A SHORT PROOF FOR OPPERMANN'S CONJECTURE

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

Abstract. We write two prove for Oppermann's conjecture using the Hardy-Wright's estimate for prime-counting function.[1]

1. Introduction

1.1. The Oppermann's conjecture (theorem).

The Oppermann's conjecture, named after Ludvig Oppermann, in 1882, relates to distribution of the prime numbers, and says that: For any integer, $n>1$, there is, at least, one prime number between n^2-n and n^2; and, at least, another prime number between n^2 and n^2+n. Algebraically speaking,

Theorem 1. (Oppermann's conjecture) [1] *Given an integer, $n>1$, there is, at least, one prime number, p_1, such that $n^2-n<p_1<n^2$; and, at least, another prime number, p_2, such that $n^2<p_2<n^2+n$.*

We utilize an alternative statement: Let $\pi(n)$ denotes the classical prime-counting function. Thus,

$$\pi(n^2-n)<\pi(n^2)<\pi(n^2+n), \tag{1}$$

or, equivalently,

$$0<\pi(n^2)-\pi(n^2-n) \tag{2}$$

and

$$0<\pi(n^2+n)-\pi(n^2). \tag{3}$$

H. Laurent [2, p. 427] noted this peculiar relation to prime numbers:

$$\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}=\begin{cases}1, \text{if } k \text{ is prime,}\\ 0, \text{if } k \text{ is composite,}\end{cases}$$

provided $k\in\mathbb{N}_{\geqslant 5}$, where $\Gamma(k)$ denotes the gamma function.

In 2013, I and the Dr. Raja Rama Gandhi [3 and 4, Theorem 2, pp. 5-8] try to demonstrate the Oppermann's conjecture based in the partial summation for prime-counting function:

$$\pi(n)=2+\sum_{k=5}^{n}\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1},$$

for $n\in\mathbb{N}_{\geqslant 5}$, here $\Gamma(n)$ is the gamma function. But, the proof was inconclusive.

In this paper, we use other strategy to demonstrate the Oppermann's conjecture once for all. This time, through the Hardy-Wright's estimative for prime-counting function, we got the desired result.

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1.2. Consequences.
Since the Oppermann's conjecture is true, then the gap size would be on the order of

$$g_n < \sqrt{p_n}.$$

This also means there would be, at least, two primes between x^2 and $(x+1)^2$; i.e., one in the range from x^2 to $x(x+1)$ and the second in the range from $x(x+1)$ to $(x+1)^2$. It is stronger than Legendre's conjecture (theorem): that there is, at least, one prime in this range. Because there is, at least, one non-prime between any two odd primes it would also imply Brocard's conjecture that there are, at least, four primes between the squares of consecutive odd primes [5, page 164]. Further, it would imply that the largest possible gaps between two consecutive prime numbers could be at most proportional to twice the square root of the numbers, as Andrica's conjecture states.

Consequently, the Oppermann's conjecture also implies that, at least, one prime number can be found in every quarter revolution of the Ulam spiral.

Given that Oppermann's conjecture implies the proof of Legendre's conjecture, Brocard's conjecture and Andrica's conjecture; it is very important for the theory of distribution of the prime numbers.

2. Preliminaries

In present, we will need of the lower bound for prime-counting function:

Theorem 2. `(Hardy-Wright's estimate)` *For $n \in \mathbb{N}_{\geqslant 1}$, then*

$$\pi(n) \geqslant \frac{\log n}{2\log 2}. \tag{4}$$

Proof. See [6, Theorem 20, page 21]. □

3. A inedited proof for theorem 1

Now, we present an new proof for Oppermann's conjecture.

Proof. For $n \in \mathbb{N}_{>1}$. We set n^2+n, n^2 and n^2-n, respectively, into n, in Eq. (4), and obtain

$$\pi(n^2+n) > \frac{\log n + \log(n+1)}{2\log 2}, \tag{5}$$

$$\pi(n^2) > \frac{2\log n}{2\log 2} \tag{6}$$

and

$$\pi(n^2-n) > \frac{\log n + \log(n-1)}{2\log 2}. \tag{7}$$

Subtracting (5) to (6), we meet

$$\pi(n^2+n) - \pi(n^2) > \frac{\log n + \log(n+1) - 2\log n}{2\log 2} = \frac{\log(n+1) - \log n}{2\log 2} > 0,$$

since $\log(n+1) > \log n$. Hence, $\pi(n^2+n) - \pi(n^2) > 0$.

On the other hand, subtracting (6) to (7), we get

$$\pi(n^2) - \pi(n^2-n) > \frac{2\log n - \log n - \log(n-1)}{2\log 2} = \frac{\log n - \log(n-1)}{2\log 2} > 0,$$

since $\log n > \log(n-1)$. Thus, $\pi(n^2) - \pi(n^2-n) > 0$. This gives us the desired result. □

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