Convergence sums at infinity with new convergence criteria

Chelton D. Evans and William K. Pattinson

Abstract

Development of sum and integral convergence criteria, leading to a representation of the sum or integral as a point at infinity. Application of du Bois-Reymond's comparison of functions theory, when it was thought that there were none. Known convergence tests are alternatively stated and some are reformed. Several new convergence tests are developed, including an adaption of L'Hopital's rule. The most general, the boundary test is stated. Thereby we give an overview of a new field we call 'Convergence sums'. A convergence sum is essentially a strictly monotonic sum or integral where one of the end points after integrating is deleted resulting in a sum or integral at a point.

1 Introduction

10.16 Raabe's tests* [10]
10.17 Generalized p-series test* [8]
10.18 Generalized ratio test* [10]
10.19 Boundary test* [8]
10.20 nth root test* [8, Theorem 5.1]

11. Miscellaneous

Before attempting to evaluate a sum or integral, we need to know if we *can* evaluate it. The aim of this paper is to introduce a new field of mathematics concerning sum convergence, with an 'aerial' view of the field, for the purpose of convincing the reader of its existence and extensive utility. In general, we are concerned with positive series.

We are building the theory of this new mathematics from the foundations of our previous papers, and by extending du Bois-Reymond's Infinitesimal and Infinitary Calculus ([2], [3], [4], [5], [6], [12]).

The introduction of the 'gossamer' number system [2, Part 1] reasons in, and defines infinitesimals and infinities, and is comparable with the hyper reals, but more user friendly.

Paper [4, Part 3] introduces an algebra that results from the comparison of functions, which is instrumental in the following development of the convergence tests.

The goal is to reproduce known tests and new tests with this theory. Advantage is taken of known tests, where a parallel test is constructed from the known test. However in some cases the new test is quite different, with an extended problem range and usage. The p-series, by application of non-reversible arithmetic at infinity, is such a test.

The tests listed above marked with * are referenced but are considered in greater detail in subsequent papers. Other ideas such as rearrangements [11], derivatives [9], and applications with infinite products are also discussed in subsequent papers.

In [4, Part 3] a method of comparing functions, generally at zero or infinity, is developed. This is the core idea for developing the convergence criterion, and subsequent tests.

However, even in this paper we will have to request faith, as not until the consideration of convergence or divergence at the boundary [8] can we understand why this maths works. For this we need du Bois-Reymond's infinitary calculus and relations.

To this point it has generally been believed, even by advocates such as G. H. Hardy, that du Bois Reymond's infinitary calculus has little application. We believe, with the discovery of sums at infinity, which we call 'convergence sums', this view may be overturned.

With over twenty tests, and the application of an infinitesimal and infinity number system, the infinireals, we believe this recognizes the infinitary calculus, produced by both du Bois-Reymond and Hardy.

Hardy himself did not believe du Bois-Reymond's theory to be of major mathematical significance. However, if our work changes this belief, both du Bois-Reymond and Hardy's path and intuition can be justified.

In [2, Part 1] we develop the infinireals and 'gossamer' number system $*G$, which is constructed from infinite integers.

When reasoning with infinitesimals and infinities, often simpler or more direct constructions are possible. We are motivated to seek this both for theoretical and practical calculations.

We have argued in our number system that infinitesimals and infinities are numbers.

In the gossamer number system, Φ are infinitesimals, Φ^{-1} are infinities and $\mathbb{R}_\infty$ are 'Infinireals' which are either infinitesimals or infinities.

Further, the numbers in $*G$ have an explicit number type such as integers, rational numbers, infinite integers, infinite transcendental numbers, etc.

Infinitary calculus is a non-standard analysis, which we see as complementing and replacing standard analysis and other non-standard analyses, where applicable.

2 What does a sum at infinity mean?

Zeno's paradoxes provide excellent reasons for us to accept infinity, as we need to consider a sum of infinite terms to obtain a finite result. The following argument from our readings on Zeno, demonstrates that a partial sum is subject to Zeno's paradoxes.

> Consider an arrow in flight. After travelling half the distance, half the distance remains. Repeating this, after each repetition, since half the distance always remains, the arrow never reaches its target.

The remaining distance $\frac{1}{2^n}|_{n=\infty}$ is of course a positive infinitesimal, smaller than any positive real number. With these kinds of problems and the discovery of infinitesimal calculus, mathematical knowledge exponentially increases.

The problem was the infinite sum: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots = 1$. By expressing $y = \frac{1}{2} + \frac{1}{4} + \ldots$ as an infinite sum, $2y = 1 + \frac{1}{2} + \frac{1}{4} + \ldots = 1 + y$, the infinitesimal is side-stepped (the infinite series are subtracted) and y is solved for $y = 1$, and the arrow hits the target. Alternatively, express as a partial sum, $y_n = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}$, then $y_n = 1 - \frac{1}{2^n}$, $\frac{1}{2^n}|_{n=\infty} \in \Phi$, $y_n|_{n=\infty} = 1$.

In fact, Zeno's paradoxes prove the existence of the infinitesimal, for such a number can always be constructed. With the arrow striking the target, an application of the transfer principle where the infinitesimal is set to zero$((*G, \Phi) \mapsto (\mathbb{R}, 0))$. We see from the two number systems $*G$ and $\mathbb{R}$, that in $*G$ the arrow has no collision with the target, but always approaches infinitesimally close to the target. The realization of the sum is by the transfer principle, transferring to a collision in $\mathbb{R}$.

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \dots$$

Typically such a sum is described as converging when the sum of the remaining terms tends to zero. $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{n-1} a_k + r_n$ where $r_n = \sum_{k=n}^{\infty} a_k$ and for convergence $r_n \to 0$ as $n \to \infty$. As this partial sum includes finite and infinite terms, we will need to construct a different partial sum, separating finite and infinite terms, as convergence/divergence is considered for the infinite terms only.

Since a sum is a sequence of partial sums, even at infinity, then the convergence of the sum is reduced to sequence theory and the convergence of the sequence. Let $s_n = \sum_{k=1}^{n} a_n$, then does $(s_n)|_{n=\infty}$ converge?

We find that all sum convergence or divergence is determined at infinity [7, Corollary 2.1], except if the sum diverges by summing a prior singularity. This is independent of the sum convergence criteria.

We apply this reasoning to sums.

Consider a sum of positive terms $\sum_{k=1}^{\infty} a_k$. Let a_k be finite so the series contains no singularities for an arbitrary number of finite terms (for example none of the terms divide by zero). Then such a sum, if it diverges, can only diverge at infinity, as a sum with a finite number of terms is always convergent.

Since a convergent series is the negation of a divergent series, such that all series are classified as either convergent or divergent, then this is determined at the singularity $n = \infty$.

Definition 2.1. *We say $\sum_{k=n_0}^{n_1} a_k$ is a 'convergence sum' at infinity. The domain of the sum is not finite; $n_0, n_1 \in \mathbb{N}_\infty$;*

We define iterating over a sum and integral with number types.

Definition 2.2.

$$\sum_{\mathbb{N}_<} a_k = a_1 + a_2 + a_k \dots \textit{ where } k \textit{ is finite.}$$

$$\sum_{\mathbb{N}_\infty} a_k = \dots, a_{n-1} + a_n + a_{n+1} + \dots \textit{ for } n \in \mathbb{N}_\infty$$

Definition 2.3. $\int_{\min(+\Phi^{-1})}^{\max(+\Phi^{-1})} a(x)\, dx = \int_{+\Phi^{-1}} a(x)\, dx = \int a(x)\, dx|_{+\Phi^{-1}}$

With the existence of infinite integers $\mathbb{J}_\infty$, integer sequences can be partitioned into finite and infinite parts, $(a_1, a_2, \ldots) = (a_1, a_2, \ldots, a_k)|_{k<\infty} + (\ldots, a_n, a_{n+1}, \ldots)|_{n=\infty}$ [7, Definition 2.4]. Consequently sequences converge at infinity, $a_m - a_n|_{\forall m,n=\infty} \simeq 0$ [7, Definition 2.16] , and since a sum is a sequence, the sum can now be deconstructed into finite and infinite parts.

Theorem 2.1. $\sum_{j=1}^{\infty} a_j = \sum_{j=1}^{j<\infty} a_j + \sum_{\mathbb{N}_\infty} a_j = \sum_{\mathbb{N}_<} a_j + \sum_{\mathbb{N}_\infty} a_j$

Proof. By the existence of infinite integers, with infinireals, $\mathbb{N}_< < \mathbb{N}_\infty$, we can extend an integer sequence to infinity.

$(1, 2, 3, \ldots) = (\mathbb{N}_<) + (\mathbb{N}_\infty)$, then $\sum_{j=1}^{\infty} a_j = \sum_{\mathbb{N}_<} a_j + \sum_{\mathbb{N}_\infty} a_j$. □

Definition 2.4. *We say $\int_{x_0}^{x_1} a(x)\, dx|_{x=\infty}$ is a 'convergence integral' at infinity. The domain of the integral is not finite; $x_0, x_1 \in +\Phi^{-1}$;*

Theorem 2.2. $\int_{\alpha}^{\infty} a(x)\, dx = \int_{\alpha}^{x<\infty} a(x)\, dx + \int_{+\Phi^{-1}} a(x)\, dx$

Proof. By the existence of infinite real numbers, $\mathbb{R}^+ < +\Phi^{-1}$, partition the integral on the domain between the finite and infinite integrals. □

Theorem 2.3. *If given the sum $\sum_{k=0}^{\infty} a_k$, a_k has no singularities for finite $k (k < \infty)$, then convergence or divergence of the sum is determined at infinity, $\sum_{\mathbb{N}_\infty} a_k$.*

Proof. Since every finite sum is convergent, only an infinite sum is divergent. As the negation of a divergent series determines convergence, only at infinity can an infinite series be determined to be convergent or divergent. □

Theorem 2.4. *If given the integral $\int_{x_0}^{\infty} a(x)\, dx$, $a(x)$ has no singularities for finite $x (x < \infty)$, then convergence or divergence of the integral is determined at infinity, $\int a(x)\, dx|_{+\Phi^{-1}}$.*

Proof. Since every definite integral without singularities is convergent, only at infinity can the integral diverge. Hence the determination of convergence or divergence occurs at infinity. □

Theorem 2.3 of course does not say what is happening at infinity, nor does it say how to use this fact; for this we need a convergence criterion. The development of a criterion is necessary for defining a sum, because the sums concerned have an infinite number of terms.

This more general view of sum convergence makes more sense when we view other sum criteria. An exotic example is in string theory [23] where $\sum_{k=1}^{n} k|_{n=\infty} = -\frac{1}{12}$. Here a sum is defined at infinity by summing shifted sequences, the finite sum is meaningless, as you need to consider an infinity of terms.

All this follows from [7, Part 6 Sequences] where an alternate definition for convergence of a sequence at infinity is defined, and since a sum is a type of sequence, the application to the definitions of sums at infinity follows.

The other aspect of sums are divergent sums, sums which do not converge. So any function which does not converge by definition diverges, and we say of such a sum that it is equal to infinity, $\sum a_n|_{n=\infty} = \infty$.

Definition 2.5. *If* $\sum a_n$ *diverges at infinity we say* $\sum a_n|_{n=\infty} = \infty$

Through logic and language, diverging sums are divided into two cases, (i) where the sum does not continually grow in magnitude, $1 - 1 + 1 - \ldots$ being an example, and (ii) where the sum continually grows in magnitude, as in $1 + 4 + 8 + 16 + \ldots$ diverging to infinity. We could classify divergence as either the divergent sum is monotonically increasing, or it is not.

The characterisation of the sum will also depend on the inner term a_n being summed. If we say a sum is monotonic we will mean the sum's sequence is monotonic. Of course we can have a sum increasing for both monotonically increasing and monotonically decreasing terms.

The sums at infinity in this paper are concerned with monotonic functions that do not plateau, which for diverging sums would be the class that continually grows.

This does not deny the many possible applications of non-monotonic series, but instead we are concerned with monotonic series as an input with our tests.

Monotonic functions, because of their guaranteed behaviour are really useful and the subject of much of infinitary calculus. The scales of infinities are examples.

In our culture, let us ask the question, are there different infinities? The idea that like different numbers, we can have different infinities is most likely unrecognised. So $x^2|_{x=\infty}$ and $x^3|_{x=\infty}$ are being seen as ∞. While this generalisation is useful, the use of infinitary calculus or little-o/big-O notation is not as readily recognised; yet it becomes advantageous to treat different infinities as different numbers.

Interpreting what a sum is generally boils down to interpreting the little dots, that is, saying what happens at infinity.

Euler, Hardy, Ramanujuan and many others have pursued and found applications in defining a sum at infinity.

In this paper another criterion is developed and compared with a non-standard analysis convergence criterion.

The new criterion separates finite and infinite arithmetic as other criteria have done. After this separation, the infinite part of the sum is considered as a point at infinity. At this point we use infinitary calculus as the mechanism to do calculations.

To help explain why this is interesting, consider a divergent sum $1+1+\ldots$, if we express the sum by $\sum 1|_{n=\infty} = \infty$ thus diverges at infinity. That is, we are in an infinite loop where one is being continually added. We could further describe the sum by the divergent function, $\sum 1 = n|_{n=\infty}$

Definition 2.6. *Given a function $f(n)$, let $\sum f(n)|_{n=\infty} = g(n)$ be interpreted as a function at infinity.*

If we construct $g(n)$ counting from a reference point then $\sum_{k=a}^{b} f(n) = \sum^{b} f(n) - \sum^{a} f(n)$.

Definition 2.7. $\sum a_n|_{n=\infty}$, $\int a(n)\, dn \in *G$;

Infinitary calculus can then be applied to the sum at infinity. Looking at convergent series we can ask what happens at infinity. In the same way a stone thrown into a still pond generates a ripple, and over time the ripples subside again leaving the still pond, so the idea of a steady state for a sum is to look at the sum at infinity and enquire about the sum's behaviour.

If the pond does not settle down, but continually vibrates, this too can be considered a steady state. A steady state looks at the behaviour of the system after an infinity of time.

From the p-series it is known that it is not enough that what is being added, tends to zero. For example $p = 1$ gives $\frac{1}{n}|_{n=\infty} = 0$, but $\sum_{k=1}^{n} \frac{1}{k}|_{n=\infty} = \infty$ diverges. This shows that summing an infinity of infinitesimals is not necessarily finite.

Interestingly, a sum that is convergent will have the terms being added, and these will no longer have an effect on the end state. We can explain this by considering the sum in a higher dimension, with infinitesimals, which when projected (by approximation) back to $\mathbb{R}$, can disappear.

By the Criterion E2 described in the next section, $\sum f(n)|_{n=\infty} = 0$ is a necessary and sufficient condition for sum convergence (provided the sum did not diverge before reaching infinity).

In [6, Part 5] classes of functions $\{\frac{1}{x^2+\pi}, \frac{1}{x^2-3x}, \ldots\}|_{x=\infty}$ could be simplified to $\frac{1}{x^2}|_{x=\infty}$ by arguments of magnitude. The same simplifications with care can be applied to sums, reducing classes of sums to particular cases. Indeed this is described later by the p-series test at infinity.

Example 2.1. *For $\sum_{n=1}^{\infty} \frac{1}{n^2-3n}$, convergence or divergence can be determined by considering the sum at infinity, $\sum \frac{1}{n^2-3n}|_{n=\infty} = \sum \frac{1}{n^2}|_{n=\infty} = 0$ is convergent by comparison with known p-series, with $p > 1$ known to converge. $n^2 - 3n = n^2|_{n=\infty}$ as $n^2 \succ 3n|_{n=\infty}$.*

The following scale can be applied to solving equations of the form $a + b = a$ (non-reversible arithmetic [6, Part 5]) when $a \neq 0$, and will be useful when solving sums.

$$(c \prec \ln(x) \prec x^p|_{p>0} \prec a^x|_{a>1} \prec x! \prec x^x)|_{x=\infty} \, [3, \text{Part } 2]$$

3 Euler's Convergence criteria

Euler on the nature of series convergence [16]:

> Series with a finite sum when infinitely continued, do not increase this sum even if continued to the double of its terms. The quantity which is increased behind an infinity of terms actually remains infinitely small. If this were not the case, the sum of the series would not be determined and, consequently, would not be finite.

Laugwitz reasons from Euler's criterion [20, p.14]:

> A series (of real numbers) has a finite sum if the values of the sum between infinitely large numbers is an infinitesimal.

Reference [1, p.212] defined a convergence Criterion E1, as a reformation of Euler's criterion in A. Robinson's non-standard analysis. Consider the tail of a sum, and a countable infinity section.

Criterion E1. *The series with general term a_k, where $a_k \geq 0$, is convergent(has a finite sum) if $\sum_{k=\omega}^{2\omega} a_k$ is an infinitesimal for any infinitely large ω.*

We have formed other criteria through minor variations, and then considered infinity as a point. The Criteria can be implemented by using an extension of du Bois-Reymond's infinitary calculus. Whereby we can define the sum at infinity as a function.

Definition 3.1. *For Criteria E2, E3, with monotonic sequence $(a_n)|_{n=\infty}$, $n \in \mathbb{N}_\infty$,*

$$\sum a_n|_{n=\infty} \in \mathbb{R}_\infty \textit{ is an infinireal.}$$

Proposition 3.1. $\sum a_n|_{n=\infty} \mapsto \{0, \infty\}$

If $\sum a_n|_{n=\infty} = \mathbb{R}_\infty$ then either $\sum a_n|_{n=\infty} = 0$ converges or $\sum a_n|_{n=\infty} = \infty$ diverges.

Proof. When the infinitesimals and infinities $\mathbb{R}_\infty$ are realized, $\Phi \mapsto 0$ and $\Phi^{-1} \mapsto \infty$, then these are the only two possible values of the sum at infinity. If the sum is undefined, by definition the sum is said to diverge and assigned ∞. $\square$

Criterion E2. *The series with general term a_k, where $a_k \geq 0$, is convergent (has a finite sum) if and only if $\sum a_n|_{n=\infty} \in \Phi$, or else the series is divergent and $\sum a_n|_{n=\infty} \in \Phi^{-1}$ or ∞.*

The integral version of Criterion E2, either $\int^n a(n)\,dn|_{n=\infty} \in \Phi$ converges. Else the integral $\int^n a(n)\,dn|_{n=\infty} \in \Phi^{-1}$ or ∞ and diverges.

For convergence, $(a)|_{n=\infty}$ must be a monotonic sequence.

The requirement that the series has a sequence of monotonic terms for convergence will later be overcome by converting the sequences to an auxiliary monotonic sequence for testing.

In developing a criterion, the convergence Criterion E2 is also justified from Euler's considerations. The same quoted criterion is realised with Criterion E2. Hence we refer to Criteria E1, E2, and E3 in the following section as Euler's convergence criteria.

In comparing convergence Criteria E1 and E2, the non-standard analysis Criterion E1 compares in a sense with an interval between two infinities, whereas the Criterion E2 compares at a point, infinity.

However, Criteria E1 and E2 can be compared. Replacing the infinitesimals in Criterion E1 with zero($\Phi \mapsto 0$) then Criterion E1 gives a similar convergence with Criterion E2. For convergence, both sums are zero.

$$\text{Criterion E1: } \sum_{k=\omega}^{2\omega} a_k = 0 \quad \text{Criterion E2: } \sum a_n|_{n=\infty} = 0$$

Example 3.1. *Showing that the harmonic series diverges by Euler convergence Criterion E1.* $\sum_{k=\Omega+1}^{2\Omega} \frac{1}{k} \approx \sum_{k=1}^{2\Omega} \frac{1}{k} - \sum_{k=1}^{\Omega} \frac{1}{k} \approx \ln 2\Omega - \ln \Omega \approx \ln 2 \neq 0$ *hence* $\sum \frac{1}{k}$ *diverges.*

Example 3.2. *Showing that the harmonic series diverges by Euler convergence Criterion E2.* $\sum \frac{1}{n}|_{n=\infty} = \int \frac{1}{n}\,dn|_{n=\infty} = \ln n|_{n=\infty} = \infty$ *diverges.*

4 A reconsidered convergence criteria

An integral or a sum as a point may seem shocking, however there turns out to be a simple explanation. By the fundamental theorem of calculus, an integral can be expressed as a difference in two integrals at a point. If one of these integrals has a much greater than magnitude, we can apply non-reversible arithmetic $a_n + b_n|_{n=\infty} = a_n$ when $a_n \succ b_n|_{n=\infty}$ and only one integral point need be tested.

$$\int_a^b f(x)\,dx = \int_a f(x)\,dx \text{ or } \int^b f(x)\,dx$$

The significance of the reduction to evaluation of the integral at a point is to reduce an integral or sum convergence test to one, instead of two function evaluations.

Hence we really are evaluating a sum or integral at a point.

As it will be advantageous to convert between sums and integrals, we can always thread a continuous monotonic function through a monotonic sequence, and conversely from a monotonic function generate a monotonic sequence. At integer values the function and sequence are equal.

$$a(n) = a_n|_{n \in \mathbb{J}_\infty}$$

Sums and integrals can sandwich each other. Either consider the criterion with a sum or integral. With the criterion having the same conditions for both sums and integrals allows for the integral test in both directions.

Criteria E3 sum and integral convergence

The following criteria E3 and E3' are linked, hence we refer to both collectively as "the E3 criteria".

E3.0 Consider an arbitrary infinite interval $[n_0, n_1]$ which can be grown to meet the conditions. $n_1 - n_0 = \infty$ is a minimum requirement; $n_0, n_1 \in \Phi^{-1}$;

E3.1 If $\int a(n)\,dn|_{n=\infty}$ cannot form a monotonic function, or the other E3 conditions fail, then the integral diverges, $\int a(n)\,dn|_{n=\infty} = \infty$.
E3.2 $\int a(n)\,dn|_{n=\infty} \in \mathbb{R}_\infty$; $n \in \Phi^{-1}$
E3.3 For divergence $\int a(n)\,dn|_{n=\infty} \in \Phi^{-1}$
E3.4 For convergence $\int a(n)\,dn|_{n=\infty} \in \Phi$
E3.5 For divergence, $\int a(n)\,dn|_{n=\infty}$ can be made arbitrarily large.
E3.6 For convergence, $\int a(n)\,dn|_{n=\infty}$ can be made arbitrarily small.

E3'.0 Consider an arbitrary infinite interval $[n_0, n_1]$ which can be grown to meet the conditions. $n_1 - n_0 = \infty$ is a minimum requirement; $n_0, n_1 \in \mathbb{J}_\infty$;
E3'.1 If $\sum a_n|_{n=\infty}$ cannot form a monotonic function, or the other E3' conditions fail, then the sum diverges, $\sum a_n|_{n=\infty} = \infty$.
E3'.2 $\sum a_n|_{n=\infty} \in \mathbb{R}_\infty$; $n \in \Phi^{-1}$
E3'.3 For divergence $\sum a_n|_{n=\infty} \in \Phi^{-1}$
E3'.4 For convergence $\sum a_n|_{n=\infty} \in \Phi$.

E3'.5 For divergence, $\sum a_n|_{n=\infty}$ can be made arbitrarily large.
E3'.6 For convergence, $\sum a_n|_{n=\infty}$ can be made arbitrarily small.

E3.0-6 satisfy the E2 criterion by adding additional properties which justify integration at a point. E3.1 overrides other E3 conditions.

Concerning E3.1: by consideration of arrangements, many classes of non-monotonic functions can be rearranged into monotonic series for testing. [11]

We do not require a strict monotonic function for the criterion E3 as a consequence of E3.5 and E3.6 conditions. A monotonic series may be tested by generating a contiguous subsequence which is strictly monotonic, removing equality [11].

Concerning E3.2: the NSA Example 3.1 integrates leaving neither an infinity or infinitesimal, then clearly such integrals exist. Our approach, however excludes this case, by defining any integral between two infinities as an infinireal $\mathbb{R}_\infty$. This was done to reduce complexity and increase usability. If a computation occurs which gives the above case ($\sum a_n|_{n=\infty} \notin \mathbb{R}_\infty$), then an assumption or condition has failed. However, the theory can be extended to include these cases, but may be more complicated. This results in Example 3.1 alternatively being evaluated in Example 3.2 by condition E3.5.

Proposition 4.1. *If $\sum a_n \notin \mathbb{R}_\infty$ or $\int a(n)\, dn \notin \mathbb{R}_\infty$ then the sum or integral is said to diverge.*

Proof. Divergence in the 'undefined sense' and not a diverging magnitude. Conditions E3.3 or E3.4 or E3'.3 or E4'.3 not met. $\square$

Conditions E3.5 and E3.6 stop the integral from plateauing. In $*G$ we could easily have a sum of infinitesimals monotonically increasing but bounded above. These same conditions allow that only one integral or sum needs to be tested.

Integration at a point approach makes sense when one of the integral evaluations has a much greater than magnitude than the other. For example, consider a diverging integral, $\int^n f(x)\, dx \succ \int^a f(x)\, dx|_{n=\infty}$ then $\int_a^n f(x)\, dx = \int^n f(x)\, dx - \int^a f(x)\, dx = \int^n f(x)\, dx = \int f(n)\, dn|_{n=\infty}$. The same order of magnitude situation could occur for the infinitely small.

Integrating a single point has in a sense decoupled integration over an interval, however this is not unexpected. The fundamental theorem of calculus itself is an expression of the difference of two integrals at a point.

$$\int_a^b f(x)\, dx = \int^b f(x)\, dx - \int^a f(x)\, dx$$

Theorem 4.1. *A sum representation of the fundamental theorem of calculus [12, Theorem*

5.1].

$$\sum_{a}^{b} f_n = \sum^{b} f_n - \sum^{a} f_n$$

While an interval integration is a concrete and tangible calculation, the theory of calculus often uses integration at a point implicitly. Other examples can be found, such as integrating using the power rule $\int x^p \, dx = \frac{1}{p+1} x^{p+1}$.

Integration at a point has theoretical advantages, the complexity of theory and calculation can be reduced. This and subsequent papers in the series have found working at infinity, with du Bois-Reymond's infinitary calculus and functional space, provides ways to build function and sequence constructions.

Proposition 4.2. *Convergence: When $n_1 > n_0$; n_0, $n_1 \in \Phi^{-1}$; there exists n_1 :*

$$\int_{n_0}^{n_1} a(n) \, dn = \int_{n_0} a(n) \, dn|_{n=\infty}$$

Proof. By arbitrarily making the infinitesimal $\int^{n_1} a(n) \, dn$ smaller(E3.6), to when the condition $\int^{n_1} a(n) \, dn \prec \int^{n_0} a(n) \, dn$ is met, then $\int^{n_1} a(n) \, dn - \int^{n_0} a(n) \, dn = -\int^{n_0} a(n) \, dn$. ☐

Proposition 4.3. *Divergence: When $n_1 > n_0$; $n_0, n_1 \in \Phi^{-1}$; there exists n_0 :*

$$\int_{n_0}^{n_1} a(n) \, dn = \int^{n_1} a(n) \, dn|_{n=\infty}$$

Proof. Since there is no least diverging infinity, we can always decrease an infinity, and from E3.5 we can construct an arbitrary smaller infinity.

Then given n_1 we can decrease n_0 till the condition $\int^{n_1} a(n) \, dn \succ \int^{n_0} a(n) \, dn|_{n=\infty}$ is met. Then $\int_{n_0}^{n_1} a(n) \, dn|_{n=\infty} = \int^{n_1} a(n) \, dn - \int^{n_0} a(n) \, dn|_{n=\infty} = \int^{n_1} a(n) \, dn|_{n=\infty}$. ☐

Theorem 4.2. *For integral convergence or divergence, we need only test one point at infinity. $\int a(n) \, dn|_{n=\infty} \in \Phi$ when convergent and $\int a(n) \, dn|_{n=\infty} \in \Phi^{-1}$ when divergent.*

Proof. By Criterion E3.1 we need only test a monotonic sequence. If E3.1 is satisfied, then either the integral is converging or diverging, as the sum is an infinireal. Both these cases are handled by Propositions 4.2 and 4.3. ☐

Both Propositions 4.2, 4.3 and Theorem 4.2 can be demonstrated.

Example 4.1. *$\int_n^{n^2} \frac{1}{x^4}\,dx|_{n=\infty} = -\frac{1}{3}\frac{1}{n^{2^3}} + \frac{1}{3}\frac{1}{n^3}|_{n=\infty} = -\frac{1}{3}\frac{1}{n^6} + \frac{1}{3}\frac{1}{n^3}|_{n=\infty} = \frac{1}{3}\frac{1}{n^3}|_{n=\infty} = 0$ converges. $\int^{n^2} \frac{1}{x^4}\,dx \prec \int^n \frac{1}{x^4}\,dx|_{n=\infty}$*

By Criterion E3, $\int^n \frac{1}{x^4}\,dx|_{n=\infty} = \frac{1}{3}\frac{1}{n^3}|_{n=\infty} = 0$ converges.

Example 4.2. *$\int_{\ln n}^n x^2\,dx|_{n=\infty} = \frac{x^3}{3}|_{\ln n}^n|_{n=\infty} = \frac{n^3}{3} - \frac{(\ln n)^3}{3}|_{n=\infty} = \frac{n^3}{3}|_{n=\infty} = \infty$ diverges. $\int^n x^2\,dx \succ \int^{\ln n} x^2\,dx|_{n=\infty}$*

By Criterion E3, $\int^n x^2\,dx|_{n=\infty} = \frac{x^3}{3}|_{n=\infty} = \frac{n^3}{3}|_{n=\infty} = \infty$ diverges.

Similar propositions are constructed for sums.

Proposition 4.4. *Convergence: When $n_1 > n_0$; n_0, $n_1 \in \mathbb{J}_\infty$; there exists n_1 :*

$$\sum_{n_0}^{n_1} a_n = \sum_{n_0} a_n|_{n=\infty}$$

Proof. By arbitrarily making the infinitesimal $\sum a_n|_{n=n_1}$ smaller(E3'.6), to when the condition $\sum a_n|_{n=n_1} \prec \sum a_n|_{n=n_0}$ is met, $\sum_{n_0}^{n_1} a_n = \sum a_n|_{n=n_1} - \sum a_n|_{n=no} = -\sum a_n|_{n=n_0}$. □

Proposition 4.5. *Divergence: When $n_1 > n_0$; $n_0, n_1 \in \mathbb{J}_\infty$; there exists n_0 :*

$$\sum_{n_0}^{n_1} a_n = \sum^{n_1} a_n|_{n=\infty}$$

Proof. Since there is no least diverging infinity, we can always decrease an infinity, and from E3'.5 we can construct an arbitrary smaller infinity.

Then given n_1 we can decrease n_0 till the condition $\sum^{n_1} a_n \succ \sum^{n_0} a_n|_{n=\infty}$ is met. Then $\sum_{n_0}^{n_1} a_n|_{n=\infty} = \sum^{n_1} a_n - \sum^{n_0} a_n|_{n=\infty} = \sum^{n_1} a_n|_{n=\infty}$. □

Theorem 4.3. *For sum convergence or divergence, we need only test one point at infinity. $\sum a_n|_{n=\infty} \in \Phi$ when convergent and $\sum a_n|_{n=\infty} \in \Phi^{-1}$ when divergent.*

Proof. By Criterion E3'.1 we need only test a monotonic sequence. If E3'.1 is satisfied, then either the integral is converging or diverging, as the sum is an infinireal. Both these cases are handled by Propositions 4.4 and 4.5. □

Consider the tail or remainder of an integral. $n_0, n_1 \in \Phi^{-1}$; $r(n_0, n_1) = \int_{n_0}^{n_1} a(n)\,dn$. Let $a(n)$ be a continuous function passing through the sequence points $(a_n)|_{n=\infty}$. The tail explains the convergence criteria for E1 and E3, and the tail explains monotonic divergence, as the tail is at infinity.

Then Criterion E1 is a criterion for the tail $r(n, 2n)|_{n=\infty}$. We find Criterion E3 also satisfies Criterion E1, but also true for any part or whole of the tail.

Criterion E3 is a more encompassing criterion than Criterion E2, as the whole tail or sum at infinity is captured. However, only a part of the tail needs to be tested, condition E0. With the reduction of rearrangement theorems [11], Criterion E3's limitation of requiring monotonic input is addressed.

5 Reflection

In comparing convergence criteria E1 and E3, both convergence criteria have a role. However, in our opinion we do not see Robinson's non-standard analysis being used everyday by engineers. Criterion E3 is better suited for this purpose and leads to a non-standard model which better supports calculation and theory, by solving in a more accessible way. That is, solving more simply, primarily by removing the "mathematical logic"- that is the field of logic, from the application. This is the specialization that is necessary to use Robinson's non-standard analysis.

The indirect logic reasoning used in the current convergence and divergence tests is similarly frustrating. While there are many uses for such logic, with direct reasoning at infinity, classes of these can be reduced. A particular example is the Limit Comparison Theorem (LCT). The act of picking the correct series to apply the test, in full knowledge that the test is going to work (as often the asymptotic series is chosen), is unnecessary (see Example 5.1). Not from a logical viewpoint, but from an application viewpoint, you have already solved the problem!

Example 5.1. *Determine using LCT that $\sum_{k=1}^{n} \frac{1}{n^2+1}$ converges. By the p-series test, we know $\sum \frac{1}{n^2}|_{n=\infty} = 0$ converges. Compare using Test 10.7, $\sum \frac{1}{n^2+1}|_{n=\infty}$ and $\sum \frac{1}{n^2}|_{n=\infty}$. $\frac{1}{n^2+1}\frac{n^2}{1}|_{n=\infty} = 1 - \frac{1}{n^2+1}|_{n=\infty} = 1$ hence both sums converge.*

The tests under the new convergence criteria have been developed to address these problems, with more direct reasoning. This will include in a later paper a concept of a derivative of a sequence[9]. The flow and mixing of these tests, and the introduction of a universal convergence test (see Test 10.19), we believe, will change the nature of convergence testing.

In introducing a new way of calculating, a focus was chosen toward the development of convergent and divergent series, theory and tests. Without further ado, the following discussion relates to the E3 convergence criterion.

Definition 5.1. *'convergence sums' use Criterion E3, 'convergence integrals' use Criterion E3.*

6 Monotonic sums and integrals

In determining convergence or divergence, if the sum or integral is well behaved, that is no singularities on the finite interval, then we only need to state this result at the end, to say whether the sum or integral has converged or diverged.

That convergence or divergence, with no finite singularities, is determined at infinity, is self evident and purely logical. If a sum or integral does not diverge for any finite values, then the only possible place where the sum or integral can diverge is at infinity. For all monotonic series and integrals, the convergence tests work at infinity.

Singularities are often caused by division by zero, so when these singularities are not present, what remains is a continuous function (see 10.5.2), which could then be tested at the singularity infinity. For a series, the continuous function is constructed by threading a continuous function through the sequence points.

Theorem 6.1. *If the integral $\int_\alpha^\infty a(n)\,dn$ has no singularities before infinity and*

$$\int a(n)\,dn|_{n=\infty} = \begin{cases} 0 & \textit{then } \int_\alpha^\infty a(n)\,dn \textit{ is convergent,} \\ \infty & \textit{then } \int_\alpha^\infty a(n)\,dn \textit{ is divergent.} \end{cases}$$

Proof. Since the finite part of the integral has no influence on convergence or divergence, the evaluation is achieved using Criterion E3, which was applied in Theorem 4.2. □

Theorem 6.2. *If the sum $\sum_{k=k_0}^\infty a_k$ has no singularities before infinity and*

$$\sum a_n|_{n=\infty} = \begin{cases} 0 & \textit{then } \sum_{k=k_0}^\infty a_k \textit{ is convergent,} \\ \infty & \textit{then } \sum_{k=k_0}^\infty a_k \textit{ is divergent.} \end{cases}$$

Proof. Since the finite part of the sum has no influence on convergence or divergence, the evaluation is achieved using Criterion E3', which was applied in Theorem 4.3. □

The given theorems are for using the infinity convergence test easily.

When the condition of no singularities for finite values occurs, the sum at infinity's purpose will be to decide and hence complete the convergence test. In such a sum it is enough to refer to the sum at infinity to determine the sum's convergence/divergence.

For example, in testing $\sum_{k=1}^\infty \frac{1}{n^2}$ we find $\sum \frac{1}{n^2}|_{n=\infty} = 0$ converges. This is the end of the test, the implication that the original series $\sum_{k=1}^\infty \frac{1}{n^2}$ converges is unnecessary.

Formally repeating the implication of convergence or divergence of the sum from the infinite partition is superfluous. Convergence or divergence at infinity logically follows from the sum.

Having said that, by convention we state $\sum a_n|_{n=\infty} = 0$ converges. Technically this symbolic statement alone says the series converges; however we have chosen to state convergence or divergence at the end to aid communication, even though it is saying the same thing twice, once symbolically and another with a word.

Through rearrangements, we believe the convergence criteria applies to all series, but the application of the criteria generally works on monotonic series. The reasons for this are that the major mathematical tools for comparing functions and the use of infinitary calculus, work best for monotonic sequences. Monotonic sequences and series are a more primitive structure than non-monotonic sequences and series, so we can develop the theory and use them as building blocks.

Testing on monotonic series makes sense if we imagine at infinity, in a similar way to the steady state, where the sum or system settles down. It is not till we have a settled state that we can determine or apply the convergence criterion. Hence the need for monotonic sequence $(a_n)|_{n=\infty}$.

While this reduces the classes of series that can be considered, in actual effect it applies a structure to the test. A method will be developed to transform or rearrange sums that are not monotonic to sums that are monotonic, and then test the transformed sum for convergence or divergence. The restriction of monotonic sequences in determining convergence or divergence will be addressed by extending the class of sequences in another paper. For example, the Alternating Convergence Theorem (Theorem 10.11) is excluded by such a restriction. So the restriction is not a restriction at all, but a problem reduction.

Consider the steady state of the sum at infinity. Let $(a_n)|_{n=\infty}$ describe the sequence at infinity. Then we can categorize the relationship cases between sum convergence and the sequence behaviour.

1. Sequence (a_n) is monotonic
2. Sequence (a_n) is not monotonic and the sum at infinity diverges.
3. Sequence (a_n) is not monotonic and the sum at infinity converges.

1.1 $\sum a_n|_{n=\infty} = 0$ converges then $(a_n)|_{n=\infty}$ is monotonically decreasing.
1.2 $\sum a_n|_{n=\infty} = \infty$ diverges and $(a_n)|_{n=\infty}$ is monotonically increasing.
1.3 $\sum a_n|_{n=\infty} = \infty$ diverges and $(a_n)|_{n=\infty}$ is monotonically decreasing.

For example, case 3 is used in the comparison test variation Theorem 10.4, and the Alternating Convergence Theorem 10.11.

Most of the tests assume a monotonic sequence, case 1. Further, just because a sequence is not case 1, does not mean that the problem cannot be transformed into a case 1 category.

Example 6.1. *1.1:* $\sum(\frac{1}{2})^n|_{n=\infty}$, *1.2:* $\sum 2^n|_{n=\infty}$, *1.3:* $\sum \frac{1}{n}|_{n=\infty} = \infty$.

7 Comparing sums at infinity

The comparison at infinity is to remove the finite part of the sum, and we seek monotonic behaviour to test for convergence/divergence. So a sequence (a_n) becomes a sequence at infinity, $(a_n)|_{n=\infty}$. In a sense the finite part of the sum is the transient.

By choosing positive monotonic sequences, primitive relations between sequences and sums, functions and integrals, can be preserved, practically allowing an interchange between the inequalities.

Theorem 7.1. *$f_n, g_n \in \mathbb{R}_\infty$; $f_n \geq 0, g_n \geq 0$. If $z \in \{<, \leq, >, \geq\}$ and the sequences f_n and g_n are monotonic then $\sum f_n \; z \; \sum g_n \Leftrightarrow f_n \; z \; g_n|_{n=\infty}$ in $*G$.*

Proof. Thread a continuous monotonic function through the sequence, which satisfies E3 criteria. Then apply Theorem 7.2. □

Theorem 7.2. *Let $f(n) \geq 0, g(n) \geq 0$ be monotonic, continuous functions at infinity. If $z \in \{<, \leq, >, \geq\}$ then $\int f(n)\, dn \; z \; \int g(n)\, dn \Leftrightarrow f(n) \; z \; g(n)|_{n=\infty}$ [7, Part 6]*

Often we solve for the relation z to include equality, so when applying the transfer principle the relation is unchanged. When ; $f, g \in \mathbb{R}_\infty$; then $(*G, <) \not\mapsto (\mathbb{R} \text{ or } \overline{\mathbb{R}}, <)$. However, $(*G, \leq) \mapsto (\mathbb{R} \text{ or } \overline{\mathbb{R}}, \leq)$.

Example 7.1. *$x^2 < x^3|_{x=\infty}$ in $*G \not\mapsto \; \infty < \infty$ in $\overline{\mathbb{R}}$. $x^2 \leq x^3|_{x=\infty}$ in $*G \mapsto \; \infty \leq \infty$ in $\overline{\mathbb{R}}$*

Example 7.2. *Compare $\sum \frac{1}{n^3} \; z \; \sum \frac{1}{n^2}|_{n=\infty}$. Solve for z.*

$$\begin{aligned}
\sum \frac{1}{n^3} \; z \; \sum \frac{1}{n^2}|_{n=\infty} & \qquad \text{(remove the sum)} \\
\frac{1}{n^3} \; z \; \frac{1}{n^2}|_{n=\infty} & \qquad \text{(cross multiply)} \\
n^2 \; z \; n^3 & \\
n^2 < n^3 & \qquad ((+\Phi^{-1}, <) \not\mapsto (\infty, <)) \\
& \qquad (*G \mapsto *G : \; < \Rightarrow \leq) \\
n^2 \leq n^3 & \qquad \text{(relation can be realized, } \infty \leq \infty) \\
\sum \frac{1}{n^3} \leq \sum \frac{1}{n^2}|_{n=\infty} & \qquad \text{(Solve for } z = \leq \text{ and substitute back)}
\end{aligned}$$

Example 7.3. *If we know $\sum \frac{1}{n^2}|_{n=\infty} = 0$ converges, then by comparison we can determine* *the convergence of* $\sum \frac{1}{n^3}|_{n=\infty}$.

$$0 \leq \sum \frac{1}{n^3} \leq \sum \frac{1}{n^2}|_{n=\infty} \qquad (\Phi \mapsto 0)$$
$$0 \leq \sum \frac{1}{n^3}|_{n=\infty} \leq 0$$
$$\sum \frac{1}{n^3}|_{n=\infty} = 0 \qquad \text{(sum converges)}$$

The idea is, that if the sum's state has reached a steady state at infinity (this could of course be a function in n), then we may compare different sums. Now, our sums at a point could simply have the summation sign sigma removed, and by infinitary calculus the inner functions compared.

If the sums $\sum f_n|_{n=\infty}$ and $\sum g_n|_{n=\infty}$ are monotonic, then at infinity their state is settled, hence $\sum f_n \; z \; \sum g_n \Rightarrow f_n \; z \; g_n|_{n=\infty}$

Showing $f_n \; z \; g_n \Rightarrow \sum f_n \; z \; \sum g_n|_{n=\infty}$ and then reversing the argument shows the implication in the other direction. $f_n \; z \; g_n|_{n=\infty}$, $f(n) \; z \; g(n)|_{n=\infty}$, $f(n)\Delta n \; z \; g(n)\Delta n|_{n=\infty}$, $\int f(n)\, dn \; z \; \int g(n)\, dn|_{n=\infty}$, $\sum f_n \; z \; \sum g_n|_{n=\infty}$.

Converting the sum to an integral and then differentiating, as this is dividing by positive infinitesimal quantities equally, the relation z will not change. $\sum f_n \; z \; \sum g_n|_{n=\infty}$, $\int f(n)\, dn \; z \; \int g(n)\, dn|_{n=\infty}$, $\frac{d}{dn} \int f(n)\, dn \; (Dz) \; \frac{d}{dn} \int g(n)\, dn|_{n=\infty}$, $f(n) \; (Dz) \; g(n)|_{n=\infty}$, $f(n) \; z \; g(n)|_{n=\infty}$.

$$\sum f_n \; z \sum g_n \; \Leftrightarrow \; f_n \; z \; g_n|_{n=\infty}$$
$$\int f(n)\, dn \; z \int g(n)\, dn \; \Leftrightarrow \; f(n) \; z \; g(n)|_{n=\infty}$$

We have chosen continuous curves $f(n)$ and $g(n)$ such that $f_n \; z \; g_n \Leftrightarrow f(n) \; z \; g(n)|_{n=\infty}$, where for integer values $f(n) = f_n$ and $g(n) = g_n$. These continuous curves maintain the relation for their interval between the points.

To summarize, comparing two sums with each other at infinity, the sum over an interval's convergence is converted to a sum at a point at infinity for both sums, and the sums at infinity are compared. By removing the sigma, this comparison now compares the sum's inner components, solving for the relation and substituting back into the original sum's comparison.

The conjecture is that all sums with integer indices can be compared at infinity. This belief will lead to considerations of arrangements, and ways the sums can be compared in subsequent papers. An example being the Alternating Convergence Theorem (Theorem 10.11), with its alternate representation given because of the test's importance.

Example 7.4. *Determine convergence of* $\sum \frac{1}{n\sqrt{n^3+1}}|_{n=\infty}$. *For interest, compare with a known convergent sum* $\sum \frac{1}{n^2}|_{n=\infty} = 0$. $\sum \frac{1}{n\sqrt{n^3+1}} \quad z \quad \sum \frac{1}{n^2}|_{n=\infty}$, $\sum \frac{1}{n(n^3+1)^{\frac{1}{2}}} \quad z \quad \sum \frac{1}{n^2}|_{n=\infty}$, $\frac{1}{n(n^3+1)^{\frac{1}{2}}} \quad z \quad \frac{1}{n^2}|_{n=\infty}$, $n^2 \quad z \quad n(n^3+1)^{\frac{1}{2}}|_{n=\infty}$, $n^2 \prec n(n^3+1)^{\frac{1}{2}}|_{n=\infty}$. *For our purposes we will not need as strong a relation. The lesser relation, less than or equal to, will suffice.* $n^2 \leq n(n^3+1)^{\frac{1}{2}}|_{n=\infty}$. *Then* $z = \leq$, *substituting this back into the original sum comparison,* $0 \leq \sum \frac{1}{n\sqrt{n^3+1}} \leq \sum \frac{1}{n^2}|_{n=\infty}$, $0 \leq \sum \frac{1}{n\sqrt{n^3+1}}|_{n=\infty} \leq 0$, $\sum \frac{1}{n\sqrt{n^3+1}}|_{n=\infty} = 0$ *converges.*

We did restrict (a_n) to being monotonic in the determination of convergence or divergence of $\sum a_n|_{n=\infty}$. If we can bound a function between two monotonic functions that are both converging or both diverging, a determination of convergence or divergence can be made by applying the sandwich principle.

Example 7.5. $\sum_{k=1}^{n} \frac{3+\sin k}{k^2}|_{n=\infty}$. *Determine convergence or divergence of* $\sum \frac{3+\sin n}{n^2}|_{n=\infty}$. $2 \leq 3+\sin n \leq 4|_{n=\infty}$, $\frac{2}{n^2} \leq \frac{3+\sin n}{n^2} \leq \frac{4}{n^2}|_{n=\infty}$, $\sum \frac{2}{n^2} \leq \sum \frac{3+\sin n}{n^2} \leq \sum \frac{4}{n^2}|_{n=\infty}$, $0 \leq \sum \frac{3+\sin n}{n^2} \leq 0|_{n=\infty}$, $\sum \frac{3+\sin n}{n^2}|_{n=\infty} = 0$ *converges.*

One of the most important comparisons is to compare against the p-series, as it greatly simplifies calculation.

Example 7.6. $\sum \frac{1}{n\sqrt{n^3+1}}|_{n=\infty} = \sum \frac{1}{n(n^3+1)^{\frac{1}{2}}}|_{n=\infty} = \sum \frac{1}{nn^{\frac{3}{2}}}|_{n=\infty} = \sum \frac{1}{n^{\frac{5}{2}}}|_{n=\infty} = 0$ *by comparison with the known p-series, hence the sum converges.* $\sum \frac{1}{n^p}|_{n=\infty} = 0$ *when* $p > 1$.

The p-series comparison is more heavily used with sums at infinity, as with infinitary calculus, arguments of magnitude can reduce the sum to a p-series test. The p-series at infinity can often replace other tests. In this sense it is different from the standard p-series test.

Example 7.7. *Rather than using the comparison test,* $\sum \frac{1}{n+n^{\frac{3}{2}}}|_{n=\infty} = \sum \frac{1}{n^{\frac{3}{2}}}|_{n=\infty} = 0$ *converges. As* $n + n^{\frac{3}{2}} = n^{\frac{3}{2}}|_{n=\infty}$ *because* $n^{\frac{3}{2}} \succ n|_{n=\infty}$.

Another comparison is when a sum varies within an interval not containing zero, as given by example 7.5 with the sum having the terms interval $[2,4]$.

As a consequence of Criterion E3, when a sum at infinity is either zero or infinity, then multiplying the sum by a constant or a bounded variable without zero leaves the sum unchanged. $0 \cdot \alpha_n = 0$ and $\alpha_n \cdot \infty = \infty$

Theorem 7.3. *If* $+\Phi < \alpha_n < +\Phi^{-1}$ *then* $\sum a_n \alpha_n = \sum a_n|_{n=\infty}$.

Proof. Since $\alpha_n \neq +\mathbb{R}_\infty$, $\exists; \beta_1, \beta_2 \in +\mathbb{R};: \beta_1 \leq \alpha_n \leq \beta_2$. $0 \leq \beta_1 \leq \alpha_n \leq \beta_2|_{n=\infty}$, $0 \leq \beta_1 a_n \leq \alpha_n a_n \leq \beta_2 a_n|_{n=\infty}$, $0 \leq \sum \beta_1 a_n \leq \sum \alpha_n a_n \leq \sum \beta_2 a_n|_{n=\infty}$, $0 \leq \beta_1 \sum a_n \leq \sum \alpha_n a_n \leq \beta_2 \sum a_n|_{n=\infty}$. If $\sum a_n|_{n=\infty} = c$, $c \in \{0, \infty\}$, then $\beta_1 c = c$, $\beta_2 c = c$. Substituting c into the above inequality, $0 \leq \beta_1 c \leq \sum \alpha_n a_n \leq \beta_2 c|_{n=\infty}$, $0 \leq c \leq \sum \alpha_n a_n \leq c|_{n=\infty}$, $\sum \alpha_n a_n|_{n=\infty} = c = \sum a_n|_{n=\infty}$. □

Theorem 7.4. *If we can deconstruct a product inside a sum to a real part and an infinitesimal, the infinitesimal may be ignored.* $\alpha \in \mathbb{R}^+$*;* $\delta \in \Phi$*;*

$$\sum a_n(\alpha + \delta)|_{n=\infty} = \sum a_n|_{n=\infty}$$

Proof. $\sum a_n(\alpha+\delta)|_{n=\infty} = \sum a_n|_{n=\infty} + \sum \delta a_n|_{n=\infty}$. If $\sum \delta a_n|_{n=\infty} = \infty$ then $\sum a_n|_{n=\infty} = \infty$ as $\sum \alpha a_n \geq \sum \delta a_n$. If $\sum \delta a_n|_{n=\infty} = 0$ then $\sum(\alpha a_n + \delta a_n)|_{n=\infty} = \sum \alpha a_n + \sum \delta a_n|_{n=\infty}$ $= \alpha \sum a_n + 0|_{n=\infty} = \sum a_n|_{n=\infty}$ □

Theorem 7.5. *If* $\Phi^+ < \alpha_n < +\Phi^{-1}$ *or* $-\Phi^{-1} < \alpha_n < -\Phi$ *then* $\sum \alpha_n a_n = \sum a_n|_{n=\infty}$.

Proof. Positive case see Theorem 7.3. Negative case, $-\Phi^{-1} < \alpha_n < -\Phi$, multiply the inequality by -1, $\Phi^{-1} > (-\alpha_n) > \Phi$, $\Phi < (-\alpha_n) < \Phi^{-1}$ which is the positive case. □

Example 7.8. $\sum \frac{3+\sin n}{n^2}|_{n=\infty} = \sum \frac{1}{n^2}|_{n=\infty} = 0$ *as* $2 \leq 3 + \sin n \leq 4|_{n=\infty}$ *is positive bounded.*

Example 7.9. *The bounded variable is used in the proof of Dirichlet's test, see convergence Test 10.14.*

When about zero, we need to be careful, because infinitesimals can appear in the calculation which contradict the use of Theorem 7.5.

Example 7.10. *An example of when not to use the positive bound. If we reason that because the* sin *function is bounded, that we can treat this as a constant, and since constants in sums are ignored the* sin *function is ignored.*

For demonstration purposes only, considering sin *as finite and simplifying as a constant (rather than the infinitesimal it is)* $\sum \frac{1}{n}|\sin\frac{1}{n}| = \sum \frac{1}{n}|_{n=\infty} = \infty$ *diverges. This is incorrect. Actually the reverse case happens and the sum converges.*

The problem is the requirement of Theorem 7.5 was not met. The infinitesimal cannot be treated as a constant, as it interacts in the test.

Considering a sin *expansion,* $|\sin\frac{1}{n}| = \frac{1}{n}|_{n=\infty}$, $\sum \frac{1}{n}|\sin\frac{1}{n}| = \sum \frac{1}{n^2}|_{n=\infty} = 0$ *converges.*

We now venture into the darker side of infinity, with non-uniqueness, where out of necessity we need to explain a counter-example. Or rather, by non-uniqueness, the counter-example is nullified.

Infinity as a space branches into other possibilities. Our convention to use left-to-right $=$ operator as a directed assignment [3, Definition 2.9] allows for exploration at infinity as, rather than one line of logic, several may need to be followed. In fact the following problem requires non-uniqueness at infinity to be understood.

Returning to $\sum a_n = \int a(n)\, dn|_{n=\infty}$, we have a counter-example where, if we directly integrate a convergent integral at infinity, Criteria E3 fails to give 0, and hence does not establish convergence.

Example 7.11. $\sum \frac{1}{8n^2+12n+4}|_{n=\infty} = 0$ *can be shown to converge by the comparison test at infinity, or reduce the sum to a known p-series. However converting the sum to an integral, without applying infinitary simplification, then integrating the sum at infinity fails to give* 0*, contradicting Criteria E3.*

$\sum \frac{1}{8n^2+12n+4}|_{n=\infty} = \int \frac{1}{8\,n^2+12\,n+4} dn|_{n=\infty} = \frac{\ln(2\,n+1)}{4} - \frac{\ln(n+1)}{4}|_{n=\infty} = \frac{\ln 2}{4} \neq 0$ *fails Criteria E3.* $\frac{\ln 2}{4} \notin \mathbb{R}_\infty$

However, had infinitary arguments been applied, we would have got the correct result.

$\int \frac{1}{8\,n^2+12\,n+4} dn|_{n=\infty} = \int \frac{1}{8\,n^2} dn|_{n=\infty} = -\frac{1}{8n}|_{n=\infty} = 0$ *satisfies Criterion E3'.*

The above, by non-uniqueness at infinity, does not contradict. Firstly, E3 found the convergence by another path, rather than directly integrating. By not applying non-reversible arithmetic, this contradicted the E3 criteria, hence it is not a valid counter-example.

Other criterion did find a solution. Again, this is not a contradiction, as different theories have different rules. Even when the two calculations had the same starting point $\int \frac{1}{8\,n^2+12\,n+4} dn|_{n=\infty}$ *but arrived at different conclusions does not contradict because they are governed by different sets of rules.*

Remark: 7.1. *Another criterion could extend E3 and not require the convergence sum to be an infinireal. Then, comparing against the boundary the above would have converged. However, we have chosen to purse the simpler Criteria E3.*

Conjecture: By application of infinitary arguments before integrating,

removing lower order terms in the integral at infinity,

we realize Criterion E3 in testing for convergence.

With the boundary test, see Test 10.19, which is the subject of the next paper, the conjecture is justified as lower order terms become additive identities and are simplified as part of the test.

What this says about infinity is interesting. Firstly, we are forced to consider infinitary arguments to explain what is going on. Secondly, the nature of expressions at infinity encodes information. In this case, persisting with the lower order magnitude terms contradicted infinitary magnitude arguments. Having $\int \frac{1}{8n^2+12n+4}\, dn|_{n=\infty}$ has perpetuated the lower order terms $12n + 4|_{n=\infty}$ at infinity, and subsequently affected the integration.

We cannot assume a rule for finite arithmetic will carry over and be the same for infinite arithmetic. Infinity is realising itself to hold different number systems, and be more complicated, and yet explain so much more.

$\int \frac{1}{8n^2+12n+4}\, dn \neq \int \frac{1}{8n^2}\, dn|_{n=\infty}$ implicitly assumes $8n^2 \not\succ 12n + 4|_{n=\infty}$, where we associated the much-greater-than relation with non-reversible arithmetic.

If we consider simplification in general, that is, where we apply arguments of magnitude, then the very process often effects other evaluations, particularly comparison. In the previous case, $n^2 + \frac{1}{n} > n^2|_{n=\infty}$, but after simplification equality is realized($n^2 = n^2$ as $\frac{1}{n}|_{n=\infty} = 0$). Since the simplification is non-Archimedean arithmetic, which exists everywhere in analysis, when limits are being taken, the realization of infinitesimals and infinities matters.

This is not a blanket statement of denying the realization operation, but that a given simplification may have consequences, such as the previous example showed. We remind ourselves that truncation is a subset of realization, so the simplification is common. Our goal is to look at such arithmetic, and manage it.

In this spirit, papers follow with further results and conjectures, at present from necessity and empirical observation. For example, the classes of sums are greatly increased by considering periodic sums at infinity; with Criterion E3 as a guide, we conjecture a rearrangement for integral sums at infinity [11].

Criterion E3 is being used as a necessary test. We can relax Criterion E3, but then other results may become less certain, or the theory becomes more complex, and may serve other purposes. (Other criteria are possible.) While we may conjecture, it is important to be able to test that conjecture.

Proposition 7.1. *If $f = \infty$ and $\ln n \prec f|_{n=\infty}$ then $\sum \frac{1}{e^f}|_{n=\infty} = 0$ and $\int \frac{1}{e^f}\, dn|_{n=\infty} = 0$ converges.*

Proof. Let $p > 1$, comparing against the convergent p-series, $\sum \frac{1}{n^p}|_{n=\infty} = 0$. Solving for relation z, $\sum \frac{1}{e^f} \; z \; \sum \frac{1}{n^p}|_{n=\infty}$, $\frac{1}{e^f} \; z \; \frac{1}{n^p}|_{n=\infty}$, $n^p \; z \; e^f|_{n=\infty}$, $p \ln n \; (\ln z) \; f|_{n=\infty}$, $p \ln n \prec f|_{n=\infty}$, $\ln z = \prec$, $z = e^{\prec} = \prec = \leq$. (Recall from [3] the left-to-right generalization of the equals operator, $\prec \Rightarrow \leq$ for positive values.) Substituting z back in the sum comparison, $0 \leq \sum \frac{1}{e^f} \leq 0$, $\sum \frac{1}{e^f}|_{n=\infty} = 0$ converges. □

Example 7.12. *Determine convergence of $\sum \frac{1}{e^x}|_{x=\infty}$. By Proposition 7.1, $f = x$, $\ln x \; z \; x|_{x=\infty}$, $\ln x \prec x|_{x=\infty}$, then $\sum \frac{1}{e^x}|_{x=\infty} = 0$ converges.*

However, it is easier to compare with a known converging p-series directly. $x^p \prec e^x|_{x=\infty}$, $\frac{1}{e^x} \prec \frac{1}{x^p}|_{x=\infty}$, $\sum \frac{1}{e^x} < \sum \frac{1}{x^p}|_{x=\infty}$, and for $p > 1$ the convergence result follows.

8 Integral and sum interchange

The integral test is often given in one direction: if a sum can be bounded below and above by a monotonic integral and if the integral converges, the sum converges, and if the integral diverges the sum diverges.

With the convergence sum Criteria E3 and E3', positive monotonic sums can be made arbitrarily large or small. Then the sum can bound an integral and determine the integral's convergence or divergence, or vice versa. Hence the integral test can be in two directions, a consequence of the E3 and E3' criteria.

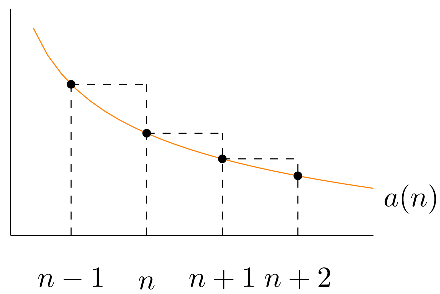

Figure 1: Strictly monotonic decreasing function

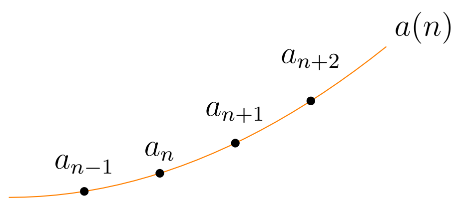

Figure 2: Strictly monotonic increasing function

Proposition 8.1. *For a monotonic function $a(x)$ in $*G$.*

$$\ldots \leq \int^{n-1} a(x)\,dx \leq \int^{n} a(x)\,dx \leq \int^{n+1} a(x)\,dx \leq \ldots |_{n=\infty}$$

For a strictly monotonic function, replace the inequality with a strict inequality.

Proof. Apply the fundamental theorem of calculus in $*G$ [12].

Divergence, $\ldots \leq \int_{n_0}^{n-1} a(x)\,dx \leq \int_{n_0}^{n} a(x)\,dx \leq \int_{n_0}^{n+1} a(x)\,dx \leq \ldots |_{n=\infty}$, $\ldots \leq \int^{n-1} a(x)\,dx - \int^{n_0} a(x)\,dx \leq \int^{n} a(x)\,dx - \int^{n_0} a(x)\,dx \leq \int^{n+1} a(x)\,dx - \int^{n_0} a(x)\,dx \leq \ldots |_{n=\infty}$, choose n_0 as an additive identity, $\ldots \leq \int^{n-1} a(x)\,dx \leq \int^{n} a(x)\,dx \leq \int^{n+1} a(x)\,dx \leq \ldots |_{n=\infty}$.

Convergence, $\ldots \geq \int_{n-1}^{n_1} a(x)\,dx \geq \int_{n}^{n_1} a(x)\,dx \geq \int_{n+1}^{n_1} a(x)\,dx \geq \ldots|_{n=\infty}$, $\ldots \geq \int^{n_1} a(x)\,dx - \int^{n-1} a(x)\,dx \geq \int^{n_1} a(x)\,dx - \int^{n} a(x)\,dx \geq \int^{n_1} a(x)\,dx - \int^{n+1} a(x)\,dx \geq \ldots|_{n=\infty}$, choose n_1 for an additive identity, $\ldots \geq -\int^{n-1} a(x)\,dx \geq -\int^{n} a(x)\,dx \geq -\int^{n+1} a(x)\,dx \geq \ldots|_{n=\infty}$. □

Proposition 8.2. *For a monotonic sequence a_n in $*G$.*

$$\ldots \leq \sum a_{n-1} \leq \sum a_n \leq \sum a_{n+1} \ldots|_{n=\infty}$$

For a strictly monotonic function, replace the inequality with a strict inequality.

Proof. Apply for sums, the mirror to the fundamental theorem of calculus in $*G$ [12].

Divergence: $\ldots \leq \sum_{k=n_0}^{n-1} a_k \leq \sum_{k=n_0}^{n} a_k \leq \sum_{k=n_0}^{n+1} a_k \ldots|_{n=\infty}$, $\ldots \leq \sum a_{n-1} - \sum a_{n_0} \leq \sum a_n - \sum a_{n_0} \leq \sum a_{n+1} - \sum a_{n_0} \ldots|_{n=\infty}$, choose n_0 for an additive identity, $\ldots \leq \sum a_{n-1} \leq \sum a_n \leq \sum a_{n+1} \ldots|_{n=\infty}$,

Convergence: $\ldots \sum_{k=n-1}^{n_1} a_k \geq \sum_{k=n}^{n_1} a_k \geq \sum_{k=n+1}^{n_1} a_k \geq \ldots$, $\ldots \geq \sum a_{n_1} - \sum a_{n-1} \geq \sum a_{n_1} - \sum a_n \geq \sum a_{n_1} - \sum a_{n+1} \ldots|_{n=\infty}$, choose n_1 as an additive identity, $\ldots \geq -\sum a_{n-1} \geq -\sum a_n \geq -\sum a_{n+1} \ldots|_{n=\infty}$ □

Proposition 8.3. *For monotonic function $a(n)$ and sequence a_n: $a(n) = a_n$ in $*G$.*

$$\sum a_n \leq \int^{n} a(x)\,dx \leq \sum a_{n+1} \leq \int^{n+1} a(x)\,dx|_{n=\infty}$$

For a strictly monotonic function, replace the inequality with a strict inequality.

Proof. From the geometric construction (see figures 1 and 2), applying the fundamental theorem of calculus and sums, with a choice of the second integrand to be an additive identity.

Divergence case: $\sum_{k=n_0}^{n} a_n \leq \int_{n_0}^{n} a(x)\,dx \leq \sum_{k=n_0}^{n+1} a_k \leq \int_{n_0}^{n+1} a(x)\,dx|_{n=\infty}$, $\sum a_n - \sum a_{n_0} \leq \int^{n} a(x)\,dx - \int^{n_0} a(x)\,dx \leq \sum a_{n+1} - \sum a_{n_0} \leq \int^{n+1} a(x)\,dx - \int^{n_0} a(x)\,dx|_{n=\infty}$, $\sum a_n \leq \int^{n} a(x)\,dx \leq \sum a_{n+1} \leq \int^{n+1} a(x)\,dx|_{n=\infty}$,

Convergence case: $\sum_{k=n}^{n_1} a_k \geq \int_{n}^{n_1} a(x)\,dx \geq \sum_{k=n+1}^{n_1} a_k \geq \int_{n+1}^{n_1} a(x)\,dx|_{n=\infty}$, $\sum a_{n_1} - \sum a_n \geq \int^{n_1} a(x)\,dx - \int^{n} a(x)\,dx \geq \sum a_{n_1} - \sum a_{n+1} \geq \int^{n_1} a(x)\,dx - \int^{n+1} a(x)\,dx|_{n=\infty}$, $-\sum a_n \geq -\int^{n} a(x)\,dx \geq -\sum a_{n+1} \geq -\int^{n+1} a(x)\,dx|_{n=\infty}$. □

Further assumptions about the steady state are that successive terms reach the same state at infinity. This may seem contradictory, but it is a property of non-uniqueness at infinity.

We can have $n < n+1 < n+2 < \dots|_{n=\infty}$ and $n = n+1 = n+2 = n+3|_{n=\infty}$ after approximation where the overriding magnitude dominates. We would say that they are just two different views of the same event.

Another example, $(n+1)^2 > n^2|_{n=\infty}$, but we can find equality in the leading coefficient, $n^2+2n+1 == n^2|_{n=\infty}$ as $n^2 \succ 2n+1$.

Such a view of magnitude leads to the following steady state interpretation at infinity.

Proposition 8.4.

$$\sum a_n = \sum a_{n+1}|_{n=\infty}$$

$$\int^n a(x)\,dx = \int^{n+1} a(x)\,dx|_{n=\infty}$$

Proof. Since the sequence of terms is monotonic, any two consecutive terms are more equal to each other than terms further way as the sequence increases. In this context we define equality even when the derivative is diverging.

Case $\sum a_n \in \Phi$, $\Phi \mapsto 0$ and consecutive sums obtain equality. Case $\sum a_n \in \Phi^{-1}$, $\Phi^{-1} \mapsto \infty$ and consecutive sums obtain equality. □

Since consecutive sums sandwich between the integral, and consecutive integrals sandwich between the sum, if one converges or diverges so does the other.

Theorem 8.1. *The integral test in both directions, interchanging sums and integrals at infinity.* $*G \mapsto \overline{\mathbb{R}}_\infty$

$$\sum a_n = \int^n a(x)\,dx|_{n=\infty}$$

Proof. Since both sequences $(\sum a_n)|_{n=\infty}$ and $(\int^n a(n)\,dn)|_{n=\infty}$ are monotonically increasing, Proposition 8.3, these inequalities show that both are bounded above or both are unbounded. Therefore, both sequences converge or both diverge.

$s \in \{0, \infty\}$; $s' \in \{0, \infty\}$; By Proposition 8.4, let $s = \sum a_n|_{n=\infty} = \sum a_{n+1}|_{n=\infty}$. By Proposition 8.3, $s \leq \int^n a(x)\,dx|_{n=\infty} \leq s$ then $\int^n a(x)\,dx|_{n=\infty} = s$. Similarly By Proposition 8.4, let $s' = \int a(x)\,dx|_{x=\infty}$. By Proposition 8.3, $s' \leq \sum a_n|_{n=\infty} \leq s'$, $\sum a_n|_{n=\infty} = s'$ □

9 Convergence integral testing

Identify the singularity points in the domain. If any of these diverge, the integral diverges. If they all converge, the integral converges.

Example 9.1. *[25, Problem 1554, p.145] Test* $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ *for convergence or divergence. Consider the singularity points* $x = \pm\infty$. *Use non-reversible arithmetic, as* $x^2 \succ 1|_{x=\infty}$ *then* $x^2 + 1 = x^2|_{x=\infty}$.

$\int^{\infty} \frac{dx}{1+x^2} = \int^{x} \frac{dx}{x^2}|_{x=\infty} = -\frac{1}{x}|_{x=\infty} = 0$ *converges. Similarly,* $\int_{-\infty} \frac{dx}{1+x^2} = \int_{-\infty} \frac{dx}{x^2} = 0$ *converges.*

Example 9.2. *[25, Example 3, p.144] Test* $\int_0^{\infty} e^{-x^2} dx$ *for convergence.*

Test the point of discontinuity $x = \infty$,

A solution by Proposition 7.1. $\ln x \;z\; x^2|_{x=\infty}$, $\ln x \prec x^2|_{x=\infty}$ *then* $\int^{x} e^{-x^2}\, dx|_{x=\infty} = 0$ *converges and the initial integral converges.*

By a comparison against a known convergent integral, $p > 1$, $\int e^{-x^2}\, dx \;\; z \;\; \int \frac{1}{x^p} dx|_{x=\infty}$, $e^{-x^2} \;\; z \;\; \frac{1}{x^p}|_{x=\infty}$, $-x^2 \;\; (\ln z) \;\; - p \ln x|_{x=\infty}$, $-x^2 \;\; (\ln z) \;\; 0|_{x=\infty}$, $-x^2 \succ 0|_{x=\infty}$, $e^{-x^2} \prec 1|_{x=\infty}$, $z = \prec = \leq$, *and the integral converges.*

Example 9.3. *[25, Example 5, p.145] Test for convergence of the elliptic integral* $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$.

The point of discontinuity of the integrand is $x = 1$. *Expand and use non-reversible arithmetic,* $(1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4|_{x=0} = 1 - 4x|_{x=0}$, *then* $(1-(x-1)^4)|_{x=0} = -4x|_{x=0}$,

$\int^{1^-} \frac{dx}{(1-x^4)^{\frac{1}{2}}} = \int^{0^-} \frac{dx}{(1-(1+x)^4)^{\frac{1}{2}}} = \int^{0^-} \frac{dx}{(-4x)^{\frac{1}{2}}} = -\int^{x} \frac{1}{2} \frac{dx}{x^{\frac{1}{2}}}|_{x=0} = x^{\frac{1}{2}}|_{x=0} = 0$ *converges.*

10 Convergence tests

The following is an exploration of sums convergence tests where the tests are rewritten with respect to the **E3** convergence criteria. These tests are derived. Known results are derived and re-written in terms of the new theory. This also helps to demonstrate the theory from a theoretical point of view.

It is assumed unless otherwise stated that the series being tested is monotonic. That is, given a series at infinity $\sum a_n|_{n=\infty}$, we can construct an associated sequence of terms from the series, $(a_n)|_{n=\infty}$. We require this sequence to be monotonic.

A notable exception is the Alternating convergence test, which has a requirement that the general term is $(-1)^n a_n$, then $(a_n)|_{n=\infty}$ is monotonic.

Mathematics often implicitly works with infinitesimals and infinities but does not declare this, for example in the calculation of limits. When a reference is made, it is usually to the extended reals $\overline{\mathbb{R}}$, however these do not explicitly declare or state infinitesimals or infinities, but $\pm\infty$ the number.

We do not necessarily mind the implicit use, however to be more descriptive, we generally reason in $*G$, and project back and state the proposition or theorem in $\mathbb{R}/\overline{\mathbb{R}}$. As a default, we have done this for the convergence tests in this paper.

However, there is a difference between transferring a result for $*G$ to $\mathbb{R}/\overline{\mathbb{R}}$, the $<$ relation.

Consider $*G: \ n^2 < n^3|_{n=\infty}$ projected to $\overline{\mathbb{R}}$ then $\infty < \infty$ is a contradiction. A similar contradiction occurs when we project an infinitesimal relation $\frac{1}{n^3} < \frac{1}{n^2}|_{n=\infty}$ then $0 < 0$ contradicts.

Pragmatically, if we use $(*G, <) \mapsto (\mathbb{R}/\overline{\mathbb{R}}, \leq)$ then such contradictions can be minimized. (See [5, Theorem 2.4])

We can geometrically understand this as $*G$ with infinitesimals and infinities, being a much more dense space. We may have an infinity of curves infinitely close to each other in $*G$ project back (by infinitesimal truncation) to a single curve in $\mathbb{R}$ [4, Example 2.19].

However, the projection of infinitesimals to 0 and positive infinities to ∞, in a contradictory way allows us to sandwich diverging infinities, or infinitesimals which are growing apart infinitesimally sandwiched to 0. And hence we say $\sum a_n|_{n=\infty} = 0$ or ∞.

10.1 p-series test

Theorem 10.1.

$$\sum \frac{1}{n^p}|_{n=\infty} = 0 \textit{ converges when } p > 1 \textit{ and}$$

$$\sum \frac{1}{n^p}|_{n=\infty} = \infty \textit{ diverges when } p \leq 1$$

$$\int \frac{1}{x^p}\, dx|_{x=\infty} = 0 \textit{ converges when } p > 1 \textit{ and}$$

$$\int \frac{1}{x^p}\, dx|_{x=\infty} = \infty \textit{ diverges when } p \leq 1$$

Proof. The p-series sum test follows from applying the integral test and considering the convergence of the analogous integral .

Since the sum has no singularity except at infinity, hence this is the only place that the sum can diverge. $\sum \frac{1}{n^p}|_{n=\infty} = \int \frac{1}{x^p}\, dx|_{x=\infty} = \frac{1}{-p+1}\frac{1}{x^{p-1}}|_{x=\infty}$ when $p \neq 1$. $p < 1$ then $\frac{1}{-p+1}\frac{1}{x^{p-1}}|_{x=\infty} = \infty$ diverges, $p > 1$ then $\frac{1}{-p+1}\frac{1}{x^{p-1}}|_{x=\infty} = 0$ converges. When $p = 1$, $\sum \frac{1}{n} = \int \frac{1}{x}\, dx|_{x=\infty} = \ln x|_{x=\infty} = \infty$ which diverges. $\square$

Example 10.1.1. *The p-test often uses infinitary calculus.* $\sum \frac{1}{n(n+1)}|_{n=\infty} = \sum \frac{1}{n^2+n}|_{n=\infty} = \sum \frac{1}{n^2}|_{n=\infty} = 0$ *converges. Perhaps a trickier but equally valid way,* $n = n+1|_{n=\infty}$ *then* $n(n+1) = n^2|_{n=\infty}$ *and the simplification follows.*

Example 10.1.2. $\sum \frac{5n+2}{n^3+1}|_{n=\infty} = \sum \frac{5n}{n^3}|_{n=\infty} = \sum \frac{5}{n^2}|_{n=\infty} = 0$ *converges. Alternatively see Example 10.7.1.*

We can convert the integral into a series, for example power series or a Riemann sum and test the integral as a series. However, we often convert from a series to an integral because it is easier work with continuous functions. For example, to an integral apply the chain rule.

10.2 Power series tests

While this is not generally called a test, possibly because $x = 0$ is almost always a solution, hence there generally is a convergence at $x = 0$. As the method uses other tests at the interval end points, and by its nature it has been included. Power series convergence sums [13] is more detailed.

Theorem 10.2. *Transform the sum* $\sum a_n x^n = \sum (b_n x)^n|_{n=\infty}$*. For convergence* $\sum (b_n x)^n|_{n=\infty} = 0$*, solving for* $|b_n x| < 1$*, the radius of convergence* $r = \frac{1}{|b_n|}|_{n=\infty}$ *If* r *exists* $x = (-r, r)$ *converges. For the interval of convergence the end points need to be tested. [13]*

Example 10.2.1. *Determine the radius of convergence and the interval of convergence of the following power series.* $\sum \frac{nx^n}{2^{n+1}}$*, the only place divergence is taking place is at the point at infinity.* $\sum \frac{nx^n}{2^{n+1}}|_{n=\infty} = \sum \frac{1}{2} n (\frac{x}{2})^n|_{n=\infty} = \sum (n^{\frac{1}{n}} \frac{x}{2})^n|_{n=\infty} = \sum (\frac{x}{2})^n|_{n=\infty} = 0$ *when* $|\frac{x}{2}| < 1$*,* $|x| < 2$*,* $r = 2$*. Investigating the end points, case* $x = 2$*,* $\sum \frac{n2^n}{2^{n+1}}|_{n=\infty} = \sum n|_{n=\infty} = \infty$ *diverges. Case* $x = -2$*,* $\sum \frac{n(-2)^n}{2^{n+1}}|_{n=\infty} = \sum n(-1)^n|_{n=\infty} = \infty$ *diverges. Interval of convergence* $x = (-2, 2)$*.*

Example 10.1. *Find the radius of convergence and convergence interval for* $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$*.*

Finding the radius of convergence at infinity, $\sum \frac{1}{n^2 3^n} x^n|_{n=\infty} = \sum \frac{1}{3^n} x^n|_{n=\infty}$ *[as* $3^n \succ\!\succ n^2|_{n=\infty}$*]* $= \sum (\frac{x}{3})^n|_{n=\infty}$*,* $|\frac{x}{3}| < 1$*,* $|x| < 3$*,* $r = 3$*. Alternatively see Example 10.16.1.*

Testing the end points of the interval, case $x = -3$*,* $\sum \frac{(-3)^n}{n^2 3^n}|_{n=\infty} = \sum \frac{(-1)^n 3^n}{n^2 3^n}|_{n=\infty} = \sum \frac{(-1)^n}{n^2}|_{n=\infty} = 0$ *converges by the ACT (Theorem 10.11) . Case* $x = 3$*,* $\sum \frac{(3)^n}{n^2 3^n}|_{n=\infty} = \sum \frac{1}{n^2}|_{n=\infty} = 0$ *converges. Interval of convergence* $x = [-3, 3]$

10.3 Integral test

See Theorem 8.1. $\sum a_n = \int^n a(x)\, dx|_{n=\infty}$

Example 10.3.1. $\sum \frac{\ln n}{n^2}|_{n=\infty} = \int \frac{\ln n}{n^2}\, dn|_{n=\infty}$. *Using the symbolic maths package Maxima as a calculator to solve the integral,* integrate(log(n)/n^2, n); $\int \frac{\ln n}{n^2}\, dn|_{n=\infty} = -\frac{\ln n}{n} - \frac{1}{n}|_{n=\infty} = 0$ *converges. Hence* $\sum \frac{\ln n}{n^2}|_{n=\infty} = 0$ *converges.*

The integral test allows the application of the chain rule. Consider when the subscript is itself a diverging function, and its derivative is a product. The sum's constant multiplier is irrelevant. $f_n \to \infty$ Determine convergence/divergence. $\sum a_{f_n} \frac{df_n}{dn}|_{n=\infty} = \int a(f(n)) \frac{df(n)}{dn} dn|_{n=\infty}$ $= \int a(f(n))\, df(n)|_{n=\infty} = \int a(n)\, dn|_{n=\infty} = \sum a_n|_{n=\infty}$

Example 10.3.2. *[14, 10.14.16, p.16]* $\sum ne^{-n^2}|_{n=\infty} = \int -2ne^{-n^2}\, dn|_{n=\infty} = \int \frac{d(-n^2)}{dn} e^{-n^2}\, dn|_{n=\infty}$ $= \int e^{-n^2} d(-n^2)|_{n=\infty} = e^{-n^2}|_{n=\infty} = 0$ *converges.*

Example 10.3.3. *[17, 3.2.31.a, p.96] Show* $\sum a_n|_{n=\infty} = 0$ *converges if the given series converges.* $\sum 3^n a_{3^n}|_{n=\infty} = 0$. $\sum 3^n a_{3^n}|_{n=\infty} = \int 3^n a(3^n) dn|_{n=\infty} = \int 3^n a(3^n) \frac{dn}{d3^n} d(3^n)|_{n=\infty} =$ $\int 3^n a(3^n) \frac{1}{\ln 3 \cdot 3^n} d(3^n)|_{n=\infty} = \int a(3^n) d(3^n)|_{n=\infty} = \int a(n) dn|_{n=\infty} = \sum a(n)|_{n=\infty} = 0$ *converges.*

The integral test can be combined with other tests, which makes it really useful.

Example 10.3.4. *[17, 2.3.4] Given* $a_n|_{n=\infty} = \infty$, *show* $\frac{1}{n}\sum_{k=1}^{n} a_k|_{n=\infty} = \infty$.

A ratio in integers is converted by the integral test to a continuous variable, and L'Hopital's rule is applied.

Since $a_n|_{n=\infty} = \infty$, $\int a_n\, dn|_{n=\infty} = \infty$, $\frac{1}{n}\sum a_n|_{n=\infty} = \frac{1}{n}\int a_n\, dn|_{n=\infty}$, *since* $\frac{\infty}{\infty}$ *form then differentiate,* $\frac{1}{n}\int a_n\, dn|_{n=\infty} = \frac{1}{\frac{d}{dn}n}\frac{d}{dn}\int a_n\, dn|_{n=\infty} = a_n|_{n=\infty} = \infty$.

10.4 Comparison test

Theorem 10.3. $0 \le a_n \le b_n|_{n=\infty}$

$$\textit{If } \sum a_n|_{n=\infty} = \infty \textit{ diverges then } \sum b_n|_{n=\infty} = \infty \textit{ diverges}$$

$$\textit{If } \sum b_n|_{n=\infty} = 0 \textit{ converges then } \sum a_n|_{n=\infty} = 0 \textit{ converges.}$$

Proof. If $0 \le a_n \le b_n|_{n=\infty}$ then $0 \le \sum a_n \le \sum b_n|_{n=\infty}$
Case $\sum a_n|_{n=\infty} = \infty$ diverges, $0 \le \infty \le \sum b_n|_{n=\infty}$ then $\sum b_n|_{n=\infty} = \infty$
Case $\sum b_n|_{n=\infty} = 0$ converges $0 \le \sum a_n \le 0$ then $\sum a_n|_{n=\infty} = 0$ □

The aim of the comparison test is to find a sum where convergence or divergence is known, and compare against that sum, component wise.

Example 10.4.1. $\sum \frac{\ln n}{n^2}|_{n=\infty}$ *An inequality approach, an indirect and not as accessible for non-maths people because of the factorization.*

$\frac{\ln n}{n^2} = \frac{\ln n}{n^{\frac{1}{2}}} \frac{1}{n^{\frac{3}{2}}}|_{n=\infty}$. *As* $\ln n \prec n^{\frac{1}{2}}|_{n=\infty}$, $\frac{\ln n}{n^{\frac{1}{2}}}|_{n=\infty} = 0$, $\frac{\ln n}{n^{\frac{1}{2}}} \frac{1}{n^{\frac{3}{2}}} \leq \frac{1}{n^{\frac{3}{2}}}|_{n=\infty}$, $0 \leq \frac{\ln n}{n^2} \leq \frac{1}{n^{\frac{3}{2}}}|_{n=\infty}$, *Summing the inequality at infinity,* $0 \leq \sum \frac{\ln n}{n^2} \leq \sum \frac{1}{n^{\frac{3}{2}}}|_{n=\infty}$, $0 \leq \sum \frac{\ln n}{n^2}|_{n=\infty} \leq 0$, $\sum \frac{\ln n}{n^2}|_{n=\infty} = 0$ *converges.*

Example 10.4.2. *The same problem above could be solved with a different comparison, where we test against the known convergent p-series sums. Let* $p > 1$, $\sum \frac{\ln n}{n^2}$ z $\sum \frac{1}{n^p}|_{n=\infty}$, $\frac{\ln n}{n^2}$ z $\frac{1}{n^p}|_{n=\infty}$, $\ln n$ z $n^{2-p}|_{n=\infty}$. *When* $2 - p > 0$ *then* $\ln n \prec n^{2-p}|_{n=\infty}$. *Let* $p = \frac{3}{2}$ *satisfies both conditions and the sum converges. Not necessary, but just to demonstrate the theory is working,* $\prec = \leq$, *substituting back into the sum,* $\sum \frac{\ln n}{n^2} \leq \sum \frac{1}{n^{\frac{3}{2}}}|_{n=\infty}$, $\sum \frac{\ln n}{n^2} \leq 0|_{n=\infty}$.

In a variation of the comparison test, we can sandwich a series which may not be monotonic between two monotonic series with the same convergence. In the sandwiched comparison test where either side of the test are monotonic sequences $(b_n)|_{n=\infty}$ and $(c_n)|_{n=\infty}$. Thereby extending the test to $\sum a_n|_{n=\infty}$ which may not be monotonic.

Theorem 10.4. *If we can sandwich a non-monotonic sequence between two monotonic sequences, which either both converge or both diverge, we can determine the non-monotonic sequence's convergence.*

Given $0 \leq a_n \leq b_n \leq c_n|_{n=\infty}$ *where* $(a_n)|_{n=\infty}$ *and* $(c_n)|_{n=\infty}$ *are monotonic sequences.*

$$\text{If } \sum a_n|_{n=\infty} = \sum c_n|_{n=\infty} \text{ then } \sum b_n = \sum c_n|_{n=\infty}$$

Proof. $0 \leq a_n \leq b_n \leq c_n|_{n=\infty}$ then $0 \leq \sum a_n \leq \sum b_n \leq \sum c_n|_{n=\infty}$. Divergent case, $\infty \leq \sum b_n \leq \infty$ then $\sum b_n|_{n=\infty} = \infty$ diverges. Convergent case, $0 \leq \sum b_n \leq 0$ then $\sum b_n|_{n=\infty} = 0$ converges. □

10.5 nth term divergence test

Theorem 10.5.

$$\text{If } a_n|_{n=\infty} \neq 0 \Rightarrow \sum a_n|_{n=\infty} = \infty \text{ diverges.}$$

Proof. By the negation of Theorem 10.6, as when not equal to 0, the negation of convergence is divergence, the sum diverges. □

Theorem 10.6. *If* $\sum a_n|_{n=\infty} = 0$ *converges then* $a_n|_{n=\infty} = 0$

Proof. A sum of terms greater than or equal to zero is positive if their exists a term greater than zero. Since the sum is zero, their exists no such term, consequently $a_n|_{n=\infty} = 0$. □

Example 10.5.1. $1 - 1 + 1 - 1 + \ldots$, $a_n|_{n=\infty} = (-1)^n \neq 0$ *hence the series diverges.*

Example 10.5.2. *An example of the second case, determine convergence/divergence of* $\sum_{n=1}^{\infty} \frac{n^3+n}{5n^3+n^2+27}$. *Since no division by zero, consider* $\sum \frac{n^3+n}{5n^3+n^2+27}|_{n=\infty}$. $a_n|_{n=\infty} = \frac{n^3+n}{5n^3+n^2+27}|_{n=\infty}$ $= \frac{n^3}{5n^3}|_{n=\infty} = \frac{1}{5} \neq 0$ *therefore divergent by nth term test. With the sum at infinity, simplifying makes it clear why the sum diverges, without need to even refer to the nth term divergence test.* $\sum \frac{n^3+n}{5n^3+n^2+27}|_{n=\infty} = \sum \frac{n^3}{5n^3}|_{n=\infty} = \sum \frac{1}{5}|_{n=\infty} = \infty$

10.6 Absolute convergence test

Theorem 10.7. *If* $\sum |a_n||_{n=\infty} = 0$ *converges then* $\sum a_n|_{n=\infty} = 0$ *is convergent.*

Proof. Let $a_k \neq 0$. $-|a_k| \leq a_k \leq |a_k|$, using $a_k = \text{sgn}(a_k)|a_k|$, where $\text{sgn}(x) = \pm 1$ when $x \neq 0$. $-|a_k| \leq |a_k|\text{sgn}(a_k) \leq |a_k|$, dividing by $|a_k|$, $-1 \leq \text{sgn}(a_k) \leq 1$ which is always true, hence $-|a_k| \leq a_k \leq |a_k|$ is true. n_0, $n_1 \in \mathbb{J}_\infty$; Summing the inequalities between two infinities n_0 and n_1, $-\sum_{k=n_0}^{n_1} |a_k| \leq \sum_{k=n_0}^{n_1} a_k \leq \sum_{k=n_0}^{n_1} |a_k|$.

The condition $\sum |a_n||_{n=\infty}$ is itself governed by Criterion E3. With a suitable choice of n_1(see Proposition 4.2), $\sum_{k=n_0}^{n_1} |a_n| = \sum_{k=n_0} |a_n|$ as the sum converges, replace n_0 by n, $\sum_n a_n|_{n=\infty} = 0$, $-\sum a_n|_{n=\infty} = 0$, $\sum a_n|_{n=\infty} = 0$.

Consider the inequality, $-\sum_{k=n_0}^{n_1} |a_k| \leq \sum_{k=n_0}^{n_1} a_k \leq \sum_{k=n_0}^{n_1} |a_k|$, $-\sum_{k=n_0} |a_k| \leq \sum_{k=n_0} a_k \leq \sum_{k=n_0} |a_k|$, $-\sum_n |a_k| \leq \sum_n a_k \leq \sum_n |a_k||_{n=\infty}$, $\sum |a_n| \leq -\sum a_n \leq -\sum |a_n||_{n=\infty}$, $0 \leq -\sum a_n|_{n=\infty} \leq 0$, $\sum a_n|_{n=\infty} = 0$ converges. □

Theorem 10.7 is a sum rearrangement theorem at infinity, summing at infinity only. However it was used in [11] to prove the more general sum theorem below.

If $\sum a_\nu$ is absolutely convergent and $\sum a'_\nu$ is an arbitrary rearrangement then $\sum a_\nu = \sum a'_\nu$ [22, Theorem 4, p.79].

10.7 Limit Comparison Theorem (LCT)

Theorem 10.8. *If* $\frac{a_n}{b_n}|_{n=\infty} = c$ *and* $c \neq 0$ *and* c *is a constant then* $\sum a_n = \sum b_n|_{n=\infty}$. *Either both series converge or both diverge.*

Proof. $\frac{a_n}{b_n}|_{n=\infty} = c$, $a_n = cb_n|_{n=\infty}$. Apply summation to both sides, $\sum a_n = c \sum b_n|_{n=\infty}$. Ignoring the constant as the sum either converges or diverges ($c \cdot \infty = \infty$ or $c \cdot 0 = 0$), $\sum a_n = \sum b_n|_{n=\infty}$. Two possibilities. Case $\sum a_n = 0 \Leftrightarrow \sum b_n = 0|_{n=\infty}$. Case $\sum a_n = \infty \Leftrightarrow \sum b_n = \infty|_{n=\infty}$ □

Example 10.7.1. $\sum_{k=0}^{n} \frac{5k+2}{k^3+1}|_{n=\infty}$, *the limit comparison test assumes the answer. If you have already worked out that the above sum tends to* $\sum \frac{1}{n^2}$, *then forming a limit is redundant. However you can verify the result by calculating the limit.*

Let $a_n = \frac{5n+2}{n^3+1}$, $b_n = \frac{1}{n^2}$. $\sum \frac{1}{n^2}|_{n=\infty} = 0$ *converges, as this is a p-series with* $p = 2 > 1$.

$\frac{a_n}{b_n}|_{n=\infty} = \frac{5n+2}{n^3+1} n^2|_{n=\infty} = \frac{5n^3}{n^3}|_{n=\infty} = 5$. *Since* $\sum \frac{1}{n^2}|_{n=\infty} = 0$ *converges then* $\sum a_n|_{n=\infty} = 0$ *converges. Alternatively see Example 10.1.2.*

See Example 5.1.

10.8 Abel's test

Theorem 10.9. *Suppose* $\sum b_n|_{n=\infty} = 0$ *converges and* (a_n) *is a monotonic convergent sequence then* $\sum a_n b_n|_{n=\infty} = 0$ *converges.*

Proof. Since (a_n) is a monotonic convergent sequence, then let $a_n|_{n=\infty} = a$, $a \prec \infty$. $a_n b_n = a b_n|_{n=\infty}$, then $\sum a_n b_n|_{n=\infty} = \sum a b_n|_{n=\infty} = a \sum b_n|_{n=\infty} = 0$ converges. □

Proof. Since $(a_n)|_{n=\infty}$ is convergent, the sequence is bounded above, say by M. $0 \leq a_n b_n \leq M b_n|_{n=\infty}$, $0 \leq \sum a_n b_n \leq \sum M b_n|_{n=\infty}$, $0 \leq \sum a_n b_n \leq M \sum b_n|_{n=\infty}$, $0 \leq \sum a_n b_n \leq 0|_{n=\infty}$, $\sum a_n b_n|_{n=\infty} = 0$ converges. □

10.9 L'Hopital's convergence test

We believe this is a new test, and have named it L'Hopital because of the similarity with L'Hopital's ratio rule [6, Part 5].

Theorem 10.10. *If* $\frac{a_n}{b_n}|_{n=\infty}$ *has a fraction of indeterminate form* $\frac{\infty}{\infty}$ *or* $\frac{0}{0}$ *then* $\sum \frac{a_n}{b_n} = \sum \frac{a'_n}{b'_n}|_{n=\infty}$. *Similarly for integrals, when* $\frac{a(n)}{b(n)}$ *is in indeterminate form,* $\int \frac{a(n)}{b(n)}\, dn = \int \frac{a'(n)}{b'(n)}\, dn|_{n=\infty}$

Proof. Since the sum or integral is an indeterminate ratio at infinity, we can use L'Hopital's rule to simplify the ratio to either 0 or ∞. If a finite number is evaluated then the criterion has failed, and the problem was ill-formed.

Since applying L'Hopital's rule to the ratio is asymptotic to the ratio, L'Hopital's rule can be used to determine convergence or divergence at infinity. □

Proof. [8, Theorem 5.2] □

Example 10.9.1. $\sum \frac{\ln n}{n^2}|_{n=\infty}$ *Since* $\frac{\ln n}{n^2}|_{n=\infty}$ *is in* ∞/∞ *form, differentiate by L'Hopital's rule,* $\sum \frac{\ln n}{n^2}|_{n=\infty} = \sum \frac{\frac{d}{dn}\ln n}{\frac{d}{dn}n^2}|_{n=\infty} = \sum \frac{1}{n}\frac{1}{2n}|_{n=\infty} = \sum \frac{1}{n^2}|_{n=\infty} = 0$ *converges.*

Example 10.9.2. *Test* $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ *for convergence or divergence.* $\sum \frac{n^2}{2^n}|_{n=\infty} = \sum \frac{n^2}{e^{n\ln 2}}|_{n=\infty}$ $= \sum \frac{2n}{\ln 2 e^{n\ln 2}}|_{n=\infty} = \sum \frac{2}{(\ln 2)^2 e^{n\ln 2}}|_{n=\infty} = \sum \frac{1}{2^n}|_{n=\infty} = 0$ *converges.*

Example 10.9.3. *[17, 3.2.17.a, p.74] Chain rule with L'Hopital's rule.* $\sum \frac{1}{2^{n^{\frac{1}{2}}}}|_{n=\infty} = \int \frac{dx}{2^{x^{\frac{1}{2}}}}|_{x=\infty}$, *let* $x = u^2$, $\frac{dx}{du} = 2u$, $\int \frac{dx}{2^{x^{\frac{1}{2}}}}|_{x=\infty} = \int \frac{dx}{du}\frac{du}{2^u}|_{u=\infty} = \int 2u\frac{du}{2^u}|_{u=\infty}$. *[* $\frac{d}{du}2^u = \frac{d}{du}e^{u\ln 2}$ $= e^{u\ln 2}\cdot \ln 2 = 2^u \cdot \ln 2$*] As* $\frac{u}{2^u}|_{u=\infty} = \frac{\infty}{\infty}$, *apply L'Hopital's rule,* $\int 2u\frac{du}{2^u}|_{u=\infty} = \int 2\frac{du}{2^u \cdot \ln 2}|_{u=\infty}$ $= \int \frac{du}{2^u}|_{u=\infty} = 0$ *converges.*

10.10 Alternating Convergence Test*

Also called the Alternating Convergence Theorem (ACT), expressing the test at infinity.

Theorem 10.11. *If* $(a_n)|_{n=\infty}$ *is a monotonic decreasing sequence and* $a_n|_{n=\infty} = 0$ *then* $\sum(-1)^n a_n|_{n=\infty} = 0$ *is convergent.*

Example 10.10.1. *[17, 3.4.25, p.96] Determine convergence of* $\sum(-1)^n \frac{n!e^n}{n^{n+p}}|_{n=\infty}$

Recognizing the n!, rearrange Stirling's formula, $n! = (2\pi n)^{\frac{1}{2}}(\frac{n}{e})^n|_{n=\infty}$, $\frac{n!}{n^n} = (2\pi n)^{\frac{1}{2}}\frac{1}{e^n}|_{n=\infty}$.

Substitute the rearranged expression into the sum. $\sum(-1)^n \frac{n!e^n}{n^{n+p}}|_{n=\infty} = \sum(-1)^n \frac{n!}{n^n}\frac{e^n}{n^p}|_{n=\infty}$ $= \sum(-1)^n (2\pi)^{\frac{1}{2}} n^{\frac{1}{2}} \frac{1}{e^n}\frac{e^n}{n^p}|_{n=\infty} = \sum(-1)^n(2\pi)^{\frac{1}{2}}\frac{1}{n^{p-1/2}}|_{n=\infty} = \sum(-1)^n\frac{1}{n^{p-1/2}}|_{n=\infty}$

When $p = \frac{1}{2}$, $\sum(-1)^n|_{n=\infty} = \infty$ *diverges. By ACT(Theorem 10.11) when* $p > \frac{1}{2}$ *then* $\frac{1}{n^{p-1/2}}|_{n=\infty} = 0$ *and the sum converges. When* $p < \frac{1}{2}$, *by the nth term test the sum diverges.*

10.11 Cauchy condensation test

Theorem 10.12. $\sum 2^n a_{2^n}|_{n=\infty}$ *converges or diverges with* $\sum a_n|_{n=\infty}$.
If $\sum 2^n a_{2^n}|_{n=\infty} = 0$ *then* $\sum a_n|_{n=\infty} = 0$. *If* $\sum 2^n a_{2^n}|_{n=\infty} = \infty$ *then* $\sum a_n|_{n=\infty} = \infty$.

Proof. Convert to the continuous domain, apply the chain rule, and convert back to the discrete domain.

$\sum 2^n a_{2^n} dn|_{n=\infty} = \int 2^n a(2^n) dn|_{n=\infty} = \int 2^n a(2^n)\frac{dn}{d2^n}d(2^n)|_{n=\infty} = \int 2^n a(2^n)\frac{dn}{d\, e^{n\ln 2}}d(2^n)|_{n=\infty}$ $= \int 2^n a(2^n)\frac{1}{2^n \ln 2}d(2^n)|_{n=\infty} = \int a(2^n)\frac{1}{\ln 2}d(2^n)|_{n=\infty} = \int a(2^n)d(2^n)|_{n=\infty} = \int a(n)dn|_{n=\infty} =$ $\sum a_n|_{n=\infty}$. Constants ignored as either converge 0 or diverge ∞. □

Example 10.11.1. *Determine convergence/divergence of* $\sum \frac{1}{n \ln n}|_{n=\infty}$.

Let $a_n = \frac{1}{n \ln n}$. $a_{2^n} = \frac{1}{2^n \ln 2^n} = \frac{1}{2^n \cdot n \cdot \ln 2}$. *Then* $\sum 2^n a_{2^n} dn|_{n=\infty} = \sum 2^n \frac{1}{2^n \cdot n \cdot \ln 2}|_{n=\infty} = \frac{1}{\ln 2} \sum \frac{1}{n}|_{n=\infty} = \infty$ *diverges.*

10.12 Ratio test*

Theorem 10.13. *[10, Theorem 2.1]* $a_n \in *G$;

$$\text{If } (\overline{\mathbb{R}}, <): \quad \frac{a_{n+1}}{a_n} < 1 \text{ then } \sum a_n|_{n=\infty} = 0 \text{ converges.}$$
$$\text{If } (\overline{\mathbb{R}}, >): \quad \frac{a_{n+1}}{a_n} > 1 \text{ then } \sum a_n|_{n=\infty} = \infty \text{ diverges.}$$

The ratio test can also be expressed as an inequality at infinity.

Theorem 10.14. *[10, Theorem 2.2]* $a_n \in *G$;

$$\text{If } (\overline{\mathbb{R}}, <): \quad a_{n+1} < a_n \text{ then } \sum a_n|_{n=\infty} = 0 \text{ converges.}$$
$$\text{If } (\overline{\mathbb{R}}, >): \quad a_{n+1} > a_n \text{ then } \sum a_n|_{n=\infty} = \infty \text{ diverges.}$$

As the application of the ratio test at infinity is in one-to-one correspondence with the same limit calculation, let us consider the modified ratio test Theorem 10.14 examples.

Example 10.12.1. *Determine convergence of* $\sum \frac{n!}{(2n)!}|_{n=\infty}$. *Let* $a_n = \frac{n!}{(2n)!}$.

$$\begin{aligned} a_{n+1} \;\; z \;\; a_n|_{n=\infty} \\ \frac{(n+1)!}{(2(n+1))!} \;\; z \;\; \frac{n!}{(2n)!}|_{n=\infty} \\ (n+1)!(2n)! \;\; z \;\; (2n+2)!n!|_{n=\infty} \\ (n+1)(2n)! \;\; z \;\; (2n+2)!|_{n=\infty} \\ (n+1) \;\; z \;\; (2n+1)(2n+2)|_{n=\infty} \\ n < 4n^2|_{n=\infty} \\ 1 < 4n|_{n=\infty} \end{aligned} \qquad \text{(By Theorem 10.14 convergent)}$$

Example 10.12.2. *A strict inequality in* $*G$ *is not a strict inequality in* $\mathbb{R}$ *when infinitely close. Consider the known divergent sum* $\sum \frac{1}{n}|_{n=\infty}$, *with a strict inequality interpretation the test fails,* $a_{n+1} \; z \; a_n|_{n=\infty}$, $\frac{1}{n+1} \; z \; \frac{1}{n}|_{n=\infty}$, $\frac{1}{n+1} < \frac{1}{n}|_{n=\infty}$, *and the sum converges, which is incorrect.*

However realizing the comparison, $\frac{1}{n+1} \; z \; \frac{1}{n}|_{n=\infty}$, $0 \; z \; 0$, $z = ==$ *equality and the test is indeterminate.*

Definition 10.1. *Define the radius of convergence r, $\frac{1}{r} = |\frac{a_n}{a_{n-1}}||_{n=\infty}$*

Example 10.12.3. *By the ratio test with a point at infinity notation, find the radius of convergence and convergence interval for $\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$. $a_n = \frac{1}{n^2 3^n}$, $|\frac{a_n}{a_{n-1}}||_{n=\infty} = |\frac{(n-1)^2 3^{n-1}}{n^2 3^n}||_{n=\infty}$ $= \frac{1}{3}|\frac{(n-1)^2}{n^2} \cdot \frac{3^{n-1}}{3^{n-1}}||_{n=\infty} = \frac{1}{3} = \frac{1}{r}$, $r = 3$. Converges when $x = (-3, 3)$.*

Test the interval's end points. When $x = 3$, $\sum \frac{3^n}{n^2 3^n}|_{n=\infty} = \sum \frac{1}{n^2} = 0$ converges. When $x = -3$, $\sum \frac{(-1)^n 3^n}{n^2 3^n}|_{n=\infty} = \sum \frac{(-1)^n}{n^2} = 0$ converges. Interval of convergence: $x = [-3, 3]$.

Alternatively see Example 10.1.

10.13 Cauchy's convergence test

The standard test. Applying Cauchy's convergent sequence test, with the partial sum as a general sequence term. If $\exists N : \forall n, m > N, |s_m - s_n| < \epsilon$ then (s_n) is a Cauchy sequence.

This test is reformed at infinity: $s_m - s_n \in \Phi$ and ; $m, n \in \Phi^{-1}$; with the condition $m - n \in \Phi^{-1}$ [7, Part 6].

By considering the convergence sums as a sequence of points, if the sequence converges then the sum converges. As a partial sum which starts counting at infinity, $(\ldots, s_n, s_{n+1}, s_{n+2}, \ldots)|_{n=\infty}$.

For the convergence test at infinity, we consider an infinite interval at infinity, and if the sum is an infinitesimal, the sum converges.

Theorem 10.15. *Consider the convergence sum, let $s_n = \sum_n a_k|_{n=\infty}$; $m, n \in \Phi^{-1}$; $m - n = \infty$. If $s_n - s_m \in \Phi$ then the sum s_n converges, else the sum diverges.*

Proof. If the sum converges, then both sums satisfy the E3 criteria, then $s_n - s_m \in \Phi$ is a difference in infinitesimals, which is also an infinitesimal. $\Phi \mapsto 0$ and the Cauchy sequence at infinity is satisfied. □

The test forms the basis of Criterion E1 convergence. The example in [1, pp.212–213] solves the same problem with a linear scale of infinities $(\omega, 2\omega, 3\omega, \ldots)|_{\omega=\infty}$. If $s_{2\omega} - s_\omega \in \Phi$ then $(s_\omega)|_{\omega=\infty}$ converges else the sequence diverges.

In constructing a Cauchy test at infinity, we can use infinities 2^n and 2^{n-1}. Example 10.13.1 uses a power of 2 scale of infinities $(2^n, 2^{n+1}, 2^{n+2}, \ldots)|_{n=\infty}$.

Proposition 10.1. *If $s_{2^n} - s_{2^{n-1}}|_{n=\infty} = 0$ then $(s_n)|_{n=\infty}$ converges, else the sum diverges.*

Proof. Consider the E3 criterion. E.0: $2^n - 2^{n-1} = \infty$ satisfied. If the sum is convergent, both the sums at a point are infinitesimals, their difference an infinitesimal $\Phi \mapsto 0$. $\square$

Example 10.13.1. *Determine convergence/divergence of* $s_n = \sum_{k=1}^{n} \frac{1}{k}$. $s_{2^n} - s_{2^{n-1}}|_{n=\infty} = \int_1^{2^n} \frac{1}{x} dx - \int_1^{2^{n-1}} \frac{1}{x} dx = \ln x|_1^{2^n} - \ln x|_1^{2^{n-1}}|_{n=\infty} = n \ln 2 - (n-1) \ln 2|_{n=\infty} = \ln 2 \neq 0$ *diverges.*

However, Criteria E3 also do this by integration at a point (see Theorems 4.2 and 4.3). This could be considered as taking the Cauchy sequence a step further with magnitude arguments.

Theorem 10.16. *By convergence of a sequence at infinity [7, Part 6], a convergence sum or integral need only have a point tested.*

Proof. A partial sum s_n, where $|n-m| = \infty$, $s_n|_{n=\infty} - s_m|_{m=\infty} = \sum a_n|_{n=\infty} - \sum a_m|_{m=\infty} = \int a(n)\, dn|_{n=\infty} - \int a(m)\, dm|_{m=\infty} = \int a(n)\, dn|_{n=\infty}$ (divergent case) or $-\int a(m)\, dm|_{m=\infty}$ (convergent case), through the choice of the second variable and the E3 criteria, or diverges.

Theorem 4.2 essentially does this in a simpler way, taking a slice of the tail with an infinite width in the domain and integrating. $\square$

10.14 Dirichlet's test

Theorem 10.17. *If* $\sum b_n|_{n=\infty} = 0$ *converges and* $(a_n)|_{n=\infty}$ *is positive and monotonically decreasing then* $\sum a_n b_n|_{n=\infty} = 0$ *converges.*

Proof. Since a_n is positive and decreasing, a_n is bounded above by a positive constant β. Let $a_n \leq \beta$, $0 \leq a_n \leq \beta$, $0 \leq a_n b_n \leq a_n \beta|_{n=\infty}$, $0 \leq \sum a_n b_n \leq \sum a_n \beta|_{n=\infty}$, but $\sum a_n \beta = \sum a_n|_{n=\infty} = 0$ converges. By the sandwich principle, $0 \leq \sum a_n b_n \leq 0|_{n=\infty}$ and $\sum a_n b_n|_{n=\infty} = 0$ converges. $\square$

10.15 Bertrand's test*

Bertrand's test [21] is included in the generalized ratio test.

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\rho_n}{n \ln n}, \quad \rho_n|_{n=\infty} = \begin{cases} > 1 & \text{then } \sum a_n \text{ is convergent,} \\ < 1 & \text{then } \sum a_n \text{ is divergent.} \end{cases}$$

10.16 Raabe's tests*

When $\frac{a_{n+1}}{a_n}|_{n=\infty} = 1$ the ratio test fails, then try Raabe's test.

Theorem 10.18. *If* $n(\frac{a_n}{a_{n+1}} - 1)|_{n=\infty} > 1$ *then* $\sum a_n|_{n=\infty} = 0$ *converges. If* $n(\frac{a_{n+1}}{a_n} - 1)|_{n=\infty} < 1$ *then* $\sum a_n|_{n=\infty} = \infty$ *diverges.*

Theorem 10.19. *If* $na_n - (n+1)a_{n+1}|_{n=\infty} > 0$ *then* $\sum a_n = 0$ *converges. If* $na_n - (n+1)a_{n+1}|_{n=\infty} < 0$ *then* $\sum a_n = \infty$ *diverges.*

Example 10.16.1. *[14, 10.16.4] Determine convergence of* $\sum \frac{3^n n!}{n^n}|_{n=\infty}$.

Let $a_n = \frac{3^n n!}{n^n}$. *The application of the Ratio test fails.* $\frac{a_{n+1}}{a_n}|_{n=\infty} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \frac{n^n}{3^n n!}|_{n=\infty} = 3(n+1)\frac{n^n}{(n+1)^n}|_{n=\infty} = 1$

$$\begin{aligned} na_n - (n+1)a_{n+1}|_{n=\infty} &\ z\ 0 && \text{(Try Raabe's Theorem 10.19)} \\ n\frac{3^n n!}{n^n} - \frac{(n+1)3^{n+1}(n+1)!}{(n+1)^{n+1}}|_{n=\infty} &\ z\ 0 \\ \frac{n!}{n^{n-1}} - \frac{3n!}{(n+1)^{n-1}}|_{n=\infty} &\ z\ 0 \\ (\frac{n+1}{n})^{n-1}|_{n=\infty} - 3 &\ z\ 0 \\ -2 &< 0 && \text{(Converges by Theorem 10.19)} \end{aligned}$$

Example 10.16.2. *[17, 3.2.16]*
Theorem 10.20. *If* $\lim_{n\to\infty} n \ln\frac{a_n}{a_{n+1}} = g$, *show* $g > 1 \Rightarrow$ *convergence and* $g < 1 \Rightarrow$ *divergence.*

Proof. Consider the case $n \ln\frac{a_n}{a_{n+1}} > 1|_{n=\infty}$, $\ln\frac{a_n}{a_{n+1}} > \frac{1}{n}|_{n=\infty}$, $\frac{a_n}{a_{n+1}} > e^{\frac{1}{n}}|_{n=\infty}$, $a_n > a_{n+1}e^{\frac{1}{n}}|_{n=\infty}$ Substitute $e = (\frac{n+1}{n})^n|_{n=\infty}$ into the inequality, $a_n > a_{n+1}((\frac{n+1}{n})^n)^{\frac{1}{n}}|_{n=\infty}$, $a_n > a_{n+1}\frac{n+1}{n}|_{n=\infty}$, $na_n - (n+1)a_{n+1} > 0|_{n=\infty}$. This is Raabe's convergence test Theorem 10.19, and hence $n \ln\frac{a_n}{a_{n+1}} > 1|_{n=\infty} \Rightarrow \sum a_n$ is convergent.

For the divergent case, after a similar substitution, $a_n < a_{n+1}((\frac{n+1}{n})^n)^{\frac{1}{n}}|_{n=\infty}$, $a_n < a_{n+1}\frac{n+1}{n}|_{n=\infty}$, $na_n - (n+1)a_{n+1} < 0$, is Theorem 10.19, divergent case. □

10.17 Generalized p-series test*

See [8, Section 2]. Known results.

Definition 10.2. *Let* $\sum \frac{1}{\prod_{k=0}^{w-1} \ln_k \cdot \ln_w^p}$ *and the corresponding integral be called the generalized p-series.*

Theorem 10.21. $\sum \frac{1}{\prod_{k=0}^{w-1} \ln_k \cdot \ln_w^p}|_{n=\infty}$ *and* $\int \frac{1}{\prod_{k=0}^{w-1} \ln_k \cdot \ln_w^p} dn|_{n=\infty}$ *diverge when* $p \leq 1$ *and converges when* $p > 1$.

10.18 Generalized ratio test*

Theorem 10.22. *Includes the ratio test, Raabe's test, Bertrand's test. See [10].*

$$\frac{a_n}{a_{n+1}} - (1 + \frac{1}{n} + \frac{1}{n \ln n} + \ldots + \frac{1}{n \ln n \ldots \ln_k n})|_{n=\infty} = \begin{cases} > 0 & \text{then } \sum a_n|_{n=\infty} \text{ is convergent,} \\ \leq 0 & \text{then } \sum a_n|_{n=\infty} \text{ is divergent.} \end{cases}$$

10.19 Boundary test*

Theorem 10.23. *See [8]. Let* w *be a fixed integer. Solve for relation* z.

$$\sum a_n \; z \; \sum \frac{1}{\prod_{k=0}^{w} \ln n}|_{n=\infty}, \quad z = \begin{cases} < & \text{then } \sum a_n|_{n=\infty} \text{ is convergent,} \\ \geq & \text{then } \sum a_n|_{n=\infty} \text{ is divergent.} \end{cases}$$

10.20 nth root test*

Theorem 10.24. *If* $|a_n|^{\frac{1}{n}}|_{n=\infty} < 1$ *then* $\sum a_n|_{n=\infty} = 0$ *converges. See [8, Theorem 5.1]*

11 Miscellaneous

When determining convergence or divergence with convergences sums we actually do a transfer [5] from an interval to a point in their construction.

$$\sum_{n=n_0}^{\infty} a_n \mapsto \sum a_n|_{n=\infty}$$

After determining convergence or divergence at infinity, we may need to translate the "convergence sum" back into a sum.

This is of course just reversing the direction which we previously used to solve for the sum's convergence or divergence.

Example 11.1. $\sum \frac{1}{n^2}|_{n=\infty} = 0 \mapsto \sum_{k=1}^{\infty} \frac{1}{n^2}$ *converges.*

The E3 criteria uses an infinite section of the tail (integration between two infinities) at infinity to determine convergence or divergence. However, this is enough to determine the whole infinite tail's convergence or divergence.

In determining convergence or divergence, we take a sum and consider the sum or integral at infinity. The reverse is possible, where we take a sum or integral at infinity and construct a sum from a sequence.

Provided the sequence is monotonic and does not contain singularities in $*G$, the same convergence or divergence properties are retained.

This is an example of a transfer from a point to an interval, as we extend from one space into another.

Theorem 11.1. *Transference from "convergence sums" to sums.*

$$\sum a_n|_{n=\infty} \mapsto \sum_{k=k_0}^{\infty} a_k$$

$$\int a(n)\,dn|_{n=\infty} \mapsto \int_{x_0}^{\infty} a(n)\,dn$$

Proof. By Theorems 11.2, 11.3, 11.4. □

Theorem 11.2. *If $\sum a_n|_{n=\infty} = 0$ converges and $(a_k)|_{k=k_0}^{\infty}$ exists and is not an infinity then $\sum_{k=k_0}^{\infty} a_k$ converges.*

Proof. Since (a_k) does not diverge, a finite sum of its terms do not diverge. Since the sum of the tail is an infinitesimal, then we can construct the stated sum, by Theorem 2.1. □

Theorem 11.3. *If $\int a(x)\,dx|_{n=\infty} = 0$ converges and function $a(x)$ exists and is not an infinity then $\int_{x_0}^{\infty} a(x)\,dn$ converges.*

Proof. Since the continuous function $a(x)$ does not diverge, its finite integral does not diverge. Since the integral of the tail is an infinitesimal, then we can construct the stated integral, by Theorem 2.2. □

Theorem 11.4. *If $\sum a_n|_{n=\infty} = \infty$ or $\int a(n)\,dn|_{n=\infty} = \infty$ diverge we can construct a respective sum $\sum_{k=k_0}^{\infty} a_k$ or integral $\int_{x_0}^{\infty} a(x)\,dx$ that diverges.*

Proof. A point that does not exist leads to a diverging sum or integral. If the sequence or interval exist, then a sum or integral with a finite part and an infinite part can be constructed. Since the tail is an infinity, and the finite part of the sum added to the tail is still an infinity, hence as expected, the sum or integral will diverge. □

We can show the theory of convergence sums includes error analysis.

Definition 11.1. *Rate of convergence of positive series is the ratio of the partial sums.*

Theorem 11.5. *Rate of convergence of the positive convergent sum* $\sum a_n|_{n=\infty}$ *is* $\frac{a_{n+1}}{a_n}|_{n=\infty}$.

Proof. Let $s_n = \sum a_n|_{n=\infty}$ be a convergent sum. $\frac{s_{n+1}}{s_n} = \frac{\sum a_{n+1}}{\sum a_n}|_{n=\infty} = \frac{\int a(n+1)\,dn}{\int a(n)\,dn}|_{n=\infty}$ is of the form $\frac{0}{0}$ as $\sum a_n|_{n=\infty} = 0$ is convergent. Use L'Hopital's [6, Part 5] rule to differentiate. $\frac{\int a(n+1)\,dn}{\int a(n)\,dn}|_{n=\infty} = \frac{a(n+1)}{a(n)}|_{n=\infty} = \frac{a_{n+1}}{a_n}|_{n=\infty}$. $\square$

Example 11.2. *Determine the rate of convergence of* $\sum \frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}}|_{n=\infty}$ *[24].*

Let $a_n = \frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}}$, $\frac{a_{n+1}}{a_n}|_{n=\infty} = \frac{(4(n+1))!(1103+26390(n+1))}{((n+1)!)^4 396^{4(n+1)}} \frac{(n!)^4 396^{4n}}{(4n)!(1103+26390n)}|_{n=\infty} = \frac{(4(n+1))!}{((n+1)!)^4 396^4} \frac{(n!)^4}{(4n)!}|_{n=\infty}$ $= \frac{(4n+4)(4n+3)(4n+2)(4n+1)(4n)!}{(n+1)^4 (n!)^4 396^4} \frac{(n!)^4}{(4n)!}|_{n=\infty} = \frac{4^4}{396^4} = \frac{1}{99^4} = 1.041020 \times 10^{-8}$ *Hence eight decimal digits per iteration.*

References

[1] L. Bibiloni, P. Viader, and J. Paradis, *On a Series of Goldbach and Euler*, The Mathematical Association of America Monthly 113, March 2006.

[2] C. D. Evans, W. K. Pattinson, *Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 1 Gossamer numbers*

[3] C. D. Evans, W. K. Pattinson, *Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 2 The much greater than relations*

[4] C. D. Evans, W. K. Pattinson, *Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 3 Comparing functions*

[5] C. D. Evans, W. K. Pattinson, *Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 4 The transfer principle*

[6] C. D. Evans, W. K. Pattinson, *Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 5 Non-reversible arithmetic and limits*

[7] C. D. Evans, W. K. Pattinson, *Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 6 Sequences and calculus in *G*

[8] C. D. Evans and W. K. Pattinson, *The Boundary test for positive series*

[9] C. D. Evans and W. K. Pattinson, *Convergence sums and the derivative of a sequence at infinity*

[10] C. D. Evans and W. K. Pattinson, *Ratio test and a generalization with convergence sums*

[11] C. D. Evans and W. K. Pattinson, *Rearrangements of convergence sums at infinity*

[12] C. D. Evans and W. K. Pattinson, *The Fundamental Theorem of Calculus with Gossamer numbers*

[13] C. D. Evans and W. K. Pattinson, *Power series convergence sums*

[14] T. M. Apostol, *Calculus*, Vol. 1, Second Edition, Xerox College Publishing.

[15] G. H. Hardy, *Orders of infinity*, Cambridge Univ. Press, 1910.

[16] L. Euler, *De progressionibus harmonicis observationes*, Opera Omnia, ser. I, vol. 14, Teubner, Leipzig, 1925, pp.87–100.

[17] W.J. Kaczor, M. T. Nowak, *Problems in Mathematical Analysis I*, AMS, Providence , 2000.

[18] E. Kreyszig, *Advanced Engineering Mathematics*, 7th edition, John Wiley & Sons.

[19] W.J.Kaczor and M.T. Norwak, *Problems in Mathematical Analysis II Continuity and Differentiation*, AMS, Providence, 2000

[20] D. Laugwitz, *Theory of Infinitesimals. An Introduction to Nonstandard Analysis*, Accademia Nazionale dei Lincei, Roma, 1980

[21] From MathWorld–A Wolfram Web Resource, *Bertrand's Test*, available at `http://mathworld.wolfram.com/BertrandsTest.html`

[22] Konrad Knopp, *Infinite sequences and series* (F. Bagemihl Trans.), New York Dover Publications, 1956

[23] *Is the sum of all natural numbers -1/12*, `http://math.stackexchange.com/questions/633285/is-the-sum-of-all-natural-numbers-frac112`

[24] *Ramanujan's approximation for* π, `http://math.stackexchange.com/questions/908535/ramanujans-approximation-for-pi`

[25] G Varanenkov, et al, *Problems in Mathematical Analysis*(B. Demidovich editorship)(G. Yankovsky Trans.), Mir 1976, Seventh printing 1989.

RMIT University, GPO Box 2467V, Melbourne, Victoria 3001, Australia
chelton.evans@rmit.edu.au