tensor alone.

2.10 Possible Negative Predictions

- The classical gravitational *Schwarzschild* solution might not be a solution in this framework, not being holomorphic, so 'event horizons' or 'Black Holes' would be ruled out.
- From the above axioms follows, that the metric tensor must always be purely diagonal, so also the current understanding of 'gravitational waves' is not supported by this framework. There shall be wave phenomena, but which might solely be identified with electromagnetic waves or otherwise.

Both should not hurt, since there is still no tangible experimental evidence of these suspected phenomena anyway.[4]

3 Outlook on a Unified Theory of Physics

We found possible gravitation and electromagnetism as fundamental forces in this model, and not more. It is currently understood that in physical reality there are four fundamental forces. So where are the weak and strong nuclear forces?

It shall be suggested, that these forces might be not so fundamental, in a sense that they do not show up in this level of description of spacetime.

But the freedom of our spacetime model does not end here. It is still subject to topological modifications. Imagine space as being elastic. When distorted, it shall relax back to a flat metric, dissipating the energy of distortion in the form of a wave.

Then imagine cutting space and 'glueing it back together in a wrong way'. It will relax to a minimum energy, which corresponds to a 'rest mass'. When 'pushed', such a 'topological center' should move according to *Newton*'s first law, behaving like a particle.

Such topological centers might decay into other topologies of lower energy. It might be worth an investigation of which topologies are possible on this 4-dimensional model and to identify the most stable ones. They might include an electron and a proton, together with atomic nuclei as stable aggregates of nucleonic topologies.

The 'strong nuclear force' might then come out as a 'topological cohesion'. Instead of the concept of quarks and gluons there might simply be a three-fold spatial structure in the topology of nucleons.

Investigating the possible topologies on a 4-space might lead a way towards a realistic unification of general relativity and quantum mechanics, not by trying to quantize everything, but analogous to digging a tunnel from both ends simultaneously.

The quaternion group is already isomorphic to the SU(2) group and able to exhibit spin, while in the algebra of topologies even an SU(3) structure can be suspected.

Still open is particularly, if and where in such a continuous realistic model *Planck*'s constant of action arises or how the question of wave-particle dualism (or *de Broglie-Bohm* pilot waves) might be resolved. If these phenomena were also included in this model, then only further investigation could reveal them.

4 Conclusions on the Spacetime Model

Historically, space had been seen as an empty stage, on which force fields act on particles, and even *Einstein* stated, that by his hypotheses time and space would be rid of the last trace of any objective reality.[5]

Contrary to that it could be argued, that indeed everything else, that is, fields and particles, become devoid of any substance in the presented view of spacetime, and

[4] Nevertheless it might hurt experimentalist who get funded for searching for such evidence.

[5] '...die Einführung dieser Hypothese, durch welche Zeit und Raum der letzten Spur objektiver Realität beraubt werden...' [3]

instead this spacetime should rather be *something*, which constitutes a universe itself, not only on a macroscopic scale, but also governing elementary particles. So fields would be stretches and contortions of spacetime and particles are topological centers on a manifold.

This implies that spacetime's mathematical behaviour might emerge from yet another underlying universal structure, which might or might not be a dynamic cellular automaton in a way *S. Wolfram* suggested [4, 9].

But a knowledge of such a yet deeper level of description may at first be unnecessary for studying its behaviour, when we know in advance that this will effectively come out as a (3+1) spacetime.

5 Operational Conclusions

The bottom line shall summarize lessons learned from the above 'mathematical experiment' and provide suggestions on how to proceed in a further search for a unified theory of physics.

- Take mathematics seriously. Ask mathematics for the behaviour of physics.
- Start with a most *general* and *realistic* mathematical model.
- Constrain the model in a most efficient way
  by the smallest conceivable set of reasonable assumptions.
- Don't invent additions to the model, instead just explore what is already there.
- Adjust the mathematical viewpoint when needed, without loss of generality.
- Consider the spacetime model presented here as a starting point
  for proceeding towards a 'grand unified theory' of physics.

References

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'Don't judge a book by the shelf'

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