Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory
Part 2 The much greater than relations

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Abstract

An infinitesimal and infinitary number system the Gossamer numbers is fitted to du Bois-Reymond's infinitary calculus, redefining the magnitude relations. We connect the past symbol relations much-less-than $\prec$ and much-less-than or equal to $\preceq$ with the present little-o and big-O notation, which have identical definitions. As these definitions are extended, hence we also extend little-o and big-O, which are defined in Gossamer numbers. Notation for an reformed infinitary calculus, calculation at a point is developed. We proceed with the introduction of an extended infinitary calculus.

1 Introduction

1. Introduction
2. Evaluation at a point
3. Infinitary calculus definitions
4. Scales of infinity
5. Little-o and big-O notation

> While the majority of mathematicians readily accepted the emancipation of analysis from geometry there were, nonetheless, powerful voices raised against the arithmetization programs. One of the sharpest critics was Paul du Bois-Reymond (1831-1889) who saw the arithmetization as a contentless attempt to destroy the necessary union between number and magnitude. [14, p.92]

The separation between geometry and number by the arithmetization of analysis has lead to the dominance of set theory. However, just as we have different languages, problems can be described with functions or set theory and other other ideas with infinity. Language for theories is both an evolution and also can be more of a choice, but 'does' have an effect on how we see the mathematics.

We believe that arguments of magnitude are essential to understand real and gossamer numbers. Without a theory from this viewpoint many things are left without explanations. Without the relations described, the symbolism and language of algebra that they describe is harder to encapsulate.

Contradictory to du Bois-Reymond, we find arithmetization in his relations that lead to a transfer principle [10, Part 4] and non-reversible arithmetic [11, Part 5]. So we claim that they are important.

Before becoming aware of du Bois-Reymond's work, we defined $\gg$ equivalently to $\succ$, as during a mathematical modelling subject a lecturer had symbolically used the symbol to describe (without definition) large differences in magnitude.

The notion of the 'order' or the 'rate of increase' of a function is essentially a relative one [5, p.2]. Consider functions $f(x)$ and $\phi(x)$, we could have functions satisfying relation $f > \phi$. However, what about their ratio? Knowing only $>$ or $\geq$ does not give a size difference of the numbers involved.

Consider monotonic functions which over time settle down, and have properties such as their ratio is monotonic too. In examining these well behaved functions, families of ratios, scales of infinities (Section 4) are considered. From these investigations, the characterisation of an infinity in size difference was discovered, and defined as a relation $\succ$ (Definition 3.2) and $\succeq$ (Definition 3.1). Here, it is not the sign of the number, but the size of the number which determines the relation.

> With the particular system of notation that he invented, it is, no doubt, quite possible to dispense; but it can hardly be denied that the notation is exceedingly useful, being clear, concise, and expressive in a very high degree. [5, p. (v)]

However, the notation was quickly superseded by little-o and big-O, primarily because the magnitude relation, instead of being expressed separately, could be packaged as a variable. E.g. $\sin x = x + O(x^3)$ instead of $x^3 \succeq -\frac{x^3}{3!} + \frac{x^5}{5!} - \dots |_{x=0}$

We believe du Bois-Reymond's relations do have a critical place, where we develop an algebra for comparing functions [9, Part 3]. For many reasons, we introduce the at-a-point notation, which is used throughout our series of papers towards the development of an alternative to non-standard analysis which we refer to as infinitary calculus.

By fitting an infinitesimal number system to du Bois-Reymond's infinitary calculus definitions (Section 3), the theory is better explained. Instead of defining a limit in $\mathbb{R}$, the limit is defined in $*G$ the extended number system.

The benefits continued with the later development of a transfer principle [10, Part 4] between $*G$ and $\mathbb{R}$, which explains mathematics that would not makes sense without infinitesimals and infinities. General limit calculations do not make sense in $\mathbb{R}$ because the number system has no infinity elements.

This theory is then used to derive a new field of mathematics 'convergence sums' [13], with applications to convergence or divergence of positive series. Where du Bois-Reymond's theory of comparing functions had been forgotten, we now believe we have we have found useful applications.

We believe this shows value and general applicability of the mathematics. The Gossamer number system's utility is demonstrated. So, we see this as a building paper.

The relation $\succ$, is defined as equivalent to little-o, and is general because it exists in a number system which includes infinities and infinitesimals, and not a modified or implicitly defined $\mathbb{R}$. The current practise implicitly uses infinitesimals and infinities, without declaring them as their own number type [8, Part 1].

In this sense we have extended du-Bois Reymond and Hardy's work. By explicitly having a number system, we can better compare functions. In a later paper on the transfer principle, we will argue that this is not just an option, but a fundamental part of calculus.

Having said the above, the objective of this paper is to introduce definitions and notations, re-state du Bois-Reymond's infinitary calculus, and connect the past with the present little-o and big-O notation.

2 Evaluation at a point

Motivation: Approximating functions by truncation in calculation is common practice. We use infinitesimals all the time.

Example 2.1. *A quick numerical check will provide evidence by approximation, where successive powers significantly reduce in magnitude, $x = 0.1$, $x^2 = 0.01$, $x^3 = 0.001$.*

$f(x) = x + x^2 + x^3 + \ldots|_{x=0}$ at $x = 0$ we mean $x \in \Phi$ (see Definition 2.5). This may be represented by $f(x) = x$, $f(x) = x + x^2$, $f(x) = x + x^2 + x^3$ or any other number of first terms at $x = 0$. As x^2 is much smaller than x, x^3 is much smaller than x^2, ...

When we send $x \to 0$, which functions that go to zero faster matters, as these may be truncated, and we can start using infinitesimals. It is this sort of reasoning and calculation that leads to the definition of the magnitude relation (see Definitions 3.1 and 3.2), and then to little-o and big-O notation which we use today.

Some people, who find negative numbers difficult to accept, will happily add and subtract positive numbers, but are unable to do so using negative numbers. In a similar way, there may occasionally be a problem where people can reason with infinity but not zero, or the other way round. Logically 0 and ∞ as numbers are very similar.

Example 2.2. *Similar reasoning can be done with infinities. Let $x = 1/n$ and assume a solution. Consider the series first three terms. $y = \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}$, $yn^3 = n^2 + n + 1$ when*

$n = \infty$. As n is much greater than 1, assume $n+1 = n$, $yn^3 = n^2 + n$, reversing, $y = \frac{1}{n} + \frac{1}{n^2}$ and $\frac{1}{n^3}$ was truncated.

Truncation non-uniqueness in calculation: Let $f(x)$ be a function of infinite terms used in function $h(x) = g(f(x))$. Since a truncated $f(x)$ can solve $h(x)$ for infinitely many truncations, we say $f(x)$ is not unique, as an infinite number of solutions may give a satisfactory result. When evaluating $h(x)$, $f(x)$ is not unique, as an infinite number of truncated evaluations can occur. It is often desirable to use the minimum number of first terms of $f(x)$ to evaluate $h(x)$. In this way, asymptotic expansions as given by $f(x)$ are said to be non-unique.

Calculation is a major part of analysis, and one of the most common evaluations is the limit of a function. This evaluation can be thought of more generally by considering the behaviour at a point, with the inclusion of infinity as a number and as a point.

When the ideas of a point are extended to include such properties as continuity, infinity, existence and divergence at a point, then it becomes clear that a point, whatever it may be, is both what we interpret and how we calculate.

With a view to realising something more general than a limit, the following definition at a point is given.

By virtue of reaching a point, we have to pass through or approach the point. The definition of evaluation at a point will also encompass approaches to the point.

This interpretation of a point accepts non-uniqueness - two parallel lines could meet at infinity or they may never meet at infinity. In particular, asymptotic expansions are not unique, but subject to orders of magnitude.

Definition 2.1. *Let $f(x)$ be an expression. Then evaluation at-a-point $f(x)|_{x=a}$ is the evaluation of $f(x)$ at $x = a$. (Optionally omit variable assignment, $f(x)|_a$)*
Case 1. All possibilities or
Case 2. Context dependent evaluation

Infinity can be considered as a point.

Case 1 concerns itself with all the different ways a point could be interpreted and calculated, and is a conceptual tool.

Given a problem either theoretical or practical, there are often different views or interpretations which may help. (For example from a programming perspective, an object orientated approach to problem modelling.)

Let C^j describe a curve continuous in the first j derivatives, then let C^0 describe a continuous curve. When building a curve that is a function, except at a point, the following possibilities may occur(see Figure 1). The curve may be discontinuous at the point, or its vector equations are continuous but the function has infinitely many values at the point, or C^0 but not C^1 continuous, or the curve is a function and also an s-curve between an interval. With a point at infinity the possibilities are endless.

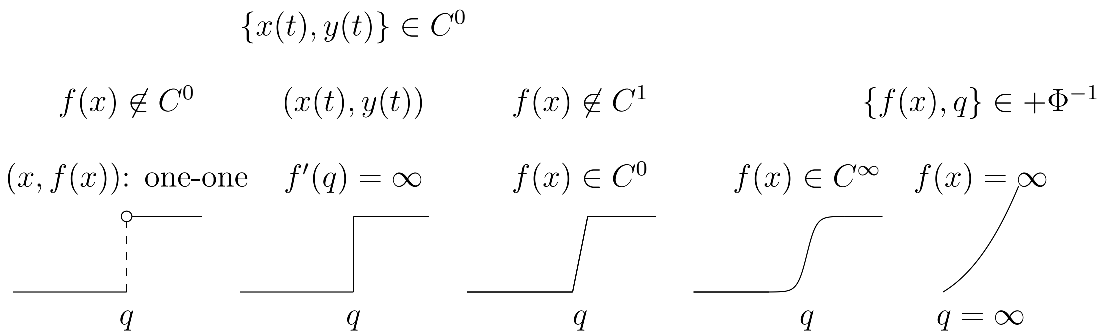

Figure 1: Examples of interpretations at a point

Case 2 is the practical aspect of calculating, where a choice of interpretation has been made, on proceeding with the "actual calculation". The context calculation separates responsibility for the justification from the theory to the point of use. This decoupling is important. If there is another way of calculating or using another branch of mathematics, the evaluation at-a-point is simply interpreted then. The trade is that less can be said, in that the definitions and theory are less exacting, but this is mitigated by the calculation being context specific, and more adaptable to our problem solving.

The consequences of decoupling can be non-trivial. For example, we do not believe in necessarily using a field when extending the reals. A trade-off for a different kind of generality may be a different number system, or chosen differently, not depending on what you want to do.

Evaluating a function at-a-point can often result in the evaluation of the limit. Indeed, this limit at a point, is a subset of the possibilities. In the context of a calculation, $\lim_{x \to a} f(x)$ can be represented by $f(x)|_{x=a}$

As mathematics is a language, a further purpose of Definition 2.1 is to communicate to the reader that other ways of calculation might be employed. For instance, where these could be incompatible with rigorous argument, one way of distinguishing differences could be through the above notation.

The notation can also be used to make existing arguments more explicit. For example $f(x) = O(g(x))|_{x=\infty}$ says that the function is being considered at infinity, not 0 or any other finite value.

Motivation for a separate notation is used so that different mathematics can work side by side with standard mathematics, and in a sense be contained. The limit concept is so 'ingrained' that doing operations that use a different paradigm, without clear communication to the reader, would be unsatisfactory.

A consequence of such flexibility is that mathematical inconsistencies can and are invariably introduced into the calculations. Where this is viable, the benefits brought to the calculation can outweigh any adverse presumptions. Designing methods to protect against inconsistencies other than narrowed definition and practice can actually make the calculus more accessible and interesting.

Definition 2.2. *Let the "at-a-point" definition, appearing on the right-hand side, apply to all functions within the expression, unless overridden by another at-a-point definition.*

Example 2.3. *$f(x)\ z\ g(x)|_{x=a}$ means $f(x)|_{x=a}\ z\ g(x)|_{x=a}$ where z is any relation. Optionally we can include round brackets around the expression, $(f(x)\ z\ g(x))|_{x=a}$.*

Definition 2.3. *We say $f(x)\ z\ g(x)|_{x=a}$ can have a context meaning, where $f(x)$ and $g(x)$ are dependent, and in some way governed by operator or relation z.*

When forming conditions for infinitely small or infinitely large, we employ a bound, which itself is going to zero or infinity; for instance, when forming definitions.

Definition 2.4. *In context, a variable x can be described at infinity $|_{x=\infty}$, then $\exists x_0, \forall x : x > x_0$*

Definition 2.5. *In context, a variable x can be described at zero $|_{x=0}$ then $\exists x_0, \forall x : |x| < x_0$*

With the transfer principle [10, Part 4], Definitions 2.4 and 2.5 which describe the neighborhood can be better expressed with infinitesimals Definition 2.7 and infinities Definition 2.6; are defined with sequences more generally in [12, Part 6].

Definition 2.6. *In context, a variable x can be described at infinity $|_{x=\infty}$, then $x \in +\Phi^{-1}$ an infinity.*

Definition 2.7. *In context, a variable x can be described at zero $|_{x=0}$ then $x \in \Phi$ an infinitesimal.*

Definition 2.8. *In context, we say $f(x)|_{x=\infty}$ then $\lim_{x\to\infty} f(x)$.*

Generally the equality "=" with respect to assignment is defined with a left-to-right ordering see Definition 2.9. Essentially as reasoning, such that the right follows from the left. This can be exact, as in one form is converted to another, or as a generalization, or rather implication. Therefore the context needs to be understood.

Definition 2.9. *In context, assignment has a left-to-right ordering.*

$$\textit{In context, instance} = \textit{generalization}$$

Example 2.4. *How to use the notation is open, some examples follow.*
1. Limit calculations $(1+\frac{1}{n})^n|_{n=\infty} = e$
2. Divergent sums $\sum_{k=1}^{n} \frac{1}{k} = \ln n|_{n=\infty}$
3. A conversion between series and integrals, read from left-to-right.
$\sum_1^n a_n = \int_1^n a(n)\,dn + c|_{n=\infty}$
4. A comparison relation. $n! > n^2|_{n=\infty}$
5. An infinitary calculus relation $f_n \prec g_n|_{n=\infty}$ *(as described in Definition 3.2).*
6. Asymptotic results

Provided there are no contradictions, the expressions at infinity can be handled algebraically in the usual ways.

Example 2.5. $n! = (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n}|_{n=\infty}$ *then times by* $e^n|_{n=\infty}$ *gives* $n!e^n = cn^{n+\frac{1}{2}}|_{n=\infty}$.

Example 2.6. $\ln(n!)|_{n=\infty} = \sum_{k=1}^{n} \ln k|_{n=\infty} = \int_1^n \ln k\,dk + \gamma|_{n=\infty} = [k\ln k \ \ - k]_1^n + \gamma|_{n=\infty}$ $= n\ln n \ \ - n|_{n=\infty}$

Definition 2.10. *Generalize the at-a-point Definition 2.2 to include condition* $c(x)$ *in relation* $f(x)|_{c(x)}$. *Where* $c(x)$ *can describe an interval.*

Example 2.7. $f(x) \ \ z \ \ g(x)|_{x=(0,1]}$ *describes the relation* z *(see [9, Part 3]) over the interval* $(0,1]$.

Example 2.8. *The vertical bar notation is more general when working across different situations. Such as when little-o and big-O notation may be cumbersome* $O(x^2)+O(x^3) = O(x^2)$ *becomes* $x^2+x^3 = x^2|_{x=0}$, *as an alternative to the approximation symbol so* $a \approx b = c$ *becomes* $a = b = c|_{n=\infty}$, $x^n e^{-n} \approx n^n e^{-n} e^{-\xi^2/2}$ *becomes* $x^n e^{-n} = n^n e^{-n} e^{-\xi^2/2}|_{n=\infty}$, $f \sim g$ *becomes* $f = g|_{n=\infty}$, *so assignment becomes consistent.*

Since non-uniqueness is accepted with the notation; $\sin x = x - x^3/3!|_{x=0}$ can be understood to mean truncation - an exact happening at $x = 0$: an approximation in $*G$ where $\Phi \mapsto 0$ see [10, Part 4]. The notation gives you the choice. If you want to be more 'exact' or explicit, then use other or further relations. $\sin x = x - x^3/3! + O(x^5)|_{x=0}$.

If say, after k or more terms, the calculation is invariant with non-reversible arithmetic, we define this as an 'exact happening'. Increasing the number of terms does not change the calculation. After a transfer, the calculations produce the same result. Such situations are common, where to few terms in the approximations give incorrect results.

The notation is also built with comparing functions in mind, where non-reversible arithmetic (see [11, Part 5]) is applied. An alternative to the limit notation, as it applies across relations and as an aid to Landau notation, can be replacing $x \to \infty$ with $|_{x=\infty}$. The concept of approach is logically equivalent to being at the value, and the notation can say this.

A notation for chaining arguments using commas as an implication and context is used. The last proposition uses the first expression. As a free algebra, this does place responsibility on both reading expressions and writing expressions. The notation is concise. When there are errors in evaluation or proofs, the chaining arguments can be rewritten one expression per line and edited. (Later, for similar reasons but a narrower purpose, in context we have defined =, with a left-to-right ordering.)

Example 2.9. $x, y \in \mathbb{R}$; $x > 0$, $y > 0$, $x + y > 0$

Definition 2.11. *Mathematical arguments can be chained with context by commas (',') and semicolons (';'), from left-to-right order. The next statement optionally has a left-to-right implication. The semicolons have a lower precedence.*

In evaluating a function at-a-point we can shift any point $x = a$ to the origin or infinity, stating the definition at ∞, is as general as stating the definition at any other point.

Proposition 2.1. *If; $f, x \in *G$; then $f(x)|_{x=a} = f(x+a)|_{x=0} = f(\frac{1}{x} + a)|_{x=\infty}$*

Hence the investigation at infinity is similar to the corresponding theory at zero. It is here that infinitesimals (near zero) and infinities (near infinity) of infinitary calculus operate.

3 Infinitary calculus definitions

The following is a summary and extension, by means of the gossamer numbers [8, Part 1], of infinitary calculus definitions and a comparison with the derived work of little-o and big-O relations. This is a calculus of magnitudes.

Occasionally $\ll$ and $\gg$ are used to indicate much smaller or larger numbers. Correspondingly $\prec$ and $\succ$ implement the idea of much smaller than and much greater than numbers by defining infinitely smaller and infinitely larger relations.

While a finite number is not infinity, a very large number treated as infinity would model the situation and allow reasoning. These relations, at zero or infinity provide an implementation of this.

The idea of much larger numbers is extended to infinity, where it becomes obvious that there are much larger numbers than others; hence du Bois-Reymond's development of the Definitions 3.2 and 3.1, which are equivalent to little-o and big-O respectively.

We demonstrate the connection between modern relations and du Bois-Reymond's relations and restate some definitions of du Bois-Reymond, referred to by G. Hardy in *Orders of Infinity* [5, pp 2–4] with their representation in Landau notation.

Definition 3.1. *We say* $f(x) \preceq g(x)|_{x=\infty}$ *if there exists* $M \in \mathbb{R}^+$*:* $|f(x)| \leq M|g(x)||_{x=\infty}$.

$$f(x) \preceq g(x) \text{ is the same as } f(x) = O(g(x))$$

Definition 3.2. *We say* $f(x) \prec g(x)|_{x=\infty}$ *then* $\frac{f(x)}{g(x)}|_{x=\infty} \in \Phi$

$$f(x) \prec g(x) \text{ is the same as } f(x) = o(g(x))$$

Definition 3.2 is equivalently defined $|f(x) \leq M_2|g(x)|$, $M_2 \in \Phi^+$. We see that this is almost the same as Definition 3.1 except $M \in \mathbb{R}^+$. One is bounded in $\mathbb{R}$, the other in $*G$.

Proposition 3.1. *If* $f(x) \prec g(x)|_{x=\infty}$ *then* $\frac{f(x)}{g(x)}|_{x=\infty} = 0$ *in* $\mathbb{R}$*. As a definition see [5, p.2].*

Proof. Apply a transfer $\Phi \mapsto 0$ (see [10, Part 4]) to Definition 3.2. □

Proposition 3.2. *If* $f(x) \prec g(x)|_{x=\infty}$ *then* $\frac{g(x)}{f(x)}|_{x=\infty} = \infty$.

Proof. By Definition 3.2 let $\delta \in \Phi$, $\frac{f}{g}|_{x=\infty} = \delta$, $\frac{g}{f}|_{x=\infty} \in \Phi^{-1} = \infty$. □

Proposition 3.3. $\delta \in \Phi^+$*; if* $\frac{f(x)}{g(x)} \leq \delta$ *then* $f(x) \prec g(x)$.

Proof. Since f and g are positive, an infinitesimal is their upper bound. Since choosing any infinitesimal in $(0, \delta]$ satisfies the much-less-than relation, $f \prec g$. □

[When applying Proposition 3.4, we will need to avoid division by zero via a transfer $0 \mapsto \Phi$ and $\infty \mapsto \Phi^{-1}$, thereby treating 0 and ∞ separately. See [10, Part 4]]

Proposition 3.4. $0 \prec \Phi$ *and* $\Phi^{-1} \prec \infty$

Proof. Since a magnitude relation, we need only consider the positive case.

$0 \prec \Phi^+$: $\delta_1, \delta_2 \in \Phi^+$ Consider $0 \; z \; \delta_1$. Choose $\delta_2 \prec \delta_1$ as there is no smallest number. $0 < \delta_2 < \delta_1$. Since 0 is smaller than δ_2 then $0 \prec \delta_1$.

$+\Phi^{-1} \prec \infty$: By inverting the relation we obtain the infinite case. Choose $\delta_2 \prec \delta_1$, $\frac{1}{\delta_2} \succ \frac{1}{\delta_1}$, $\frac{1}{\delta_1} \prec \frac{1}{\delta_2}$, however $\frac{1}{\delta_2} < \infty$ then $\frac{1}{\delta_1} \prec \infty$. □

Definition 3.3.

$$\textit{When } f(x) \prec g(x) \textit{ we say } f(x) \textit{ is much-less-than } g(x)$$

$$\textit{When } f(x) \succ g(x) \textit{ we say } f(x) \textit{ is much-greater-than } g(x)$$

Definition 3.4.

$$g(x) \prec f(x) \text{ is the same as } f(x) \succ g(x)$$
$$g(x) \preceq f(x) \text{ is the same as } f(x) \succeq g(x)$$

Definition 3.5.

$$f(x) \succ g(x) \text{ is the same as } f(x) = \omega(g(x))$$
$$f(x) \succeq g(x) \text{ is the same as } f(x) = \Omega(g(x))$$

Because little-o and big-O are defined on a right-hand side order for $\prec$ and $\preceq$, additional symbols are needed for $\succ$ and $\succeq$. Here, infinitary calculus has a notational advantage.

Definition 3.6. *We say $f(x) \asymp g(x)|_{x=\infty}$ if $f(x) \succeq g(x)|_{x=\infty}$ and $f(x) \preceq g(x)|_{x=\infty}$. (See Definition 3.11 and Proposition 3.6.)*

$$f(n) \asymp g(n) \text{ is the same as } f(n) = \Theta(g(n))$$

Definition 3.7. *We say $a \simeq b$ then a and b are infinitesimally close, $a - b \in \Phi \cup \{0\}$ [15, p.57]*

Definition 3.8. *We say $f(x) \sim g(x)|_{x=\infty}$ then $\frac{f(x)}{g(x)}|_{x=\infty} \simeq 1$*

We may consider the asymptotic relation $\sim$ as an equality with respect to the product, and the infinitesimally close relation $\simeq$ as an equality with respect to addition.

Definition 3.9. *We say $f(x) \propto g(x)|_{x=\infty}$ if $f(x)/g(x)|_{x=\infty} \simeq c$. This uses a different relation symbol from Hardy's in [5, pp 2–4].*

The functions $f(x)/g(x)$ may not necessarily be compared, particularly if oscillating between categories at infinity occurs. [5, p.4] $f \succ g$ and $f \preceq g$ are not each other's logical negations in general.

Example 3.1. *A counter example demonstrating logical negation does not imply a much less than or equal to relation.*

$$\begin{aligned}
(f_n/g_n)|_{n=\infty} &= (0, \infty, 0, \infty, \ldots) && \text{(Sequence at infinity)} \\
\textit{Assume } f_n \not\succ g_n \textit{ implies } f_n &\preceq g_n|_{n=\infty} && \text{(Dividing by } g_n\text{)} \\
(f_n/g_n) &\preceq (g_n/g_n) && \text{(Component-wise comparison)} \\
(f_n/g_n) &\preceq (1, 1, 1, \ldots) && \\
(0, \infty, 0, \infty, \ldots) &\preceq (1, 1, 1, \ldots) && \text{(A component-wise contradiction)}
\end{aligned}$$

Proposition 3.5. *If the relation between f and g are $\{f \prec g, f \asymp g, f \succ g\}$, then the negation of one of these relations would imply one of the other two relations.*

Proof. $f \succ g$ and $f \prec g$ are disjoint. Since $\asymp$: $f \succeq g$ or $f \preceq g$ covers the remaining cases. Since this is given as disjoint, only one of the three cases can occur. □

Further theorems follow: if $f \succ g$, $g \succeq h$, then $f \succ h$. This is interesting from an application perspective as the ratio f/g has settled down into one of the three relations.

In comparing relations $\{o(), O(), \omega(), \Omega()\}$ with $\{\prec, \preceq, \succ, \succeq\}$, while the variable relation and symbols are equivalent, the symbols can be easier to manage and understand in comparison.

However the relation variables of Landau's notation have a major advantage over infinitary calculus relation symbols in that the relation is packaged as a variable in the equation. $\frac{1}{1-x} = 1 + x + x^2 + O(x^3)$.

Consequently the definitions of infinitary calculus symbols and Landau notation can be viewed as complementary.

We add further definitions to infinitary calculus that extend its use as an infinitesimal calculus analysis.

Definition 3.10. *We say $c(x) \prec \infty$ when $c(x) \neq \pm\infty$ and that $c(x)$ is bounded.*

$a \prec \infty$ is not the same as $a \in \mathbb{R}$ as the bound includes infinitesimals. While the function $c(x)$ has finite bounds, $c_0 < c(x) < c_1$, these functions do not need to converge. E.g. $f(x) = \sin x|_{x=\infty} \prec \infty$, $f(x)|_{x=\infty} \prec \infty$. At times such functions behave similarly to constants. However an infinitesimal when realized is 0 and not positive, hence the need to exclude infinitesimals from a finite positive bound definition.

Definition 3.11. *A variable has a "finite positive bound" if $a < x < b$, $\{a, b\} \in \mathbb{R}^+$.*

Example 3.2. *$(\frac{1}{n}, 1)|_{n=\infty}$ is not a finite positive bound as the infinitesimal $\frac{1}{n}|_{n=\infty} \notin \mathbb{R}^+$, $\frac{1}{n}|_{n=\infty}$ is not finite. Further when realizing the infinitesimal, the interval is not all positive, as it includes 0. $(\frac{1}{n}, 1)|_{n=\infty} = [0, 1)$ Similarly $\sin\frac{1}{n} = \frac{1}{n}|_{n=\infty} = 0$ does not have finite positive bound.*

Proposition 3.6. *If $f \asymp g$ then $\frac{f}{g}$ has a finite positive bound.*

Proof. From Definition 3.6 if $f \asymp g$ then $f \preceq g$ and $f \succeq g$. $M, M_2 \in \mathbb{R}^+$; from Definition 3.11 if $f \succeq g$ then $Mf \geq g$. $f \preceq g$ then $f \leq M_2 g$, $\frac{f}{M_2} \leq g \leq Mf$, $\frac{1}{M_2} \leq \frac{g}{f} \leq M$. Inverting, $M_2 \geq \frac{f}{g} \geq \frac{1}{M}$. □

Hardy in *Orders of Infinity* [5, p.4] states several theorems with the much greater than relations and their transitivity. The infinitesimal numbers developed earlier can be used as a tool to prove these theorems. When proving, without loss of generality, consider positive functions.

While Hardy states the reader will be able to prove the theorems without difficulty, here a new number system is used for that purpose.

Proposition 3.7. *$f \succ \phi$, $\phi \succeq \psi$, then $f \succ \psi$*

Proof. $\delta \in \Phi^+$;

$$\begin{aligned} f \succ \phi \text{ then } \phi = \delta f & \qquad \text{(Definition 3.2)} \\ \phi \succeq \psi \text{ then } M\phi \geq \psi & \qquad \text{(Definition 3.1)} \\ M\delta f \geq \psi & \qquad \text{(Redefine } \delta \text{ to absorb } M) \\ \delta f \geq \psi & \\ \delta \geq \frac{\psi}{f} & \qquad \text{(Proposition 3.3)} \\ f \succ \psi & \end{aligned}$$

□

Definition 3.12. *Let $\neg$ be the negation operation, and z the binary relation. $\neg\neg(f\ z\ g) = (f\ z\ g)$.*

$$\neg(f\ z\ g) \ = \ (f\ (\neg z)\ g)$$

Example 3.3. *Examples of negation in $\mathbb{R}$ or $*G$. $(\neg <) \ = \ \geq$. $(\neg ==) \ = \ \neq$.*

Theorem 3.1. *$a, b, c \in *G\backslash\{0\}$; If $ab \neq c$ then $a \neq cb^{-1}$*

Proof. $(ab \neq c) = \neg(ab = c) = \neg(a = cb^{-1}) = (b \neq cb^{-1})$ □

Proposition 3.8. *$f \succeq \phi$ implies the negation of $f \prec \phi$, $\neg(f \prec \phi)$. However $\neg(f \prec \phi)$ does not imply $f \succeq \phi$.*

Proof. Without loss of generality, consider positive f and ϕ. Since $f \succeq \phi$, $\exists M : M \in \mathbb{R}^+$ then $Mf \geq \phi$, $M \geq \frac{\phi}{f}$. Since $\frac{\phi}{f}$ is positive and bounded above, and $\frac{\phi}{f} \in \mathbb{R}^+ + \Phi$.

If we consider the negation relation, $\delta \in \Phi$, $\neg(f \prec \phi)$, $\neg(\frac{f}{\phi} = \delta)$ $\frac{f}{\phi} \neq \delta$, which excludes infinitesimals, but not infinities. Since $\frac{f}{\phi}$ is positive, then expressed as an interval $\frac{\phi}{f} \in [\mathbb{R}^+ + \Phi, +\Phi^{-1}]$.

We can see the first interval is a subinterval in the second, hence implication is confirmed, but that the second interval is not contained in the first, then the 'not implied' confirmed. □

4 Scales of infinity

In music a scale ordered by increasing pitch is an ascending scale, while descending scales are ordered by decreasing pitch. Indeed everyone has heard musicians going through the scales in rehearsal before a performance.

In an analogous way mathematics has its scales where families of functions ascend and descend. Because of a property of the numbers zero and infinity, the scales are defined at these points, giving a number system at zero and at infinity.

Since these scales are intimately involved with the evaluation of a function at a point(extended sense), the scales apply to any function evaluation. A simple example is that when $a \neq 0$, $x^2|_{x=a} = (x+a)^2|_{x=0} = x^2 + 2ax + a^2|_{x=0}$, we also see $x^2 \prec 2ax \prec a^2|_{x=0}$, correlates to the scale $x^2 \prec x \prec 1|_{x=0}$

Hardy discusses in detail the rates of growth of functions, and compares two functions where different functions could be ordered. Hence I believe the title of his book is fittingly "*Orders of Infinity*". A new function can always be inserted between two ordered functions. Different families of functions possess different orderings. This is similar to the real number system where we can always find a number between two other numbers.

The notion of numbers being much greater than ($\succ$) or much smaller than ($\prec$) other numbers makes sense for numbers that are infinitely large or infinitely small.

Consider the family of functions x^k as $x \to \infty$. Moving away from the origin, each function with increasing exponent k, becomes steeper.

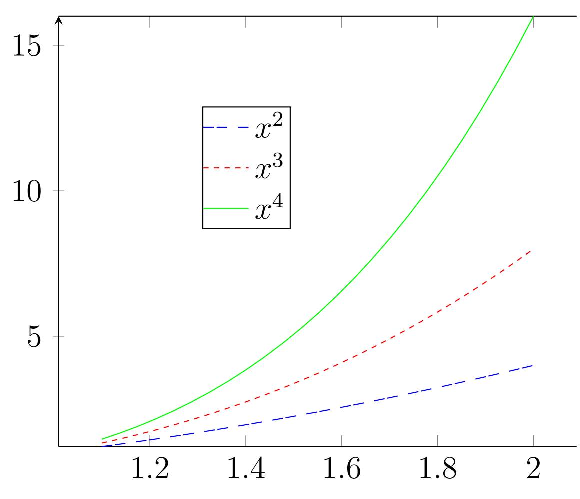

Figure 2: powers at infinity

From the considerations of infinitely large functions, relational scales can be developed. The progression of these functions form a scale of higher infinities.

Let $x \to \infty$ then $x^2/x \to \infty, x^3/x^2 \to \infty, \ldots$

By defining a measure of the magnitude as the absolute value of the ratio between two functions at infinity, the scales of infinity are more easily expressed. See the $\succ$ relation (Definition 3.2).

$(\ldots \succ x^3 \succ x^2 \succ x^1 \succ x^0 \succ x^{-1} \succ \ldots)|_{x=\infty}$

This relation is conveniently symmetrical such that by swapping the function's sides, the arrow reverses in direction in the same way $3 < 5$ becomes $5 > 3$.
$(x \prec x^2 \prec x^3 \prec \ldots)|_{x=\infty}$

Other examples: $(e^x \prec e^{e^x} \prec e^{e^{e^x}} \prec \ldots)|_{x=\infty}$ and importantly the logarithmic scale $(n \succ \ln n \succ \ln\ln n \succ \ln\ln\ln n \succ \ldots)|_{n=\infty}$

Definition 4.1. *Let k-powers of e be represented by $e_k(x)$: $e_0(x) = x$, $e_k(x) = e^{e_{k-1}(x)}$*

Definition 4.2. *Let k-nested natural logarithms be represented by $\ln_k(x)$: $\ln_{-1} x = 1$, $\ln_0 x = x$, $\ln_k x = \ln(\ln_{k-1} x)$*

Definition 4.3. *As a convention, when $\ln_k$ has no argument, we define $\ln_k = \ln_k n$.*

Consider $\ln_k|_{n=\infty}$. If n reaches infinity before the k-nested log functions, $\ln_k = \infty$ is guaranteed. Looking at this another way, let k be finite. This avoids the possibility of the logarithm becoming negative or complex. Each of these infinities belongs to a family scale.
$(\ln x \succ \ln_2 x \succ \ln_3 x \succ \ldots)|_{x=\infty}$

Conjecture 4.1. *Given $f(x)|_{x=\infty} = \infty$, $k = \infty$, $\ln_k f(x) = \infty$ when x reaches infinity before k.*

While Conjecture 4.1 is usually expressed as a definition, the possibility of an ordering of variables at infinity should be expected, and this may provide much further investigation. A variable reaching infinity before another variable could better explain partial differential equations, where other variables are held constant, and the target variable differentiated.

Definition 4.4. *Given initial relation $\phi_1 \succ \phi_2$, and function $\phi : \phi_{n+1} = \phi(\phi_n)$ with the property $\phi_n \succ \phi_{n+1}$, the relations $(\phi_1 \succ \phi_2 \succ \ldots \succ \phi_n \succ \ldots)|_{n=\infty}$, are referred to as 'scales of infinity' [5, p.9]. Similarly with the much-less-than relation $\prec$.*

With the definition of much less than and much greater than, multiplying the scale by constants has no effect.

Proposition 4.1. *$f, g \in *G$; $\alpha_1, \alpha_2 \in \mathbb{R}\backslash\{0\}$; then $f \succ g \Leftrightarrow \alpha_1 f \succ \alpha_2 g$*

Proof. $f \succ g$ then $\frac{g}{f} = \Phi$, $\frac{\alpha_2 g}{f} = \alpha_2 \Phi = \Phi$, $\frac{\alpha_2 g}{\alpha_1 f} = \frac{1}{\alpha_1}\Phi = \Phi$, then $\alpha_1 f \succ \alpha_2 g$ □

Corollary 4.1.

$$\textit{Given } (\phi_1 \succ \phi_2 \succ \ldots)|_{n=\infty}$$

$$\textit{If } a_n \in \mathbb{R}\backslash\{0\}, \textit{ then } (a_1\phi_1 \succ a_2\phi_2 \succ a_3\phi_3 \ldots)|_{n=\infty}$$

Proof. Apply Proposition 4.1 to each relation. □

Infinitely small magnitude scales can likewise be considered. For the powers of x^k this has the effect of reversing the relation when evaluating x at 0.
$(\ldots \prec x^4 \prec x^3 \prec x^2 \prec \ldots)|_{x=0}$

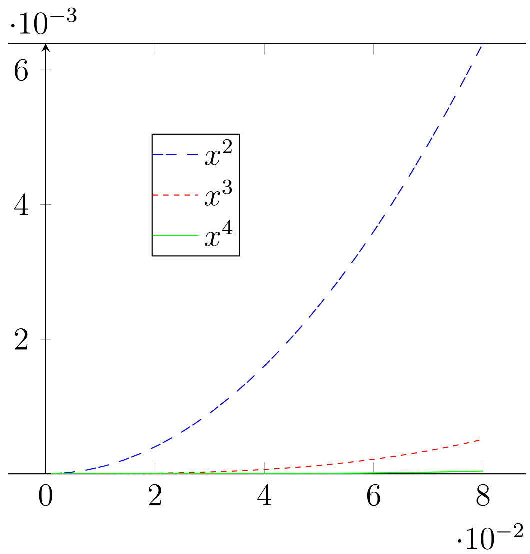

Figure 3: powers at zero

The scales of infinity are often used implicitly in calculations. For example, truncating the Taylor series, or with limit calculations by ignoring the infinitesimals which effectively sets the infinitesimals to zero.

Since scales of infinity describe infinitesimals, calculus can be constructed with these ideas. A most useful scale in algebraic simplification orders different families of curves, whereby different types of infinitesimals and infinities are compared.

$$(c \prec \ln(x) \prec x^p|_{p>0} \prec a^x|_{a>1} \prec x! \prec x^x)|_{x=\infty}$$
$$(\ldots \prec x^{-2} \prec x^{-1} \prec 1 \prec x \prec x^2 \prec \ldots)|_{x=\infty}$$
$$(\ldots \succ x^{-2} \succ x^{-1} \succ 1 \succ x \succ x^2 \succ \ldots)|_{x=0}$$
$$(\ldots \prec e^{e^{-x}} \prec e^{-x} \prec 1 \prec e^x \prec e^{e^x} \prec \ldots)|_{x=\infty}$$
$$(x \succ \ln x \succ \ln_2 x \succ \ldots)|_{x=\infty}$$
$$(v \prec \ln v \succ \ln_2 v \succ \ln_3 v \succ \ldots)|_{v=0^+}$$
$$(hf' \succ \tfrac{h^2}{2!}f^{(2)} \succ \tfrac{h^3}{3!}f^{(3)} \succ \ldots)|_{h=0^+} \text{ when } f^{(k)} \prec \infty$$

Table 1: Summary of scales

5 Little-o and big-O notation

Since infinitary calculus has equivalent definitions for little-o and big-O notation, it can be used to do the same things. It can describe function growth, compare functions, and derive theorems.

Where little-o and big-O notation surpasses the infinitary calculus notation, we see both notations as complementary. In particular, the Landau notation's strength is that it contains the relation as an end term to a formula. That is, a relation is packaged and managed as a variable.

$$e^x = 1 + x + \frac{1}{2}x^2 + O(x^3)$$

The infinitary calculus symbols are not "side dependent", $f(x) \succ g(x)$ is the same as $g(x) \prec f(x)$, which can give the algebra a sense of freedom. The Landau notation introduced $\omega(x)$ and $\Omega(x)$ to express the relations on the "other side", see Definition 3.5.

Before proceeding, properties of the magnitude relations $\{\prec, \preceq, \succ, \succeq\}$ are derived using $*\overline{G}$, thus demonstrating its usefulness in proofs. These properties are then used to prove theorems with little-o and big-O, demonstrating an equivalence with the magnitude relations.

To simplify the proofs, from Proposition 5.3, we can make the arguments positive. Therefore with assumptions regarding a and b, we can always transform the problem to one with sequences positive or greater than zero, since these relations are not affected by the sign of the elements of the sequence.

Since a and b are positive numbers, either infinitesimals, infinities, or real numbers except 0, then we can multiply and divide a and b, before realizing the infinitesimal or infinity.

Proposition 5.1. $b \succ a \Leftrightarrow \frac{1}{b} \prec \frac{1}{a}$

Proof. $a \neq 0$, $b \neq 0$, let $\delta \in \Phi$, $b \succ a$ then $\frac{a}{b} = \delta$, $\frac{1}{\frac{a}{b}} = \frac{1}{\delta}$, $\frac{\frac{1}{a}}{\frac{1}{b}} = \frac{1}{\delta}$, $\frac{\frac{1}{b}}{\frac{1}{a}} = \delta$, $\frac{1}{b} \prec \frac{1}{a}$. Similarly if $\frac{1}{b} \prec \frac{1}{a}$ then $\frac{\frac{1}{b}}{\frac{1}{a}} = \delta$, $\frac{a}{b} = \delta$, $a \prec b$. □

Proposition 5.2. $b \succ a \Leftrightarrow cb \succ ca$, $c \in *G \backslash \{0\}$

Proof. $a \neq 0$, $b \neq 0$, let $\delta \in \Phi$, $b \succ a$ then $\frac{a}{b} = \delta$, $\frac{ca}{cb} = \delta$, $cb \succ ca$. Similarly if $cb \succ ca$ then $\frac{ca}{cb} = \delta$, $\frac{a}{b} = \delta$, $b \succ a$. □

Proposition 5.3. $a \prec b \Leftrightarrow -a \prec b$

Proof. $a \neq 0$, $b \neq 0$, let $\delta \in \Phi$, $a \prec b$ then $\frac{a}{b} = \delta$, $\frac{-a}{b} = -\delta$, $-a \prec b$. Similarly if $-a \prec b$ then $\frac{-a}{b} = \delta$, $\frac{a}{b} = -\delta$, $a \prec b$. □

Proposition 5.4. $a \succ b \Leftrightarrow a + \lambda \succ b + \lambda$ *when* $\lambda \prec a$ *and* $\lambda \prec b$.

Proof. $\frac{a+\lambda}{b+\lambda} = \frac{a}{b+\lambda} = \frac{a}{b} \in \Phi^{-1}$ then $a + \lambda \succ b + \lambda$. Reversing the argument, $\frac{a}{b} = \frac{a+\lambda}{b} = \frac{a+\lambda}{b+\lambda}$ $\in \Phi^{-1}$ then $a + \lambda \succ b + \lambda$. □

Proposition 5.5. $a \succeq b \Leftrightarrow \frac{1}{a} \preceq \frac{1}{b}$

Proof. $a \neq 0$, $b \neq 0$, $a \succeq b$, $\exists \alpha : \alpha|a| \geq |b|$, $\alpha \geq \frac{|b|}{|a|}$, $\alpha \frac{1}{|b|} \geq \frac{1}{|a|}$, $\frac{1}{b} \succeq \frac{1}{a}$, $\frac{1}{a} \preceq \frac{1}{b}$. □

Proposition 5.6. $a \succeq b \Leftrightarrow ca \succeq cb$

Proof. Let $c \in *G \backslash \{0\}$. Consider $a \succeq b$, $\exists \alpha : \alpha|a| \geq |b|$, $\alpha|c||a| \geq |c||b|$, $\alpha|ca| \geq |cb|$, $ca \succeq cb$ then $a \succeq b \Rightarrow ca \succeq cb$, reversing the argument gives the implication in the other direction. □

Proposition 5.7. $a \preceq b \Leftrightarrow -a \preceq b$

Proof. Consider $a \succeq b$, $\exists \alpha : \alpha|a| \geq |b|$, $\alpha|-a| \geq |b|$, $-a \succeq b$, then $a \succeq b \Rightarrow -a \succeq b$, reversing the argument gives the implication in the other direction. □

Proposition 5.8. $\lambda \prec \infty$, $\;a \succeq b \Rightarrow a + \lambda \succeq b + \lambda$

Proof. $a \succeq b$, $\exists \alpha : \alpha|a| \geq |b|$, $\alpha|a| + \lambda \geq |b| + \lambda$, Assume $\alpha > 1$ as we can always increase α. Case $\lambda > 0$, $\alpha|a| + \alpha\lambda \geq |b| + \lambda$, $\alpha|a + \lambda| \geq |b| + |\lambda| \geq |b + \lambda|$, $a + \lambda \succeq b + \lambda$. Case $-\lambda$ then $\alpha|a| - \lambda \geq |b| - \lambda$, $\alpha|a| + \lambda \geq |b| + \lambda$, the above positive case. Hence $a \succeq b \Rightarrow a + \lambda \succeq b + \lambda$ □

Example 5.1. *For proving the following big-O theorem we found infinitary calculus to be easier to reason with than the solution given in [7, Theorem 2.(8)] .*

$$\text{If } g(x) = o(1) \text{ then } \frac{1}{1 + O(g(x))} = 1 + O(g(x))$$

Proof. Let $v(x) = O(g(x))$

$$\begin{aligned}
1 + v(x) &\succeq 1 \\
\frac{1}{1+v(x)} &\preceq 1 && \text{(from Proposition 5.5)} \\
\frac{v(x)}{1+v(x)} &\preceq v(x) && \text{(from Proposition 5.6)} \\
\frac{-v(x)}{1+v(x)} &\preceq v(x) && \text{(from Proposition 5.7)} \\
\frac{-v(x)-1+1}{1+v(x)} &\preceq v(x) \\
\frac{1}{1+v(x)} - 1 &\preceq v(x) \\
\frac{1}{1+v(x)} &\preceq 1 + v(x) && \text{(from Proposition 5.8)} \\
\frac{1}{1+O(g(x))} &= 1 + O(g(x)) && \square
\end{aligned}$$

Verification: rather than building the inequality, the inequality can be verified directly. $\frac{1}{1+v(x))} \preceq 1 + v(x)$, $\frac{1}{1} \preceq 1 + v(x)$ is true, since $1 + v(x)|_{x=a} = 1$ and $v(x) \preceq g(x) \prec 1|_{x=a}$

Example 5.2. *Consider the proof of the following theorem from [7, Theorem 2.(8)] .*

$$\textit{If } g(x) = o(1) \textit{ then } \frac{1}{1+o(g(x))} = 1 + o(g(x))$$

Using an inequality in infinitary calculus to prove the theorem. Let $h(x) = o(g(x))|_{x=a}$, $h(x) \prec g(x)|_{x=a}$, $1 + h(x) \succeq 1|_{x=a}$, $\frac{1}{1+h(x)} \preceq 1|_{x=a}$, $\frac{h(x)}{1+h(x)} \preceq h(x)|_{x=a}$, $\frac{-h(x)}{1+h(x)} \preceq h(x)|_{x=a}$, $\frac{-h(x)-1+1}{1+h(x)} \preceq h(x)|_{x=a}$, $\frac{1}{1+h(x)} - 1 \preceq h(x)|_{x=a}$, $\frac{1}{1+h(x)} - 1 \prec g(x)|_{x=a}$, $\frac{1}{1+h(x)} - 1 = o(g(x))|_{x=a}$, $\frac{1}{1+o(g(x))} = 1 + o(g(x))$

From [7] the theorem is derived in the standard way by taking the limit. Applying the little-o definition directly. $\lim_{x\to a} \frac{\frac{1}{1+h(x)}-1}{g(x)} = \lim_{x\to a} \frac{\frac{1-(1+h(x))}{1+h(x)}}{g(x)} = -\lim_{x\to a} \frac{h(x)}{g(x)} \frac{1}{1+h(x)} = -\lim_{x\to a} \frac{h(x)}{g(x)} \frac{\frac{1}{g(x)}}{\frac{1}{g(x)}+\frac{h(x)}{g(x)}}$ $= -\lim_{x\to a} \frac{h(x)}{g(x)} \frac{1}{1+\frac{h(x)}{g(x)}g(x)} = -\lim_{x\to a} 0 \cdot \frac{1}{1+0\cdot g(x)} = 0$

The same calculation with infinitary calculus evaluation at the point and applying the definition. $\frac{\frac{1}{1+h(x)}-1}{g(x)}|_{x=a} = \frac{\frac{1-(1+h(x))}{1+h(x)}}{g(x)}|_{x=a} = \frac{\frac{-h(x)}{1+h(x)}}{g(x)}|_{x=a} = -\frac{h(x)}{g(x)}\frac{1}{1+h(x)}|_{x=a} = -\frac{h(x)}{g(x)}|_{x=a} = 0$ *as* $1 + h(x)|_{x=a} = 1$ *and* $h(x) \prec g(x)|_{x=a}$

The infinitary calculus expresses scales of infinities with more intuitive meaning. Writing the scales with big-O notation, using the left side to right side definition, big-O notation is defined where $f = O(g)$ is not the same as $O(g) = f$. Let $a > 0$, $b > 0$, $k > 0$. We can write $O(e^{-ax}) = O(x^{-b}) = O(\ln(x)^{-k})$ which has a left-to-right definition of $O()$ and express in infinitary calculus as $e^{-ax} \preceq x^{-b} \preceq \ln(x)^{-k}|_{x=\infty}$.

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