Acceleration of the Infinite Series for the Sine and Cosine Functions

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"then know this, you and all the people of Israel: It is by the name of Jesus Christ of Nazareth, whom you crucified but whom God raised from the dead, that this man stands before you healed. Salvation is found in no one else, for there is no other name under heaven given to mankind by which we must be saved." - Acts 4:10,12.

Abstract

I proved some accelerations of the infinite series for the sine and cosine functions.

1 Introduction

In this paper, I demonstrated the following expansion of the infinite series:

$$\sin(z) = \sum_{k=0}^{\infty} \frac{z^{4k+1}}{(4k+1)!}\left[1 - \frac{z^2}{4k+2} + \frac{z^2}{4k+3}\right]$$

and

$$\cos(z) = \sum_{k=0}^{\infty} \frac{z^{4k}}{(4k)!}\left[1 - \frac{z^2}{4k+1} + \frac{z^2}{4k+2}\right],$$

which converge rapidly. Choosing $z = 1$ for the formulas above, I have

$$\sin(1) = \sum_{k=0}^{\infty} \frac{1}{(4k+1)!}\left[1 - \frac{1}{4k+2} + \frac{1}{4k+3}\right]$$

and

$$\cos(1) = \sum_{k=0}^{\infty} \frac{1}{(4k)!}\left[1 - \frac{1}{4k+1} + \frac{1}{4k+2}\right].$$

2 Acceleration of the Infinite Series for the Sine and Cosine Functions

2.1 Sine Function

Theorem 1. *For* $z \in \mathbb{R}$, *then*

$$\sin(z) = \sum_{k=0}^{\infty} \frac{z^{4k+1}}{(4k+1)!}\left[1 - \frac{z^2}{4k+2} + \frac{z^2}{4k+3}\right],$$

where $\sin(z)$ *denotes the sine function and* $k!$ *denotes the factorial function.*

Proof. I know that

$$\sin(z) = \sum_{k=0}^{\infty} \frac{\cos\left(\frac{\pi k}{2}\right)}{(k+1)!} z^{k+1}. \tag{1}$$

On the other hand, I notice that

$$\cos\left(\frac{\pi k}{2}\right) = \begin{cases} 1, k = 0, 4, 8, 12, 16, 20, \ldots \\ 0, k = 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots \\ -1, k = 2, 6, 10, 14, 18, \ldots \end{cases} \tag{2}$$

From Eq. (1) and Eq. (2), it follows that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\cos\left(\frac{\pi k}{2}\right)}{(k+1)!} z^{k+1} &= \sum_{k=0}^{\infty} \frac{1}{(4k+1)!} z^{4k+1} - \sum_{k=0}^{\infty} \frac{1}{(4k+3)!} z^{4k+3} \\ &= \sum_{k=0}^{\infty} z^{4k+1} \left[\frac{1}{(4k+1)!} - \frac{z^2}{(4k+3)!} \right] \\ &= \sum_{k=0}^{\infty} \frac{z^{4k+1}}{(4k+1)!} \left[1 - \frac{z^2}{(4k+3)(4k+2)} \right] \\ &= \sum_{k=0}^{\infty} \frac{z^{4k+1}}{(4k+1)!} \left[1 - \frac{z^2}{4k+2} + \frac{z^2}{4k+3} \right]. \end{aligned}$$

□

Remark 2. The infinite series in question converge rapidly. For example, choosing $z = 1/2$ in Theorem 1, I have

$$\sin\left(\frac{1}{2}\right) = 0.479425538604203000273287935215 57\ldots$$

$$\sum_{k=0}^{1} \frac{1}{2^{4k+1}(4k+1)!} \left[1 - \frac{1}{4(4k+2)} + \frac{1}{4(4k+3)} \right] = 0.47942553\mathbf{32341269841269841269841 2698}\ldots$$

$$\sum_{k=0}^{2} \frac{1}{2^{4k+1}(4k+1)!} \left[1 - \frac{1}{4(4k+2)} + \frac{1}{4(4k+3)} \right] = 0.479425538604\mathbf{183420264149430816097 48}\ldots$$

$$\sum_{k=0}^{3} \frac{1}{2^{4k+1}(4k+1)!} \left[1 - \frac{1}{4(4k+2)} + \frac{1}{4(4k+3)} \right] = 0.4794255386042030002\mathbf{51853888903326734}\ldots$$

$$\sum_{k=0}^{4} \frac{1}{2^{4k+1}(4k+1)!} \left[1 - \frac{1}{4(4k+2)} + \frac{1}{4(4k+3)} \right] = 0.4794255386042030002732879\mathbf{258870750912}\ldots$$

2.2 Cosine Function

Theorem 3. *For* $z \in \mathbb{R}$, *then*

$$\cos(z) = \sum_{k=0}^{\infty} \frac{z^{4k}}{(4k)!}\left[1 - \frac{z^2}{4k+1} + \frac{z^2}{4k+2}\right],$$

where $\cos(z)$ *denotes the cosine function and* $k!$ *denotes the factorial function.*

Proof. I know that

$$\cos(z) = \sum_{k=0}^{\infty} \frac{\cos\left(\frac{\pi k}{2}\right)}{k!} z^k. \tag{3}$$

On the other hand, I note that

$$\cos\left(\frac{\pi k}{2}\right) = \begin{cases} 1, k = 0, 4, 8, 12, 16, 20, \ldots \\ 0, k = 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots \\ -1, k = 2, 6, 10, 14, 18, \ldots \end{cases} \tag{4}$$

From Eq. (3) and Eq. (4), it follows that

$$\cos(z) = \sum_{k=0}^{\infty} \frac{z^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{z^{4k+2}}{(4k+2)!} = \sum_{k=0}^{\infty} \frac{z^{4k}}{(4k)!}\left[1 - \frac{z^2}{(4k+2)(4k+1)}\right]$$

$$= \sum_{k=0}^{\infty} \frac{z^{4k}}{(4k)!}\left[1 - \frac{z^2}{4k+1} + \frac{z^2}{4k+2}\right].$$

□

Remark 4. The infinite series in question converge rapidly. For example, choosing $z = 1/2$ in Theorem 3, I obtain

$$\cos\left(\frac{1}{2}\right) = 0.877582561890372716116281582603829651991645 19\ldots$$

$$\sum_{k=0}^{1} \frac{1}{2^{4k}(4k)!}\left[1 - \frac{1}{4(4k+1)} + \frac{1}{4(4k+2)}\right] = 0.877582\mathbf{465277777777777777777777777}\ldots$$

$$\sum_{k=0}^{2} \frac{1}{2^{4k}(4k)!}\left[1 - \frac{1}{4(4k+1)} + \frac{1}{4(4k+2)}\right] = 0.8775825618\mathbf{898637290564373897707 2}\ldots$$

$$\sum_{k=0}^{3} \frac{1}{2^{4k}(4k)!}\left[1 - \frac{1}{4(4k+1)} + \frac{1}{4(4k+2)}\right] = 0.877582561890372716115\mathbf{387586651847564}\ldots$$

$$\sum_{k=0}^{4} \frac{1}{2^{4k}(4k)!}\left[1 - \frac{1}{4(4k+1)} + \frac{1}{4(4k+2)}\right] = 0.877582561890372716116281\mathbf{1908254146}\ldots$$