The Boundary test for positive series

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Abstract

With convergence sums, a universal comparison test for positive series is developed, which compares a positive monotonic series with an infinity of generalized p-series. The boundary between convergence and divergence is an infinity of generalized p-series. This is a rediscovery and reformation of a 175 year old convergence/divergence test.

1 Introduction

1. Introduction
2. Generalized p-series
3. The existence of the boundary and tests
4. The boundary test Examples
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We see the boundary test as the most general and powerful of all convergence tests. A tool for other tests and theory, large or small.

The boundary test we believe to be the most general positive series test. Both du Bois-Reymond's theory, and described by Hardy, the comparison of functions were forgotten largely because the theory had no perceived applications. The rediscovery of this test should place it as something which has been missing from convergence and divergence theory.

Definition 1.1. $L_w = \prod_{k=0}^{w} \ln_j$, $L_w = 1$ *when* $w < 0$

> Du Bois-Reymond seems to have been led to consider this ordering of functions by way of an attempt to construct an ideal series or integral which would serve as a boundary between convergent and divergent series or integrals, based on BERTRAND'S series. These are sometimes called ABEL'S series ... series of the form $\sum \frac{1}{L_{w-1} \ln_w^p}$ [12, p.103].

Generally the convergence sums [5] and comparison is in $*G$. When we say a sum at infinity is 0, the transfer $\Phi \mapsto 0$ has been applied. When we say a sum at infinity is infinity, the transfer $\Phi^{-1} \mapsto \infty$ was applied if the sum was defined.

2 Generalized p-series

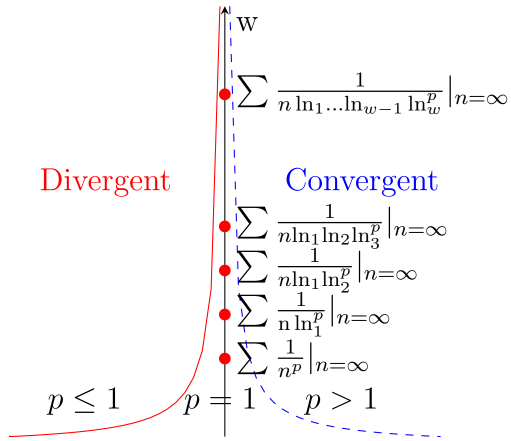

Figure 1: Generalized p-series between convergence/divergence

We discussed the possibility of two straight lines approaching each other, with two possibilities at infinity. The lines never meet, or the lines meet at infinity. We find a more complex case where two classes of sums about $p = 1$, having their functions close, never meet, one sums to infinity, the other zero.

The following discussion is on the generalization of the p-series through an observation and investigation. While the results are known, they are expressed with infinitary calculus at infinity.

It so happens that the p-series has for all values $p > 1$ the series converges, and all values $p \leq 1$ the series diverges.

Theorem 2.1. *If* $p \leq 1 \Rightarrow \sum \frac{1}{n^p}|_{n=\infty} = \infty$ *diverges. If* $p > 1 \Rightarrow \sum \frac{1}{n^p}|_{n=\infty} = 0$ *converges.*

Definition 2.1. *Let* k*-nested natural logarithms be represented by* $\ln_k(x)$*:* $\ln_0 x = x$, $\ln_k x = \ln(\ln_{k-1} x)$*. For convenience, let* $\ln_k$ *without an argument mean* $\ln_k(n)$*.*

Theorem 2.2. *Let* $f(w,n) = \int \frac{1}{\prod_{k=0}^{w-1} \ln_k(n)} \cdot \frac{1}{(\ln_w(n))^p} dn$*. When* $w \geq 1$ *then* $f(w,n) = f(w-1, \ln n)$ *and* $(f(w,n) = f(0,n) = \int \frac{1}{t^p} dt)|_{n=\infty}$ *where* $t = \ln_w n|_{n=\infty}$*.*

Proof. Let $n = e^v$, $\frac{dn}{dv} = e^v$. $f(w,n) = \int \frac{1}{n} \frac{1}{\prod_{k=1}^{w-1} \ln_k(n)} \cdot \frac{1}{(\ln_w(n))^p} \frac{dn}{dv} dv = \int \frac{1}{n} \frac{1}{\prod_{k=1}^{w-1} \ln_k(e^v)} \cdot \frac{1}{(\ln_w(e^v))^p} e^v dv = \int \frac{1}{n} \frac{1}{\prod_{k=1}^{w-1} \ln_{k-1}(v)} \cdot \frac{1}{(\ln_{w-1}(v))^p} n dv = \int \frac{1}{\prod_{k=1}^{w-1} \ln_{k-1}(v)} \cdot \frac{1}{(\ln_{w-1}(v))^p} dv = \int \frac{1}{\prod_{k=0}^{w-2} \ln_k(v)} \cdot \frac{1}{(\ln_{w-1}(v))^p} dv = f(w-1, v)$

Apply $f(w,n) = f(w-1, \ln n)$, w times to remove $\frac{1}{\prod_{k=0}^{w-1} \ln_k}$ inside the integral. $(f(w,n) = f(0,n) = \int \frac{1}{t^p} dt)|_{n=\infty}$, $t = \ln_w n|_{n=\infty}$. □

Theorem 2.3.

$$\int \frac{1}{L_w} dn = \ln_{w+1}|_{n=\infty}$$

Proof. Substitute $p = 0$ and $w = w + 1$ into Theorem 2.2 then $t = \ln_{w+1} n|_{n=\infty}$ and $f(0, n)|_{n=\infty} = \int \frac{1}{t^p} dt|_{n=\infty} = t|_{n=\infty} = \ln_{w+1}|_{n=\infty}$ □

Since the divergence of the sum is the same as the divergence of the integral, by Theorem 2.3 the following generalization of the harmonic series always diverges.

$$(\sum \frac{1}{n}, \sum \frac{1}{n \ln n}, \sum \frac{1}{n \ln n \ln_2}, \sum \frac{1}{n \ln \ln_2 \ln_3}, \ldots)|_{n=\infty} \text{ sums diverge}$$

We could have reasoned the above by considering scales of infinities [2]. $L_0 \prec L_1 \prec L_2 \prec L_3 \prec \ldots|_{n=\infty}$, $\frac{1}{L_0} \succ \frac{1}{L_1} \succ \frac{1}{L_2} \succ \frac{1}{L_3} \succ \ldots|_{n=\infty}$, assuming $f \succ g$ then $\sum f \succ \sum g$, $\sum \frac{1}{L_0} \succ \sum \frac{1}{L_1} \succ \sum \frac{1}{L_2} \succ \sum \frac{1}{L_3} \succ \ldots|_{n=\infty}$ which shows the sums diverging more slowly.

Definition 2.2. *Let* $\sum \frac{1}{\prod_{k=0}^{w-1} \ln_k \cdot \ln_w^p}$ *or* $\int \frac{1}{\prod_{k=0}^{w-1} \ln_k \cdot \ln_w^p} dn|_{n=\infty}$ *be called the generalized p-series.*

Theorem 2.4.

$$\sum \frac{1}{\prod_{k=0}^{w-1} \ln_k \cdot \ln_w^p}|_{n=\infty} = \begin{cases} 0 & \textit{converges when } p > 1 \\ \infty & \textit{diverges when } p \leq 1 \end{cases}$$

Proof. By the integral theorem the convergence and divergence of the series is the same as the respective integral. $\sum \frac{1}{\prod_{k=0}^{w-1} \ln_k \cdot \ln_w^p} = \int \frac{1}{\prod_{k=0}^{w-1} \ln_k(n)} \cdot \frac{1}{(\ln_w(n))^p} dn|_{n=\infty} = f(w, n)$

Apply Theorem 2.2, $f(w, n) = f(w-1, \ln n)$, w times to remove $\frac{1}{\prod_{k=0}^{w-1} \ln_k(n)}$ inside the integral.

$(f(w, n) = f(0, n) = \int \frac{1}{t^p} dt)|_{n=\infty}$, $t = \ln_w n$, which is known to converge when $p > 1$ and diverge when $p \leq 1$. Hence the generalized p-series proved. □

3 The existence of the boundary and tests

Arguments for and against the boundary have existed from its inception, when it was realized that no one series could partition all positive series into either converging or diverging series.

> KNOPP says, for example, that it is clear "that it is quite useless to attempt to introduce anything of the nature of a boundary between convergent and divergent series, as was suggested by P. DU BOIS-REYMOND ... in whatever manner we may choose to render it precise, it will never correspond to the actual circumstances" [KNOPP 1954, 313 or 1951, 304; his emphasis]. [12, p.135]

While teaching (see [11]) comments.

> Recently one of our students remarked that the harmonic series $\sum \frac{1}{n}$ acts as a boundary between convergence and divergence.

This was not that different from the historical account where new tests emerged, and was contradicted by Abel, $\sum \frac{1}{n \ln n}$ diverges.

The series was generalized to the generalized p-series (Section 2), and considered as a boundary between convergence and divergence.

However, ironically it was du Bois-Reymond who disproved the boundary [12, p.103].

> One can ask whether or not the convergence or divergence of all series with positive terms can be settled by comparison with a real multiple of one of these series. The answer is no, and this was first shown by DU BOIS-REYMOND [DU BOIS-REYMOND 1873, 88-91].

See Theorem 6.3 and Theorem 6.1. Hardy summarizes (see [10, pp.67–68]).

> Given any divergent series we can always find one more slowly divergent. ... given any convergent series, we can find one more slowly convergent.

A. Pringsheim was of the same opinion.

> The analogy with the irrational numbers is a logical blunder; one can insert between the elements of the two classes defined by $x^2 < 2$ and $x^2 > 2$ a new thing corresponding to the relation $x^2 = 2$, but between the convergence and divergence of positive series, there is no such "third". [12, p.151]

However, the concept of an ideal function did not go away.

> BOREL says "we know that there is no function of n, in the ordinary sense of the word, which has this property; that is, we call $O(n)$ an ideal function; it is not a true function" [BOREL 1946, 148]. But there it is, nevertheless. PRINGSHEIM'S attempts to argue and ridicule it out of existence failed to sway BOREL. [12, p.147]

Conjecture 3.1. *While one series cannot be the boundary, what is to say that an infinite collection of series cannot form the boundary.*

While one series alone cannot separate all convergent and divergent series, Theorem 6.1 and Theorem 6.3 do not say anything about an infinity of such series. Indeed the generalized p-series keeps decreasing its terms in size. In other words an appeal that there is no smallest series fails on a collection of infinitely smaller term series, is invalid.

Surprisingly, after deriving a general boundary test which is the subject of this paper, it was found in fact to be a rediscovery.

The test was referenced in the appendix (not an important place for a major test) in Hardy's Orders of Infinity, with no example and limited description. Hardy does cite other references, referring to the 'logarithmic criteria' by De Morgan, attributing the criteria in 1839 [10, p.67]. Further, the test does not appear as a general test in the current known convergent tests.

However, we believe this is the long sort after universal test for a positive series either converging or diverging.

The test given by Hardy was incorrect for the divergent series case, possibly a typing error.

Theorem 3.1. ***The logarithmic test*** *[10, Appendix II p.66] with correction.*

The series $\sum a_n \ \ (a_n \geq 0)$

$$\textit{is convergent if } a_n \preceq \frac{1}{\prod_{k=0}^{w-1} \ln_k \cdot (\ln_w)^{1+\alpha}} \textit{ where } \alpha > 0,$$

$$\textit{and divergent if } a_n \succeq \frac{1}{\prod_{k=0}^{w} \ln_k}$$

The integral $\int^{\infty} f(x)\,dx \ \ (f \geq 0)$

$$\textit{is convergent if } f(x) \preceq \frac{1}{\prod_{k=0}^{w-1} \ln_k x \cdot (\ln_w x)^{1+\alpha}} \textit{ where } \alpha > 0,$$

$$\textit{and divergent if } f(x) \succeq \frac{1}{\prod_{k=0}^{w} \ln_k x}$$

We developed a detailed convergence criteria E3 [5] where comparing a sum we are able to remove the sum/integral and compare the corresponding monotonic functions. In the reformed test, a direct comparison with the boundary is made.

$$\sum a_n \ z \ \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$$

$$a_n \ z \ \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$$

By expressing the boundary and comparison differently, in the original test the two functions (1): $\frac{1}{\prod_{k=0}^{w-1} \ln_k x \cdot (\ln_w x)^{1+\alpha}}$ convergent and asymptotic to the boundary, and (2): $\frac{1}{\prod_{k=0}^{w} \ln_k}$ the p-series on the boundary in Theorem 3.1 are rephrased to compare against the p-series function (2) only, see Theorem 3.2. Also, the relation being solved for is a simpler relation. Only one relation needs to be solved for, not two.

We also needed an algebra for comparing functions to make the boundary test usable.

Some further differences in the theorems, $\succeq$ in the logarithmic test is replaced by $\geq$ in the boundary test. This can be explained where the original test is considered from a finite perspective, and $\succeq$ removes the transient sum terms before reaching infinity. By considering the series at infinity and requiring a monotonic sequence of terms this was avoided.

Remark: 3.1. *By considering different rearrangements and intervals, we can deform a_n into a monotonic sequence [8] comparable with the generalized p-series. If this is not possible then by definition the series or integral diverges. Similarly with functions, $a(n)$ can be deformed.*

Theorem 3.2. ***The boundary test***

$$\sum a_n \ z \sum \frac{1}{\prod_{k=0}^{w} \ln_k n}\Big|_{n=\infty}, \quad z = \begin{cases} < & \text{then } \sum a_n|_{n=\infty} = 0 \text{ is convergent,} \\ \geq & \text{then } \sum a_n|_{n=\infty} = \infty \text{ is divergent.} \end{cases}$$

$$\int a(n)\,dn \ z \int \frac{1}{\prod_{k=0}^{w} \ln_k n}\,dn\Big|_{n=\infty}, \quad z = \begin{cases} < & \text{then } \int a(n)\,dn|_{n=\infty} = 0 \text{ is convergent,} \\ \geq & \text{then } \int a(n)\,dn|_{n=\infty} = \infty \text{ is divergent.} \end{cases}$$

Since w is a fixed integer, there is an infinity of tests. By removing the sum and solving the comparison of functions a unique w is found.

Proof. $\sum a_n \ z \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$, $\int a(n)\,dn \ z \int \frac{1}{\prod_{k=0}^{w} \ln_k} dn|_{n=\infty}$, differentiate, $a(n) \ z \ \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$, by Theorem 3.3 solve for $z = \{<, \geq\}$. □

Theorem 3.3. ***The boundary test comparison***

$$a_n \ z \ \frac{1}{\prod_{k=0}^{w} \ln_k n}\Big|_{n=\infty}, \quad z = \begin{cases} < & \text{then } \sum a_n|_{n=\infty} = 0 \text{ is convergent,} \\ \geq & \text{then } \sum a_n|_{n=\infty} = \infty \text{ is divergent.} \end{cases}$$

$$a(n) \ z \ \frac{1}{\prod_{k=0}^{w} \ln_k n}\Big|_{n=\infty}, \quad z = \begin{cases} < & \text{then } \int a(n)\,dn|_{n=\infty} = 0 \text{ is convergent,} \\ \geq & \text{then } \int a(n)\,dn|_{n=\infty} = \infty \text{ is divergent.} \end{cases}$$

Solving the comparison of functions a unique w is found.

Proof. In [7, Corollary 3.1] we show the equivalence between the boundary test and the generalized ratio test. If we consider the generalized ratio test is proved, since we establish the equivalence the boundary test would consequently be proved.

We believe the generalized ratio test is proved, though Knopp [13, p.129] did not cite the proof. Instead he referred to the generalized p-series as criteria. Since they are proved, this is not contradictory, as a criteria are the assumptions, that which is held to be true. □

Proof. The boundary is equivalent to the generalized p-series test Theorem 2.1. By solving for the relation, the appropriate p-series may be found and tested against.

In general, non-reversible arguments of magnitude are used to reduce lower order terms. This is required to meet the convergence criteria E3 [5]. □

Remark: 3.2. *If a circular argument occurs, then additional information, perhaps an identity may be required to be found to solve for the relation. Similarly further conditions may be input if the problem is ill posed or incomplete.*

Remark: 3.3. *If a positive sum is less than the boundary the sum converges; else the sum diverges.*

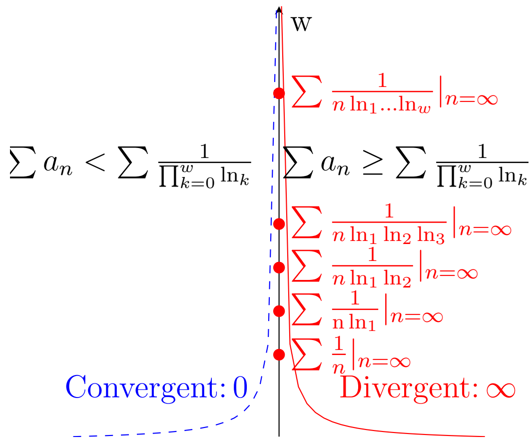

Figure 2: Series $\sum a_n|_{n=\infty}$ compared with the boundary

The boundary test has a convergence criterion specifically for evaluating sums as either 0 or ∞. The logarithmic test compares the inner components.

Proposition 3.1. *The Logarithmic test Theorem 3.1 implies the Boundary test Theorem 3.2*

Proof. By removing the transients, the conditions are simplified.

Divergent case. $a_n \succeq \frac{1}{L_w}$, $M_1 \in \mathbb{R}^+$, $\exists M_1$: $M_1 a_n \geq \frac{1}{L_w}$, $M_1 a_n L_w \geq 1|_{n=\infty}$, $\ln M_1 + \ln a_n + \ln L_w \geq 0|_{n=\infty}$, $\ln a_n + \ln L_w \geq 0|_{n=\infty}$ as $\ln M_1 \prec \ln a_n|_{n=\infty}$, $a_n L_w \geq 1|_{n=\infty}$, $a_n \geq \frac{1}{L_w}|_{n=\infty}$.

Convergent case. Let $\alpha > 0$, $a_n \preceq \frac{1}{L_{w-1} \ln_w^{1+\alpha}}$, $M_2 \in \mathbb{R}^+$, $\exists M_2$: $a_n \leq M_2 \frac{1}{L_{w-1} \ln_w^{1+\alpha}}$, $a_n M_2^{-1} L_{w-1} \ln_w^{1+\alpha} \leq 1$, $\ln a_n - \ln M_2 + \ln L_{w-1} + \ln \ln_w^{1+\alpha} \leq 0$, $\ln a_n + \ln L_{w-1} + \ln \ln_w^{1+\alpha} \leq 0$ as $\ln a_n \succ \ln M_2$, $a_n \leq \frac{1}{L_{w-1} \ln_w^{1+\alpha}}$, since $\frac{1}{L_{w-1} \ln_w^{1+\alpha}} < \frac{1}{L_{w-1} \ln_w^{1}}$ then $a_n < \frac{1}{L_w}$.

Both cases are disjoint and cover the line. The conditions correspond exactly with the boundary test. □

The way the logarithmic test was structured shows that there was no explicit separation between finite and infinite numbers. With the removal of the transients there is no need to solve for M_1 and M_2.

> It is a reasonable question to ask how the most important and general positive series convergence and divergence test was found and lost.

Hardy was of the belief that du Bois-Reymond's theory was highly original, but served no purpose. Du Bois-Reymond was looking for a theory of the continuum. It may well be a case of once again, a theoretical piece of mathematics that appears utterly useless, even after having been considered, becomes essential to our understanding.

We have arrived at this view by several steps, a separation between finite and infinite numbers [4], a number system with infinities [1], an algebra for comparing functions [3] and convergence sums [5].

On the importance of the logarithmico-exponential scales, to argue for the generality of the test, Hardy comments:

> No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms. [10, p.48]

Since the boundary test is a comparison against the whole logarithmico-exponential scale, then if what Hardy said is true; that all such testable functions can be expressed in logarithmico-exponential scale, then as the boundary test compares against this scale, all functions as said are able to be compared with the boundary by the boundary test. The boundary test, with the monotonic constraint is theoretically complete.

As discussed in [8], the theory of convergence sums is extended to include testing for non-monotonic sequences. By considering arrangements, convert non-monotonic series to monotonic series for convergence testing, thereby increasing the classes of series which can be tested. It follows that this makes the boundary test more general.

As a consequence of Theorem 3.2: If a sum is less than the boundary at infinity then the sum converges. Symbolically, if $\sum a_n < \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$ then $\sum a_n|_{n=\infty} = 0$ converges. Therefore when solving $\sum a_n \; z \; \frac{1}{\prod_{k=0}^{w} \ln_k}$, if we solve for z and find $z = <$, it immediately follows that $\sum a_n|_{n=\infty} = 0$ converges.

4 The boundary test Examples

Example 4.1. $\sum \frac{1}{n^2}|_{n=\infty}$ *is the p-series with* $p = 2$ *and is known to converge. Testing this series against the boundary.*

$$\sum \frac{1}{n^2} \;\; z \;\; \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$$
$$\frac{1}{n^2} \;\; z \;\; \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$$
$$\prod_{k=0}^{w} \ln_k \;\; z \;\; n^2|_{n=\infty}$$
$$\sum_{k=0}^{w} \ln_{k+1} \;\; (\ln z) \;\; 2\ln n|_{n=\infty}$$
$$0 \;\; (\ln z) \;\; 2\ln n|_{n=\infty} \qquad (\text{as } \ln n \succ \ln_{k+1})$$
$$0 < 2\ln n|_{n=\infty} \qquad (\ln z = <,\ z = e^{<} = <)$$
$$\sum \frac{1}{n^2} < \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty} \qquad (\textstyle\sum a_n|_{n=\infty} = 0 \text{ converges})$$

The boundary test can handle products such as $n!$ by converting them to sums via the log operation, and integrating the function in the continuous domain.

Example 4.2. *Determine convergence or divergence of* $\sum \frac{1}{n!}|_{n=\infty}$.

$\sum \frac{1}{n!}\ z\ \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$, $\prod_{k=0}^{w} \ln_k\ z\ n!|_{n=\infty}$, $\sum_{k=1}^{w+1} \ln_k\ (\ln z)\ \sum_{k=1}^{n} \ln k|_{n=\infty}$, $\sum_{k=1}^{w+1} \ln_k\ (\ln z)\ \int^n \ln n\, dn|_{n=\infty}$, $\sum_{k=1}^{w+1} \ln_k\ (\ln z)\ n \ln n|_{n=\infty}$, $\ln n\ (\ln z)\ n\ln n|_{n=\infty}$, $\ln z = <$, $z = e^{<} = <$, $\sum \frac{1}{n!}|_{n=\infty} = 0$ *converges.*

Example 4.3. *Determine the convergence/divergence of* $\sum \frac{n!}{(2n)!}|_{n=\infty}$

This problem would normally be done more simply with the ratio or comparison test.

$$
\begin{aligned}
\sum \frac{n!}{(2n)!} \; z \; \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty} & \qquad \text{(Compare against the boundary)}\\
\frac{n!}{(2n)!} \; z \; \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty} & \\
n! \prod_{k=0}^{w} \ln_k \; z \; (2n)!|_{n=\infty} & \\
\ln(n! \prod_{k=0}^{w} \ln_k) \; (\ln z) \; \ln (2n)!|_{n=\infty} & \\
\sum_{k=1}^{n} \ln k + \sum_{k=1}^{w+1} \ln_k \; (\ln z) \; \sum_{k=1}^{2n} \ln k|_{n=\infty} & \qquad \text{(Apply log law, convert products to sums)}\\
\int_1^n \ln x dx + \sum_{k=1}^{w+1} \ln_k \; (\ln z) \; \int_1^{2n} \ln x dx|_{n=\infty} & \qquad \text{(discrete to a continuous domain)}\\
\int^n \ln x dx + \sum_{k=1}^{w+1} \ln_k \; (\ln z) \; \int^{2n} \ln x dx|_{n=\infty} & \\
n \ln n + \sum_{k=1}^{w+1} \ln_k \; (\ln z) \; (2n)\ln(2n)|_{n=\infty} & \\
n \ln n + \ln n \; (\ln z) \; (2n)\ln(2n)|_{n=\infty} & \qquad \text{(largest boundary)}\\
n \ln n \; (\ln z) \; (2n)\ln(2n)|_{n=\infty} & \qquad (\ln n \prec n \ln n, \; 2n \prec 2n \ln(2n)|_{n=\infty})\\
\ln z = \prec, \quad z = e^{\prec} = < \quad \textit{converges} &
\end{aligned}
$$

Example 4.4. *Determine convergence or divergence of* $\frac{1}{\ln_2 3} + \frac{1}{\ln_2 4} + \frac{1}{\ln_2 5} + \dots$

$\frac{1}{\ln_2 n} \; z \; \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$, $\prod_{k=0}^{w} \ln_k \; z \; \ln_2|_{n=\infty}$, $\sum_{k=1}^{w+1} \ln_k \; (\ln z) \; \ln_3|_{n=\infty}$, $\ln_1 \geq \ln_3|_{n=\infty}$, $\ln z = \geq$, $z = e^{\geq} = \geq$ *and the series diverges.*

Example 4.5. *Ramanujan gave the following* $\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$. *Although the following does not calculate the sum, by comparing against the boundary we show the sum converges.*

$$\sum \frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}} \; z \; \sum \frac{1}{\prod_{k=0}^{w} \ln_k n}|_{n=\infty} \qquad \text{(remove sum, cross multiply)}$$

$$(4n)!(26390n) \prod_{k=0}^{w} \ln_k n \; z \; (n!)^4 396^{4n}|_{n=\infty} \qquad \text{(undo multiplication)}$$

$$\ln(4n)! + \ln n + \sum_{k=1}^{w+1} \ln_k n \; (\ln z) \; 4\ln(n!) + 4n \ln 396|_{n=\infty} \qquad (w = 0 \text{ largest left side})$$

$$\sum_{k=1}^{4n} \ln k + 2\ln n \; (\ln z) \; 4\sum_{k=1}^{n} \ln k + 4n \ln 396|_{n=\infty} \qquad (\textstyle\int \ln n \, dn = n \ln n|_{n=\infty})$$

$$4n \ln 4n + 2\ln n \; (\ln z) \; 4n \ln n + 4n \ln 396|_{n=\infty}$$

$$4n \ln n + \ln 4 \cdot 4n + 2\ln n \; (\ln z) \; 4n \ln n + 4n \ln 396|_{n=\infty}$$

$$\ln 4 \cdot 4n + \; (\ln z) \; 4 \cdot \ln 396 \cdot n|_{n=\infty}$$

$$(\ln z) = <, \; z = < \qquad \text{(sum converges)}$$

5 Convergence tests

The boundary test can be used to prove other convergence tests. In particular the generalized ratio test [7]: which uses the boundary test to prove the ratio test, Raabe's test, Bertrand's test, as a consequence of proving the more general test, the generalized ratio test.

These other tests require a rearrangement of the sequence at infinity, and hence is a discussion for another paper, where the sequence will be rearranged before input into the general boundary test. The nth root test(see [5, Section 8.20])

Theorem 5.1. *If* $|a_n|^{\frac{1}{n}}|_{n=\infty} < 1$ *then* $\sum a_n|_{n=\infty} = 0$ *converges.*

Proof. Comparing against the boundary with the known convergent condition, $\sum |a_n| < \sum \frac{1}{\prod_{k=0}^{w} \ln_{\mathrm{k}}}|_{n=\infty}$ then $\sum |a_n||_{n=\infty} = 0$ converges. Removing the sum. $|a_n| < \frac{1}{\prod_{k=0}^{w} \ln_{\mathrm{k}}}|_{n=\infty}$, $|a_n|^{\frac{1}{n}} < \frac{1}{(\prod_{k=0}^{w} \ln_{\mathrm{k}})^{\frac{1}{n}}}|_{n=\infty}$, $|a_n|^{\frac{1}{n}} < \frac{1}{\prod_{k=0}^{w} (\ln_{\mathrm{k}}{}^{\frac{1}{n}})}|_{n=\infty}$. Since $n^{\frac{1}{n}} = (\ln_0)^{\frac{1}{n}} = 1$ then $(\ln_{\mathrm{k}})^{\frac{1}{n}}|_{n=\infty} = 1$, $|a_n|^{\frac{1}{n}} < 1|_{n=\infty}$ □

L'Hopital's convergence test [5, Section 8.9].

Theorem 5.2. *If* $\frac{f}{g} = \frac{f'}{g'}|_{n=\infty}$, *where* f *and* g *are in indeterminate form* ∞/∞ *or* $0/0$ *then* $\sum \frac{f}{g} = \sum \frac{f'}{g'}|_{n=\infty}$

Proof. $\sum \frac{f}{g} \; z \; \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$, $\frac{f}{g} \; z \; \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$, $\frac{f'}{g'} \; z \; \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$, $\sum \frac{f'}{g'} \; z \; \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$. Since both sums have the same relation then $\sum \frac{f}{g} = \sum \frac{f'}{g'}|_{n=\infty}$ □

Theorem 5.3. *The power series test(see [9]). If $|x| < 1$ then $\sum x^n|_{n=\infty} = 0$ converges.*

Proof. By the absolute value convergence test, consider positive x only. Compare against the boundary. $\sum x^n \; z \; \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$, $x^n \; z \; \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$, $x^n \prod_{k=0}^{w} \ln_k|_{n=\infty} \; z \; 1$, $n \ln x + \sum_{k=1}^{w+1} \ln_k|_{n=\infty} \; (\ln z) \; 0$, $(\ln z) = <$ only when $\ln x$ is negative, $0 < x < 1$. Reversing the process, $n \ln x + \sum_{k=1}^{w+1} \ln_k|_{n=\infty} < 0$, $x^n \prod_{k=0}^{w} \ln_k|_{n=\infty} < 1$, $x^n < \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$, $\sum x^n < \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$, as less than the boundary the sum converges. □

Theorem 5.4. *If $a \succ\!\succ b$ then $\sum ab|_{n=\infty} = \sum a|_{n=\infty}$*

Proof. By definition $a \succ\!\succ b$ then $\ln a + \ln b = \ln a$. Consider $\sum ab \; z \; \sum \frac{1}{\prod_{k=0}^{w}}|_{n=\infty}$, $ab \prod_{k=0}^{w} \ln_k \; z \; 1|_{n=\infty}$, $\ln a + \ln b + \sum_{k=1}^{w+1} \ln k \;\; (\ln z) \;\; 0|_{n=\infty}$, $\ln a + \sum_{k=1}^{w+1} \ln k \;\; (\ln z) \;\; 0|_{n=\infty}$, reverse the process, $\ln(a \prod_{k=0}^{w} \ln_k) \; (\ln z) \; \ln 1|_{n=\infty}$, $a \prod_{k=0}^{w} \ln_k \; z \; 1|_{n=\infty}$, $\sum a \; z \; \sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty}$. □

Example 5.1. $\sum \frac{n^{\frac{1}{n}}}{n^2}|_{n=\infty}$ *We can immediately observe $n^{\frac{1}{n}} \prec\!\prec n^2|_{n=\infty}$ if we recognise $n^{\frac{1}{n}}|_{n=\infty}$ as a constant. Then apply the theorem $\sum \frac{n^{\frac{1}{n}}}{n^2}|_{n=\infty} = \sum \frac{1}{n^2}|_{n=\infty} = 0$ converges. Alternatively, consider $n^{\frac{1}{n}} \; z \; n^2|_{n=\infty}$, $\frac{1}{n} \ln n \; (\ln z) \; 2 \ln n|_{n=\infty}$, $\frac{1}{n} \; (\ln z) \; 2|_{n=\infty}$, $\frac{1}{n} \prec 2|_{n=\infty}$, then $n^{\frac{1}{n}} \prec\!\prec n^2|_{n=\infty}$.*

The following are references to the boundary test being applied to solve convergence tests in other papers.

- Alternating convergence test (see [5, 8.10]), proved in [6, Convergence tests].
- Generalized ratio test [7]

6 Representing convergent/divergent series

As the boundary test is complete (Theorem 3.2), since the boundary separates between convergence and divergence, for a given series, a lower bound for divergent series and an upper bound for convergent series exists.

Although this may seem a trivial rearrangement of the boundary test, we show how we may isolate and describe classes of convergence and divergence from the original test.

For example, the idea of a class of series that is convergent to the boundary, but is also convergent may seem counter intuitive.

Example 6.1. *$\frac{1}{n^2}$ is bounded above, $\frac{1}{n^2} < \frac{1}{n^p}$ when $1 < p < 2$, which is known to converge. Applying a comparison, $\frac{1}{n^2} < \frac{1}{n^p}|_{n=\infty}$, $\sum \frac{1}{n^2} < \sum \frac{1}{n^p}|_{n=\infty}$, $\sum \frac{1}{n^2} \leq 0$, $\sum \frac{1}{n^2} = 0$.*

Example 6.2. *$\frac{1}{n\,(\ln n)^{\frac{1}{2}}}$ is bounded below, $\frac{1}{n \ln n} < \frac{1}{n\,(\ln n)^{\frac{1}{2}}} < \frac{1}{n}$ then $\frac{1}{n \ln n}$ is the first discrete lower bound of the boundary, $w = 1$. Applying a comparison, $\frac{1}{n \ln n} < \frac{1}{n\,(\ln n)^{\frac{1}{2}}}|_{n=\infty}$, $\sum \frac{1}{n \ln n} < \sum \frac{1}{n\,(\ln n)^{\frac{1}{2}}}|_{n=\infty}$, $\infty \leq \sum \frac{1}{n\,(\ln n)^{\frac{1}{2}}}|_{n=\infty}$, $\sum \frac{1}{n\,(\ln n)^{\frac{1}{2}}}|_{n=\infty} = \infty$ diverges.*

Remark: 6.1. *All positive monotonic divergent series are bounded by the boundary below.*

Lemma 6.1. *For monotonic and diverging series $\sum a_n|_{n=\infty} = \infty$, for some w,*

$$\sum \frac{1}{\prod_{k=0}^{w} \ln_k}|_{n=\infty} \leq \sum a_n|_{n=\infty}$$

Proof. Boundary test Theorem 3.2: swap sides and inequality direction for the divergent case. □

For convergent series $\sum a_n|_{n=\infty} = 0$ either the series is asymptotic to 'below the boundary' or below the boundary$(a_n < \frac{1}{n})|_{n=\infty}$.

Remark: 6.2. *All positive monotonic convergent series are bounded above by a series less than the boundary.*

Lemma 6.2. *When $\sum a_n|_{n=\infty} = 0$ converges then for some w and $p > 1$,*

$$\sum a_n|_{n=\infty} \leq \sum \frac{1}{\prod_{k=0}^{w} \ln_k \cdot \ln_{w+1}^{p}}|_{n=\infty}$$

Proof. Boundary test Theorem 3.2: convergent case. □

These bounds are available as another tool. Applying the lower and upper bound at the boundary to derive the following theorems.

Remark: 6.3. *There always exists a series that converges more slowly.*

Theorem 6.1. *If $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms then there exists a monotonic sequence $(b_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} b_n = \infty$ and series $\sum_{n=1}^{\infty} a_n b_n$ converges.*

Proof. Reform as the convergence sum, solve by Theorem 6.2, then apply the transfer principle, transferring a convergence sum to a sum [5, Theorem 11.1]. □

Theorem 6.2. *If $\sum a_n|_{n=\infty} = 0$ is a convergent series with positive terms then there exists a monotonic sequence $(b_n)|_{n=\infty}$ such that $b_n|_{n=\infty} = \infty$ and $\sum a_n b_n|_{n=\infty} = 0$ converges.*

Proof. Since $\sum a_n|_{n=\infty} = 0$ then by Theorem 6.2, $\exists c, p > 1$: $a_n \leq \frac{1}{L_{c-1}(\ln_c)^p}$. For positive b_n, $a_n b_n \leq \frac{1}{L_{c-1}(\ln_c)^p} b_n$, $(\sum a_n b_n \leq \sum \frac{1}{L_{c-1}(\ln_c)^p} b_n)|_{n=\infty}$.

Compare $\sum \frac{1}{L_{c-1}(\ln_c)^p} b_n|_{n=\infty}$ against the boundary. $(\sum \frac{1}{L_{c-1}(\ln_c)^p} b_n \; z \; \sum \frac{1}{L_w})|_{n=\infty}$, $(\frac{1}{L_{c-1}(\ln_c)^p} b_n \; z \; \frac{1}{L_w})|_{n=\infty}$, $(L_w b_n \;\; z \;\; L_{c-1}(\ln_c)^p)|_{n=\infty}$, $(\ln(L_w b_n) \; (\ln z) \; \ln(L_{c-1}(\ln_c)^p))|_{n=\infty}$, $(\sum_{k=1}^{w+1} \ln_k + \ln b_n \;\; (\ln z) \;\; \sum_{k=1}^{c} \ln_k + p \ln_{c+1})|_{n=\infty}$.

Choose $b_n = \ln_{w+1}$, then $\sum_{k=1}^{w+1} \ln_k + \ln b_n = \sum_{k=1}^{w+1} \ln_k|_{n=\infty}$ as $\ln_{w+1} \succ \ln_{w+2}|_{n=\infty}$.

Reversing the process, $(\sum_{k=1}^{w+1} \ln_k \; (\ln z) \; \sum_{k=1}^{c} \ln_k + p \ln_{c+1})|_{n=\infty}$, $(\ln(L_w) \; (\ln z) \; \ln(L_{c-1}(\ln_c)^p))|_{n=\infty}$, $(L_w \; z \; L_{c-1}(\ln_c)^p)|_{n=\infty}$, $(\frac{1}{L_{c-1}(\ln_c)^p} \; z \; \frac{1}{L_w})|_{n=\infty}$, $(\sum \frac{1}{L_{c-1}(\ln_c)^p} \; z \; \sum \frac{1}{L_w})|_{n=\infty}$.

Since the left hand side sum $\sum \frac{1}{L_{c-1}(\ln_c)^p}$ is always convergent, and the right hand sum $\sum \frac{1}{L_w}$ is always divergent, $z = <$. $\sum \frac{1}{L_{c-1}(\ln_c)^p} b_n|_{n=\infty} = 0$ then $\sum a_n b_n|_{n=\infty} = 0$ converges. □

Remark: 6.4. *There always exists a series that diverges more slowly.*

Theorem 6.3. *If $\sum_{n=1}^{\infty} a_n$ is a divergent series with positive terms then there exists a monotonic sequence $(b_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} b_n = 0$ and the series $\sum_{n=1}^{\infty} a_n b_n$ diverges.*

Proof. Reform as the convergence sum, solve by Theorem 6.4, then apply the transfer principle, transferring a convergence sum to a sum [5, Theorem 11.1]. □

Theorem 6.4. *If $\sum a_n|_{n=\infty} = \infty$ is a divergent series with positive terms then there exists a monotonic sequence $(b_n)_{n=\infty}$ such that $b_n|_{n=\infty} = 0$ and $\sum a_n b_n|_{n=\infty} = \infty$ diverges.*

Proof. Since $\sum a_n$ is diverging there exists a boundary sequence that acts as a lower bound. $\exists w : \frac{1}{L_w} \leq a_n|_{n=\infty}$, $\frac{1}{L_w} b_n \leq a_n b_n|_{n=\infty}$, let $b_n = \frac{1}{\ln_{w+1}}$, $\frac{1}{L_w \ln_{w+1}} \leq a_n b_n|_{n=\infty}$, $\frac{1}{L_{w+1}} \leq a_n b_n|_{n=\infty}$, $\sum \frac{1}{L_{w+1}} \leq \sum a_n b_n|_{n=\infty}$, $\infty \leq \sum a_n b_n|_{n=\infty}$, then $\sum a_n b_n|_{n=\infty} = \infty$ diverges. □

References

[1] C. D. Evans, W. K. Pattinson, *Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 1 Gossamer numbers*

[2] C. D. Evans, W. K. Pattinson, *Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 2 The much greater than relations*

[3] C. D. Evans, W. K. Pattinson, *Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 3 Comparing functions*

[4] C. D. Evans, W. K. Pattinson, *Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 6 Sequences and calculus in* $*G$

[5] C. D. Evans, W. K. Pattinson, *Convergence sums at infinity a new convergence criteria*

[6] C. D. Evans, W. K. Pattinson, *Convergence sums and the derivative of a sequence at infinity*

[7] C. D. Evans, W. K. Pattinson, *Ratio test and a generalization with infinitary calculus*

[8] C. D. Evans and W. K. Pattinson, *Rearrangements of convergence sums at infinity*

[9] C. D. Evans and W. K. Pattinson, *Power series convergence sums*

[10] G. H. Hardy, *Orders of infinity*, Cambridge Univ. Press, 1910.

[11] F. Hartmann and D. Sparows, *Investigating Possible Boundaries Between Convergence and Divergence*, available at `http://www42.homepage.villanova.edu/frederick.hartmann/Boundaries/Boundaries.pdf`

[12] G. Fisher, *The infinite and infinitesimal quantities of du Bois-Reymond and their reception* , 27. VIII. 1981, Volume 24, Issue 2, pp 101–163, Archive for History of Exact Sciences

[13] Konrad Knopp, *Infinite sequences and series* (F. Bagemihl Trans.), New York Dover Publications, 1956

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