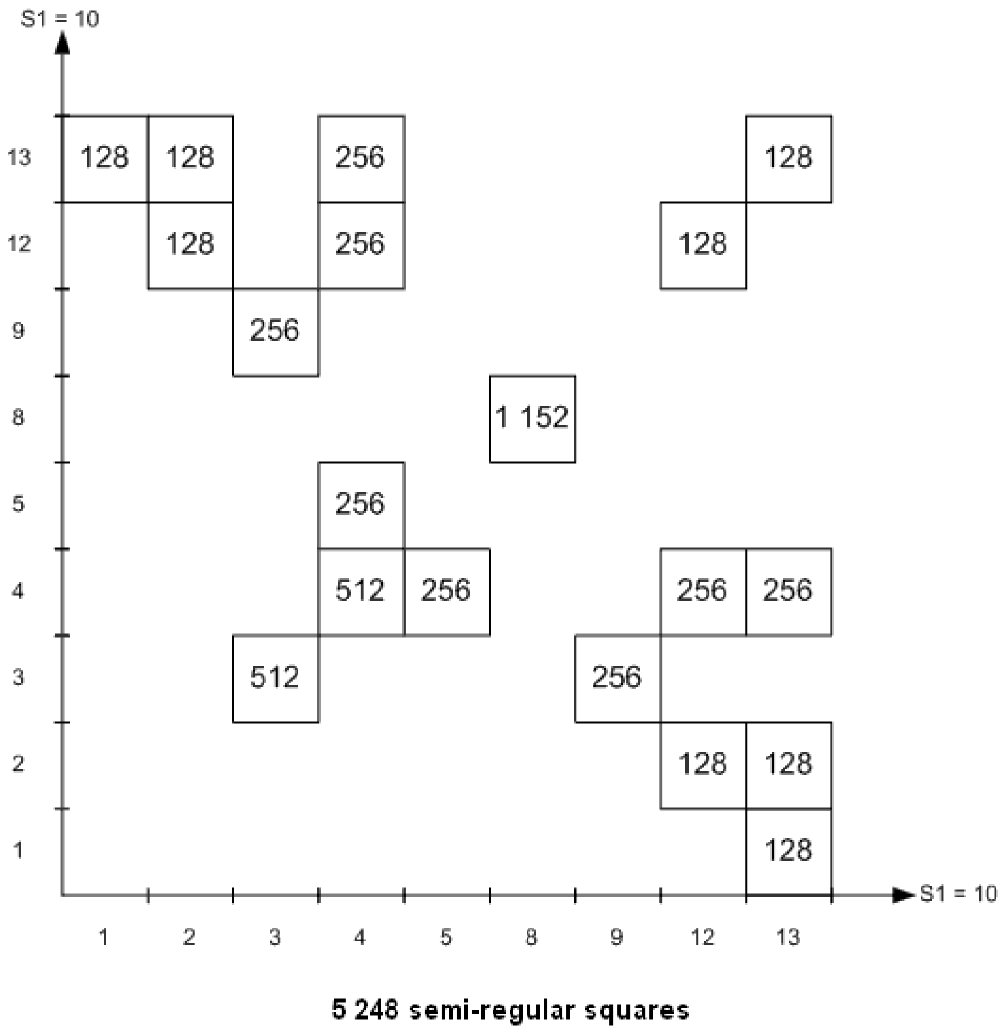
Exceptional Lie Algebra Magic Square Series

By John Frederick Sweeney



Abstract

The Exceptional Lie Algebra contains a series of Magic Squares, shown above. This paper presents this series, especially the series E6 – E7 and E8. In addition a Magic Square Series related to the Octonions, Fano Plane, the Klein Quartic and PS / 2 has been found.

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Introduction

In the minds of most, Magic Squares constitute little more than a trivial parlour game, and in no way constitute serious mathematics. Magic Squares have long been considered curiosities, the subject of mathematical puzzles for middle school students, to keep them interested in math yet not overwhelming them with consequential mathematical topics such as trigonometry or calculus. Few have understood that Magic Squares play a key role in particle physics, and for the most part, Magic Squares have been neglected by serious academics.

The Luo Shu Magic Square, for example, constitutes a mathematical curiousity that merely helps to add the word inscrutable to the list of adjectives used to describe the Chinese: why would an entire culture spend millenia obsessed with a simple parlour game? The question itself appears too trivial to warrant serious academic attention. This lack of curiousity about such matters had led western math and physics to overlook a key function in the production of the wide variety of nuclear particles.

Instead, western mathematical physics has taken certain constructs and declared them "magical" when in fact they are not magical in any sense. "Magic Numbers" and the Freudenthal – Tits Magic Square provide two cases in point – neither of them truly magical in the sense of magic squares. They simply appear magical in certain aspects, in the sense of Arthur C. Clarke, who wrote that we describe things as magical when we fail to understand them.

The Freudenthal – Tits Magic Square is in fact a Pascal Triangle on its side, or in reverse, and was given that monniker when mathematicians failed to understand the relationship between RCHOST and Exceptional Lie Algebras, which remains relatively misunderstood today, especially on the latter end of RCHOST – the Octonions, Sedenions and Trigintaduonions.

The magic square series described within was found on the internet by this author, but not described in that way. In fact, the composer of the website still does not believe that this series is what the author of this paper believes.

In an earlier paper on Vixra, the author of this paper showed how Magic Squares play a key role in the formation of matter. Chief among those Magic Squares, and at the heart of Qi Men Dun Jia, is the 3 x 3 Luo Shu Magic Square of Chinese metaphysics.

Beyond this example exist many more, which are presented in this paper.

John Baez on Exceptional Lie Algebras

On October 18th, 1887, Wilhelm Killing wrote a letter to Friedrich Engel saying that he had classified the simple Lie algebras. In the next three years, this revolutionary work was published in a series of papers [59]. Besides what we now call the `classical' simple Lie algebras, he claimed to have found 6 `exceptional' ones -- new mathematical objects whose existence had never before been suspected. In fact he only gave a rigorous construction of the smallest of these. In his 1894 thesis, Cartan [11] constructed all of them and noticed that the two 52-dimensional exceptional Lie algebras discovered by Killing were isomorphic, so that there are really only 5.

The Killing-Cartan classification of simple Lie algebras introduced much of the technology that is now covered in any introductory course on the subject, such as roots and weights. We shall avoid this technology, since we wish instead to see the exceptional Lie algebras as Octonionic cousins of the classical ones -- slightly eccentric cousins, but still having a close connection to *geometry*, in particular the Riemannian geometry of projective planes. For this reason we shall focus on the compact real forms of the simple Lie algebras.

The classical simple Lie algebras can be organized in three infinite families:

$$\begin{aligned} \mathfrak{so}(n) &= \{x \in \mathbb{R}[n]\colon\ x^* = -x,\ \mathrm{tr}(x) = 0\}, \\ \mathfrak{su}(n) &= \{x \in \mathbb{C}[n]\colon\ x^* = -x,\ \mathrm{tr}(x) = 0\}, \\ \mathfrak{sp}(n) &= \{x \in \mathbb{H}[n]\colon\ x^* = -x\}. \end{aligned}$$

The corresponding Lie groups are

$$\begin{aligned} \mathrm{SO}(n) &= \{x \in \mathbb{R}[n]\colon\ xx^* = 1,\ \det(x) = 1\}, \\ \mathrm{SU}(n) &= \{x \in \mathbb{C}[n]\colon\ xx^* = 1,\ \det(x) = 1\}, \\ \mathrm{Sp}(n) &= \{x \in \mathbb{H}[n]\colon\ xx^* = 1\}. \end{aligned}$$

These arise naturally as symmetry groups of projective spaces over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, respectively. More precisely, they arise as groups of **isometries**: transformations that preserve a specified Riemannian metric. Let us sketch how this works, as a warmup for the exceptional groups.

First consider the projective space $\mathbb{RP}^n$. We can think of this as the unit sphere in $\mathbb{R}^{n+1}$ with antipodal points x and $-x$ identified. It thus inherits a Riemannian metric from the sphere, and the obvious action of the rotation group $\mathrm{O}(n+1)$ as isometries of the sphere yields an action of this group as isometries of $\mathbb{RP}^n$ with this metric. In fact, with this metric, the group of all isometries of $\mathbb{RP}^n$ is just

$$\mathrm{Isom}(\mathbb{RP}^n) \cong \mathrm{O}(n+1)/\mathrm{O}(1)$$

where $\mathrm{O}(1) = \{\pm 1\}$ is the subgroup of $\mathrm{O}(n+1)$ that acts trivially on $\mathbb{RP}^n$. The Lie algebra of this isometry group is

$$\mathfrak{isom}(\mathbb{RP}^n) \cong \mathfrak{so}(n+1).$$

The case of $\mathbb{CP}^n$ is very similar. We can think of this as the unit sphere in $\mathbb{C}^{n+1}$ with points x and αx identified whenever α is a unit complex number. It thus inherits a Riemannian metric from this sphere, and the unitary group $\mathrm{U}(n+1)$ acts as isometries. If we consider only the connected component of the isometry group and ignore the orientation-reversing isometries coming from complex conjugation, we have

$$\mathrm{Isom}_0(\mathbb{CP}^n) \cong \mathrm{U}(n+1)/\mathrm{U}(1)$$

where $\mathrm{U}(1)$ is the subgroup that acts trivially on $\mathbb{CP}^n$. The Lie algebra of this isometry group is

$$\mathfrak{isom}(\mathbb{CP}^n) \cong \mathfrak{su}(n+1).$$

The case of $\mathbb{HP}^n$ is subtler, since we must take the non - commutativity of the quaternions into account. We can think of $\mathbb{HP}^n$ as the unit sphere in $\mathbb{H}^{n+1}$ with points x and αx identified whenever α is a unit quaternion, and as before, $\mathbb{HP}^n$ inherits a Riemannian metric. The group $\mathrm{Sp}(n+1)$ acts as isometries of $\mathbb{HP}^n$, but this action comes from *right* multiplication, so

$$\mathrm{Isom}(\mathbb{HP}^n) \cong \mathrm{Sp}(n+1)/\{\pm 1\},$$

since not $\mathrm{Sp}(1)$ but only its center $\{\pm 1\}$ acts trivially on $\mathbb{HP}^n$ by right multiplication. At the Lie algebra level, this gives

$$\mathfrak{isom}(\mathbb{HP}^n) \cong \mathfrak{sp}(n+1).$$

For lovers of the Octonions, it is tempting to try a similar construction starting with $\mathbb{OP}^2$. While non - associativity makes things a bit tricky, we show in Section 4.2 that it can in fact be done. It turns out that $\mathrm{Isom}(\mathbb{OP}^2)$ is one of the exceptional Lie groups, namely F_4. Similarly, the exceptional Lie groups E_6, E_7 and E_8 are in a certain subtle sense the isometry groups of projective planes over the algebras $\mathbb{C} \otimes \mathbb{O}$, $\mathbb{H} \otimes \mathbb{O}$ and $\mathbb{O} \otimes \mathbb{O}$. Together with F_4, these groups can all be defined by the so-called `magic square' construction, which makes use of much of the algebra we have described so far. We explain three versions of this construction in Section 4.3. We then treat the groups $\mathrm{E}_6, \mathrm{E}_7$ and E_8 individually in the following sections. But first, we must introduce G_2: the smallest of the exceptional Lie groups, and none other than the automorphism group of the Octonions.

Complexification Method of Exceptional Lie Algebras

By Frank Tony Smith

At www.**tony5m17h.net**/ichgene6.html

Frank Tony Smith writes:

Chinese cosmology begins with the undivided **Tai** Chi,
then separating into Yin-Yang, ... :
Let o represent the undivided **Tai** Chi, a scalar point
of origin:

```
          |     |
          |     |
     _____|_____|_____
          |     |
          |  o  |
     _____|_____|_____
          |     |
          |     |
          |     |
```

Then add 4 vector directions of Physical Spacetime:
   1, i, j, k of the quaternions
to get the 5 Elements:

```
          |     |
          |  i  |
     _____|_____|_____
          |     |
       j  |  o  |  1
     _____|_____|_____
          |     |
          |  k  |
          |     |
```

Then add 4 vector directions of Internal Symmetry Space:
E, I, J, K of the octonions,
which are the basis for the D4-D5-E6-E7 physics model,
to get 9 directions:

```
     |     |
  J  |  i  |  I
_____|_____|_____
     |     |
  j  |  o  |  1
_____|_____|_____
     |     |
  K  |  k  |  E
     |     |
```

The 10th direction is Yin-Yang reflection
of the 8 vector directions 1, i, j, k, E, I, J, K.

Now, identify the 3x3 square with the Magic Square

```
     |     |
  4  |  9  |  2
_____|_____|_____
     |     |
  3  |  5  |  7
_____|_____|_____
     |     |
  8  |  1  |  6
     |     |
```

whose central number, 5, is
central in the sequence 1,2,3,4, 5, 6,7,8,9
which sequence corresponds
to the octonions 1,i,j,k, 0, E,I,J,K

whose total number for each line is 15,
the dimension of the largest Hopf fibration
and the dimension of the imaginary sedenions.

If you take into account the direction in which you add each
of the 8 ways, and add all directed ways together
you get a total of 16x15 = 240
which is the number of vertices of a Witting polytope.

The total of all 9 numbers of the Magic Square is 45,
the dimension of the D5 Lie algebra Spin(10)
that is used in the D4-D5-E6-E7 physics model
in which
the D4 Spin(8) subgroup of Spin(10) corresponds
to 28 bivector gauge bosons
and the 16-dimensional homogeneous space
Spin(10) / Spin(8)xU(1)
corresponds to an 8-dimensional complex domain
whose Shilov boundary is RP1 x S7
corresponding to an 8-dimensional spacetime.

The ternary number arrangement is similar to
the Fu Xi binary number arrangement of the I **Ching**.

The 81 tetragrams correspond to the 81 verses
of the **Tao Te Ching and the Celestial Pivot of the Yellow Emperor's Internal Canon.**

The **Tai** X**uan** J**ing** may be at least as
old as the King Wen arrangement of the I **Ching**,
since such tetragrams have been
found on Shang and Zhou dynasty oracle bones.

To construct the **Tai** X**uan** J**ing**,
start with the 3x3 I **Ching** Magic Square

4	9	2
3	5	7
8	1	6

whose central number, 5, is also
central in the sequence 1,2,3,4, 5, 6,7,8,9
which sequence corresponds
to the octonions 1,i,j,k, 0, E,I,J,K

whose total number for each line is 15,
the dimension of the largest Hopf fibration
and the dimension of the imaginary sedenions.

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whose Shilov boundary is RP1 x S7
corresponding to an 8-dimensional spacetime.

Notice that the 3x3 Magic Square gives
the gauge bosons and the spacetime
of the D4-D5-E6-E7 physics model
but
does not contain the spinor fermions.

The 3 generations of spinor fermions
corresond to a Lie Algebra Magic Square.

The **Tai** X**uan** J**ing** construction will
give us the spinor fermions,
and therefore corresponds to
the complete D4-D5-E6-E7 physics model.

To construct the **Tai** Hsaun **Ching**,
consider the Magic Square sequence as a line

3 8 4 9 5 1 6 2 7

with central 5 and opposite pairs at equal distances.

If you try to make that, or a multiple of it, into a 9x9 Magic Square whose central number is the central number 41 of 9x9 = 81 = 40+1+40, you will fail because 41 is not a multiple of 5.

However, since 365 = 5x73 is the central number of 729 = 364+1+364 , you can make a 9x9x9 Magic Cube with 9x9x9 = 729 entries, each 9x9 square of which is a Magic Square.
The Magic Cube of the **Tai** Hsaun **Ching**
gives the same sum for all lines parallel to an edge, and for all diagonals containing the central entry.

The central number of the Magic Cube, 365, the period of a Maya Haab.

The total number for each line is 3,285 = 219 x 15.
The total of all numbers is 266,085 = 5,913 x 45.

Since 729 is the smallest odd number greater than 1 that is both a cubic number and a square number, the 729 entries of the 9x9x9 Magic Cube with central entry 365
can be rearranged to form
a 27x27 Magic Square with 729 entries and central entry 365.

27 = 3x3x3 = 13+1+13 is a cubic number with central number 14,
and there is a 3x3x3 Magic Cube with central entry 14
(14 is the dimension of the exceptional Lie algebra G2)
and sum 42:

10	24	8	26	1	15	6	17	19
23	7	12	3	14	25	16	21	5
9	11	22	13	27	2	20	4	18

The lowest dimensional non-trivial representation of the Lie algebra E6 is 27-dimensional, corresponding to the 27-dimensional Jordan algebra of 3x3 Hermitian octonionic matrices.

E6 is the 78-dimensional Lie algebra that is used in the D4-D5-E6-E7 physics model in which the 32-dimensional homogeneous space E6 / Spin(10)xU(1) corresponds to a 16-dimensional complex domain whose Shilov boundary is two copies of RP1 x S7 corresponding to Spin(8) spinors, representing 8 fermion particles and 8 fermion antiparticles.

All 4 components of the D4-D5-E6-E7 model, arising from the 4 fundamental representations of Spin(8), are contained within E6:
8 half-spinor fermion particles;
8 half-spinor fermion antiparticles;
8-dimensional spacetime
(4 Physical Spacetime dimensions and
4 Internal Symmetry dimensions);
and 28 gauge bosons
(12 for the Standard Model,
15 for Conformal Gravity and the Higgs Mechanism, and
1 for propagator phase).

The Lie algebra E6 is 72+6 = 78-dimensional, and has Weyl group of order 72x6! = 51,840 which is the symmetry group of the 6-dimensional polytope 2_21 with 27 vertices and 27+72 faces which is also the symmetry group of the 27 line configuration:

The 78 dimensions of E6 correspond to the 78 Tarot cards.

Since E6 as used in the D4-D5-E6-E7 physics model represents the two half-spinor representations of Spin(8),

For Spin(n) up to n = 8,
here are is their Clifford algebra structure
as shown by the Yang Hui (Pascal) triangle
and the dimensions of their spinor representations

```
  n                                                       Total
                            Spinor
                                                       Dimension
                           Dimension

0                   1                  2^0 =   1= 1x1          1
1                 1   1                2^1 =   2= 1+1          1
2               1   2   1              2^2 =   4= 2x2        2 = 1+1
3             1   3   3   1            2^3 =   8= 4+4          2
4           1   4   6   4   1          2^4 =  16= 4x4        4 = 2+2
5         1   5  10  10   5   1        2^5 =  32=16+16         4
6       1   6  15  20  15   6   1      2^6 =  64= 8x8        8 = 4+4
7     1   7  21  35  35  21   7   1    2^7 = 128=64+64         8
8   1   8  28  56  70  56  28   8   1  2^8 = 256=16x16      16 = 8+8
```

Since each row of the Yang Hui (Pascal) triangle corresponds to the graded structure of an exterior algebra
with a wedge product, call each row a wedge string.

In this pattern, the 28 and the 8 for n = 8 correspond
to the 28 gauge bosons of the D4 Lie algebra
and to the 8 spacetime (4 physical and 4 internal symmetry)
dimensions that are added when you go to the D5 Lie algebra.

The 8+8 = 16 fermions that are added when you go to E6,
corresponding to spinors, do not correspond to any single
grade of the n = 8 Clifford algebra with graded structure
1   8   28   56   70   56   28   8   1
but correspond to the entire Clifford algebra as a whole.

The total dimension of the Clifford algebra
is given by the Yang Hui (Pascal) triangle
pattern of binary expansion (1 + 1)^n,
which
corresponds to the number of vertices of
a hypercube of dimension n.

The spinors of the Clifford algebra of dimension n
are derived from the total matrix algebra of dimension 2^n
with pattern

```
n

0                    1
1                  2
2                4
3              8
4           16
5         32
6       64
7   128
8 256
```

This can be expanded to a pattern

```
n

0                   1
1                 2   1
2               4   2   1
3             8   4   2   1
4          16   8   4   2   1
5        32  16   8   4   2   1
6      64  32  16   8   4   2   1
7   128  64  32  16   8   4   2   1
8 256 128  64  32  16   8   4   2   1
```

in the same form as the Yang Hui (Pascal) triangle.

Call each row a spinor string.

For a given row in the binary (1+1)^n Yang Hui (Pascal) triangle
the string product of a spinor string and a wedge string

(2^N, 2^(N-1), 2^(N-2), ... , 2^(N-J), ... , 4, 2, 1)
(1 , N , N(N-1)/2,...,N^k J^(N-k)/(k!(N-k)!)J),...,N(N-1)/2,N,1)

gives the rows of the ternary (1+2)^n power of 3 triangle

```
n

0                    1                          3^0 =     1
1                  2    1                       3^1 =     3
2               4    4    1                     3^2 =     9
3             8    12   6    1                  3^3 =    27
4          16   32   24    8    1               3^4 =    81
5        32   80   80   40   10    1            3^5 =   243
6      64  192  240  160   60   12    1         3^6 =   729
7    128  448  672  560  280   84   14    1     3^7 = 2,187
8  256 1024 1792 1792 1120  448  112   16    1  3^8 = 6,561
```

Just as the binary (1+1)^n triangle corresponds to the I **Ching**,
the ternary (1+2)^n triangle corresponds to the **Tai Hsuan Ching**.
The ternary triangle also describes
the sub-hypercube structure of a hypercube.

The ternary power of 3 triangle is not only
used in representations of the spinors in the D4-D5-E6-E7 model,
it was also by Plato in describing cosmogony and music.

Complexification of Exceptional Lie Algebras

Frank Tony Smith

Aperiodic Tilings in 2, 3, and 4 dimensions **can be thought of as Irrational Slices of an** **8-dimensional E8 Lattice** **and its sublattices, such as** **E6**. **The 2-dimensional** **Penrose Tiling** **in the above image was generated by** **Quasitiler** **as a section of a 5-dimensional cubic lattice based on the 5-dimensional** **HyperCube** **shown in the center above the Penrose Tiling plane. The above-plane geometric structures in the above image are, going from left to right:**

4-dimensional 24-cell, whose 24 vertices are the root vectors of the 24+4 = 28-dimensional D4 Lie algebra;

two 4-dimensional HyperOctahedra, lying (in a 5th dimension) above and below the 24-cell, whose 8+8 = 16 vertices add to the 24 D4 root vectors to make up the 40 root vectors of the 40+5 = 45-dimensional D5 Lie algebra;

5-dimensional HyperCube, half of whose 32 vertices are lying (in a 6th dimension) above and half below the 40 D5 root vectors, whose 16+16 = 32 vertices add to the 40 D5 root vectors to make up the 72 root vectors of the 72+6 = 78-

dimensional E6 Lie algebra;

two 27-dimensional 6-dimensional figures, lying (in a 7th dimension) above and below the the 72 E6 root vectors, whose 27+27 = 54 vertices add to the 72 E6 root vectors to make up the 126 root vectors of the 126+7 = 133-dimensional E7 Lie algebra; and

two 56-dimensional 7-dimensional figures, lying (in an 8th dimension) above and below the the 126 E7 root vectors, and two polar points also lying above and below the 126 E7 root vectors, whose 56+56+1+1 = 114 vertices add to the 126 E7 root vectors to make up the 240 root vectors of the 240+8 = 248-dimensional E8 Lie algebra.

The 240 E8 root vectors form a Witting Polytope. They are related to the 256 elements of the Cl(1,7) Clifford Algebra of the D4-D5-E6-E7-E8 VoDou Physics model as follows:

Cl(1,7) has 256 = 2^8 elements, corresponding to the 2^8 vertices of an 8-dimensional HyperCube and having the graded structure

1 8 28 56 70 56 28 8 1

with even part 1 28 70 28 1 and odd part 8 56 56 8

An 8-dimensional HyperCube decomposes into 2 half-HyperCubes, each with 128 vertices, and each corresponding to one of the 2 mirror-image half-spinor representations of the D8 Lie algebra whose Euclidean-signature spin group is 120-dimensional Spin(16) and whose half-spinor representations have dimension (1/2) (2^(16/2)) = 256/2 = 128;

One **half-HyperCube** corresponds to the 128 even elements 1 28 70 28 1 and the other to the 128 odd elements 8 56 56 8;

The128 vertices of the odd 8-dimensional **half-HyperCube** with graded structure 8 56 56 8 correspond to 128 of the 240 E8 root vectors as follows:

8 to the 8 Octonion vector space basis elements, with positive sign;

8 to the 8 Octonion vector space basis elements, with negative sign;

56+56 = 112 to the 112 vectors with non-zero components on the Octonion real axis;

The other 240-128 = 112 do not directly correspond to the 128 vertices of the even 8-dimensional HyperCube of the even half-spinor representation of the D8 Spin(16) Lie algebra, but correspond to 112 of the 120 generators of Spin(16), the adjoint bivector representation of D8.

Exceptional Lie Algebra

Magic Squares

For order 4, it is possible to show exactly how the 7,040 magic squares are generated from capital and lower-case letter squares. For that, it is not necessary to enumerate all the possible Gallic squares, you need only to identify the elementary lower-case letter Gallic squares (and the induced squares by group G of 32 and by permutations). You search after the orthogonal squares to these elementary Gallic squares. You have to search only Gallic squares with semi-regular rows and columns because only diagonals can be irregular (you verify this property a posteriori on all the 7 040 squares and all the basis; maybe it is possible to demonstrate directly that rows and columns cannot be irregular). If you classify the Gallic squares according to the value of the parameter S1, sum of the first diagonal, you verify on the 7 040 squares that S1 can take only some values (then the sum of the second diagonal is S2=2σ-S1). And among these Gallic squares, you have to take only those which have orthogonal squares.

I give hereafter the result of my research.
With the first basis (1,2,3,4;0,4,8,12), I found:

there are 9 elementary squares S1=10 in a normalized position (the numbers hereafter are those of my program of enumeration), and each one has orthogonal squares:

1 1 2 3 4 2413 1 2 3 4 34

314243212) 3) 4 1221434321

) 5) 1 2 4 3 231

432414312 1

2 4 3 4312

12434312 1

3 4 2 13244

2314213

8 1 3 4 2 4213 1 3 4 2 43
2112344213

) 243131249) 12) 1 4 3 2 41
2323413214

13) 1 4 4 1 32
2323324114

(squares # 1, 2 and 12 are Latin and magic, square # 8 is Latin diagonal)
there are 7 elementary squares S1=6, but only 6 with orthogonal squares:

1) 1 1 4 4 2 1 1 4 4

233332244 32322323

112) 4) 4411 1 2

3 4 3124

24314321

5) 1 2 3 4 4 1 2 4 3

123143243 41321423

218) 18) 4312 1 4

2 3 4132

14234132

(squares # 1, 2 and 12 are Latin and magic, square # 8 is Latin diagonal)
there are 9 elementary squares S1=9, but only 7 with orthogonal squares:

1) 1 2 3 4 23143 1 2 3 4 332

24143213) 6) 7) 221434411 1

2 4 3 32142

3414312 1 2

4 3 4213134

24312

11) 1 3 4 2 32142 1 3 4 2 42
1314324123

431412313) 23) 1 4 4 1 223
333224114

(square # 11 is Latin but not magic).

For each elementary square, I searched the number of induced squares and the number of orthogonal squares:

S1=10	# of elementary square	Number of induced squares (group 32 * permutations)	Number of orthogonal squares for each	Total number of orthogonal squares
	1	4*2= 8	16	8*16= 128
	2	4*2= 8	32	8*32= 256
	3	32*2=64	12	64*12= 768
	4	16*2=32	40	32*40= 1 280
	5	32*1=32	8	32*8= 256
	8	16*3=48	24	48*24=

				1 152
	9	32*1=32	8	32*8= 256
	12	16*1=16	32	16*32= 512
	13	16*1=16	40	16*40= 640
256				5248

(There are 80 Latin magic squares, coming from the elementary squares

Numbers - 1, 2, 8 and 12. \l

There are 48 Latin diagonal squares coming from the elementary square # 8.\l
You see that Latin magic squares haven't the maximum number of orthogonal squares.

The total number of orthogonal squares to Latin magic squares is 2 048 - with 1 152 orthogonal to Latin diagonal squares -).

S1=6	# of elementary\ square	Number of induced squares\ (group 32 * permutations)	Number of orthogonal\ squares for each	Total number\ of orthogonal squares
	1	16*1=16	16	16*16=256
	2	16*1=16	16	16*16=256
	4	32*1=32	12	32*12=384
	5	32*1=32	14	32*14=448
	8	32*1=32	6	32* 6=192
	18	16*1=16	16	16*16=256
144				1 792

S1=10	# of elementary\ square	Number of induced squares\ (group 32 * permutations)	Number of orthogonal\ squares for each	Total number\ of orthogonal squares
	1	32*2=64	4	64*4=256
	3	32*2=64	4	64*4=256
	6	32*2=64	4	64*4=256

	7	32*1=32	6	32*6=192
	11	32*3=96	4	96*4=384
	13	32*2=64	3	64*3=192
	23	32*1=32	8	32*8=256
416				1 792

Then, I built two figures: one for the 5 248 semi-regular squares and another for the 1 792 irregular squares:

The 7 040 magic squares are distributed in blocks of squares. In each block, all the squares come from the same couple of two elementary Gallic squares, each one in a normalized position. In fact, for the capital letters values (0,4,8,12), I used the same program as for lower-case letters values (1,2,3,4) with:

0→1	**4→2**	**8→3**	**12→4**

In that way, the Gallic squares are written in a normalized form.

The figures give the number of magic squares in each block, but naturally I identified each magic square in the list of 7,040.

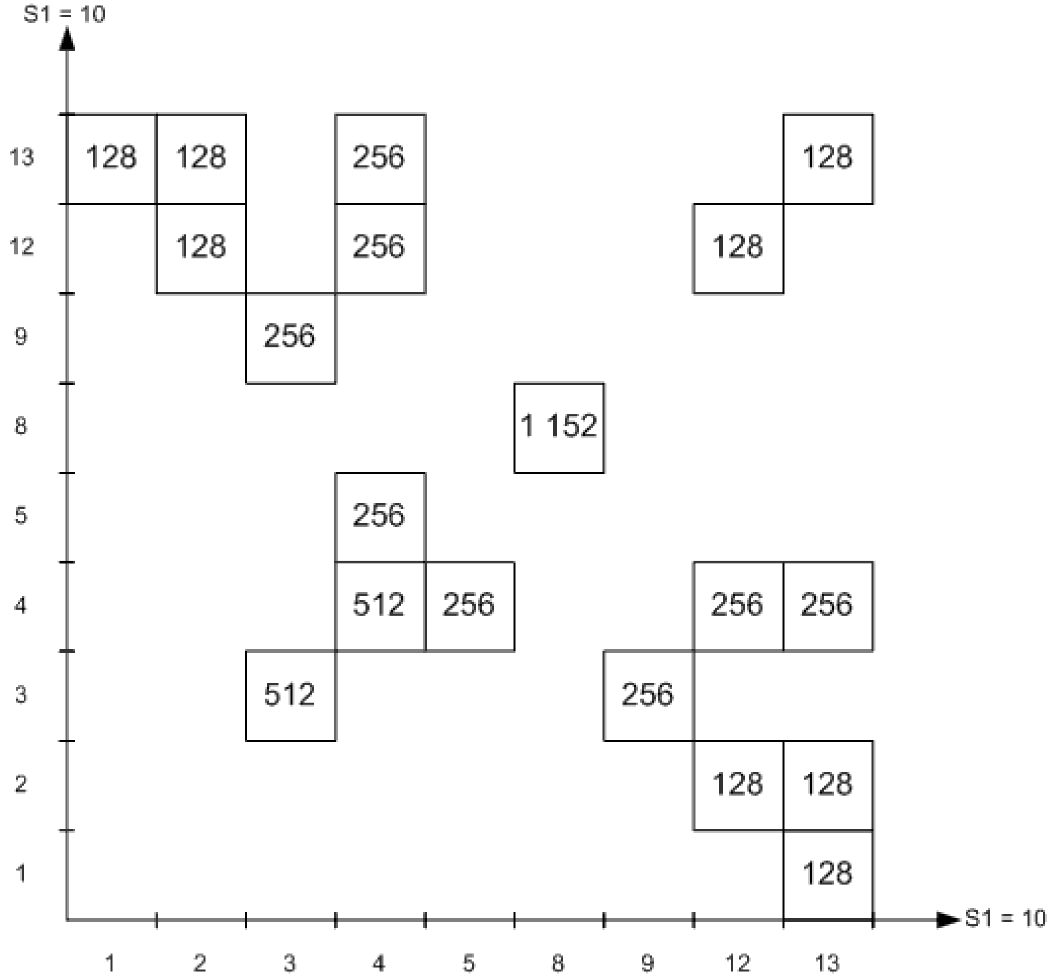
All the 7,040 magic squares can be built from 9+6+7=22 elementary Gallic squares (with the first basis).

I also drove the decomposition of the 7,040 squares with the second and the third basis (with the 3 other basis, the diagrams are obtained by permutation of capital and lower-case letters, i.e. by symmetry about the first bisecting line). \Jgive the results:

	Basis # 1	**Basis # 2**	**Basis #3**
Number of semi-regular squares	5,248\J (with 1,152 regular)	4,224\J (with 1,152 regular)	4,736\J (with 1,152 regular)
Number of irregular squares	1 ,792	2 ,816	2, 304
Number of elementary squares	22	32	45
Number of blocks of squares	36	41	41

\JThe set of 1,152 regular squares is different in the 3 basis. The 3 sets of 1,152 constitute the set of 3,456 semi-pandiagonal squares. \J

There are 4,224 squares which are always semi-regular in all the basis, and 1,280 squares which are always irregular in all the basis.\J There are 5,760 semi-regular squares with at least one basis.\J
\J
Note: for orders superior to 4, it is naturally impossible to do the same task up to the very end.



5 248 semi-regular squares

Octonion Magic Squares

On June 21st, 2004 an email came in from engineer **Francis Gaspalou**. Here is a part of his email:

"... I can confirm to you that I have found the same results, by a very similar method. The number of essentially different squares (3,365,114) can be divided by 2 : there are 96 transpositions for a given square, and not only 48. With a Duron 1,4 Ghz computer and a program in C language, I found in 12 days the 1,682,557 essentially different squares."

The New Transposition W

Francis Gaspalou called the new transposition W. He found it in August, 2000 when he examined regular ultramagic 7x7-squares. Each ultramagic square is transformed to another ultramagic square by applying transposition W. W is an involutory transposition, like S, R^2 and K^3. That means, you get the original square by applying W twice.

11	36	31	45	51	56	71
63	33	27	42	67	53	23
13	72	22	47	62	12	73
54	24	74	44	14	64	34
15	76	26	41	66	16	75
65	35	21	46	61	55	25
17	32	37	43	57	52	77

--->
W

11	36	31	45	51	56	71
63	33	27	42	67	53	23
13	72	22	47	62	12	73
54	24	74	44	14	64	34
15	76	26	41	66	16	75
65	35	21	46	61	55	25
17	32	37	43	57	52	77

--->
W

11	36	31	45	51	56	71
63	33	27	42	67	53	23
13	72	22	47	62	12	73
54	24	74	44	14	64	34
15	76	26	41	66	16	75
65	35	21	46	61	55	25
17	32	37	43	57	52	77

The numbers in the white cells are fixed. Each other number is interchanged with a number in a cell of the same colour.

The New Transposition-Group

The transpositions R, S, K and W generate a group of 96 different transpositions. Thus there are only

20,190,684 · 8 / 96 = 1,682,557

essentially different ultramagic order-7 squares.
Now the computation should take only half the time.

Are there more general transpositions?

No! As 1,682,557 is a prime, the set of essentially different ultramagic order-7 squares can't be invariant relative to any further transposition.
(Otherwise this transposition would generate a complete set of essentially different ultramagic order-7 squares and any pair of such squares would define such a transposition - that's not true.)

The transpositions R, S, K and W generate a group of 96 different transpositions.

Thus there are only 20,190,684 · 8 / 96 = 1,682,557 essentially different ultramagic order-7 squares.

Discussion

The key here is 1,682,557 essentially different squares.

Recall that the writer here is describing Order 7 Magic Squares. The Fano Plane consists of seven points and seven lines, and since it forms the Octonion Multiplication table, Octonions form in groups of seven; the Pythagorean Music Scale consists of seven notes in a scale, etc. In this respect, it appears that the writer has discovered all of the Order 7 Magic Squares which relate to the Octonions and their isomorphs.

Lydian	1	9/8	81/64	729/512	3/2	27/16	243/128
Phrygian	1	256/243	32/27	4/3	3/2	128/81	16//9
Dorian	1	9/8	32/27	4/3	3/2	27/16	16/9
Hypolydian	1	9/8	81/64	4/3	3/2	27/16	243/128
Hypophrygian	1	256/243	32/27	4/3	1024/729	128/81	16/9
Hypodorian	1	9/8	32/27	4/3	3/2	128/81	16/9
Mixolydian	1	9/8	81/64	4/3	3/2	27/16	16/9

From S.M. Philipps website

We know as well that the Octonions contain 168 permutation groups. Perhaps the author has discovered that the number 168 really should be considered as 168.2557 in relation to the Octonions and the group PS – 2. This number might provide more accurate equations when dealing with Octonions formulations, such as with Fermions, where such minute differences make for drastic changes at the nuclear scale.

In addition, Octonionic particles vibrate (please see Octonion Song paper on Vixra by John Sweeney), so the more specific 168.2557 figure may correspond precisely to frequencies produced by Octionic particles. In Article 13 S.M. Philips discusses what may be vibrations but Chinese censorship has prevented the author of this paper from downloading that article.

Freudenthal – Tits Magic Square

Frank Dodd Tony Smith posted this graphic on his website many years ago:

This chart depicts a Clifford Path, which is similar to a Knight Tour, through advanced numbers, with Real, Complex, Quaternion at the lower left, and Octonions represented by H @ H. Compare this chart with the Freudenthal – Tits Magic Square, which is related to Triality:

A \ B	R	C	H	O
R	A_1	A_2	C_3	F_4
C	A_2	$A_2 \times A_2$	A_5	E_6
H	C_3	A_5	D_6	E_7
O	F_4	E_6	E_7	E_8

In some way that remains unclear to the author, a connection exists between the Reece Harvey textbook chart of Matrix Algebras, (from Smith's website) the Freudenthal – Tits Magic Square, and the Exceptional Lie Algebra and Octonion Magic Squares described herein. Further research will be needed to fully understand these connections and implications.

Conclusion

This paper has shown origins and descriptions of the series of Exceptional Lie Algebras by John Baez, then a method of complexifiying each level, from an email to Baez from Frank “Tony” Smith. Then, the paper has given the true Magic Squares for Exceptional Lie Algebras by an engineer, as well as the group of Magic Squares related to the Octonions, the Fano Plane, the Klein Quartic, the Sephirot of the Cabala, and related structures, including the following:

SL(3,2), the automorphism group of the Fano plane and\ Octonions, of order 168; PSL(2,7): group of 168 of Klein Quartic automorphisms, isomorphic to SL(3,2). All of these relate in some way to different elements of the series of Exceptional Lie Algebras.

The series of Exceptional Lie Algebras, as well as Jordan Algebras, are derived from matrices, which are 2 x 2, 3 x 3, 4 x 4, etc. square arrangements of numbers. These Magic Squares are important in that they provide a variable function for determining the matrices which comprise nuclear particles.

Vedic Nuclear Particle theory, and Vedic Geometry, posit the existence of some 22 Hyper Spheres in the series of nuclear particles, which provide generic shells for potential particles. The type of particle which eventually emerges is determined by a Magic Square. Frank “Tony” Smith has described this process in relationship to the Tai Hsuan Ching (Tai Xuan Jing) and the Sedenions, while John Sweeney has included the same segment in a 2014 paper on Vixra.

Specifically, H7 and H8 combine to form the Exceptional Lie Algebra E8, as well as the nuclear maxima number of 33.2, with the assistance of one additional particle. A wide variety of particles may emerge from this process, and the actual particle which emerges depends upon which permutation of Exceptional Lie Algebra Magic Square is selected for that process. In the theory of Vedic Particle Physics, the aspect which selects that specific Magic Square permutation would be either the god Vishnu or the spiritual element which is part of the nuclear growth process.

The point is that a choice exists at this stage, and the specific Exceptional Lie

Algebra Magic Square permutation chosen at this stage ultimately determines the final result.

Confirmation of the importance of Magic Squares comes from the Rig Veda, the oldest book known to humanity, probably some 13,000 years old. See the author's paper on Vixra, “Rig Veda Magic Squares” for details.

Note: Gaspalou remains uncertain of these findings, yet the author of this paper feels confident of this research to publish a preliminary pre – print paper on Vixra to invite further input on this topic.

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Appendix I Octonions (from Wikipedia)

The above definition though is not unique, but is only one of 480 possible definitions for octonion multiplication with $e_0 = 1$. The others can be obtained by permuting and changing the signs of the non-scalar basis elements. The 480 different algebras are isomorphic, and there is rarely a need to consider which particular multiplication rule is used.

Each of these 480 definitions is invariant up to signs under some 7-cycle of the points (1234567), and for each 7-cycle there are four definitions, differing by signs and reversal of order. A common choice is to use the definition invariant under the 7-cycle (1234567) with $e_1e_2 = e_4$ as it is particularly easy to remember the multiplication. A variation of this sometimes used is to label the elements of the basis by the elements , 0, 1, 2, ..., 6, of the projective line over the finite field of order 7.

The multiplication is then given by e = 1 and $e_1e_2 = e_4$, and all expressions obtained from this by adding a constant (mod 7) to all subscripts: in other words using the 7 triples (124) (235) (346) (450) (561) (602) (013). These are the nonzero codewords of the quadratic residue code of length 7 over the field of 2 elements. There is a symmetry of order 7 given by adding a constant mod 7 to all subscripts, and also a symmetry of order 3 given by multiplying all subscripts by one of the quadratic residues 1, 2, 4 mod 7.[4][5]

The multiplication table can be given in terms of the following 7 quaternionic triples (omitting the identity element): (I*jk*), (*i*J*k*), (*ij*K), (IJK), (I*im*), (J*jm*), (K*km*) in which the lowercase items are vectors (mathematics and physics) and the uppercase ones are bivectors.

Integral octonions[edit]

There are several natural ways to choose an integral form of the octonions. The simplest is just to take the octonions whose coordinates are integers. This gives a nonassociative algebra over the integers called the Gravesian octonions. However it is not a maximal order, and there are exactly 7 maximal orders containing it. These 7 maximal orders are all equivalent under automorphisms. The phrase "integral octonions" usually refers to a fixed choice of one of these seven orders.

These maximal orders were constructed by Kirmse (1925), Dickson and Bruck as follows. Label the 8 basis vectors by the points of the projective plane over the field with 7 elements. First form the "Kirmse integers" : these consist of octonions whose coordinates are integers or half integers, and that are half odd integers on one of the 16 sets

(124) (235) (346) (450) (561) (602) (013) (0123456)
(0356) (1460) (2501) (3612) (4023) (5134) (6245)
of the extended quadratic residue code of length 8 over the field of 2 elements, given by , (124) and its images under adding a constant mod 7, and the complements of these 8 sets. (Kirmse incorrectly claimed that these form a maximal order, so thought there were 8 maximal orders rather than 7, but as Coxeter (1946) pointed out they are not closed under multiplication; this mistake occurs in several published papers.) Then switch infinity and any other coordinate; this gives a maximal order. There are 7 ways to do this, giving 7 maximal orders, which are all equivalent under cyclic permutations of the 7 coordinates 0123456.

The Kirmse integers and the 7 maximal orders are all isometric to the E8 lattice rescaled by a factor of 1/ 2. In particular there are 240 elements of minimum nonzero norm 1 in each of these orders, forming a Moufang loop of order 240.

The integral octonions have a "division with remainder" property: given integral octonions *a* and *b* 0, we can find *q* and *r* with *a* = *qb* + *r*, where the remainder *r* has norm less than that of *b*.

In the integral octonions, all left ideals and right ideals are 2-sided ideals, and the only 2-sided ideals are the principal ideals *nO* where *n* is a non-negative integer.

The integral octonions have a version of factorization into primes, though it is not straightforward to state because the octonions are not associative so the product of octonions depends on the order in which one does the products. The irreducible integral octonions are exactly those of prime norm, and every integral octonion can be written as a product of irreducible octonions. More precisely an integral octonion of norm *mn* can be written as a product of integral octonions of norms *m* and *n*.

The automorphism group of the integral octonions is the group $G_2(\mathbf{F}_2)$ of order 12096, which has a simple subgroup of index 2 isomorphic to the unitary group ${}^2A_2(3^2)$. The isotopy group of the integral octonions is the perfect double cover of the group of rotations of the E8 lattice.

In mathematics, the **Freudenthal magic square** (or **Freudenthal–Tits magic square**) is a construction relating several Lie algebras (and their associated Lie groups). It is named after Hans Freudenthal and Jacques Tits, who developed the idea independently. It associates a Lie algebra to a pair of division algebras *A*, *B*. The resulting Lie algebras have Dynkin diagrams according to the table at right. The "magic" of the Freudenthal magic square is that the constructed Lie algebra is symmetric in *A* and *B*, despite the original construction not being symmetric, though Vinberg's symmetric method gives a symmetric construction; it is not a magic square as in recreational mathematics.

The Freudenthal magic square includes all of the exceptional Lie groups apart from G_2, and it provides one possible approach to justify the assertion that "the exceptional Lie groups all exist because of the octonions": G_2 itself is the automorphism group of the octonions (also, it is in many ways like a classical Lie group because it is the stabilizer of a generic 3-form on a 7-dimensional vector space – see prehomogeneous vector space).

The last row and column here are the orthogonal algebra part of the isotropy algebra in the symmetric decomposition of the exceptional Lie algebras mentioned previously.

These constructions are closely related to hermitian symmetric spaces – cf. prehomogeneous vector spaces.

Symmetric spaces[edit]

Riemannian symmetric spaces, both compact and non-compact, can be classified uniformly using a magic square construction, in (Huang & Leung 2011). The irreducible compact symmetric spaces are, up to finite covers, either a compact simple Lie group, a Grassmannian, a Lagrangian Grassmannian, or a double Lagrangian Grassmannian of subspaces of for normed division algebras **A** and **B**. A similar construction produces the irreducible non-compact symmetric spaces.

History[edit]

Rosenfeld projective planes[edit]

Following Ruth Moufang's discovery in 1933 of the Cayley projective plane or "octonionic projective plane" **P**^{2}(**O**), whose symmetry group is the exceptional Lie group F4, and with the knowledge that G2 is the automorphism group of the octonions, it was proposed by Rozenfeld (1956) that the remaining exceptional Lie groups E6, E7, and E8 are isomorphism groups of projective planes over certain algebras over the octonions:[1]

the **bioctonions**, **C** **O**,
the **quateroctonions**, **H** **O**,
the **octooctonions**, **O** **O**.

This proposal is appealing, as there are certain exceptional compact Riemannian symmetric spaces with the desired symmetry groups and whose dimension agree with that of the putative projective planes (dim(**P**^{2}(**K** **K**)) = 2dim(**K**)dim(**K**)), and this would give a uniform construction of the exceptional Lie groups as symmetries of naturally occurring objects (i.e., without an a priori knowledge of the exceptional Lie groups). The Riemannian symmetric spaces were classified by Cartan in 1926 (Cartan's labels are used in sequel); see classification for details, and the relevant spaces are:

- the octonionic projective plane – FII, dimension 16 = 2 × 8, F_4 symmetry, Cayley projective plane **P**^{2}(**O**),
- the bioctonionic projective plane – EIII, dimension 32 = 2 × 2 × 8, E_6 symmetry, complexified Cayley projective plane, **P**^{2}(**C** **O**),
- the "**quateroctonionic projective plane**"[2] – EVI, dimension 64 = 2 × 4 × 8, E_7 symmetry, **P**^{2}(**H** **O**),
- the "**octooctonionic projective plane**"[3] – EVIII, dimension 128 = 2 × 8 × 8, E_8 symmetry, **P**^{2}(**O** **O**).

The difficulty with this proposal is that while the octonions are a division algebra, and thus a projective plane is defined over them, the bioctonions, quarteroctonions and octooctonions are not division algebras, and thus the usual definition of a projective plane does not work. This can be resolved for the bioctonions, with the resulting projective plane being the complexified Cayley plane, but the constructions do not work for the quarteroctonions and octooctonions, and the spaces in question do not obey the usual axioms of projective planes,[1] hence the quotes on "(putative) projective plane". However, the tangent space at each point of these spaces can be identified with the plane (**H** **O**)^2, or (**O** **O**)^2

further justifying the intuition that these are a form of generalized projective plane.[2][3] Accordingly, the resulting spaces are sometimes called **Rosenfeld projective planes** and notated as if they were projective planes. More broadly, these compact forms are the **Rosenfeld elliptic projective planes**, while the dual non-compact forms are the **Rosenfeld hyperbolic projective planes**. A more modern presentation of Rosenfeld's ideas is in (Rosenfeld 1997), while a brief note on these "planes" is in (Besse 1987, pp. 313–316).[4]

The spaces can be constructed using Tit's theory of buildings, which allows one to construct a geometry with any given algebraic group as symmetries, but this requires starting with the Lie groups and constructing a geometry from them, rather than constructing a geometry independently of a knowledge of the Lie groups.[1]

Magic square[edit]

While at the level of manifolds and Lie groups, the construction of the projective plane **P**^{2}(**K** **K**) of two normed division algebras does not work, the corresponding construction at the level of Lie algebras *does* work. That is, if one decomposes the Lie algebra of infinitesimal isometries of the projective plane **P**^{2}(**K**) and applies the same analysis to **P**^{2}(**K** **K**), one can use this decomposition, which holds when **P**^{2}(**K** **K**) can actually be defined as a projective plane, as a *definition* of a "magic square Lie algebra" *M*(**K**,**K**) This definition is purely algebraic, and holds even without assuming the existence of the corresponding geometric space. This was done independently circa 1958 in (Tits 1966) and by Freudenthal in a series of 11 papers, starting with (Freudenthal 1954) and ending with (Freudenthal 1963), though the simplified construction outlined here is due to (Vinberg 1966).[1]

Dedication

Some men see things the way they are, and ask, “Why?”

I see things that have never been, and ask “Why not?”

So let us dedicate ourselves to what the Greeks wrote so long ago:

To tame the savageness in man and make gentle the life of this world.

Robert Francis Kennedy