Advanced Difference Equation Theory

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Introduction to Finite Differences:

Definitions and initial problem statement:

Consider the operator

$$Q_w[f(x)] = \lim_{h \to w} \frac{f(x+h) - f(x)}{h} : \lim_{h \to w} \frac{f(u+2h) - f(u)}{2h} : x - h = u, w$$

$$Q_w[f(x)]$$

Naturally:

$$Q_0[f(x)] = \frac{df}{dx},$$

This operator, which will be referred to as a difference quotient, can be utilized to construct all finite-step recurrence equations by considering a proper value w as well proper iterations of the operator.

We are concerned with solving the equation

$$a_0(x) + a_1(x)f + a_2(x)Q_w[f] + a_3(x)Q_w^2[f] \ldots a_n(x)Q_w^{n-1}[f] = 0$$

Where: $Q_w^n[f]$ denotes the 'nth' iteration of our Q operator, with general complex number functions $a_i(x)$ and complex numbers w.

The Product Rule for Finite Differences:

To do so we begin by noting given two functions $a(x), b(x)$ the expressions

$$Q_w[a(x)b(x)] = \lim_{h \to w} \left[\frac{a(x+h)b(x+h) - a(x)b(x)}{h} \right] =$$

$$w * Q_w[a(x)] * Q_w[b(x)] + Q_w[a(x)]b(x) + a(x)Q_w[b(x)]$$

This is the generalized product rule for finite differences. It breaks down to the familiar product rule in calculus when w = 0 but is also well defined for other values of w.

Chain Rule for Finite Differences:

We note the following chain rule

$$Q_w\left[f\big(g(x)\big)\right] = Q_w[g(x)]\, Q_{wQ_w[g(x)]}[f(g)]$$

Which can be expanded out algebraically to yield:

$$Q_w[g(x)]\frac{\left(f(g+w\,Q_w[g(x)])-f\big(g(x)\big)\right)}{w\,Q_w[g(x)]}=\frac{\left(f\left(g+w\frac{\big(g(x+w)-g(x)\big)}{w}\right)-f\big(g(x)\big)\right)}{w}=$$

$$=\frac{f\big(g(x+w)\big)-f\big(g(x)\big)}{w}$$

The Exponential Rule for Finite Differences:

Furthermore consider the equation:

$$Q_w[f]=f$$

The solution to this equation is not exactly trivial until one realizes that the function in equation must be either periodic of exponential in some sense. Naturally therefore if we assume:

$$f(x)=C_1a^x$$

The equation becomes:

$$\lim_{h\to w}\frac{C_1a^{x+h}-C_1a^x}{h}=C_1a^x$$

Dividing both sides by the constant in front and noting that the limit can be extended we have:

$$\lim_{h\to w}\frac{a^{x+h}-a^x-ha^x}{h}=0$$

Noting that this can all be divided by a common factor we have:

$$\lim_{h\to w}\frac{a^h-1-h}{h}=0$$

Now solving for a yields:

$$a=\lim_{h\to w}(1+h)^{\frac{1}{h}}$$

And therefore

$$f=C_i(1+w)^{\frac{x}{w}}$$

For complex constants C_i over all possible numbers $(1+w)^{\frac{1}{w}}$ are solutions to the equation

$$Q_w[f]=f$$

Naturally in the limiting case of 0

$$\lim_{h\to 0}(1+h)^{\frac{1}{h}} = e$$

Thereby implying that $\frac{d}{dx}[e^x] = e^x \rightarrow Q_0[e^x] = e^x$

Consider now the expression

$$Q_w\left[(1+w)^{\frac{g(x)}{w}}\right] = \frac{(1+w)^{\frac{g(x+w)}{w}} - (1+w)^{\frac{g(x)}{w}}}{w} =$$

$$(1+w)^{\frac{g(x)}{w}}\left(\frac{\frac{(1+w)^{\frac{g(x+w)}{w}}}{(1+w)^{\frac{g(x)}{w}}} - 1}{w}\right) = (1+w)^{\frac{g(x)}{w}}\left(\frac{(1+w)^{\frac{g(x+w)-g(x)}{w}} - 1}{w}\right)$$

Naturally we can observe $\frac{g(x+w)-g(x)}{w} = Q_w[g(x)]$ and thereby find:

$$(1+w)^{\frac{g(x)}{w}}\left(\frac{(1+w)^{Q_w[g(x)]} - 1}{w}\right)$$

Again:

$$\lim_{w\to 0}\left[\frac{(1+w)^{Q_w[g(x)]} - 1}{w}\right] = g'(x) \rightarrow Q_0\left[e^{g(x)}\right] = g'(x)e^{g(x)}$$

Linearity and Inverse Finite Differences

Furthermore it is trivial to show that

$$Q_w[a(x) + b(x)] = Q_w[a(x)] + Q_w[b(x)]$$

And:

$$Q_w[c * a(x)] = c * Q_w[a(x)]\ \forall\ c \in \mathbb{C}$$

Thus we can establish that we are indeed working with a linear operator. We will define the inverse function accordingly as $Q_w^{-1}[f]$ such that $Q_w\left[Q_w^{-1}[f]\right] = f$. Naturally

$$Q_0[f(x)] = \int f(x)\,dx$$

We will stick with indefinite inverses for now as to make them definite is not necessary for our purposes.

The Chain Rule for Finite Differences

We introduce two chain Rules for finite Differences that are of use to us. We modify our notation to

$$Q_k[h,x] \to \frac{h(x+k) - h(x)}{k}$$

$$(that\ is\ x\ is\ now\ in\ the\ Q\ operator\ and\ we\ can\ change\ the\ argument)$$

$$Q_k\big[h\big(g(x)\big),x\big] = \frac{h\big(g(x+k)\big) - h\big(g(x)\big)}{k} = \frac{h(g(x) + kQ_k[g(x)]) - h\big(g(x)\big)}{k}$$

Now from this tool it becomes possible to solve a variety of tricky difference equations as well as create a tool analogous to U-substitution. An example:

$$2y\, Q_k[y,x] + Q_k[y,x]^2 = f(x) \to$$

$$(y + Q_k[y,x])^2 - y^2 = f(x) \to$$

$$y^2 = Q_k^{-1}[f(x),x] + C \to$$

$$y = \sqrt{Q_k^{-1}[f(x),x] + C}$$

For a u-subsitution example consider the following

$$Q_1^{-1}(x^{2n},x){:}\, u = x^n \to \frac{Q_1[u]}{Q_1[x]} = \sum_{i=1}^{n} \binom{n}{i} x^{n-i}$$

$$Q_1^{-1}(f(x),x) \to F(x) : Q_k[F(x)] = \frac{F\big(x + Q_k(x)\big) - F(x)}{k} = Q_k[x]\ \frac{Q_{kQ_k[x]}[F(x)]}{Q_{kQ_k[x]}[x]}$$

Solving the first order General Finite Difference Equation:

Reduction of form:

From here the next natural step is to solve the equation:

$$a_0(x) + a_1(x)f + a_2(x)Q_w[f] = 0$$

In the general form. We will use a process reminiscent of Duhamel's formula for those familiar with the technique from Differential Equations. We begin by dividing the entire expression by $a_2(x)$ and renaming the constant and linear terms: $\frac{a_0(x)}{a_2(x)} = b_0(x)$ & $\frac{a_1(x)}{a_2(x)} = b_1(x)$

$$b_0(x) + b_1(x)f + Q_w[f] = 0$$

Now subtract the constant functional term to find:

$$b_1(x)f + Q_w[f] = -b_0(x)$$

Finding Inversion Factors:

At this point we note that a natural strategy is to find a pair of functions $z(x), \lambda(x)$ such that

$$Q_w[z(x)f] = \lambda(x)\,(\,b_1(x)f + Q_w[f])$$

This is a generalization of the concept of a integration factor from differential equations. This implies from our product rule established earlier that:

$$w * Q_w[f]Q_w[z(x)] + Q_w[f]z(x) + fQ_w[z(x)] = \lambda(x)b_1(x)f + \lambda(x)Q_w[f]$$

We can disassemble this into a system of equations:

$$\begin{Bmatrix} fQ_w[z(x)] = \lambda(x)b_1(x)f \\ \big(w * Q_w[z(x)] + z(x)\big)Q_w[f] = \lambda(x)Q_w[f] \end{Bmatrix}$$

We divide out the 'f' terms that both equations share in common on both sides to find:

$$\begin{Bmatrix} Q_w[z(x)] = \lambda(x)b_1(x) \\ \big(w * Q_w[z(x)] + z(x)\big) = \lambda(x) \end{Bmatrix}$$

We can now solve for $\lambda(x)$ by noting based on the top equation and then considering the bottom:

$$w * \lambda(x)b_1(x) + z(x) = \lambda(x) \to \lambda(x)(w * b_1(x) - 1) + z(x) = 0 \to \lambda(x) = \frac{z(x)}{1 - w * b_1(x)}$$

Now of course we can take a single equation of our choice and substitute this value of $\lambda(x)$ to yield a difference equation alone in $z(x)$. I say single since both equations when given this substitution will yield the same equation (After being simplified).

$$Q_w[z(x)] + z(x) = \frac{z(x)}{1 - w * b_1(x)}$$

We now rearrange terms:

$$Q_w[z(x)] = \frac{z(x)}{1 - w * b_1(x)} - z(x) = \frac{z(x)}{1 - w * b_1(x)} - \frac{1 - b_1(x)}{1 - w * b_1(x)}z(x) \to$$

$$Q_w[z(x)] = \frac{wb_1(x)}{1 - wb_1(x)}z(x)$$

The natural choice of function to fit this would be some exponential form. Recall:

$$Q_w\left[(1+w)^{\frac{g(x)}{w}}\right] = (1+w)^{\frac{g(x)}{w}}\left(\frac{(1+w)^{Q_w[g]} - 1}{w}\right)$$

Thus we are tasked with solving the equation:

$$\left(\frac{(1+w)^{Q_w[g]}-1}{w}\right)=\frac{wb_1(x)}{1-wb_1(x)}$$

Solving this:

$$(1+w)^{Q_w[g]}-1=\frac{w^2b_1(x)}{1-wb_1(x)}\rightarrow$$

$$(1+w)^{Q_w[g]}=\frac{1-wb_1(x)+w^2b_1(x)}{1-wb_1(x)}$$

$$Q_w[g]=\log_{1+w}\left[\frac{1-wb_1(x)+w^2b_1(x)}{1-wb_1(x)}\right]\rightarrow$$

$$g=Q_w^{-1}\left(\log_{1+w}\left[\frac{1-wb_1(x)+w^2b_1(x)}{1-wb_1(x)}\right]\right)$$

Therefore:

$$z(x)=(1+w)^{\frac{1}{w}Q_w^{-1}\left(\log_{1+w}\left[\frac{1-wb_1(x)+w^2b_1(x)}{1-wb_1(x)}\right]\right)}$$

And furthermore:

$$\lambda(x)=\frac{(1+w)^{\frac{1}{w}Q_w^{-1}\left(\log_{1+w}\left[\frac{1-wb_1(x)+w^2b_1(x)}{1-wb_1(x)}\right]\right)}}{1-w*b_1(x)}$$

Solving the Original Equation:

Now consider the original equation

$$Q_w[f]+b_1(x)f=-b_0(x)$$

We multiply both sides by $\lambda(x)$ and then perform and inversion of Q_w to yield:

$$(1+w)^{\frac{1}{w}Q_w^{-1}\left(\log_{1+w}\left[\frac{1-wb_1(x)+w^2b_1(x)}{1-wb_1(x)}\right]\right)}*f(x)$$
$$=Q_w^{-1}\left(-b_0(x)\frac{(1+w)^{\frac{1}{w}Q_w^{-1}\left(\log_{1+w}\left[\frac{1-wb_1(x)+w^2b_1(x)}{1-wb_1(x)}\right]\right)}}{1-w*b_1(x)}\right)$$

We divide out negative signs and then isolate $f(x)$ from the left hand side to find:

$$f(x)=(1+w)^{-\frac{1}{w}Q_w^{-1}\left(\log_{1+w}\left[\frac{1-wb_1(x)+w^2b_1(x)}{1-wb_1(x)}\right]\right)}Q_w^{-1}\left(b_0(x)\frac{(1+w)^{\frac{1}{w}Q_w^{-1}\left(\log_{1+w}\left[\frac{1-wb_1(x)+w^2b_1(x)}{1-wb_1(x)}\right]\right)}}{w*b_1(x)-1}\right)$$

Or in different formatting:

$$f(x) = \lim_{h \to w} \frac{Q_h^{-1}\left(b_0(x)\frac{(1+h)^{\frac{1}{h}Q_h^{-1}\left(\log_{1+h}\left[\frac{1-hb_1(x)+h^2b_1(x)}{1-hb_1(x)}\right]\right)}}{h * b_1(x) - 1}\right)}{(1+h)^{\frac{1}{h}Q_h^{-1}\left(\log_{1+h}\left[\frac{1-hb_1(x)+h^2b_1(x)}{1-hb_1(x)}\right]\right)}}$$

The limit makes this expression well defined for the case of $w = 0$. Again we now substitute the appropriate values for b_0 and b_1 to find:

Final Solution of First Order:

$$f(x) = \lim_{h \to w} \frac{Q_h^{-1}\left(\frac{a_0(x)}{a_2(x)}\frac{(1+h)^{\frac{1}{h}Q_h^{-1}\left(\log_{1+h}\left[\frac{1-h\frac{a_1(x)}{a_2(x)}+h^2\frac{a_1(x)}{a_2(x)}}{1-h\frac{a_1(x)}{a_2(x)}}\right]\right)}}{h * \frac{a_1(x)}{a_2(x)} - 1}\right)}{(1+h)^{\frac{1}{h}Q_h^{-1}\left(\log_{1+h}\left[\frac{1-h\frac{a_1(x)}{a_2(x)}+h^2\frac{a_1(x)}{a_2(x)}}{1-h\frac{a_1(x)}{a_2(x)}}\right]\right)}}$$

The General Nth Order Equation:

Matrix Reduction:

Now stepping this forward to arbitrarily high order again consider the original problem:

$$a_0(x) + a_1(x)f(x) + a_2(x)Q_w[f(x)] + a_3(x)Q_w^2[f(x)] \dots a_n(x)Q_w^{n-1}[f(x)] = 0$$

Begin by dividing the entire equation by $a_n(x)$ and subtracting all but the $Q_w^{n-1}[f]$ term to the other side:

$$Q_w^{n-1}[f(x)] = -\frac{a_0(x)}{a_n(x)} - \frac{a_1(x)}{a_n(x)}f(x) - \frac{a_2(x)}{a_n(x)}Q_w[f(x)] \dots - \frac{a_{n-1}(x)}{a_n(x)}Q_w^{n-2}[f(x)]$$

Consider now the vectors:

$$L_1(x) = \begin{bmatrix} f(x) \\ Q_w[f(x)] \\ \vdots \\ Q_w^{n-2}[f(x)] \end{bmatrix}, L_2(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -a_0(x) \end{bmatrix}$$

And the matrix

$$B(x) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{a_1(x)}{a_n(x)} & -\frac{a_2(x)}{a_n(x)} & -\frac{a_3(x)}{a_n(x)} & \dots & -\frac{a_{n-1}(x)}{a_n(x)} \end{bmatrix}$$

It then becomes clear our system can be encoded by

$$Q_w[L_1(x)] = B(x)L_1(x) + L_2(x)$$

The solution is then obtained by subtracting over to find:

$$Q_w[L_1(x)] - B(x)L_1(x) = L_2(x)$$

Non-Commutative Integration Factors:

And now searching for an appropriate pair of integration factors. This time we need to keep the non commutative nature of matrices in check (our answer will be of the form $z(x)f(x)$ we assume) and therefore note

$$\lambda(x) = Q_w[z(x)] + z(x)$$

$$-\lambda(x)B(x) = Q_w[z(x)]$$

$$\lambda(x) + \lambda(x)B(x) = z(x)$$

$$\lambda(x)\big(I_{n-1} + B(x)\big) = z(x) \rightarrow \lambda(x) = z(x)\big(I_{n-1} + B(x)\big)^{-1}$$

$$z(x)\big(I_{n-1} + B(x)\big)^{-1} = Q_w[z(x)] + z(x) \rightarrow$$

$$z(x)\left(\big(I_{n-1} + B(x)\big)^{-1} - I_{n-1}\right) = Q_w[z(x)]$$

The solution to this functional equation will require ordered exponentials. We begin by noting that that there exist a pair of sets of functions (the number of functions per set depending on the value of w)

$$(1+w)^{\frac{g(x)}{w}}$$

Such that

$$(1+w)^{\frac{g(x+1)}{w}} - (1+w)^{\frac{g(x)}{w}}$$

$$Q_w\left[(1+w)^{\frac{g(x)}{w}}\right] = \frac{(1+w)^{\frac{g(x+1)}{w}} - (1+w)^{\frac{g(x)}{w}}}{w} = \left(\frac{(1+w)^{Q_w[g(x)]} - I_{n-1}}{w}\right)(1+w)^{\frac{g(x)}{w}}$$

OR in a different scheme:

$$Q_w\left[(1+w)^{\frac{g(x)}{w}}\right] = (1+w)^{\frac{g(x)}{w}}\left(\frac{(1+w)^{Q_w[g(x)]} - I_{n-1}}{w}\right)$$

These are the left and right ordered exponentials respectively. Clearly:

$$z(x)\left(\left(I_{n-1} + B(x)\right)^{-1} - I_{n-1}\right) = (1+w)^{\frac{g(x)}{w}}\left(\frac{(1+w)^{Q_w[g(x)]} - I_{n-1}}{w}\right) \rightarrow$$

$$\left(\left(I_{n-1} + B(x)\right)^{-1} - I_{n-1}\right) = \left(\frac{(1+w)^{Q_w[g(x)]} - I_{n-1}}{w}\right) \rightarrow$$

$$w\left(\left(I_{n-1} + B(x)\right)^{-1} - I_{n-1}\right) + I_{n-1} = (1+w)^{Q_w[g(x)]} \rightarrow$$

$$Q_w^{-1}\left[\log_{1+w}\left(w\left(\left(I_{n-1} + B(x)\right)^{-1} - I_{n-1}\right) + I_{n-1}\right)\right] = g(x)$$

Thus

$$z(x) = (1+w)^{\frac{1}{w}Q_w^{-1}\left[\log_{1+w}\left(w\left(\left(I_{n-1}+B(x)\right)^{-1} - I_{n-1}\right) + I_{n-1}\right)\right]}$$

$$\lambda(x) = (1+w)^{\frac{1}{w}Q_w^{-1}\left[\log_{1+w}\left(w\left(\left(I_{n-1}+B(x)\right)^{-1} - I_{n-1}\right) + I_{n-1}\right)\right]}\left(I_{n-1} + B(x)\right)^{-1}$$

And therefore the general solution to the equation is

Final Solution:

$$L_1(x) = -(1+w)^{-\frac{1}{w}Q_w^{-1}\left[\log_{1+w}\left(w\left(\left(I_{n-1}+B(x)\right)^{-1} - I_{n-1}\right) + I_{n-1}\right)\right]} Q_w^{-1}\left[(1+w)^{\frac{1}{w}Q_w^{-1}\left[\log_{1+w}\left(w\left(\left(I_{n-1}+B(x)\right)^{-1} - I_{n-1}\right) + I_{n-1}\right)\right]}\left(I_{n-1} + B(x)\right)^{-1} L_2\right]$$

Notice here we are only interested in the top element of the vector $L_1(x)$ for our solution.

Connections between the different operators:

The natural question of expressing the different Q operators among themselves can be made. One may grow curious: Is there a way to write certain functions of one of the operators among the other? Is there a well defined Algebra of Q operators?

Motivation by the Gamma Function:

Consider the function:

$$A(x) = (x-1)! = \Gamma(x)$$

It's clear that the given function obeys the functional equation

$$A(x+1) - A(x) = x! - (x-1)! = (x-1)(x-1)! = (x-1)A(x)$$

Which can be refactored into:

$$Q_1[A(x)] - (x-1)A(x) = 0$$

Naturally we have a solution we can generate with the traditional method. We define integration factors λ_1, λ_2 such that

$$\lambda_2\, Q_1[A(x)] - \lambda_2(x-1)A(x) = Q_1(\lambda_1 A) = Q_1[\lambda_1]A + (Q_1[\lambda_1] + \lambda_1)\, Q_1[A]$$

$$\begin{Bmatrix} (1-x)\lambda_2 = Q_1[\lambda_1] \\ \lambda_2 = Q_1[\lambda_1] + \lambda_1 \end{Bmatrix}$$

Let: $\lambda_1 = 2^{g(x)}$ then it follows

$$\begin{Bmatrix} (1-x)\lambda_2 = 2^{g(x)}\left(2^{Q[g(x)]} - 1\right) \\ \lambda_2 = 2^{g(x)}2^{Q[g(x)]} \end{Bmatrix}$$

Thus:

$$(1-x)2^{g(x)}2^{Q[g(x)]} = 2^{g(x)}\left(2^{Q[g(x)]} - 1\right)$$

Yielding:

$$g(x) = -Q_1^{-1}[\ln(x)]$$

Then the solution is:

$$\frac{A}{2^{Q_1^{-1}[\ln(x)]}} = C \to A = C2^{Q_1^{-1}[\ln(x)]}$$

But the expression:

$$2^{Q_1^{-1}[\ln(x)]}$$

Is still unknown to us! It does turn out that the gamma function has an alternative representation:

$$\Gamma(x) = \int_0^{\infty} e^{-y}y^{1-x}\, dy \to$$

$$Q_{0,y}^{-1}[e^{-y}y^{1-x}]|_0^{\infty} = 2^{Q_{1,y}^{-1}[\ln(y)]|_0^x}$$

Which is an identity that reveals deep connections between the exponential and logarithmic function due to the Gamma function problem. Perhaps more complex identities could exist that allow us to transform different Q identities into each other. A sort of Functional Algebra.

Additional Tools and Techniques:

Naturally finding exact closed forms for most of these solutions will not be possible so a method of approximation and conversion of forms would indeed be useful.

We begin by noting that in general there is a binomial theorem style formula

The Binomial Iteration Law:

$$Q_w^n[f(x)] = \sum_{i=0}^{\infty}\left[\left(-\frac{1}{w}\right)^i \binom{n}{i} f(x+iw)\right] = \sum_{i=0}^{\infty}\left[\left(-\frac{1}{w}\right)^i \frac{\Gamma(n+1)}{\Gamma(n-i+1)\,\Gamma(i+1)} f(x+iw)\right]$$

Which allows us to convert one type of finite difference into a multitude of other types. Furthermore in general

$$Q_w\left[\frac{1}{(n+1)!}(x)(x-w)\dots(x-nw)\right] =$$

$$\frac{1}{w}\left(\frac{1}{(n+1)!}(x+w)(x)\dots(x-(n-1)w) - \frac{1}{(n+1)!}(x)(x-w)\dots(x-nw)\right)$$

$$= \frac{1}{w(n+1)!}(x)\dots(x-(n-1)w)\,(x+w-x+nw) = \frac{1}{w(n+1)!}(x)\dots(x-(n-1)w)(n+1)w$$

$$= \frac{1}{n!}(x)\dots(x-(n-1)w)$$

This allows us to generalize tools such as Taylor series to the form of

Generalized Taylor Series:

$$f(x) \approx \sum_{i=0}^{\infty}\left[\frac{Q_w^i[f(a)]}{i!}\prod_{j=0}^{i}[x-a-\,jw\,]\right]$$

Yielding infinitely many new unique ways of describing functions in series form.

For example:

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i \ (found\ using\ w=0, a=0\ in\ taylor\ series)$$