Investigations on the Chebyshev functions: two lower bound for first Chebyshev function and a conjecture for prime numbers

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"But he was pierced for our transgressions, he was crushed for our iniquities; the punishment that brought us peace was on him, and by his wounds we are healed. We all, like sheep, have gone astray, each of us has turned to our own way; and the Lord has laid on him the iniquity of us all." - Isaiah 53:5-6.

Abstract. I proved two lower bound for first Chebyshev function and leave a conjecture on prime numbers.

1. Introduction

The first Chebyshev function, $\vartheta(x)$ or $\theta(x)$, is defined by

$$\vartheta(x) \triangleq \sum_{n \leqslant x} \ln(p_n). \tag{1}$$

The second Chebyshev function, $\psi(x)$, is given by

$$\psi(x) \triangleq \sum_{p_n^k \leqslant x} \ln(p_n) = \sum_{n \leqslant x} \Lambda(n) = \sum_{n \leqslant x} \lfloor \log_{p_n}(x) \rfloor \ln(p_n),$$

where $\Lambda(n)$ denotes the von Mangoldt function, wich is defined as

$$\Lambda(n) = \begin{cases} \ln(p_n), \text{if } n = p_n^k \text{ for some prime } p_n \text{ and integer } k \geqslant 1; \\ 0, \text{otherwise.} \end{cases}$$

An direct relationship between them is given by

$$\psi(x) = \sum_{n=1}^{\infty} \vartheta\left(x^{1/n}\right). \tag{2}$$

In this paper, I demonstrated the following lower bounds for first Chebyshev function, that is,

$$-\int_0^{\infty} \mathrm{Ei}(-t)\, \frac{e^t + (x-1) e^{-t(x-1)} - x e^{-t(x-2)}}{(e^t - 1)^2} \mathrm{dt} < \vartheta(x)$$

and

$$-\int_0^{\infty} \mathrm{Ei}(-t) \frac{e^{-tx}\left[e^{t(x+1)} + x - (x+1)\, e^t\right]}{(e^t - 1)^2} \mathrm{dt} < \vartheta(x).$$

This last integral representation is the closest to the first Chebyshev function.

2. Two lower bound for first Chebyshev function

Theorem 1. *For all* $x \in \mathbb{R}_{>0}$, *then*

$$-\int_0^\infty \mathrm{Ei}(-t)\,\frac{e^t + (x-1)e^{-t(x-1)} - x\,e^{-t(x-2)}}{(e^t-1)^2}\mathrm{dt} < \vartheta(x),$$

where $\vartheta(x)$ *denotes the first Chebyshev function and* $\mathrm{Ei}(x)$ *denotes the exponential integral function.*

Proof. I know [1] that

$$\frac{\ln z}{z-1} = \int_0^1 \frac{\mathrm{du}}{u\,(\ln u - 1)(\ln u - z)}, \tag{3}$$

for $\Re(z) > 0$. I substitute (3) in (1), and encounter

$$\vartheta(x) = \sum_{n \leqslant x} \ln(p_n) = \sum_{n \leqslant x} \int_0^1 \frac{(p_n - 1)\mathrm{du}}{u(\ln u - p_n)(\ln u - 1)}$$

$$\begin{aligned}
&= \int_0^1 \frac{1}{u(\ln u - 1)} \sum_{n \leqslant x} \frac{p_n - 1}{\ln u - p_n}\mathrm{du} \\
&= \int_0^1 \frac{1}{u(\ln u - 1)} \sum_{n \leqslant x} (p_n - 1) \int_0^{-\infty} e^{-t(\ln u - p_n)}\,\mathrm{dtdu} \\
&= \int_0^1 \frac{1}{u(\ln u - 1)} \int_0^{-\infty} e^{-t\ln u} \sum_{n=1}^{x} (p_n - 1)e^{t p_n}\,\mathrm{dtdu} \\
&= \int_0^{-\infty} \int_0^1 \frac{u^{-t-1}\mathrm{du}}{\ln u - 1} \sum_{n \leqslant x} (p_n - 1)e^{t p_n}\mathrm{dt} \\
&= -\int_0^\infty e^t\,\mathrm{Ei}(-t) \sum_{n \leqslant x} (p_n - 1)e^{-t p_n}\mathrm{dt} \\
&> -\int_0^\infty e^t \mathrm{Ei}(-t) \sum_{n \leqslant x} (n - 1)e^{-t n}\mathrm{dt} \\
&= -\int_0^\infty e^t \mathrm{Ei}(-t)\,\frac{1 + (x-1)e^{-t x} - x\,e^{-t(x-1)}}{(e^t - 1)^2}\mathrm{dt} \\
&= -\int_0^\infty \mathrm{Ei}(-t)\,\frac{e^t + (x-1)e^{-t(x-1)} - x\,e^{-t(x-2)}}{(e^t - 1)^2}\mathrm{dt},
\end{aligned}$$

which proves the result. □

Theorem 2. *For all* $x \in \mathbb{R}_{>0}$, *then*

$$-\int_0^\infty \mathrm{Ei}(-t)\frac{e^{-t x}\left[\,e^{t(x+1)} + x - (x+1)\,e^t\right]}{(e^t - 1)^2}\mathrm{dt} < \vartheta(x),$$

where $\vartheta(x)$ *denotes the first Chebyshev function and* $\mathrm{Ei}(x)$ *denotes the exponential integral function.*

Proof. I know [1] that

$$\frac{\ln z}{z-1} = \int_0^1 \frac{\mathrm{du}}{u\,(\ln u - 1)(\ln u - z)}, \tag{4}$$

for $\Re(z)>0$. I replace (4) in (1), and obtain

$$\vartheta(x)=\sum_{n\leqslant x}\ln(p_n)=\sum_{n\leqslant x}\int_0^1\frac{(p_n-1)\mathrm{du}}{u(\ln u-p_n)(\ln u-1)} \tag{5}$$

$$=\int_0^1\frac{1}{u(\ln u-1)}\sum_{n\leqslant x}\frac{p_n-1}{\ln u-p_n}\mathrm{du}$$

$$=\int_0^1\frac{1}{u(\ln u-1)}\sum_{n\leqslant x}(p_n-1)\int_0^{-\infty}e^{-t(\ln u-p_n)}\,\mathrm{dtdu}$$

$$=\int_0^1\frac{1}{u(\ln u-1)}\int_0^{-\infty}e^{-t\ln u}\sum_{n=1}^{x}(p_n-1)e^{tp_n}\,\mathrm{dtdu}$$

$$=\int_0^{-\infty}\int_0^1\frac{u^{-t-1}\mathrm{du}}{\ln u-1}\sum_{n\leqslant x}(p_n-1)e^{tp_n}\mathrm{dt}$$

$$=-\int_0^{\infty}e^t\,\mathrm{Ei}(-t)\sum_{n\leqslant x}(p_n-1)e^{-tp_n}\mathrm{dt}$$

$$>-\int_0^{\infty}e^t\,\mathrm{Ei}(-t)\sum_{n\leqslant x}n\,e^{-tn}\mathrm{dt}$$

$$=-\int_0^{\infty}\mathrm{Ei}(-t)\frac{e^{-tx}\left[e^{t(x+1)}+x-(x+1)\,e^t\right]}{(e^t-1)^2}\mathrm{dt},$$

which proves the result. $\square$

3. A conjecture for prime numbers

Conjecture. *For x sufficiently large, then*

$$\sum_{n=1}^{\infty}\frac{p_n-1}{e^{p_nx}}\sim\frac{1}{2(3-4\cosh(x)+\cosh(2x))},$$

rapidly, where p_n denotes the nth prime number, e^x denotes the exponential function and $\cosh(x)$ denotes the cosine hyperbolic function.

Warning 3. In truth, for $x>3$, then

$$\sum_{n=1}^{\infty}\frac{p_n-1}{e^{p_nx}}\approx\frac{1}{2(3-4\cosh(x)+\cosh(2x))},$$

as attested by the numerical calculations. For example, for $x=4$, thus, $\sum_{n=1}^{\infty}\frac{p_n-1}{e^{4p_n}}=0.000347759...$ and $\frac{1}{2(3-4\cosh(4)+\cosh(8))}=0.000361207...$; for $x=5$, thus, $\sum_{n=1}^{\infty}\frac{p_n-1}{e^{5p_n}}=0.000046011...$ and $\frac{1}{2(3-4\cosh(5)+\cosh(10))}=0.000046644...$; and so on.

References

[1] Guedes, Edigles, *The natural logarithm and applications*, to appear.