Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 3 Comparing functions

Chelton D. Evans and William K. Pattinson

Abstract

An algebra for comparing functions at infinity with infinireals, comprising of infinitesimals and infinities, is developed: where the unknown relation is solved for. Generally, we consider positive monotonic functions f and g, arbitrarily small or large, with relation z: $f \; z \; g$. In general we require f, g, $f-g$ and $\frac{f}{g}$ to be ultimately monotonic.

1 Introduction

In extending du Bois-Reymond's theory, we have discovered a new number system [1, Part 1], and used this to rephrase du Bois-Reymond's much greater than relations [2, Part 2]. However, at the heart of Reymond's theory is the comparison of functions.

Today this may seem of little interest because there are no applications which directly require this. Even Hardy, through writing and extending du Bois-Reymond's work [8] thought this. Others better incorporated the theory: little-o and big-O notation have the same definitions as relational operators $\{\prec, \preceq\}$ and similarly other relations [2, Part 2].

Instead, du Bois-Reymond's work became a catalyst for other higher mathematics and itself as an operational calculus was largely forgotten. In this era of immense change and other issues which they faced, this is not surprising. For just one example, the theory on divergent sums and functions was being established.

Our aim is to open the field of infinitary calculus through the development of another infinitesimal and infinitary calculus that is derived from du Bois-Reymond's work. The method solves for relations between functions. Subsequent papers giving applications for comparing functions are found. (E.g. sum convergence [6])

Comparing functions is the key discovery in the theory's development.

With the method of comparing functions at infinity described in this paper, we believe there is a significant improvement over the method described by Hardy, which either computes the relation with a limit, which is fine, or uses logarithmico-exponential scales [8, pp.31–33].

By comparing functions at infinity, du-Bois Reymond showed the existence of curves infinitely close to each other. For example, the construction of infinitely many curves infinitesimally close to a straight line (see Example 2.20). Thereby demonstrating the curves exist at infinity.

However, the number system in which the curves reside includes infinitesimals and infinities, and hence is a non-standard analysis.

While du Bois-Reymond did not define a number system as Abraham Robinson had done with Non-standard analysis (NSA), the constructions prove that such a number system exists.

That Abraham made little reference to du Boise-Reymond's work is unexplained. Though he did use similar notation. For example in [11, p.97]: if $a \in \mathbb{J}$ and $b \in *J$ then $a \prec b$. Elements of $*J$ he called infinite.

As this is a reference paper in the sense that it contains propositions which we have collected, while working through problems. Subsequent papers reference this paper.

We can consider comparison in terms of addition or multiplication. Where z is a relation.

$f \; z \; g$	Comparison
$f - g \; z \; 0$	additive sense
$\frac{f}{g} \; z \; 1$	multiplicative sense

Table 1: Binary relation comparison

2 Solving for a relation

While we are very familiar with solving for variables as values, in general we do not solve variables for relations. However there is no real reason not to do so.

In the course of devising an alternative way to compare functions at infinity, a new way of comparing functions has been developed, where the primary focus is to solve for the relation.

Definition 2.1. *Let $f(x) \; z \; g(x)$ be a comparison of the functions $f(x)$ and $g(x)$ where z is the variable relation.*

When possible, we could then solve for a relation z, for example $z \in \{<, \leq \prec, \preceq, >, \geq, \succ, \succeq, =, \simeq, \neq, \prec\!\prec, \succ\!\succ\}$.

Definition 2.2. *Given $f(x)\ z\ g(x)$, with relation z, then applying function or operator ϕ to one or both sides of the variable relation z results in a new relation $(\phi(z))$.*

The brackets about the relation are an aid to distinguish the relation as a variable.

$$f(x)\ z\ g(x), \quad \phi(f(x))\ (\phi(z))\ \phi(g(x))$$

The function is also applied to the middle as changing either f or g can change the relation. For example, applying an exponential function to both sides and the middle gives $e^{f(x)}\ (e^z)\ e^{g(x)}$, where (e^z) is the new relation. Applying the logarithm function to all parts, $\ln f(x)\ (\ln z)\ \ln g(x)$, where $(\ln z)$ is the new relation. Applying differentiation to all parts, $Df(x)\ (Dz)\ Dg(x)$, where (Dz) is the new relation. D is a shorthand operator for differentiation $\frac{d}{dx}$.

Differentiating or integrating positive monotonic functions, with the condition that their difference is monotonic, preserves the $<$ and $\leq$ relations: $Df(x)\ (Dz)\ Dg(x) \Leftrightarrow f(x)\ z\ g(x)$ (see [5, Part 6]), and by notation, relations $(Dz) = (z)$, $(\int z) = (z)$.

Definition 2.3. *We say $z \in \mathbb{B}$ to mean that z is a binary relation.*

Definition 2.4. *When $f\ (\phi(z))\ g$ and $(\phi(z)) = z_2$, where ; $z, z_2, (\phi(z)) \in \mathbb{B}$; we may propose $\phi(z) = z_2$, provided that there is no contradiction.*

In practice, when solving for a relation, we presume such a relation exists and then proceed to solve for it. The method of brackets about the relation is a label of applied operations. If this is reversible, a solution exists to unravel the said operations. Definition 2.4 allows you to proceed with the solution process without having to formerly say so.

$$\text{If } (D^n z) = \succ \text{ then solve } D^n z = \succ$$

Example 2.1. $2x^2\ z\ 5x|_{x=\infty}$, $4x\ (D\,z)\ 5|_{x=\infty}$, $(D\,z) = \succ$, *removing the brackets and solving for z,* $D\,z = \succ$, $\int D\,z = \int \succ$, $z = \int \succ = \succ$ *(see Table 2).*

Definition 2.5. *Let a "finite relation" be a relation without consideration of infinitesimal or infinitary arithmetic.*

Example 2.2. *The relations $\forall n > n_0 : n + 1 > n$, for positive n, then $\frac{1}{n} > 0$ is not finite relations, as they include infinite arithmetic. No lower bound exists. The infimum (greatest lower bound) 0 exists, but is another type of number. Positive n has no upper bound either, but has an infimum ∞. The "bounds" do exist, but involve infinite arithmetic, and in a sense are numbers of another dimension.*

Definition 2.6. *Let a relation that is not finite have infinitesimal or infinite arithmetic.*

Any operation on the relation produces a new comparison, or new z_k, however such a system allows us to solve for the initial z, through the myriad of possibilities.

Example 2.3. *Compare in a multiplicative-sense n^2 and n. See Figure 1. In solving for $\succ$, the relations z_1, z_2, z_3 did not change after division, $(\ldots, n^2 \succ n, n \succ 1, 1 \succ n^{-1} \ldots)|_{n=\infty}$, by $b \succ a \Leftrightarrow cb \succ ca$ [2, Proposition 5.2].*

$$\begin{tikzcd}
 & & & n^2 \; z_1 \; n \arrow[dl, dash, "\div n"'] \arrow[dr, dash, "\frac{n^2}{n}|_{n=\infty} = \infty"] & \\
 & & n \; z_2 \; 1 \arrow[dl, dash, "\div n"'] \arrow[dr, dash, "\frac{n}{1}|_{n=\infty} = \infty"] & & n^2 \succ n \\
 & 1 \; z_3 \; \frac{1}{n} \arrow[dl, dash, "\frac{1}{n} \in \Phi"'] \arrow[dr, dash, "\frac{1}{n}|_{n=\infty} = 0"] & & n \succ 1 & \\
1 \succ \frac{1}{n} & & \mathbb{R} : \; 1 > 0 & &
\end{tikzcd}$$

Figure 1: Example calculation of relations connecting $*G$ and $\mathbb{R}$

Definition 2.7. *Given relations z_1 and z_2,*
If $f(x) \;\; z_1 \;\; g(x) \Rightarrow \phi(f(x)) \;\; z_2 \;\; \phi(g(x))$ then we say $\phi(z_1) = z_2$.

If $f(x) \;\; z_1 \;\; g(x) \Rightarrow e^{f(x)} \;\; z_2 \;\; e^{g(x)}$ then we say $e^{z_1} = z_2$.
If $f(x) \;\; z_1 \;\; g(x) \Rightarrow \ln f(x) \;\; z_2 \;\; \ln g(x)$ then we say $\ln z_1 = z_2$.

Use as an aid in calculation as a left to right operator when solving relations, where all functions in the relations are positive, $e^{>} = \succ$, $\ln \succ = >$, $\succ = >$.

The following compares the rate of increase of functions in infinitary calculus, which we find computationally easier than that described by Hardy, and at the least is an alternate way of performing such calculations.

> The Caterpillar was the first to speak. “What size do you want to be?” it asked. “Oh, I’m not particular as to size,” Alice hastily replied; “only one doesn’t like changing so often, you know.” “I don’t know,” said the Caterpillar. [10, pp 72–73]

After eating the mushroom in her right hand, Alice was shrunk, and eating from the left hand she was magnified.

In an analogous way, by applying powers and logarithms we can magnify or shrink aspects of the function comparison. Combined with non-reversible arithmetic [4, Part 5] (for example $n^2 + n = n^2|_{n=\infty}$), we can solve for the relation.

Powers and logarithms are mutual inverses. While logs of different bases can undo any powers, the natural logarithm ln and e are the most useful. In solving relations, it is often convenient to apply these functions to both sides of a relation, in a similar manner to solving equations. Then apply infinitary arithmetic with non-reversible algebra to simplify the relation.

Consider raising both sides of a finite inequality to a power. E.g. $3 > 2$, $e^3 > e^2$ and the relation symbol did not change.

Example 2.4. *Now consider a relation where both numbers are diverging to infinity. For example,* $3x > 2x|_{x=\infty}$*, then* $e^{3x} > e^{2x}|_{x=\infty}$ *but more importantly* $e^{3x} \succ e^{2x}|_{x=\infty}$ *as* $e^{3x}/e^{2x} = e^x|_{x=\infty} = \infty$.

If two numbers are positive and one is much larger than the other, then the weaker relation that one of the numbers is greater than the other, must be true.

Theorem 2.1. $f = \infty$, $g = \infty$, *if* $f \succ g$ *then* $f > g$

Proof. $f \succ g$ then $\frac{g}{f} = \delta$, $\delta \in \Phi$, $g = f\delta$. Comparing, $f \; z \; g$, $f \; z \; f\delta$, $1 \; z \; \delta$, $z = >$. □

Magnifying a less than or greater than relationship magnifies the inequality, provided their difference is not finite (see Theorems 2.2 and 2.3). Demonstrating this, consider a condition with an infinitesimal difference, so the inequality exists in $*G$, but not $\mathbb{R}$.

Example 2.5. *Show :* $f < g$ *does not imply* $e^f \prec e^g$. $\delta \in \Phi$; $g = f + \delta$, $f = \infty$, $f < g$, $f < f + \delta|_{\delta=0}$, $e^f \; z \; e^{f+\delta}|_{\delta=0}$, $1 \; z \; e^{\delta}|_{\delta=0}$, $1 \; z \; 1 + \delta + \frac{1}{2}\delta^2 + \ldots|_{\delta=0}$, $0 \; z \; \delta + \frac{1}{2}\delta^2 + \ldots|_{\delta=0}$, $0 \; z \; \delta|_{\delta=0}$, $z = <$, *but* z *is not* $\prec$.

If we realize the infinitesimals $*G \mapsto \mathbb{R}$ *and we have equality.* $1 \; z \; 1 + \delta|_{\delta=0}$, $\delta \mapsto 0$, $1 \;==\; 1$.

Theorem 2.2. $f = \infty$, $g = \infty$, *If* $f < g$ *and* $f - g \prec \infty \;\; \Rightarrow \;\; e^f < e^g$

Proof. $f < g$, $0 < g - f$, $e^0 < e^{g-f}$, $1 < e^{g-f}$, $e^f < e^g$ □

Theorem 2.3. $f = \infty$, $g = \infty$, *if* $f < g$ *and* $g - f = \infty \;\; \Rightarrow \;\; e^f \prec e^g$

Proof. $e^f \; z \; e^g$, $e^0 \; z \; e^{g-f}$, $1 \; z \; e^{\infty}$, $z = \prec$. □

Theorem 2.4. $f = \infty$, $g = \infty$, *if* $f \prec g$ *then* $g - f = \infty$

Proof. $\delta \in \Phi$; since $f \prec g$, let $f = \delta g$. Consider $g - f = g - g\delta = g(1 - \delta) \simeq g = \infty$. □

Theorem 2.5. $f = \infty$, $g = \infty$, *if* $f \prec g$ *then* $e^f \prec e^g$

Proof. $f \prec g$, $g - f = \infty$. $e^f \; z \; e^g$, $1 \; z \; e^{g-f}$, $1 \; z \; e^{\infty}$, $1 \; z \; \infty$, $z = \prec$ □

Proposition 2.1. *$f = \infty$, $g = \infty$, if $f \succ g$ then* $\ln f - \ln g = \infty$

Proof. $f \succ g$ then let $f\delta = g$ where $\delta \in +\Phi$; $f = \delta^{-1}g$. Consider $\ln f - \ln g = \ln(\delta^{-1}g) - \ln g$ $= \ln \delta^{-1} + \ln g - \ln g = \ln \delta^{-1} = \infty$. $\square$

In reducing a large number, the log function applied to both sides of a relation can decrease the inequality.

Theorem 2.6. *$f = \infty$, $g = \infty$ If $f < g$ then* $\ln f < \ln g$

Proof. $f < g$, at any point let $g = f^{1+\delta}$, $\delta > 0$. Compare $\ln f \ z \ \ln g$, $\ln f \ z \ \ln f^{1+\delta}$, $\ln f \ z \ (1+\delta)\ln f$, $0 \ z \ \delta \ln f$, $z = <$ then $\ln f < \ln g$. $\square$

Theorem 2.7. *$f = \infty$, $g = \infty$ If $f \prec g$ then* $\ln f < \ln g$

Proof. $f \prec g$ then $\frac{f}{g} = \delta$; $\delta \in \Phi^+$; $f = \delta g$ Since g is not an infinitesimal, multiplying by an infinitesimal decreases the number, then $f < g$. Alternatively $f \ z \ g$, $\delta g \ z \ g$, $\delta \ z \ 1$, $z = <$ as an infinitesimal is smaller than any positive number. As ln preserves the relation, $f < g \Rightarrow \ln f < \ln g$ $\square$

In reducing an infinite number with a log from a much less than relation does not imply another much less than relation, nor does it exclude it.

Proposition 2.2. *$f = \infty$, $g = \infty$, $f \prec g \not\Rightarrow$* $\ln f \prec \ln g$

Proof. Counter example: Example 2.6 $\square$

Theorem 2.8. *$f = \infty$, $g = \infty$, if $f \succ g$ then $Df \succ Dg$.*

Proof. $Df \ z \ Dg$, $1 \ z \ \frac{Dg}{Df}$, but $\frac{Dg}{Df} = \frac{g}{f}$ then $1 \ z \ \frac{g}{f}$. $g = \delta f$; $\delta \in \Phi^+$; $1 \ z \ \delta$, $z = \succ$ $\square$

Example 2.6. *$e^{3x} \succ e^{2x}|_{x=\infty} \not\Rightarrow 3x \succ 2x|_{x=\infty}$ as $\frac{3x}{2x}|_{x=\infty} = \frac{3}{2} \neq \infty$.*

When asking what happens when we reduce an infinity, in a similarly way to magnifying the relationship, we can consider the two complete cases $g - f \prec \infty$ and $g - f = \infty$, thereby showing these conditions to be necessary and sufficient for determining what happens when reducing the relation.

Proposition 2.3. *$f = \infty$, $g = \infty$, $f \propto g \Leftrightarrow \ln g - \ln f \prec \infty$,*

Proof. $f \propto g$ then $\frac{f}{g} = \alpha_n \prec \infty$, $\ln\frac{f}{g} = \ln \alpha_n$, $\ln f - \ln g \prec \infty$. $\square$

Proposition 2.4. *$f = \infty$, $g = \infty$, $f \asymp g \Leftrightarrow \ln g - \ln f \prec \infty$,*

Proof. By definition $f \asymp g$ then $f \preceq g$ and $f \succeq g$ [2, Part 2]. $a, b \in \mathbb{R}^+$; If $f \asymp g$ then $a < \frac{f}{g} < b$. Case $a < \frac{f}{g}$, $ag < f$, $\ln a + \ln g$ $(\ln\ <)$ $\ln f$ $\ln a$ $(\ln\ <)$ $\ln f - \ln g$. Similarly, $\frac{f}{g} < b$, $f < bg$, $\ln f$ $(\ln\ <)$ $\ln b + \ln g$, $\ln f - \ln g$ $(\ln\ <)$ $\ln b$. Putting the two conditions together, $\ln a$ $(\ln\ <)$ $\ln f - \ln g$ $(\ln\ <)$ $\ln b$. By Theorem 2.6, $\ln\ < = <$. $\ln a < \ln f - \ln g < \ln b$. Similarly ; $a', b' \in \mathbb{R}^+$; then $a' < \frac{g}{f} < b'$ gives finite bounds. □

The development of solving for the unknown relation as a variable comes about through comparing functions, which includes the calculation of limits, see [4, Part 5]. After developing the theory, while investigating infinitesimals in *Orders of Infinity* [8], similar problems were found and some of du Bois-Reymond's known theorems were rediscovered. At this point the alternate calculation was already useful, rather than trying to follow Hardy's calculations.

Using the equality symbol as an operator reading from left to right, define $z_1 = z_2$, from Definition 2.7 a table of relation implications, the equals symbol is interpreted from left to right, generally leading to the right side being a generalization from the left. As an operator analogy; Mercedes = car; BMW = car. The right-hand side is the generalisation of the left.

When using equality operator = for generalization, place the variable relation being solved for, on the left side of the equals sign. For example, writing $\ln z = \succ$ instead of $\succ = \ln z$. Then the generalisation can be combined with solving the variable. $\ln z = \succ$, $e^{\ln z} = e^{\succ} = \succ$, $z = \succ$. Examples of exponential and log functions:

In $*G$, given $f = \infty$, $g = \infty$, $f\ z\ g$ and $\phi(f)$ $(\phi(z))$ $\phi(g)$, then $\phi(z) = z_2$. The functions are continuous and monotonic in $*G$.

$\phi(\{<, \prec\})$	$\phi(\{>, \succ\})$	Condition	Reference
$e^{<} = <$	$e^{>} = >$	$f - g \prec \infty$	Th. 2.2
$e^{<} = \prec$	$e^{>} = \succ$	$f - g = \infty$	Th. 2.3
$\prec = <$	$\succ = >$		Th. 2.1
$e^{\prec} = \prec$	$e^{\succ} = \succ$		Th. 2.5
$\ln < = <$	$\ln > = >$		Th. 2.6
$\ln \prec = <$	$\ln \succ = >$		Th. 2.7
$\ln \prec = \prec$		$f \prec\!\!\prec g$	[4, Part 5]
$\int \prec = \prec$	$\int \succ = \succ$	Ignore integration constants	[5, Part 6]
$D\prec = \prec$	$D\succ = \succ$		Th. 2.8
$\int < = <$	$\int > = >$		[5, Part 6]
$\int \leq = \leq$	$\int \leq = \leq$		[5, Part 6]
$D< = <$	$D> = >$	$Df - Dg$ is not constant	[5, Part 6]
$D\leq = \leq$	$D\geq = \geq$	$Df - Dg$ is not constant	[5, Part 6]

Table 2: Relation simplification for positive divergent functions

Example 2.7. *Decreasing/reducing the infinities*

$$\begin{aligned} 3x > 2x|_{x=\infty} \\ \ln(3x) \;\; (\ln >) \;\; \ln(2x)|_{x=\infty} \\ \ln 3 + \ln x \;\; (\ln >) \;\; \ln 2 + \ln x|_{x=\infty} \\ \ln 3 \;\; (\ln >) \;\; \ln 2 \\ \ln 3 > \ln 2 \textit{ then } \ln > = > \end{aligned}$$

Example 2.8. *Increasing/magnifying the infinities*

$$\begin{aligned} 3x > 2x|_{x=\infty} \\ e^{3x} \;\; (e^{>}) \;\; e^{2x}|_{x=\infty} \\ e^{3x} \succ e^{2x}|_{x=\infty} \textit{ then } e^{>} = \succ \end{aligned} \qquad (\tfrac{e^{3x}}{e^{2x}}|_{x=\infty} = e^{x}|_{x=\infty} = \infty)$$

To show how all this works, take an example problem from [8, p.8]. Solve for z the following, where Δ is an arbitrarily large but fixed value.

Example 2.9.

$$\begin{aligned} e^{x} \;\; z \;\; x^{\Delta}|_{x=\infty} \\ \ln e^{x} \;\; (\ln z) \;\; \ln x^{\Delta}|_{x=\infty} \\ x \;\; (\ln z) \;\; \Delta \ln x|_{x=\infty} \\ \ln z = \succ, z = e^{\succ} = \succ \\ e^{x} \succ x^{\Delta}|_{x=\infty} \end{aligned}$$

At first, raising a relation to a power may seem silly, but it is useful when applied as a notational aid in the solution; understood as a magnification it makes sense. However, the solution is not always unique, $z = e^{\succ} = >$ is true too, as $\succ = >$ with the left-to-right reading.

The comparison at infinity can ignore added constants, that is, the comparison is with infinite elements.

Proposition 2.5. *If $f = \infty$, $g = \infty$, $f \succ \alpha$, $g \succ \beta$, $z \in \{>, \geq, \succ, \succeq\}$ then*

$$f + \alpha \;\; z \;\; g + \beta \quad \Rightarrow \quad f \;\; z \;\; g$$

Proof. $f + \alpha \; z \; g + \beta$, $f \; z \; g + \beta$ because $(f + \alpha = f$ as $f \succ \alpha)$, $f \; z \; g$ as $g \succ \beta$ $\square$

Example 2.10. *Demonstrated by example, another problem from [8, p.8]. Given* $P_m(x) = \sum_{k=0}^{m} p_k x^k$, p_k *is positive and* $Q_n(x) = \sum_{k=0}^{n} q_k x^k$, q_k *is positive. Show* $\ln\ln P_m(x) \sim \ln\ln Q_n(x)|_{x=\infty}$

$$
\begin{aligned}
\ln\ln P_m(x) \ z\ \ln\ln Q_n(x)|_{x=\infty} & \\
\ln\ln \sum_{k=0}^{m} p_k x^k \ z\ \ln\ln \sum_{k=0}^{n} q_k x^k|_{x=\infty} & \quad (\text{Apply } x^k \succ x^{k-1},\, p_k x^k + p_{k-1}x^{k-1} = p_k x^k|_{x=\infty}) \\
\ln\ln p_m x^m \ z\ \ln\ln q_n x^n|_{x=\infty} & \\
\ln(\ln p_m + m\ln x) \ z\ \ln(\ln q_n + n\ln x)|_{x=\infty} & \\
\ln(m\ln x) \ z\ \ln(n\ln x)|_{x=\infty} & \\
\ln m + \ln\ln x \ z\ \ln n + \ln\ln x|_{x=\infty} & \\
\ln\ln x \ z\ \ln\ln x|_{x=\infty} & \qquad (\text{from } \ln_2 x \succ \ln m,\ \ln_2 x \succ \ln n) \\
z = \sim &
\end{aligned}
$$

Contrast the above with an example where the highest order diverging terms are simplified (subtracting equally infinite quantities); the next highest order diverging terms determine the relation.

Example 2.11. *Solve for* z*, for the comparison* $n^n n \ z\ e^n n!|_{n=\infty}$. $n\ln n + \ln n \ (\ln z)\ n + \sum_{k=1}^{n} \ln_k|_{n=\infty}$. *Given* $\sum_{k=1}^{n} \ln_k = n\ln n - n|_{n=\infty}$ *[*$\sum_{k=1}^{n} \ln_k n = \int_1^n \ln_k n\, dn = n\ln n - n|_{n=\infty}$*] then,* $n\ln n + \ln n \ (\ln z)\ n + n\ln n - n|_{n=\infty}$, $\ln n \ (\ln z)\ 0|_{n=\infty}$, $\ln z = \succ$, $z = e^{\succ} = \succ$, $n^n n \succ e^n n!|_{n=\infty}$

The application of the logarithm has simplified the problem from products of functions to sums of functions.

Example 2.12. *Consider the following theorems from [8, pp.300–302] Theorem 7.11. If* $a > 0,\ b > 0$ *we have*

$$\lim_{x\to\infty} (\ln x)^b / x^a = 0, \quad \lim_{x\to\infty} x^b / e^{ax} = 0$$

Proof. $\ln x \prec x|_{x=\infty}$, $\ln x \ z\ x^a|_{x=\infty}$, $\ln_2 x \ (\ln z)\ a\ln x|_{x=\infty}$, $\ln_2 x \prec a\ln x|_{x=\infty}$, $z = e^{\prec} = \prec$, $\ln x \prec x^a|_{x=\infty}$, $(\ln x)^b \ z_2\ x^a|_{x=\infty}$, $b\ln_2 x \ (\ln z_2)\ a\ln x|_{x=\infty}$, $b\ln_2 x \prec a\ln x|_{x=\infty}$, $(\ln x)^b \prec x^a|_{x=\infty}$. $(\ln x)^b/x^a|_{x=\infty} = 0$ □

Proof. $e^x \succ x|_{x=\infty}$, $e^x \ z\ x^b|_{x=\infty}$, $x \ (\ln z)\ b\ln x|_{x=\infty}$, $x \succ b\ln x|_{x=\infty}$, $z = e^{\succ} = \succ$, $e^x \succ x^b|_{x=\infty}$, $e^{ax} \ z_2\ x^b|_{x=\infty}$, $ax \ (\ln z_2)\ b\ln x|_{x=\infty}$, $ax \succ b\ln x|_{x=\infty}$, $e^{ax} \succ x^b|_{x=\infty}$. $x^b/e^{ax}|_{x=\infty} = 0$ □

Applying infinitary calculus to problems can result in choosing whether to use a theorem, or solving by calculating directly.

Example 2.13. *Find* $\lim_{x\to 0^+} x^\alpha \ln x$*, where* $\alpha > 0$*,* $x^\alpha \ln x|_{x=0^+}$*,* $x^{-\alpha}\ln x^{-1}|_{x=\infty}$*,* $\ln x^{-1} \;\; z \;\; x^\alpha|_{x=\infty}$*,* $\ln x^{-1} \prec x^\alpha|_{x=\infty}$*,* $x^{-\alpha}\ln x^{-1}|_{x=\infty} = 0$*,* $x^\alpha \ln x|_{x=0^+} = 0$*.*
Another way. Let $y = x^\alpha \ln x|_{x=0^+} = x^{-\alpha}\ln x^{-1}|_{x=\infty}$*,* $\ln y = \ln(x^{-\alpha}\ln x^{-1})|_{x=\infty} = \ln x^{-\alpha} + \ln_2 x^{-1}|_{x=\infty} = \ln x^{-\alpha}|_{x=\infty}$ *as* $\ln x \succ \ln_2 x|_{x=\infty}$*,* $y = x^{-\alpha}|_{x=\infty} = 0$

Example 2.14. *[8, p.31] Compare the rate of increase of* $f = (\ln x)^{(\ln x)^\mu}$ *and* $\phi = x^{(\ln x)^{-v}}$*.*

$$\begin{aligned} f \;\; z \;\; \phi|_{x=\infty} \\ (\ln x)^{(\ln x)^\mu} \;\; z \;\; x^{(\ln x)^{-v}}|_{x=\infty} \\ (\ln x)^\mu \ln_2 x \;\; (\ln z) \;\; (\ln x)^{-v}\ln x|_{x=\infty} \\ \mu \ln_2 x + \ln_3 x \;\; (\ln_2 z) \;\; -v\ln_2 x + \ln_2 x|_{x=\infty} \\ (\mu+v)\ln_2 x + \ln_3 x \;\; (\ln_2 z) \;\; \ln_2 x|_{x=\infty} \end{aligned}$$

Case $\mu + v = 1$*,* $\ln_3 x \;\; (\ln_2 z) \;\; 0|_{x=\infty}$*,* $f \succ \phi$*.*
Case $\mu + v < 1$*,* $(\mu+v)\ln_2 x \;\; (\ln_2 z) \;\; \ln_2 x|_{x=\infty}$*,* $0 \prec (1-\mu-v)\ln_2 x|_{x=\infty}$*,* $f \prec \phi$*.*
Case $\mu + v > 1$*,* $(\mu+v)\ln_2 x \;\; (\ln_2 z) \;\; \ln_2 x|_{x=\infty}$*,* $(\mu+v-1)\ln_2 x|_{x=\infty} \succ 0$*,* $f \succ \phi$*.*

In solving relations of infinite magnitude, another case occasionally arises where both sides of the relation are infinite, but opposite in sign. Raise all parts to a power, with the effect of pushing the positive infinity further to infinity, and the negative infinity to zero, effectively pulling the relation further apart.

Example 2.15. $n = \infty$

$$\begin{aligned} -n \;\; (\ln z) \;\; \ln n \\ e^{-n} \;\; z \;\; e^{\ln n} \\ e^{-(n+1)} \;\; z \;\; e^{\ln(n+1)} \\ e^{-(n+2)} \;\; z \;\; e^{\ln(n+2)} \\ 0 \leftarrow \;\; z \;\; \rightarrow \infty \end{aligned}$$

Proposition 2.6.

$$\textit{If } f = \infty,\ g = \infty,\ -f \; z \; g \textit{ then } e^{-f} \prec e^g$$

Proof. Solving for z, $e^{-f} \; z \; e^g$, $\frac{1}{e^f} \; z \; e^g$, $0 \prec \infty$, as $e^f = +\infty$ then $e^{-f} = 0$. $z = \prec$. □

Example 2.16. *Solve* $n \ln x \;\; (\ln z) \;\; \ln n|_{n=\infty}$ *when* $x = (0, 1)$*. Within this interval* $\ln x$ *is negative.* $n \ln x \succ \ln n|_{n=\infty}$*,* $e^{n \ln x} \;\; (e^\succ) \;\; e^{\ln n}|_{n=\infty}$*,* $x^n \;\; (e^\succ) \;\; n|_{n=\infty}$*,* $x^n \prec n|_{n=\infty}$*,* $z = e^\succ = \prec$*.*

Definition 2.8. *Let* $==$ *mean an equality relation.*

Definition 2.9. *Let* $z = ==$ *mean equality is assigned to the variable* z

We have further introduced a use of assignment as a left-to-right generalisation. As maths is a language, this decision was made to chain together implications.

Having the context of the problem being solved for is important.

Multiplying by -1 reverses the orders direction, the order relations $\{<, \leq, >, \geq\}$ are not effected by positive multiplication or addition.

Comparison in an additive sense 'is' effected by adding and subtracting terms. Not in the sense of the order as described, but the magnitudes can be shifted, hence the magnitude relations can change direction. If $a \succ b$, $a + -a \prec b - a$, $0 \prec b - a$. [2, Part 2]

When adding the same value to both sides the much-greater-than relation can change direction, unlike inequalities $\{<, \leq\}$ which are invariant.

Example 2.17.

$$\begin{aligned} x^2 &\succ x|_{x=\infty} \\ x^2 - x^2 &\prec -x^2 + x|_{x=\infty} && \text{(direction changes)} \\ 0 &\prec -x^2 + x|_{x=\infty} \end{aligned}$$

In solving a relation, say relation z, if the aim in solving is to satisfy all the expressions involving z, then adding to both sides can introduce contradictory solutions. If after addition, the highest terms magnitude is removed, the next highest order determines the magnitude relation.

Example 2.18. *Solve z_1 and z_2 for the same relation.*

$$\begin{aligned} n^2 + 3 \; z_1 \; n^2 + n|_{n=\infty} && \text{(subtract } n^2\text{)} \\ 3 \; z_2 \; n|_{n=\infty} \\ z_1 = <, \quad z_2 = < \end{aligned}$$

However, for usability, we really do not wish to be this formal.

Example 2.19.

$$\begin{aligned} n^2 + 3 \; z \; n^2 + n|_{n=\infty} && \text{(subtract } n^2\text{)} \\ 3 \; z \; n|_{n=\infty} && \text{(solving } z = \prec \text{ contradicts the first expression)} \\ z = < && \text{(True for both expressions)} \end{aligned}$$

This management of variables is part of the mathematics. At times, it is only necessary to solve with forward implications. Other times, for example in constructing proofs, reversibility, having implications in both directions is required. Or more generally, we want to solve not for several binary relation variables, when we can do so with one.

Up till now we have explored some of the mechanics for solving the relation. Of course, the premise was simple, solve for the relation as a variable.

However, infinitary calculus concerns itself with functions and curves, and continuous families of curves; we imagine curves between curves as real numbers between other real numbers. Just as the relation separated and defined different numbers, relations again separate and define different curves.

In discussing the infinity of curves near an existing curve, G. Fisher [9, pp 109–110] comments and beautifully quotes du Bois-Reymond in developing an infinity of curves close to $y = x$, with the following relationship at infinity. $x^{\frac{p}{p+1}} < x^{\ln_2(x)/\ln_2(x+1)} < x|_{x=\infty}$, $p \in \mathbb{N}$. That is, there is a function between $x^{\frac{p}{p+1}}$ and x, and there are an infinitely many such functions, $x^{\frac{p}{p+1}} < \phi(x) < x|_{x=\infty}$. Therefore, there are an infinitely many functions close to $y = x$ at infinity, where the space of real valued functions diverge, $f(x)|_{x=\infty} = \infty$.

Example 2.20. *There exists an infinity of curves infinitesimally close to the straight line* $y = x$*. Show* $x^{\frac{p}{p+1}} < x^{\frac{\ln_2 x}{\ln_2(x+1)}} < x^1|_{x=\infty}$.

We will use an indirect inequality approach, where we introduce another inequality. Undoing the base x*, show* $\frac{p}{p+1} < \frac{x}{x+1} < \frac{\ln_2 x}{\ln_2(x+1)}|_{x=\infty} < 1$.

$$\frac{p}{p+1} \; z \; \frac{x}{x+1}|_{x=\infty}$$
$$p(x+1) \; z \; (p+1)x|_{x=\infty}$$
$$px + p \; z \; px + x|_{x=\infty}$$
$$p \; z \; x|_{x=\infty}, \quad z = <$$
$$x^{\frac{x}{x+1}}|_{x=\infty} \; z \; x^{\ln_2(x)/\ln_2(x+1)}|_{x=\infty}$$
$$\frac{x}{x+1} \cdot \ln x \; (\ln z) \; \ln_2(x)/\ln_2(x+1) \cdot \ln x|_{x=\infty}$$
$$\frac{x}{x+1} \; (\ln z) \; \ln_2(x)/\ln_2(x+1)|_{x=\infty}$$
$$x \ln_2(x+1) \; (\ln z) \; (x+1) \ln_2 x|_{x=\infty}$$
$$\ln x + \ln_3(x+1) \; (\ln_2 z) \; \ln(x+1) + \ln_3 x|_{x=\infty}$$
$$\ln x \; (\ln_2 z) \; \ln(x+1)|_{x=\infty} \qquad (\text{as } \ln x \succ \ln_3 x|_{x=\infty})$$
$$x \; (\ln z) \; x+1|_{x=\infty}$$
$$\ln z = <, \quad z = e^{<} \; = <$$

The last relation trivially follows, $\frac{\ln_2 x}{\ln_2(x+1)} \; z \; 1|_{x=\infty}$, $\ln_2 x \; z \; \ln_2(x+1)|_{x=\infty}$, $z = <$. *Then* $x^{\frac{p}{p+1}} < x^{\frac{\ln_2 x}{\ln_2(x+1)}} < x^1|_{x=\infty}$.

We did not need to introduce the additional inequality. However, like algebra in general, we may not get the minimal solution. The same calculation, without the added inequality, and using an asymptotic approximation could have been used, see [7, Example 2.5].

As a rule of thumb, think of infinity as being as large as the reals, so that you could construct any graph there, and then know that the space is larger still. However what is striking is the existence of the relations themselves, and this property, to not just partition, but develop algorithms at infinity. It is obvious then that if you can iterate in that space, have relations existing in that space, then you can construct algorithms and mathematical reasoning in that space.

In comparing sequences and functions, monotonic sequences and functions, that is sequences and functions which are either equal to or increasing, or equal to and decreasing, are of great interest.

Since a monotonic sequence can be made into a monotonic function, and a monotonic function back into a monotonic sequence, the theory of functions given is true for sequences.

Roughly, if we can determine that two functions are monotonic, we can do other things such as compare their ratio, and other mathematics.

Hence, the interest with Hardy's L-functions, which are monotonic functions, that comprise of a finite combination of $\{+, -, \div, \times, \ln, \mathrm{e}\}$ operations.

Also, with monotonic functions, the association of a sequence of points to a curve would allow sequences and functions to be connected, via the same relations.

As sequences are indexed, a connection between the discrete and continuous can be made.

Proposition 2.7. *Given a relation z, and functions (f, g), then $(z, (f, g))$ is a relation if for all values in the domain, there is no contradiction. If there is a contradiction, $(z, (f, g))$ is not a relation, and is disproved as one.*

Proof. Transforming the functions into sets, since the set generated by $(z, (f, g))$ is not an exact subset of the relation set generated from z, by definition $(z, (f, g))$ is not a relation. $\square$

Example 2.21. *Show $n^2 > e^n|_{n=\infty}$ contradicts. By using L'Hopital [4, Part 5], $\infty > \infty$, differentiate, $2n > e^n|_{n=\infty}$, $2 > e^n|_{n=\infty}$, contradicts as $2 \prec e^n|_{n=\infty}$.*

Another approach is by solving, and showing that the symbols are contradictory.

$n^2 \ z \ e^n|_{n=\infty}$, $\ln(n^2) \ (\ln z) \ \ln(e^n)|_{n=\infty}$, $2 \ln n \ (\ln z) \ n|_{n=\infty}$, $(\ln z) = \prec$, $z = e^{\prec} = \prec$. Since both arguments are positive, $\prec$ implies $<$ which contradicts $>$.

Propositions and theorems for the infinitely small can be similarly constructed.

Proposition 2.8. *If ; $f, g \in +\Phi$; and $z \in \{<, \leq, =, >, \geq\}$,*

$$f \ z \ g \ \Leftrightarrow e^f \ z \ e^g$$

Proof. Assuming the partial sums of the exponential function with an infinitesimal are asymptotic to the infinite sum, $e^{\delta} = 1 + \delta + \frac{\delta^2}{2} + \ldots \sim 1 + \delta + \sum_{i=0}^{w} \frac{f^{(i)}}{i!}$.

We have the general partial sum comparisons z_i, $1 + f \; z \; 1 + g$, $1 + f + \frac{f^2}{2} \; z_2 \; 1 + g + \frac{g^2}{2}$, $1 + f + \frac{f^2}{2} + \frac{f^3}{3!} \; z_3 \; 1 + g + \frac{g^2}{2} + \frac{g^3}{3!}$, ...

Case $e^f \; z \; e^g$ implies $f \; z \; g$: $e^f \; z \; e^g$, $1 + f + \frac{f^2}{2} + \ldots \; z \; 1 + g + \frac{g^2}{2} + \ldots$, apply non-reversible addition [4, Part 5 Theorem 2.1] to f and g partial sums, for example $\frac{f^i}{i!} + \frac{f^{i+1}}{(i+1)!} = \frac{f^i}{i!}$ as $f^i \succ f^{i+1}$ and $z_{i+1} = z_i$. $1 + f \; z \; 1 + g$, $f \; z \; g$.

Reversing the process to prove the implication in the other direction. Adding the infinitesimal does not change the relation as the next term is much less than in magnitude to the previous term. $f \; z \; g$, $1 + f \; z \; 1 + g$, $1 + f + \frac{f^2}{2!} \; z \; 1 + g + \frac{g^2}{2!}$, $1 + f + \frac{f^2}{2} + \frac{f^3}{3!} \ldots \; z \; 1 + g + \frac{g^2}{2} + \frac{g^3}{3!} + \ldots$, recognising the expression as the exponential functions, $e^f \; z \; e^g$. □

For comparing of functions, we do need well behaved functions, hence the monotonic requirements. The classes of functions may appear to be restricted, but this can be expanded in many ways. Non-reversible arithmetic can be used to remove transient terms. For an additive comparison, we may only need $f - g$ to be ultimately monotonic, and not the ratio.

So important is the determination of these classes of functions that they lead to the following definition and conjecture.

In accordance with redefining other infinitary calculus relations, we redefine the L-function in $*G$.

Definition 2.10. *Define an L-function in $*G$ without implicit complex numbers as a finite combination of* $\{+, -, \div, \times, \ln, \mathrm{e}\}$ *operations.*

The following is given by Hardy as a theorem [8, p.24 Appendix I], but here stated as a conjecture because we re-defined the L-functions in $*G$ instead of $\mathbb{R}$. For example, the limit in $*G$ [5, Part 6] can be a function, as it can contain infinitesimals and infinities. A transfer of the theorem in $*G \mapsto \overline{\mathbb{R}}$ [3, Part 4] would result in the theorem stated by Hardy.

Conjecture 2.1. *Any L-functions at infinity is ultimately continuous and monotonic.*

If f and g are L-functions then at $f \; z \; g|_{n=\infty}$, z is unique and $z \in \{\prec, \succ, \propto\}$

Additionally the L-functions have the property that if f and g are L-functions, so is their ration f/g. Truncated infinite series can also be L-functions.

Example 2.22. $x^2 = e^{2 \ln x}|_{x=\infty}$ *is an L-function.*

3 M-functions an extension of L-functions

Surprisingly, considering infinity as a point in the comparison theory at infinity is not enough. On occasion, it is beneficial to compare between two infinities.

This idea of comparison between two infinities was indirectly taken and adapted from NSA where convergence was determined by integrating at infinity: integrated between two infinities to determine convergence or divergence. Similarly for comparison, we can compare at infinity by comparing at infinity, over an infinite interval.

Conjecture 3.1. *L-functions can be compared over any infinite interval. f, $g \in$ L-functions; a, $b, \in \Phi^{-1}$; $b - a = \pm\infty$; $\{\prec, \succ, \propto\} \in z$: $f \; z \; g|_{x=[a,b]} \equiv f \; z \; g|_{x=\infty}$*

We would like to extend Conjecture 2.1 to include other functions such as $n!|_{n=\infty}$ which do not contain a finite number of multiplication operations, but is monotonic, and either ever increasing or ever decreasing. The following is an attempt to capture this.

What is the M-word? Marriage. We define a marriage of properties from Conjecture 2.1 the monotonic L-functions and 'infinite term functions'.

By the following definition, assuming the conjecture is true, all L-functions are M-functions.

Definition 3.1. *M-functions satisfy one of the following comparisons: $\propto$, $\prec$ or $\succ$ for any infinite interval.*

If the functions have the same behaviour at infinity, that is $f, g \in \Phi^{-1}$; then only consider one infinite interval.

$$f(n),\; g(n) \in \text{M-functions};\; a,\; b,\; b - a \in \Phi^{-1};$$

$$\text{If } f \prec g|_{[a,b]} \text{ then by definition } f \prec g|_{n=\infty}$$

Consider the following problem, first easily solved with Stirling's formula, then without.

With Stirling's formula $e^n n! = cn^{n+\frac{1}{2}}|_{n=\infty}$, the comparison is obvious.

$$\begin{aligned} n^n \; z \; e^n n!|_{n=\infty} \\ n^n \; z \; cn^{n+\frac{1}{2}}|_{n=\infty} \\ 1 \; z \; cn^{\frac{1}{2}}|_{n=\infty} \\ z = \prec \end{aligned}$$

Example 3.1. *Without Stirling's formula, the problem appears more difficult. Firstly, re-organize the comparison.*

$$n^n \; z \; e^n n!|_{n=\infty}$$
$$n^n \; z \; e^n \prod_{k=1}^{n} k|_{n=\infty}$$
$$n^n \; z \; \prod_{k=1}^{n} ek|_{n=\infty}$$

Consider the comparison between two infinities. For example $(n, 2n]|_{n=\infty}$.

$$\prod_{k=n+1}^{2n} n \;\; z \;\; \prod_{k=n+1}^{2n} ek$$
$$\prod_{k=n+1-n}^{2n-n} n \;\; z \;\; \prod_{k=n+1-n}^{2n-n} e(k+n)$$
$$n^n \;\; z \;\; \prod_{k=1}^{n}(e(n+k))|_{n=\infty}$$
$$n^n \;\; z \;\; \prod_{k=1}^{n}(ne(1+\frac{k}{n}))|_{n=\infty}$$
$$n^n \;\; z \; n^n e^n \prod_{k=1}^{n}(1+\frac{k}{n})|_{n=\infty}$$
$$1 \; z \; e^n \prod_{k=1}^{n}(1+\frac{k}{n})|_{n=\infty}$$
$$z = \prec$$

In Example 3.1, it is assumed that $n^n, e^n n! \in$ M-functions.

The theory of functions is very important, as if we can guarantee certain properties, the analysis can be developed in powerful ways. In whatever form that the theory finally takes, identifying and restricting the functions will allow the application of comparison algebra to be more consistent and exacting. The development of the applications theory which we believe is a new field of mathematics rests on the comparison function theory. It is no longer a question of 'will it work', but 'how does it work'. Getting this right potentially means being independent from NSA for solving large classes of problems, a goal worth striving for. A language of functions rather than a language of sets for solving theory with functions is required.

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RMIT University, GPO Box 2467V, Melbourne, Victoria 3001, Australia
chelton.evans@rmit.edu.au