Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory
Part 1 Gossamer numbers

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Abstract

The discovery of what we call the gossamer number system $*G$, as an extension of the real numbers includes an infinitesimal and infinitary number system; by using 'infinite integers', an isomorphic construction to the reals by solving algebraic equations is given. We believe this is a total ordered field. This could be an equivalent construction of the hyperreals. The continuum is partitioned: $0 < \Phi^+ < \mathbb{R}^+ + \Phi < +\Phi^{-1} < \infty$. A one-one correspondence over an infinity of infinite intervals is interpreted differently resulting in the infinite-integers having a higher cardinality than the countable integers, and a likely consequence that $*G$ and the hyperreals have a higher cardinality than the real numbers.

1 Introduction

An infinite integer is an integer at infinity, larger than any finite integer. The set of integers $\mathbb{J}$ is a countable infinity. However, by representing this countability as an infinite integer $n|_{n=\infty}$, in Non-Standard-Analysis (NSA) ω; using evaluation at a point notation Definition 3.1; if we take the reciprocal $\frac{1}{n}|_{n=\infty}$ we have an infinite number that is positive and not rational, or even a real number, but a positive infinitesimal.

This number system is of interest to anyone working with calculus. Infinities and infinitesimals in mathematics are universal.

Despite this, infinitesimals and infinities largely are not recognised as actual numbers. That is, they are not declared, like integers or reals. Limits are used all the time, and these "are" infinitesimals, that what we daily use as shorthand is not acknowledged. Infinitesimals are so successful that they are accepted as known and are applied so often. (Since we use them, why is there any need to study them)

We have gone back in time to a mathematician who systematically considered infinitesimals and infinities. Paul du Bois-Reymond, like everyone else, was at first uncomfortable with infinity. By comparing functions [9, Part 3 Example 2.20], he investigated the continuum and realized the space which we would call Non-Standard-Analysis today. Our subsequent papers in this series consider this in some detail.

The purpose of this paper is to construct a number system that both contains infinitesimals and infinities, and a more natural extension to the real numbers than other alternatives.

A fact of significant importance is that the real numbers are composed of integers. Rational numbers are built from the ratio of two integers, algebraic numbers from solving an equation of rational coefficients, and so other numbers with infinite processes are constructed.

With the discovery of infinite integers [14, p.1] and A. Robinson's enlargement of the integers $\mathbb{J}$: *J [13, p.97]), by a similar process to the construction of the real numbers from integers we can construct infinite and infinitesimal numbers. We introduce a number system which is comparable to, and an alternative to hyperreal and surreal numbers.

By having 'infinity' itself as a number, like i for complex numbers, we can follow a standard construction analogous to building the real numbers from integers.

We believe that no such simple construction has been given that provides the usability without the complexity. (i.e. avoids the complicated mathematics of logic and set theory)

With time, we would like to see the claimed misuse of infinitesimals as 'not being rigorous', start to be reconsidered. A number system which directly supports their use in the traditional sense is possible. In [10, Part 4] and [11, Part 5] we argue that rigour should not be the only goal of a number system.

We take a constructionist's approach to the development of the infinitesimal and infinitary number system. A series of papers for a larger theory, based on du Bois-Reymond's ideals and our extensions to them, is developed.

Working with infinitesimals and infinities, we find a need for better representation. Indeed, the more we consider calculus, the greater is the need for this.

By adding infinities and infinitesimals to the real numbers, and declaring these additions to be of these types, this primarily is an extension of the real number line. Rather than saying a number becomes infinitely small, or infinitely large, the numbers are able to be declared as infinitesimals and infinities respectively. Theorems with this characterisation naturally follow.

Why should we be interested in infinitesimals or infinities? We are indeed, using them all the time through limits, applied mathematics and theory. Why should we want to further characterise them?

In Flatland [7], set in a world of two dimensions, where a 3D creature, a sphere, attempting to communicate, describes its world to a flatlander inhabitant, the square. Proving its existence by removing an item from a closed 2D cupboard, and materializing it elsewhere.

If we imagine the infinitesimals and infinities as working in a higher dimension, then we project back to the reals afterwards. See [10, Part 4 The transfer principle].

The name we give to the number system which we construct is the "gossamer numbers", for, analogous to a spider's web, there can be ever finer strands between the real numbers. About any real number, there are infinitely many numbers asymptotic to a given real number. Similarly about any curve, there are infinitely many curves asymptotic to it.

What this view gives is a deeper fabric of space to work with. Propositions in $\mathbb{R}$ may be better explained as propositions in the higher dimensional space.

We look at the problem from an equation's perspective. Just as complex numbers solve the equation $x^2 = -y^2$ in terms of x, $y = xi$, we can then put forward an equation at infinity which likewise needs new ideas and new numbers.

Historically, the development of a number system preceded with an equation or problem which needed to be solved. By looking at ancient mathematics such as Diophantine equations, words were used in place of a mathematical language such as our modern algebra.

Without the language of algebra, it took a genius to solve by today's measure routine problems. Similarly with Newton's calculus, the English mathematicians lagged behind the continent until the adaption of Leibniz's calculus. In short, the mathematical language can greatly impact on both the development and use of mathematics affecting the culture.

While it can be argued that infinitesimals and infinities are known and managed with other mathematics such as asymptotic notation and the hyperreal number system, by producing two fields of mathematics, we argue that there is a need for change to include new mathematical language. Similarly, the application of hyperreal numbers justified their existence.

2 The infinite integer

We believe that what an atom is to a physicist, an infinite integer is to a mathematician, well that is how it should be! With this belief, we go on to derive infinitesimals and infinities, building on the existence of the infinite integer $\mathbb{J}_\infty$ (Definition 4.15).

An infinite integer's existence does not contradict Euclid's proof that there is no largest integer because the existence of infinite integers is not a single largest integer, but a class of integers.

Certain properties become apparent. We define a partition between finite and infinite numbers, but there is no lowest or highest infinite number, as there is no lowest or highest infinite integer.

$$(\ldots, n-2, n-1, n, n+1, n+2, \ldots)|_{n \in \mathbb{J}_\infty}$$

In the construction of the extended numbers, what distinguishes these new numbers is the inclusion of an infinity within the number. While the real numbers are constructed created by infinite processes, *the real numbers do not contain an infinity.* We can imagine such a number with the inclusion of an infinity as ever changing, and not static. We need such a construction to explain limits.

That we are continually being asked to explain the existence of real numbers, let alone numbers with infinities is a challenge. However, these numbers provide immeasurable insight into the mechanics of calculus. Would we deny the existence of quantum physics because we felt only the need for classical physics? This is similar to the denial of the infinitesimal and infinitary numbers because we already have the real numbers.

We are accustomed to the use of complex numbers which have a separate component, the imaginary part. Similarly with the extension of the real number, an infinitesimal or infinitary component is added. However, this will generally be seen as part of the number, and not represented as separate components. Having said that, in this paper we do represent the components separately, to help prove the basic properties of the number system.

Once the infinite integer is accepted, an infinity not in $\mathbb{R}$, the infinitesimal follows as its reciprocal $\frac{1}{n} \notin \mathbb{R}$.

While the set of integers is a countable infinity, we need to distinguish between infinite and finite numbers.

Definition 2.1. *We say* $\mathbb{J}_{<}$ *to mean a finite integer,* $k \in \mathbb{J}$ *and* k *is finite.* $\mathbb{J}_{<} \subset \mathbb{J}$ *We similarly define* $\mathbb{N}_{<} \subset \mathbb{N}$ *the finite natural numbers,* $\mathbb{Q}_{<} \subset \mathbb{Q}$ *the finite rational numbers,* $\mathbb{A}_{<} \subset \mathbb{A}$ *the finite algebraic numbers.*

Definition 2.2. *We say* $\mathbb{J}_{<} < \mathbb{J}_{\infty}$ *to mean than for all finite integers* $i \in \mathbb{J}$*, and all 'infinite integers'* $j \in \mathbb{J}_{\infty}$ *then* $i < j$*.*

We note that the set of finite numbers is infinite, but any given finite number is not infinite. We can consider any finite integer as large as we please, but once we do it is in existence it is without infinity.

3 Preliminaries

This paper is one of six in this series, which we believe is establishing a new field of mathematics. The proofs and notation contain mathematics in [8, Part 2] and [9, Part 3], however if $f \succ g$ then $\frac{f}{g} \in \Phi$ is equivalent to $f = o(g)$, where Φ is an infinitesimal (Definition 4.4). The proofs can solve for a relation, which is studied in Part 3.

We introduce a notation at infinity which can express both a limit and the realization of being at infinity. We later develop and justify this choice more fully [8, Part 2], [10, Part 4], [11, Part 5].

Definition 3.1. *$f(x)|_{x=\infty}$ represents the notion of the "order" of $f(x)$ as a function "at infinity", or a limit at infinity. See [8, Evaluation at-a-point Definition 2.1].*

Remark: 3.1. *We will later see that formal objects can be formed with a number system that reflects the requirements of our notion of infinitesimals and infinities.*

Now we give an example of an infinitary equation, which can only be solved by an infinitary number.

Solve $(x^2+1)y|_{x=\infty} = x^3|_{x=\infty}$. $y = \frac{x^3}{x^2+1}|_{x=\infty}$ is a solution, another is $y = x|_{x=\infty}$.

We already use infinitesimals and infinities with little-o and big-O notation. However, unlike real numbers, we often do not declare them. And this may be a problem, because, unlike real numbers, they can have radically different properties.

So here is the contradiction, you use these numbers ubiquitously, for example, calculating limits, but you do not declare them as such.

Now we are not entirely adverse to the utility of this approach, things need to be done and calculations performed.

However, not even relatively modern mathematicians such as Hardy are immune. While having extensive powers of computation, he never defined an infinitesimal as a number.

Examples of infinitesimal and infinitary numbers are everywhere.

Zeno's paradoxes [2] assume a truth and argue to a paradox. Hence disproving the assumed truth. In assuming continuity as an infinity of divisibility for physical questions of motion, logical contradictions follow. Achilles, when chasing the tortoise, never catches up, as, when Achilles travels towards the tortoise, there is always some remaining distance to reach it [3].

The distance left over is of course a positive infinitesimal, a number smaller than any positive real numbers [12, Section 2: What does a sum at infinity mean?]

From [1], to counter Zeno's arguments, Atomism, while ascribed to physical theory, the indivisibility of matter and presumably time. However, Aristotle believed in the continuity of magnitude.

> Aristotle identifies continuity and discreteness as attributes applying to the category of Quantity. As examples of continuous quantities, or continua, he offers lines, planes, solids (i.e. solid bodies), extensions, movement, time and space; among discrete quantities he includes number and speech [1]

Infinities and infinitesimals are crucial in the development of calculus, before their ostracism (19th century) from mainstream calculus. We believe this is a case of 'throwing the baby out with the bath water'.

Hence, the banishment of infinitesimals was a rejection which sent them underground. The Greeks, with their geometric methods, had followed a similar story. Where they could not describe numbers, as this had to wait for the discovery of zero and the positional number system, reasoning was replaced by other arguments (Egyptian fractions for calculations and Roman numerals for date calculations remained).

In modern times, the first person to systematically study the infinitesimals and infinities was du Bois-Reymond, whose writing around 1870 onwards, on the scales of infinities [5, pp.9–21], for example $(\ldots, x^{-2}, x^{-1}, 1, x, x^2, x^3, \ldots)|_{x=\infty}$, and the ratios of infinities, which are instances of comparing functions.

> ... certain problems have "familiarized mathematicians with the use of scales of comparison other than those of powers of a variable .. This extension goes back above all to the works of P. du Bois-Reymond who was the first to approach systematically the problems of the comparison of functions in the neighborhood of a point, ... [6, Bourbaki p.157]

Cantor had a competing theory which represented the continuum with sets and he believed infinitesimals did not fit in. So our development may have been skewed by the rise of set theory, which became dominant and is heavily present in Abraham's NSA.

From the work of Abraham Robinson, infinitesimals have in recent years been made more rigorous; however they have not been made accessible, that is, easy to use. To address the inaccessibility, reformations of NSA have been constructed. However, NSA by its nature is very technical and used for high-end mathematics.

Note that the gossamer number system is also technical. However, the gossamer number system we present is more accessible. It has been built to be used with functions. We would have no idea how to implement many of the applications that later follow such as the rearrangement theorems with NSA, or the theory which is later developed. This is not a trivial distinction: the tools that a worker uses to do a job matter. We may not wish to deal with low level set theory logic while working with functions, particularly if it is unnecessary.

Our concerns are NSA's debasement of meaning. That is, the meaning and use of infinitesimals and infinities is lost or "muddied" during applications and proofs.

This is not to invalidate NSA. Indeed, later we do use NSA which as a reference is invaluable. Just as we have axiomatic geometry, does not mean that we have to reason with it. NSA is specialized, and requires more knowledge to use it, if you want to use it in the first place.

Instead, we will look where possible for an alternative. From the premise of this and later papers, it follows that Robinson's NSA is not the only way.

The paper has two primary tasks; describe the construction of the number system, and then work towards proving $*G$ is an ordered field. We also believe others can forward this work with the benefits being that the number system can be used in many ways.

These are separate goals, the justification of introducing additional complexities as we have separated the components. There is likely other ways to go about this.

This is not the same as using this mathematics. The real numbers themselves are used in so many different ways, that we expect a generalisation of them to be even more diverse.

4 Infinitesimals and infinities

A more general way of considering infinity is the realization of reaching infinity.

Infinitesimals and infinities naturally occur in any description at infinity. E.g. an infinitesimal, $\frac{1}{n}|_{n=\infty}$. The inverse is an infinity. $1/(1/n) = n|_{n=\infty} = \infty$. It would be fair to say that calculus (for the continuous variable) without infinitesimals and infinities would not exist (for example, a limit or a derivative could not exist).

If x is infinite and we realize x to infinity, then at infinity x becomes an extended real. If x is an *infinite integer*, then it is an integer at infinity. Considering Robinson's NSA, it becomes clear that with infinite integer ω, we can have an infinity of infinite integers $(\omega, \omega+1, \omega+2, \ldots)$. In this context, infinity is its own number system, where we have arrived at infinity, and it is a very large space. A lower case n at infinity will be understood as an infinite integer, the same as ω. (Though with the notation any variable can be used.) With infinitesimals the situation is similar, since given an infinity we can always construct an infinitesimal by dividing 1 by the infinity.

We can similarly construct *infinite rational numbers*. The numbers themselves need not all be infinite, but a composition of infinities and reals. E.g. $\frac{n^2+1}{n^3-2}|_{n=\infty}$. Similarly, there are *infinite surds*. E.g. $\sqrt{2}n|_{n=\infty}$. What about *infinite reals*? That is, a number which is as dense as a real, but at infinity. The infinite numbers will in many respects behave similarly to their finite counterparts.

Meaning can be attributed to expressions at infinity. We may consider $n|_{n=\infty}$ as the process of repeatedly adding 1. Similarly $\ln n|_{n=\infty}$ corresponds with summing the harmonic series. If divergent series, which are ubiquitous, are asymptotic to divergent functions, then the functions can have a geometric meaning. Another example, $\sin x|_{x=\infty}$ continually generates a sine curve at infinity.

The next jump is that from the reals to du Bois-Reymond's infinitesimals and infinities at infinity, realizing the space at infinity. To do this we separate finite and infinite numbers, therefore separating finite and infinite space. For example, integers $\mathbb{J}_<$ and infinite integers $\mathbb{J}_\infty$ at infinity. $\mathbb{J}_< < \mathbb{J}_\infty$ Definition 2.2.

Infinities and infinitesimals form their own number system. As a number, extending the reals with infinities, the infinities are a supremum for the reals, with an extended Dedekind cut, as an infinity is larger than any finite number.

Definition 4.1. *Define a positive infinite number to be "larger" than any number in* $\mathbb{R}$

Definition 4.2. *Define a negative infinite number to be "smaller" than any number in* $\mathbb{R}$

Definition 4.3. *Define an infinite number, 'an infinity' as a positive infinity or negative infinity, the set of all these being denoted* Φ^{-1}. *However exclude* $\pm\infty \notin \Phi^{-1}$ *for reversible multiplication.*

Definition 4.4. *We say a number is an "infinitesimal"* Φ *if the reciprocal of the number is an infinity. If* $\frac{1}{x} \in \Phi^{-1}$ *then* $x \in \Phi$.

Definition 4.5. *If* x *is a positive infinitesimal then* $x \in +\Phi$ *or* $x \in \Phi^+$.

Definition 4.6. *If* x *is a negative infinitesimal then* $x \in -\Phi$ *or* $x \in \Phi^-$.

Definition 4.7. *If* x *is a positive infinity or negative infinity then* $x \in +\Phi^{-1}$ *or* $-\Phi^{-1}$ *respectively.*

Definition 4.8. *Let* $\frac{1}{0} = \infty$, 0 *and* ∞ *are mutual inverses.*

Corollary 4.1. 0 *is not an infinitesimal.*

Proof. $\frac{1}{0} = \infty \notin \Phi^{-1}$. ∞ is 'infinity', not 'an' infinity (see Definition 4.3). □

0 is finite, but not an infinitesimal. 0 is a special number, which could be called a super infinitesimal. The reason for not including 0 as an infinitesimal is to make reasoning clearer by avoiding division by zero, having $R\backslash\{0\} \cup R_\infty$ (see Definition 4.21) with respect to multiplication form an abelian group.

Proposition 4.1. *A infinitesimal is less than any finite positive number.*

Proof. Solving for a comparison relation [9, Part 3]. A negative number is less than a positive number, then only the positive infinitesimal case remains. $x \in \mathbb{R}^+$; $\delta \in +\Phi$; then $\frac{1}{\delta} \in +\Phi^{-1}$, compare the infinitesimal and real number, $\delta \; z \; x$, $1 \; z \; \frac{x}{\delta}$, $\frac{1}{x} \; z \; \frac{1}{\delta}$, $\mathbb{R}^+ \; z \; +\Phi^{-1}$, by Definition 4.1 $z = <$ then $\delta < x$, $\Phi < \mathbb{R}^+$. □

Definition 4.9. *We say* $\mathbb{R}_\infty$ *is an 'infinireal' number if the number is an infinitesimal or an infinity. If* $x \in \mathbb{R}_\infty$ *then either* $x < \mathbb{R}^-$ *or* $x > \mathbb{R}^+$*.*

$$x \in \mathbb{R}_\infty \text{ then } x \in \Phi \text{ or } x \in \Phi^{-1},\ \mathbb{R}_\infty = \Phi \cup \Phi^{-1}$$

Definition 4.10. $\overline{\mathrm{R}}_\infty = \{-\infty, 0, \infty\}$ *(A realization of* $\mathbb{R}_\infty$*)*

Definition 4.11. *If* $a \notin b$ *then no element in* a *is in set* b*. Equivalent to* $\{b\}\backslash\{a\}$ *or* $\{b\} - \{a\}$*.*

For example, $\Phi \notin f$ means the variable or function contains no infinitesimals. Similarly $\Phi^{-1} \notin f$ means f contains no infinities.

$$\notin \mathbb{R}_\infty \equiv \mathbb{R} + \Phi[\mathbb{R} \neq 0]$$

Definition 4.12. *Define the numbers* 0 *and* ∞*:* $0 < |\Phi|$ *and* $|\Phi^{-1}| < \infty$*.*

Theorem 4.1. *Partitioning the number line,* $0 < \Phi^+ < \mathbb{R}^+ \cup \Phi < +\Phi^{-1} < \infty$*.*

Proof. $\delta_1, \delta_2, \delta_3 \in \Phi$; $x \in \mathbb{R}^+$; $z \in \mathbb{B}$ a binary relation. Consider $\delta_1\ z\ x + \delta_2$, $\delta_1 - \delta_2\ z\ x$. Since $(\Phi, +)$ is closed, let $\delta_3 = \delta_1 - \delta_2$. $\delta_3\ z\ x$. By $\Phi < \mathbb{R}^+$ (Proposition 4.1) $z = <$, since adding and subtracting on both sides does not change the inequality. $\square$

Given the existence of the infinite integers, a constructive definition $*G$ Definition 4.21 of an extended real number system follows. The number system at infinity is isomorphic with the real number system, defining integers, rational numbers, algebraic numbers, irrational numbers and transcendental numbers all at infinity.

Definition 4.13. *Let* $\mathbb{A}_<$ *be the symbol for the finite algebraic numbers.*

Definition 4.14. *Let* $\mathbb{A}'_<$ *be the symbol for the finite transcendental numbers.*

Definition 4.15. *Define an "infinite integer"* $\mathbb{J}_\infty$ *at infinity, larger in magnitude than any finite integer.*

Definition 4.16. *Define an "infinite natural number"* $\mathbb{N}_\infty$ *as a positive infinite integer.*

Definition 4.17. *Define an "infinite rational number"* $\mathbb{Q}_\infty$ *as a ratio of finite integers and "infinite integers".*

$$\{\frac{\mathbb{J}_\infty}{\mathbb{J}_\infty}, \frac{\mathbb{J}_<}{\mathbb{J}_\infty}, \frac{\mathbb{J}_\infty}{\mathbb{J}_<}\} \in \mathbb{Q}_\infty$$

Definition 4.18. *Define an "infinite algebraic number"* $\mathbb{A}_\infty$ *which is the root of a non-zero finite polynomial in one variable with at least one infinite rational number coefficient.*

Definition 4.19. *Define an "infinite irrational number"* $\mathbb{Q}'_\infty$ *as an infinity that is not an infinite rational number.*

Definition 4.20. *Define an "infinite transcendental number" $\mathbb{A}'_\infty$ as an infinity which is not an infinite algebraic number.*

Definition 4.21. *Define the gossamer numbers, $*G$ as numbers that comprise of Definitions 4.15–4.20.*

Example 4.1. *Given a fraction of the form $\frac{\mathbb{J}_\infty}{\mathbb{J}_\infty}$, $\frac{2(n+1)}{n+1}|_{n=\infty} = \frac{2}{1}$ cancelling like $\mathbb{J}_\infty$ terms leaves a fraction of the form $\frac{\mathbb{J}_<}{\mathbb{J}_<} \in \mathbb{J}_<$.*

As a byproduct of the $*G$ construction, $\mathbb{R}$ is embedded within Definition 4.21, as there exist $\frac{\mathbb{J}_\infty}{\mathbb{J}_\infty} \in \mathbb{J}_<$. We could choose to define $\mathbb{Q}_\infty$ to exclude $\mathbb{Q}_<$. Then all $\{\mathbb{J}_\infty, \mathbb{Q}_\infty, \mathbb{Q}'_\infty, \mathbb{A}_\infty, \mathbb{A}'_\infty\}$ would contain an explicit infinity within the number. By definition all infinireals $\mathbb{R}_\infty$ contain an infinity as an element within the number.

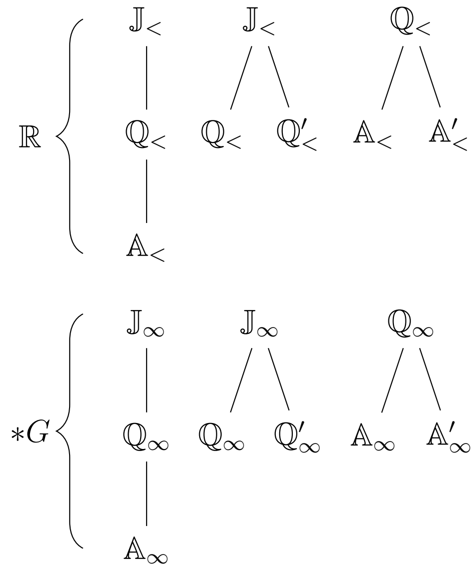

Figure 1: real and gossamer number composition

A hierarchy diagram for the number systems shows the parent number system which is used to build the child number system (see Figure 1). In this way, all reals are composed of integers and gossamer numbers are composed of integers and infinite integers. The reals are embedded within the gossamer numbers.

Remark: 4.1. *We have constructed the real numbers not with $\mathbb{J}$ but $\mathbb{J}_<$ as the real numbers cannot contain infinities. In this way, we have also provided a construction of the real numbers. Restating with an explicit construction: Define a rational number $\mathbb{Q}_<$ as a ratio of $\mathbb{J}_<$ without division by zero. Define an irrational number $\mathbb{Q}'_<$ which is not a rational number and finite. Define the finite algebraic number $\mathbb{A}_<$ as a root of a non-zero polynomial in one variable with at least one rational coefficient $\mathbb{Q}_<$ and finite. Define a transcendental number $\mathbb{A}'_<$ which is not algebraic $\mathbb{A}_<$ and finite.*

Remark: 4.2. *While a finite number is not an infinity, the collection of finite numbers is an infinity, as there is no greatest finite number. Hence, the countability of the finite numbers and the finite numbers is separated. This apparent paradox is explained by any instance of a finite number begin less than the whole. (infinity is non-unique and has other possibilities)*

Definition 4.22. *We define extended gossamer numbers* $*\overline{G}$, *where* $*\overline{G} = *G \cup \pm\infty$

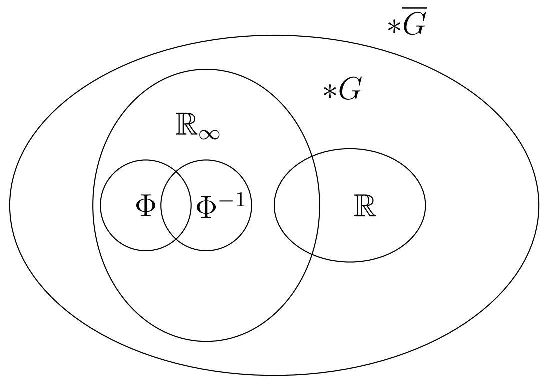

Figure 2: gossamer numbers and infinireals composition (Not a Venn diagram)

The infinitary component generally is not unique. Where the intersection in Figure 2 represents adding different components, between the numbers there exist overlaps. Infinity dominates, with any combination of non-infinity numbers with infinity resulting in an infinity.

Definition 4.23. *Let* Φ_*^{-1} *be a number type, the infinireal infinity without reals or infinitesimals, under addition in the base. If* $x \in \Phi^{-1} \backslash \{\Phi, \mathbb{R}\}$ *then* $x \in \Phi_*^{-1}$.

$$\Phi_*^{-1} = \Phi^{-1} - \mathbb{R} - \Phi$$

The following sets are disjoint and partition Φ^{-1}.

$$\{\Phi_*^{-1} + \Phi\} \cup \{\Phi_*^{-1} + \mathbb{R}\backslash\{0\}\} \cup \{\Phi_*^{-1} + \mathbb{R}\backslash\{0\} + \Phi\} \cup \{\Phi_*^{-1}\} = \Phi^{-1}$$

Example 4.2. *Example of numbers and their relation to Figure 2.* $n + \frac{1}{n}|_{n=\infty} \in \Phi^{-1} + \Phi$ *has an infinitary and infinitesimal component;* $\sqrt{2} + \frac{1}{n} \in \mathbb{R} + \Phi$ *has both a real and an infinitesimal component.*

If we compare $\mathbb{R}_\infty$ and $\mathbb{R}$, $\mathbb{R}_\infty$ have an infinity within the number itself, an infinite variable, and the reals do not, hence the reals are constant. We could then conceive of the numbers $\mathbb{R}_\infty$ and $\mathbb{R} + \Phi$ as ever changing and containing an infinity, something of which the reals are not allowed.

Having gossamer numbers, it would logically follow to define complex gossamer numbers ($a, b \in *G$; $a + bi$). Given how useful complex numbers are, this would enable complex numbers with infinitesimals and infinities.

Converting a given element of $*G$ into an algebraic form can help identify what type the number is. If this is not possible, and the number is not of the other simpler types, then the number is generally an infinite transcendental number.

Example 4.3. *Let* $k \in \mathbb{J}_<$, $k = \frac{k}{1} \in \mathbb{Q}_<$. *Let* $n \in \mathbb{J}_\infty$, $n = \frac{n}{1} \in \mathbb{Q}_\infty$.

Example 4.4. $n|_{n=\infty}$ *is an integer at infinity in* Φ^{-1}. $n \in \mathbb{J}_\infty$. *We can compare infinite integers with each other in the same way we can compare integers with other integers.* $n < n+1 < n+2 < \ldots |_{n=\infty}$

Example 4.5. $n^2 + \frac{1}{n}|_{n=\infty} \in \mathbb{Q}_\infty$ *because* $n^2 + \frac{1}{n} = \frac{n^3+1}{n}|_{n=\infty}$ *of form* $\frac{\mathbb{J}_\infty}{\mathbb{J}_\infty} \in \mathbb{Q}_\infty$.

Example 4.6. $y^2+2y+1 = 2n^2|_{n=\infty}$ *is of form* $\mathbb{J}_<\, y^2 + \mathbb{J}_<\, y + \mathbb{J}_< = \mathbb{J}_\infty$, $\mathbb{Q}_<\, y^2 + \mathbb{Q}_<\, y + \mathbb{Q}_< = \mathbb{Q}_\infty$ *is in algebraic form,* $y \in \mathbb{A}_\infty$.

Example 4.7. *Show* $\sqrt{2}n|_{n=\infty} \in \mathbb{A}_\infty$. *Let* $y = 2^{\frac{1}{2}}n$, $y^2 = 2n^2$, $2n^2 = \frac{2n^2}{1} \in \mathbb{Q}_\infty$, $y^2 + 0y^1 - \frac{2n^2}{1}y^0 = 0$, $\mathbb{Q}_<\, y^2 + \mathbb{Q}_<\, y^1 + \mathbb{Q}_\infty y^0 = 0$, $y \in \mathbb{A}_\infty$.

If we consider a number in $*G$ as composed of three components, infinitesimals, reals, and infinities while the number is unique, this composition is not. $n+2|_{n=\infty} \in \Phi^{-1}$ is an infinity, but is composed of both an infinity and an integer. This is because $\mathbb{R}^+ < n+2|_{n=\infty}$, and partitions the reals, hence the number is an infinity, but $2 \in \mathbb{J}_<$, $2+n \in \mathbb{J}_< + \mathbb{J}_\infty \in +\Phi^{-1}$.

As a consequence of the infinitesimals Φ and infinities Φ^{-1} definitions, they are not symmetrically defined. $\frac{1}{n}+3|_{n=\infty} \notin \Phi$, but $n^2+1|_{n=\infty} \in \Phi^{-1}$. An infinity can have an arbitrary real or infinitesimal part, but an infinitesimal cannot have either a real or an infinity part.

Proposition 4.2. *The gossamer number system comprises of reals and infinirreal numbers.*

$$*G = \Phi + \mathbb{R} + \Phi^{-1}$$

However, this does not mean that $*G$ as a number cannot be uniquely represented. A unique representation may aid with theorems and proofs. We define the following infinity to exclude infinitesimals and real numbers in an addition definition. Using Φ_*^{-1}, a gossamer number can be uniquely represented as three components, Φ, $\mathbb{R}$, Φ_*^{-1}. As a vector of three independent components.

Definition 4.24. *Uniquely represent* $*G$ *by three independent components.*

$$a \in *G \text{ then } a = (\Phi, \mathbb{R}, \Phi_*^{-1})$$

Any number in $*G$ is in one of the following seven forms: Φ, $\Phi + \mathbb{R}$, $\Phi + \Phi_*^{-1}$, $\Phi + \mathbb{R} + \Phi_*^{-1}$, $\mathbb{R}$, $\mathbb{R} + \Phi_*^{-1}$, Φ_*^{-1}

For proofs, with the component representation, we can define addition and multiplication between two unique numbers in $*G$, for the components which we know and do not know.

Definition 4.25. *Addition $a + b$; $a, b \in *G$;*

$$a + b = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + a_2, b_1 + b_2, a_3 + b_3)$$

We know $\mathbb{R}\backslash\{0\} \cdot \Phi \in \Phi$ (Proposition 6.44), $\mathbb{R}\backslash\{0\} \cdot \Phi^{-1} \in \Phi^{-1}$ (Proposition 6.45), $\mathbb{R} \cdot \mathbb{R} \in \mathbb{R}$, $\Phi \cdot \Phi \in \Phi$ (Proposition 6.14), and $\Phi^{-1} \cdot \Phi^{-1} \in \Phi^{-1}$ (Proposition 6.15), hence for these multiplications we can determine which components they belong to.

Definition 4.26. *Multiplication $a \cdot b$; $a, b \in *G$;*

$$a \cdot b = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3)$$

$$a \cdot b = (a_1 b_2 + a_2 b_1 + a_1 b_1, a_2 b_2, a_3 b_3 + a_3 b_2 + a_2 b_3) + a_1 b_3 + a_3 b_1$$

Remark: 4.3. *We do not know the product type of $a_1 b_3$ and $a_3 b_1$ as the numbers could be in any or multiple categories. This is the familiar indeterminate case $0 \cdot \infty$. $\frac{5}{n+1} \cdot n|_{n=\infty}$ $= 5 - \frac{5}{n+1} \in \mathbb{R} + \Phi$*

We have constructed a number system, and from this construction believe the following properties, while not proved, are true. This would require a proof that $*G$ is a total order field Proposition 6.43, in perhaps a similar way to a proof of $\mathbb{R}$ as a total order field.

5 Properties

We believe the $*G$ construction is valid is actually a field, but we are less certain about the cardinality. This could be a consequence of the representations and definitions, or possible errors, or that by doing the mathematics in another way, some insight may have been gained.

The following may be of interest to the set theoretical mathematicians. We do not consider one-one correspondence over an infinity of infinite intervals to be correct, but over a finite number of infinite intervals to be true. A finite number of infinite sets is not an infinite number of infinite sets.

To explain why the one-one correspondence is different, we believe that by including infinities within the set we are effectively increasing the cardinality.

Definition 5.1. *For determining cardinality, a one-one correspondence over an infinite interval is a one-to-one correspondence between a finite number of infinite intervals.*

Theorem 5.1. *The cardinality of $\mathbb{J}_\infty$ is larger than the cardinality of $\mathbb{J}$*

Proof. $1 \ldots n$ cannot be put in one-one correspondence with $\mathbb{J}_\infty$, as one element of $\mathbb{J}_\infty$ alone can be put into one-one correspondence with $\mathbb{J}$. Let $k \in \mathbb{J}_\infty$, placing k in one-one correspondence with $\mathbb{J}$, $(\ldots, (-2, k-2), (-1, k-1), (0, k), (1, k+1), \ldots)$. Since there is an infinity of such k elements, unlike the diagonalization argument with 2 infinite axes (which proves rational numbers have the same cardinality as the integers), this case has no one-one correspondence. E.g. Consider $(k) = (n, n^2, n^3, n^4, \ldots)|_{n=\infty}$. If this sequence were finite it could be put into a one-one correspondence, but it is not.

A simpler way, countable infinities describe, $1 \ldots n$, $1 \ldots n^2$, $\ldots$, $1 \ldots n^w|_{n=\infty}$ for finite w, but not infinite w. □

This may have consequences for the continuum hypothesis: that there is no cardinality between the integers and the real numbers. However, if the continuum hypothesis is true, the infinite-integers would have greater than or equal to cardinality than the real numbers.

Theorem 5.1 is a different one-one correspondence from Cantor. With non-uniqueness at infinity, there may be other mathematics here, so the truth or falsehood is relative to the mathematical system chosen, if one is chosen at all.

This also has a consequence for the cardinality of the hyperreals which by the current theory has the same cardinality as the real numbers [17]. We believe that this is incorrect, and hence the current theory is not modelling the situation. In contrast, our theory would result in the reals having a lower cardinality than that of the hyperreals.

We really believe that the hyperreals and $*G$ have the same cardinality.

Intuitively, we do not agree that $*G$ has the same cardinality as $\mathbb{R}$, particularly as we have defined $\mathbb{R}_<$ as the real numbers, which are devoid of any infinity n terms.

The nature of the gossamer numbers gives reason to believe infinitesimals and infinities are much more dense than reals, for about any real number there is an infinity of infinitesimal numbers, which map back to a unique real number. [10, Part 4]

> The hyperreal point of view is that the geometric line is capable of sustaining a much richer and more intricate set than the real line. [16, p.14]

The preceding quote does seem to contradict the current position of the hyperreals having the same cardinality as the reals. Why would this property not be reflected in the cardinality?

Conjecture 5.1. *The cardinality of the gossamer numbers is larger than the cardinality of the reals.*

Since the construction of the number system of reals and gossamer numbers is identical except with different number types, and the cardinality of the infinite integers $\mathbb{J}_\infty$ is larger than the cardinality of the finite integers $\mathbb{J}_<$, then the input having different cardinality results in an output with different cardinality.

Another approach is to consider the solution space, by comparing the transcendental solution space with the infinite transcendental solution space. Let $|\mathbb{N}_\infty|$ describe the infinite integers cardinality.

$a_k \in \mathbb{Q}$; $\sum_{\mathbb{N}_\infty} a_k x^k = 0$ has $\aleph_0^{|\mathbb{N}_\infty|}$ solution space.
$b_k \in \mathbb{Q}_\infty$; $\sum_{\mathbb{N}_\infty} b_k x^k = 0$ has $|\mathbb{N}_\infty|^{|\mathbb{N}_\infty|}$ solution space.
Solving with comparison algebra shows $|\mathbb{N}_\infty|^{|\mathbb{N}_\infty|} \succ \aleph_0^{|\mathbb{N}_\infty|}$ and the gossamer numbers have a larger cardinality.

After scales of infinities (also see [8]), another geometric example at infinity involves the visualization of infinitesimally close curves. We can have infinitely many curves, infinitesimally close to a single curve, separated by infinitely small distances, $f(x) - g(x) \in \Phi$ (see [9, Example 2.20]).

Infinitesimals are required to describe such a space, in the same way that real numbers are needed to extend rational numbers, and rational numbers to extend integers.

Infinitesimal and infinity exclusion can be compared with removing other classes of numbers, and is staggering. Not even Apostol [4] discusses infinitesimals or infinities as their own number system, but implicitly uses infinitesimals and infinities when convenient. Given the banishment of infinitesimals from calculus in general, for example, their exclusion as a number from Apostol and generally every modern calculus text (we cited Apostle as this is highly regarded, but does not describe infinitesimals as numbers). Such a choice may be understandable, but it is not complete.

What follows is a construction of an extended real number system, we call $*G$, which includes infinitesimals and infinities, which is similar to the hyperreals, but different.

We differentiate between infinitesimals and zero. Similarly we differentiate between an infinity such as $n^2|_{n=\infty}$ and the "number" ∞.

6 Field properties

Unproven propositions. Other propositions assume their truth.

Proposition 6.1 $(+\Phi^{-1},+)$ is closed
Proposition 6.2 $(+\Phi_*^{-1},+)$ is closed
Proposition 6.3 $(+\Phi,+)$ is closed
Proposition 6.4 $(\Phi\cup\{0\},+)$ is closed
Proposition 6.5 $(\Phi\cup\{0\},+)$ is commutative #
Proposition 6.6 $(\Phi\cup\{0\},+)$ is associative #
Proposition 6.7 $(\Phi_*^{-1}\cup\{0\},+)$ is closed
Proposition 6.8 $(\Phi_*^{-1}\cup\{0\},+)$ is commutative #
Proposition 6.9 $(\Phi_*^{-1}\cup\{0\},+)$ is associative #
Proposition 6.10 $(*G,+)$ is closed
Proposition 6.11 $(*G,+)$ is commutative
Proposition 6.12 $(*G,+)$ is associative
Proposition 6.13 $(+\Phi,\cdot)$ is closed
Proposition 6.14 $(\Phi,\cdot)$ is closed
Proposition 6.15 $(\Phi^{-1},\cdot)$ is closed
Proposition 6.16 $(+\Phi_*^{-1},\cdot)$ is closed
Proposition 6.17 $(\Phi_*^{-1},\cdot)$ is closed
Proposition 6.18 $\Phi\cdot\Phi_*^{-1}\in *G\backslash\{0\}$ #
Proposition 6.19 $\Phi_*^{-1}\cdot\Phi\in *G\backslash\{0\}$ #
Proposition 6.20 $(\mathbb{R}_\infty,\cdot)\in *G\backslash\{0\}$
Proposition 6.21 $(*G,\cdot)$ is closed
Proposition 6.22 $(\Phi^{-1},\cdot)$ is commutative #
Proposition 6.23 $(\Phi_*^{-1},\cdot)$ is commutative
Proposition 6.24 $(\Phi,\cdot)$ is commutative
Proposition 6.25 $(\Phi,\Phi_*^{-1},\cdot)$ is commutative #
Proposition 6.26 $(*G\backslash\{0\},\cdot)$ is commutative
Proposition 6.27 $(*G\backslash\{0\},\cdot)$ is associative
Proposition 6.30 $(*G,+,\cdot)$ is distributive, $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$
Proposition 6.31 $(\Phi\cup\{0\},+)$ has identity 0
Proposition 6.32 $(\Phi_*^{-1}\cup\{0\},+)$ has identity 0
Proposition 6.33 $(*G,+)$ has identity $(0,0,0)$
Proposition 6.34 $(\Phi^{-1}\cup\{0\},+)$ has an inverse
Proposition 6.35 $(\Phi_*^{-1}\cup\{0\},+)$ has an inverse
Proposition 6.36 $(\Phi\cup\{0\},+)$ has an inverse
Proposition 6.37 $(*G,+)$ has an inverse
Proposition 6.38 $(*G,+)$ is an abelian group
Proposition 6.39 $(*G\backslash\{0\},\cdot)$ has an identity
Proposition 6.40 $(*G\backslash\{0\},\cdot)$ has inverse
Proposition 6.41 $(*G\backslash\{0\},\cdot)$ is a group
Proposition 6.42 $*G$ is a field
Proposition 6.43 $*G$ is a total order field. $a,b,c\in *G$; If $a\leq b$ then $a+c\leq b+c$. If $0\leq a$ and $0\leq b$ then $0\leq a\times b$. #
Proposition 6.44 $\mathbb{R}\backslash\{0\}\cdot\Phi\in\Phi$ #
Proposition 6.45 $\mathbb{R}\backslash\{0\}\cdot\Phi^{-1}\in\Phi^{-1}$ #
Proposition 6.46 $\mathbb{R}\backslash\{0\}\cdot\mathbb{R}_\infty\in\mathbb{R}_\infty$

Proposition 6.47 Let $\theta \in *G\backslash\{0\}$, $\theta > 0$, $z \in \{<, \leq, ==, >, \geq\}$. $f\ (z)\ g \Leftrightarrow \theta \cdot f\ (z)\ \theta \cdot g$ #
Proposition 6.48 Let $z \in \{<, \leq, ==, >, \geq\}$. The relations z invert when multiplied by a negative number in $*G$. For the equality case the left and right sides of the relation are interchanged.

Proposition 6.1. *$(+\Phi^{-1}, +)$ is closed*

Proof. $a, b \in +\Phi^{-1}$; since $b > 0$, $a + b > a$, $a + b \in +\Phi^{-1}$ □

Proposition 6.2. *$(+\Phi_*^{-1}, +)$ is closed*

Proof. Since the components are separate, by definition; $a, b \in +\Phi_*^{-1}$; $a + b = (0, 0, a_3) + (0, 0, b_3) = (0, 0, a_3 + b_3) \in +\Phi_*^{-1}$ since adding two positive numbers are also positive. □

Proposition 6.3. *$(+\Phi, +)$ is closed*

Proof. $a, b \in +\Phi^{-1}$; $\frac{1}{a}, \frac{1}{b} \in \Phi$; $\frac{1}{a} + \frac{1}{b} = \frac{b+a}{ab}$, since $ab \succ a + b$ then $\frac{a+b}{ab} \in \Phi$. □

Proposition 6.4. *$(\Phi \cup \{0\}, +)$ is closed*

Proof. $a, b \in +\Phi^{-1}$; For both positive number and negative numbers are closed by Proposition 6.3. Consider case adding infinitesimals of opposite sign, $\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$. If $a = b$ then $\frac{1}{a} - \frac{1}{b} = \frac{0}{ab} = 0$ is closed. If $a \neq b$ then $b - a \prec ab$ and $\frac{b-a}{ab} \in \Phi$ is closed. □

Proposition 6.5. *$(\Phi \cup \{0\}, +)$ is commutative*
Proposition 6.6. *$(\Phi \cup \{0\}, +)$ is associative*
Proposition 6.7. *$(\Phi_*^{-1} \cup \{0\}, +)$ is closed*

Proof. Since the components are separate, by definition; $a, b \in \Phi_*^{-1}$; $a + b = (0, 0, a_3) + (0, 0, b_3) = (0, 0, a_3 + b_3) \in \Phi_*^{-1}$ or $(0, 0, 0)$. □

Proposition 6.8. *$(\Phi_*^{-1} \cup \{0\}, +)$ is commutative*
Proposition 6.9. *$(\Phi_*^{-1} \cup \{0\}, +)$ is associative*
Proposition 6.10. *$(*G, +)$ is closed*

Proof. Since $(\Phi, +)$ is closed, $(\mathbb{R}, +)$ is closed, and $(\Phi_*^{-1}, +)$ is closed, then, as an addition in $*G$ is the sum of three independent components are all closed. □

Proposition 6.11. *$(*G, +)$ is commutative*

Proof. Since the components (Propositions 6.5, 6.8 and $(\mathbb{R}, +)$) are independent, and commutative. □

Proposition 6.12. *$(*G, +)$ is associative*

Proof. Since the components (Propositions 6.6, 6.9 and $(\mathbb{R}, +)$) are independent, and associative. $\square$

Proposition 6.13. $(+\Phi, \cdot)$ *is closed*

Proof. $a, b \in +\Phi^{-1}$; $\frac{1}{a}, \frac{1}{b} \in +\Phi$; $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab} \in +\Phi$ as $ab \in +\Phi^{-1}$. $\square$

Proposition 6.14. $(\Phi, \cdot)$ *is closed*

Proof. Since the sign can be factored out, consider the positive case only, by Proposition 6.13 this is true. $\square$

Proposition 6.15. $(\Phi^{-1}, \cdot)$ *is closed*

Proof. Since a negative sign can be factored out, only need to consider positive infinities. $x, y \in +\Phi^{-1}$; choose $x \leq y$, then $x \leq y \leq xy$, as $x > 1$. $xy \in \Phi^{-1}$ by Definition 4.1. $\square$

Proposition 6.16. $(+\Phi_*^{-1}, \cdot)$ *is closed*

Proof. $a, b \in +\Phi^{-1}$; $\frac{1}{a}, \frac{1}{b} \in +\Phi$; $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab} \in +\Phi$ as $ab \in +\Phi^{-1}$. $\square$

Proposition 6.17. $(\Phi_*^{-1}, \cdot)$ *is closed*

Proof. Factor out the negative sign, since $(+\Phi_*^{-1}, \cdot)$ Proposition 6.16 then closed. $\square$

Proposition 6.18. $\Phi \cdot \Phi_*^{-1} \in *G\backslash\{0\}$
Proposition 6.19. $\Phi_*^{-1} \cdot \Phi \in *G\backslash\{0\}$
Proposition 6.20. $(\mathbb{R}_\infty, \cdot) \in *G\backslash\{0\}$

Proof. From Propositions 6.18 and 6.19. $\square$

Proposition 6.21. $(*G, \cdot)$ *is closed*

Proof. Any product with 0 results in 0. Consider non-zero products, since multiplying by the components is closed, and multiplying by an infinitesimal and infinity is in $*G\backslash\{0\}$ by Propositions 6.18 and 6.19, then by the multiplication of components Definition 4.26, the product is closed. $\square$

Proposition 6.22. $(\Phi^{-1}, \cdot)$ *is commutative*
Proposition 6.23. $(\Phi_*^{-1}, \cdot)$ *is commutative*

Proof. By Proposition 6.22, a subset is also commutative. $a, b \in \Phi$; $a', b' \in \Phi_*$; If $a \cdot b = b \cdot a$, and let the unique expressions be $a' = a$, $b' = b$, then it follows by substitution $a' \cdot b' = b' \cdot a'$. $\square$

Proposition 6.24. $(\Phi, \cdot)$ *is commutative*

Proof. $a, b \in \Phi^{-1}$; By Proposition 6.22 the infinities are commutative, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab} = \frac{1}{ba}$ $= \frac{1}{b} \cdot \frac{1}{a}$ ☐

Proposition 6.25. $(\Phi, \Phi_*^{-1}, \cdot)$ *is commutative*

Proposition 6.26. $(*G \backslash \{0\}, \cdot)$ *is commutative*

Proof. Since the components are commutative, and multiplication between infinitesimals and infinities is commutative, then general multiplication is commutative. ☐

Proposition 6.27. $(*G \backslash \{0\}, \cdot)$ *is associative*

Proof. Using Maxima symbolic mathematics package. Implement multiplication, with the fourth element in an unknown status.

```
f4(a1,a2,a3,a4,b1,b2,b3,b4) := [expand(a1*b2+a2*b1+a1*b1), expand( a2*b2), expand(a3*b3+a3*b2+a2*
expand( a1*b3+a3*b1 + a4*(b1+b2+b3) + b4*(a1+a2+a3)+a4*b4)];
f5(a,b) := f4( a[1], a[2], a[3], a[4], b[1], b[2], b[3], b[4] );
```

The following gave the same output, proving the associativity.

```
f5( f5( [a1,a2,a3,0], [b1,b2,b3,0]), [c1,c2,c3,0] );
f5( [a1,a2,a3,0], f5( [b1,b2,b3,0], [c1,c2,c3,0] ) );
```

$[a1\, b2\, c2 + a2\, b1\, c2 + a1\, b1\, c2 + a2\, b2\, c1 + a1\, b2\, c1 + a2\, b1\, c1 + a1\, b1\, c1,\ a2\, b2\, c2, a3\, b3\, c3 + a2\, b3\, c3 + a3\, b2\, c3 + a2\, b2\, c3 + a3\, b3\, c2 + a2\, b3\, c2 + a3\, b2\, c2,\ a1\, b3\, c3 + a1\, b2\, c3 + a3\, b1\, c3 + a2\, b1\, c3 + a1\, b1\, c3 + a1\, b3\, c2 + a3\, b1\, c2 + a3\, b3\, c1 + a2\, b3\, c1 + a1\, b3\, c1 + a3\, b2\, c1 + a3\, b1\, c1]$ ☐

Proposition 6.28. $(\Phi_*^{-1} \cup \{0\}, +, \cdot)$ *is distributive*

Proposition 6.29. $(\Phi \cup \{0\}, +, \cdot)$ *is distributive*

Proposition 6.30. $(*G, +, \cdot)$ *is distributive,* $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Proof. $a(b+c) = (a_1, a_2, a_3) \cdot (b_1 + c_1, b_2 + c_2, b_3 + c_3) = (a_1(b_2 + c_2) + a_2(b_1 + c_1) + a_1(b_1 + c_1), a_2(b_2+c_2), a_3(b_3+c_3)+a_3(b_2+c_2)+a_2(b_3+c_3))\ +a_1(b_3+c_3)+a_3(b_1+c_1) = (a_1b_2+a_1c_2+a_2b_1+a_2c_1+a_1b_1+a_1c_1, a_2b_2+a_2c_2, a_3b_3+a_3c_3+a_3b_2+a_3c_2+a_2b_3+a_2c_3)\ +a_1b_3+a_1c_3+a_3b_1+a_3c_1$

$ab + ac = (a_1, a_2, a_3)(b_1, b_2, b_3) + (a_1, a_2, a_3)(c_1, c_2, c_3)$
$= [(a_1b_2 + a_2b_1 + a_1b_1, a_2b_2, a_3b_3 + a_3b_2 + a_2b_3) + a_1b_3 + a_3b_1] + [(a_1c_2 + a_2c_1 + a_1c_1, a_2c_2, a_3c_3 + a_3c_2 + a_2c_3) + a_1c_3 + a_3c_1]$
$= (a_1b_2 + a_2b_1 + a_1b_1 + a_1c_2 + a_2c_1 + a_1c_1, a_2b_2 + a_2c_2, a_3b_3 + a_3b_2 + a_2b_3 + a_3c_3 + a_3c_2 + a_2c_3) + a_1b_3 + a_3b_1 + a_1c_3 + a_3c_1$

Consider that the three components are distributive, Propositions 6.28 and 6.29 and $\mathbb{R}$. They are also commutative, Propositions 6.8 and 6.5 and $\mathbb{R}$. Then the expressions are equal. ☐

Proposition 6.31. $(\Phi \cup \{0\}, +)$ *has identity* 0

Proof. $a \in +\Phi^{-1}$; $\frac{1}{a} + 0 = \frac{1}{a} + \frac{0}{a} = \frac{1+0}{a} = \frac{1}{a}$. ☐

Proposition 6.32. $(\Phi_*^{-1} \cup \{0\}, +)$ *has identity* 0

Proof. By Proposition 6.31 since true for any Φ^{-1}. $\square$

Proposition 6.33. *$(*G,+)$ has identity $(0,0,0)$*

Proof. Solve $a+b=a$ for identity b. $(a_1,a_2,a_3)+(b_1,b_2,b_3)=(a_1,a_2,a_3)$ then by the components independence, $a_1+b_1=a_1$, $b_1=0$ by Proposition 6.12, $a_2+b_2=a_2$, $b_2=0$ as a real number, $a_3+b_3=a_3$, $b_3=0$ by Proposition 6.32. $\square$

Proposition 6.34. *$(\Phi^{-1}\cup\{0\},+)$ has an inverse*

Proof. By assumption that any number in $*G$ can have an integer coefficient multiplier of ± 1; $x\in\Phi^{-1}$; consider $x+(-x)=x(1+-1)=x\cdot 0=0$ $\square$

Proposition 6.35. *$(\Phi_*^{-1}\cup\{0\},+)$ has an inverse*

Proof. By Proposition 6.34 as true for any Φ^{-1}. $\square$

Proposition 6.36. *$(\Phi\cup\{0\},+)$ has an inverse*

Proof. $a,b\in+\Phi^{-1}$; $\frac{1}{a}+\frac{1}{b}=0$, $\frac{b}{ba}+\frac{a}{ba}=\frac{b+a}{ba}=0$ by Proposition 6.34 when $b=-a$, and inverse of $\frac{1}{a}$ is $-\frac{1}{a}$. $\square$

Proposition 6.37. *$(*G,+)$ has an inverse*

Proof. Since each component has an inverse, Propositions 6.35 and 6.36, $a+b=(a_1,a_2,a_3)+(b_1,b_2,b_3)=(0,0,0)$, $b=(-a_1,-a_2,-a_3)$. $\square$

Proposition 6.38. *$(*G,+)$ is an abelian group*

Proof. Group properties: closure Proposition 6.10, associativity Proposition 6.12, identity Proposition 6.33, inverse Proposition 6.37. Abelian group by Proposition 6.11. $\square$

Proposition 6.39. *$(*G\backslash\{0\},\cdot)$ has an identity 1*

Proof. $(a_1,a_2,a_3)\cdot(0,1,0)=(a_1\cdot 1+a_2\cdot 0+a_1\cdot 1,a_2\cdot 1,a_3\cdot 0+a_3\cdot 1+a_2\cdot 0)+a_1\cdot 0+a_3\cdot 0$ $=(a_1+0+0,a_2,0+a_3+0)+0+0=(a_1,a_2,a_3)$ $\square$

Proposition 6.40. *$(*G\backslash\{0\},\cdot)$ has inverse*

Proof. If we consider the 7 forms of a_1b_3 and the 7 forms of a_3b_1 there are 49 combinations, leading to 49 sets of equations of the form $(a_1,a_2,a_3)\cdot(b_1,b_2,b_3)=(0,1,0)$. Since this is 3 linear equations with 3 unknowns, providing the equations do not contradict, there is always a unique solution.

Contradictory solutions are a consequence of the different cases. For example, $a_2 \neq 0$ for the following set of equations. $a_1 a_3 \in \Phi$; $a_3 b_1 \in \Phi$; $a_1 b_2 + a_2 b_1 + a_1 b_1 + a_1 b_3 + a_3 b_1 = 0$; $a_2 b_2 = 1$; $a_3 b_3 + a_3 b_2 + a_2 b_3 = 0$; □

Proposition 6.41. $(*G \backslash \{0\}, \cdot)$ *is a group*

Proof. Properties: closure Proposition 6.21, associativity Proposition 6.27, identity Proposition 6.39, inverse Proposition 6.40. □

Proposition 6.42. $*G$ *is a field*

Proof. Properties: Proposition 6.38, $(*G, +)$ is a group. Proposition 6.41, $(*G \backslash \{0\}, \cdot)$ is a group. Proposition 6.30, $(*G, +, \dot{)}$ is distributive. □

Proposition 6.43. $*G$ *is a total order field [15].* $a, b, c \in *G$*; If* $a \leq b$ *then* $a + c \leq b + c$*. If* $0 \leq a$ *and* $0 \leq b$ *then* $0 \leq a \times b$*.*

Scalar multiplication by a real number except 0 is closed.

Proposition 6.44. $\mathbb{R} \backslash \{0\} \cdot \Phi \in \Phi$
Proposition 6.45. $\mathbb{R} \backslash \{0\} \cdot \Phi^{-1} \in \Phi^{-1}$
Proposition 6.46. $\mathbb{R} \backslash \{0\} \cdot \mathbb{R}_\infty \in \mathbb{R}_\infty$

Proof. $\mathbb{R}_\infty$ is the above infinitesimal and infinity cases, Propositions 6.44 and 6.45. □

Proposition 6.47. $\theta \in *G \backslash \{0\}$*; if* θ *is positive,* $z \in \{<, \leq, ==, >, \geq\}$*.*

$$f\ (z)\ g \Leftrightarrow \theta \cdot f\ (z)\ \theta \cdot g$$

Proposition 6.48. $z \in \{<, \leq, ==, >, \geq\}$*; The relations* z *invert when multiplied by a negative number in* $*G$*. For the equality case the left and right sides of the relation are interchanged.*

Proof. $a, b \in *G$; $a\ z\ b$, $-a\ (-z)\ -b$, $-a + b\ (-z)\ 0$, $b\ (-z)\ a$. □

7 Conclusion

It is with disbelief that so many practising mathematicians have little explicit application of infinitesimals; that the online forums have so little discussion and that only the specialized few use NSA. This must change. Without explicitly partitioning the finite and infinite, there is a world of analysis that will not see the light of day.

The gossamer number system structure is the simplest explanation (Occam's razor) for infinitesimals and infinities. The construction is identical with the implicit equation construction of real numbers except where the integers, the building blocks of the real number system, are replaced with 'infinite integers'.

Since $*G$ (we believe) is a field, the theory is general. For example, you could perhaps plug $*G$ into Fourier analysis theory, and extend the theory.

Du Bois-Reymond expressed the numbers like we do today as functions. Largely to avoid accounts of the numbers being fiction and devoid of real world meaning.

However, had he realised the number system (which we believe needed the discovery of the infinite integers), he could have expressed his theory with two number systems $\mathbb{R}$ and $*G$ or $*R$. Many others, including Newton, Leibniz, Cauchy, Euler and Robinson have considered these questions. A two-tiered number system $\mathbb{R}$ and $*G$ has evolved, not as an option, but as a necessity in explaining mathematics.

We have attempted to find a rigorous formulation to "traditional non-rigorous extensions", which we believe has not been considered. By construction, the zero divisors problem relevant to the hyperreals is avoided, as in $*G$ there is only one zero.

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In response, better communication with numerous suggestions and extensions were subsequently applied, and the construction of the gossamer number system was found.

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