Pretty Good State Transfer Over Arbitrary Distances

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It has been shown that for a graph X with n vertices, the correlation between a state at vertex v_S and a state at vertex v_r at time t is given by

$$P(t) = |\langle v_S|e^{-iAt}|v_R\rangle|^2 \tag{1}$$

where A is the adjacency matrix of the graph X.

If $P(t) = 1$, we say there is perfect state transfer at time t between v_S and v_R. In recent years, a relaxation of perfect state transfer has been introduced known as pretty good state transfer, or PGST. This occurs when the output probability $P(t) < 1$, but where it reaches a desired threshold such as $P(t) = 0.9$ or greater. It turns out that there is a very simple family of graphs that exist in 2-dimensions, where PGST is realized in short time, over arbitrary distances. By simply adding star structures to the third and third to last vertices of a path graph, PGST is made possible over any distance.

The first example of this was a graph nicknamed "Monstro" which is depicted in Figure 1. It is simply a P16 graph with stars of size 4 added to the third and third to last vertices.

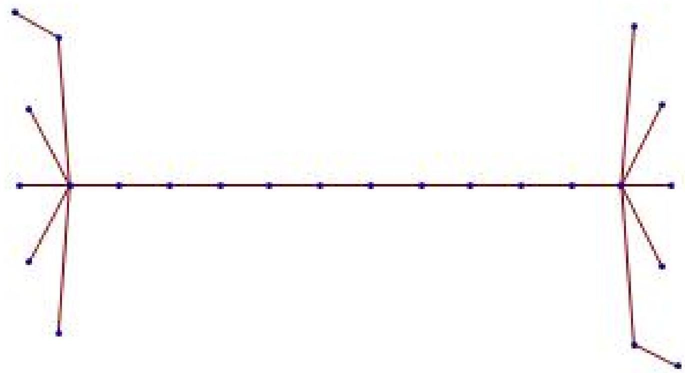


Figure 1: This figure shows network nicknamed "Monstro", that has 24 vertices and admits $> 90\%$ PGST over a diameter of 15 in an absolute time of 65.9. In comparison to P16 which has a diameter of 15 and admits 90% PGST in an absolute time of 38,093.1, the quantum walk across Monstro takes 0.17% of the time.

The output probability between the end vertices (the ones separated by the diameter) is periodic in all cases of these networks. For Monstro, this is depicted below in Figures 2 and 3.

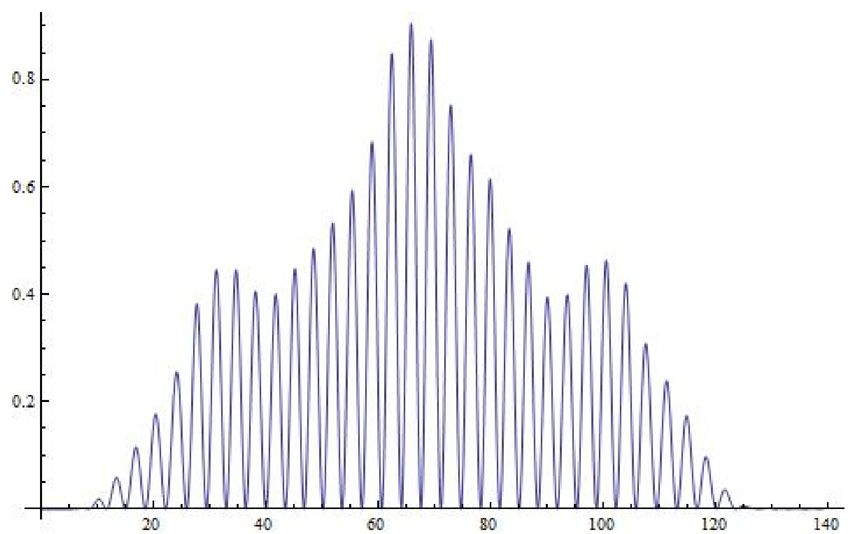


Figure 2: This figure shows the output probability between the end vertices of Monstro as a function of time. The vertical axes is the output probability and the horizontal is time. We see that the $P(t)$ is beautifully periodic.

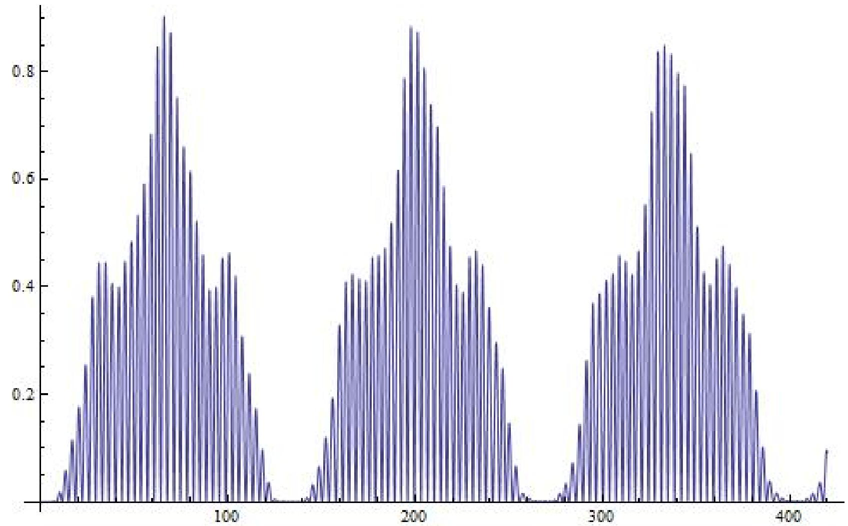
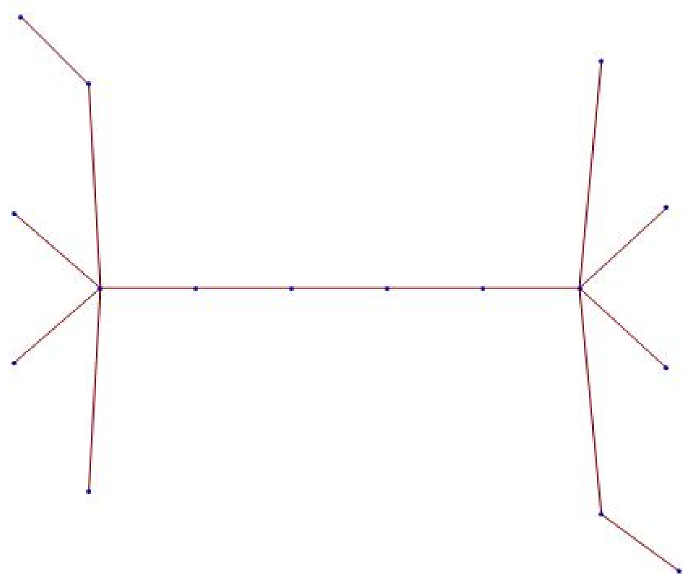
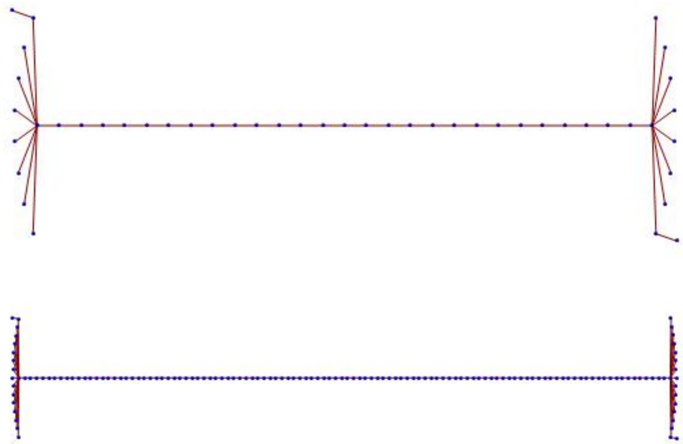


Figure 3: This figure also shows the output probability between the end vertices of Monstro as a function of time, but for a longer period.

As the size of the added star increases, so does the diameter of the network which admits PGST. To gain a visual understanding of what is going on, here are a few more graphs in the family.





It turns out that for every size star there is a corresponding distribution of path graphs that can be modified to achieve PGST. This distribution increases as the size of the star increases. For example, we can see in Figure 4 that when we attach 60, 100 and 200 star sizes, that the number of paths that may be modified to admit PGST increases. We see a very nice U shape distribution with one side slightly higher than the other. It also turns out, that for consecutive star sizes, there is always overlap in the regime of path sizes that can be modified. For example, stars of sizes 28 and 27 can both modify a path of size 207, but 28 can modify larger ones and alternatively 27 can modify smaller ones.

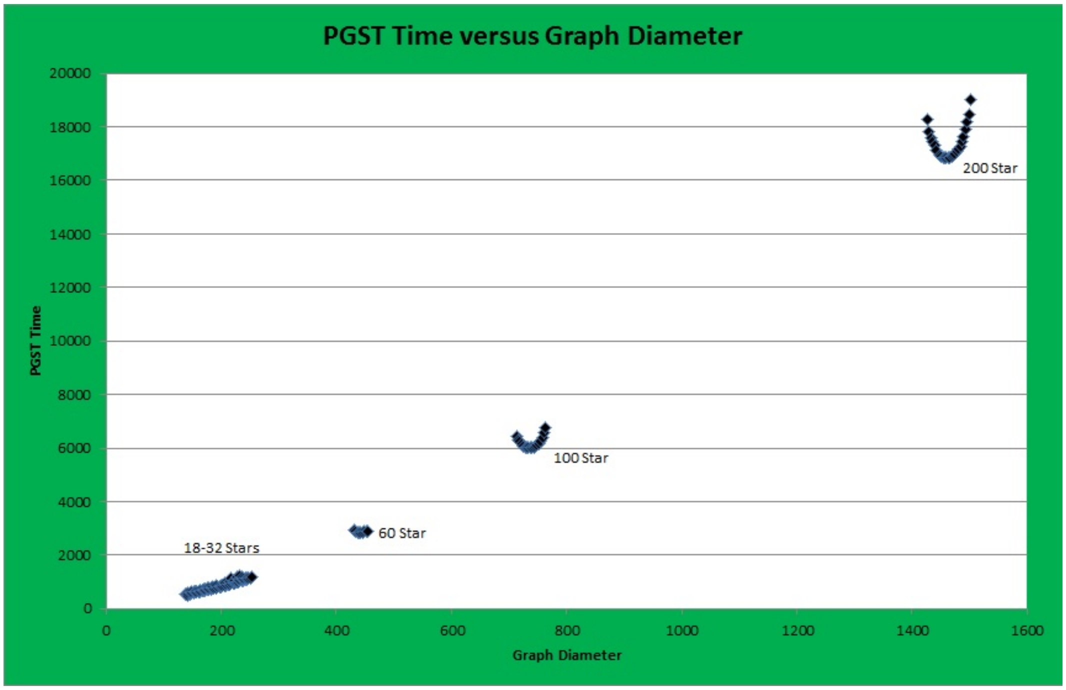


Figure 4: This figure shows the distribution of graph diameters that a star can modify. We see that as a star size increases, the amount of corresponding graphs also increases. It is not immediatly apparent from this Figure, but close analysis of the data shows that the distribution pattern holds for smaller star sizes such as 18-32, and that there is overlap of the regimes. Another intersting feature is that the path lengths that can be modified for a given star size always grows by three within that regime.

Within every regime for a given star size, there is always a graph that has a minimum time per hop across it's diameter. Using only these graphs, the relationship between the star size and the corresponding graph diameter can be determined easily. We can see this in Figure 5.

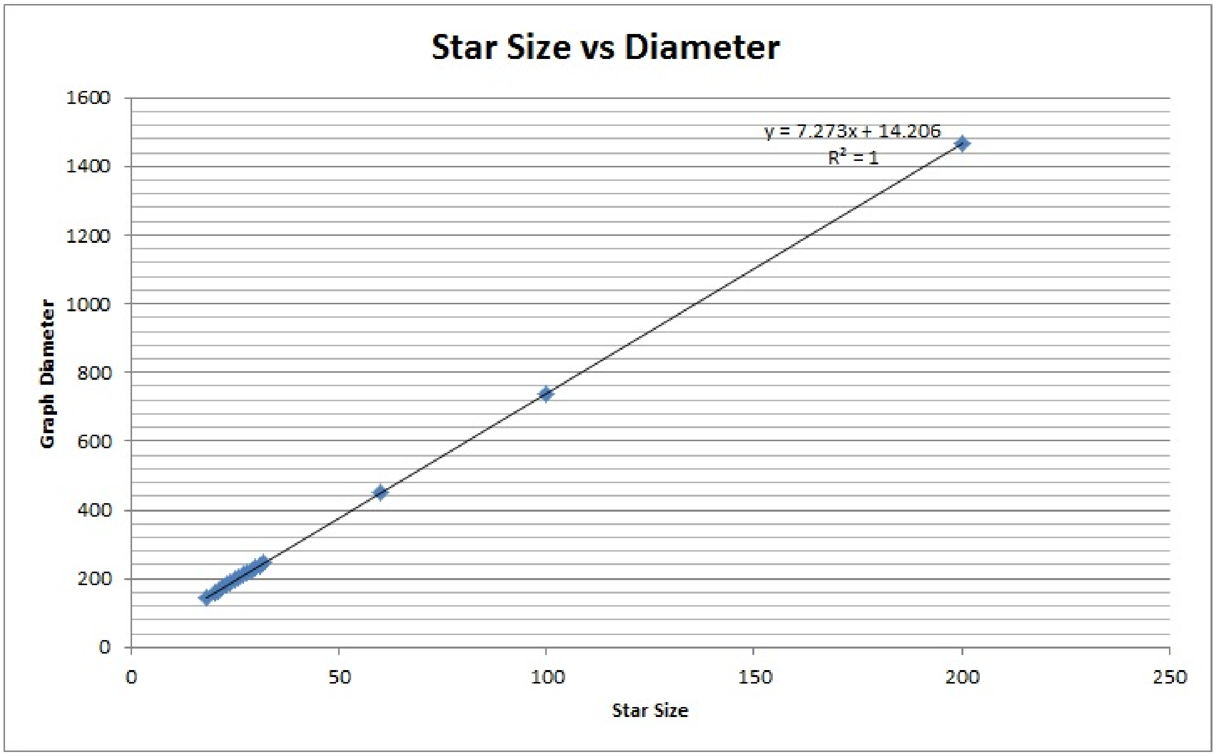


Figure 5: This figure shows how star size is related to graph diameter. Using the networks corresponding to the minimum time per hop, we get the relationship that Diameter$\approx 7.273 \times \text{StarSize} + 14.206$.

Using the linear relationship given by the equation in Figure 5, the corresponding diameter to a given star is easily predicted. This method has worked without fail to predict the approximate diameter that corresponds to any size star.

Another important result is the ability to predict the approximate time for PGST to occur across a given graph diameter. This is also done very easily. Using the graphs corresponding to minimum time per hop again, we can see in Figure 6 that there is a very apparent trend.

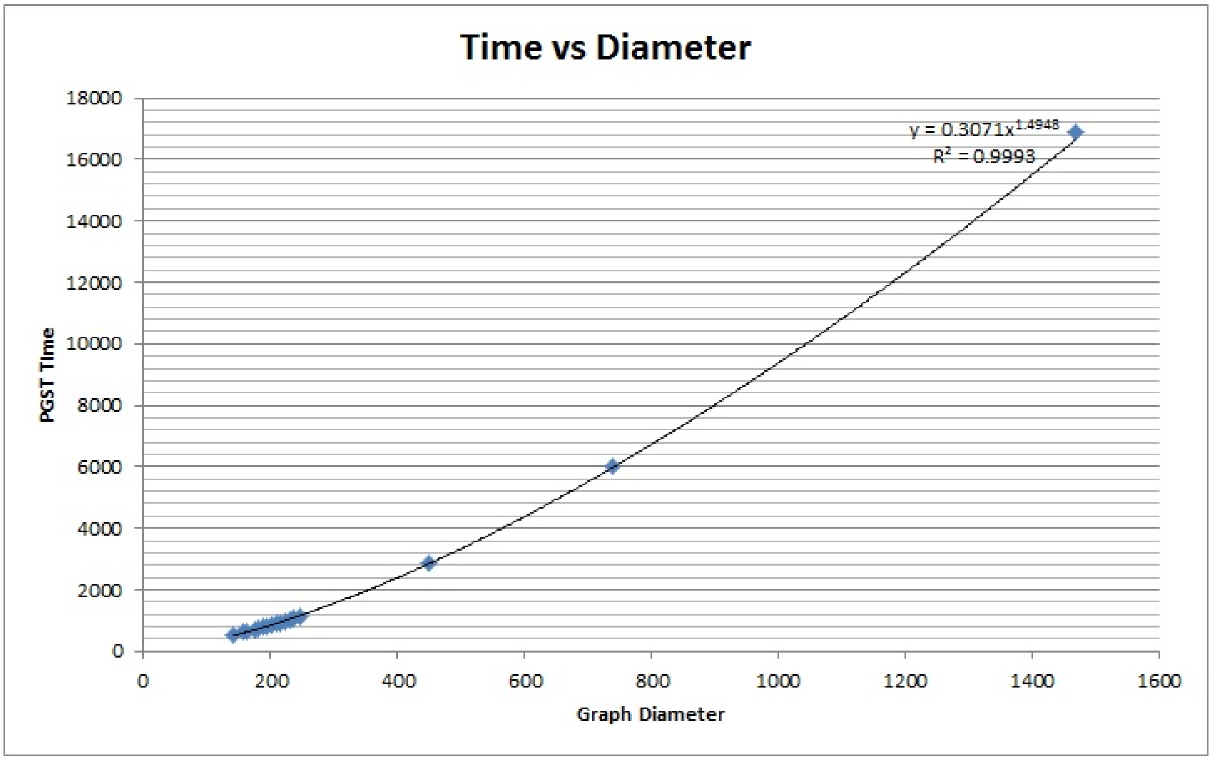


Figure 6: This figure shows how PGST time is related to graph diameter. The R^2 value can be improved by taking more data, but nevertheless, the relationship that $t_{expected} = 0.3071 \times \text{Diameter}^{1.4948}$ has proven to be very accurate computationaly.

We can see that the time required to achieve PGST in this family of networks does increase exponentially, but at a rate that is easily predictable. The fact that the time increases exponentially as the diameter increases is a slight downside to these networks as the grow very large, but there is also an upside as they grow. It turns out the the output probability also increases as the network size increases. Lets look at a few plots of $P(t)$ as a function of t for networks as they grow.

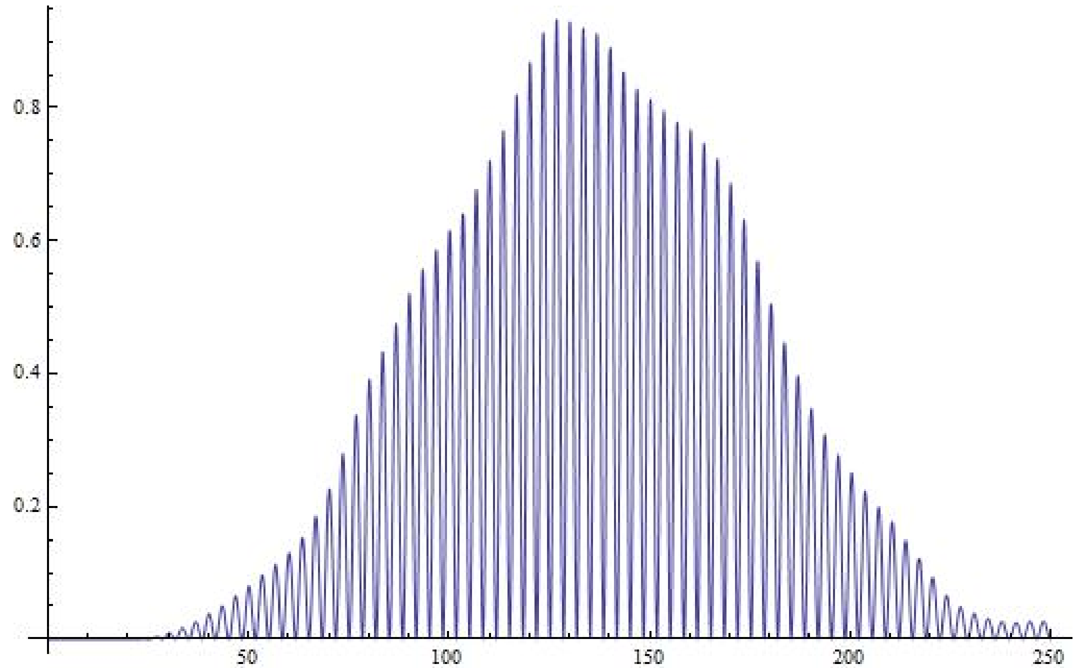


Figure 7: This figure shows $P(t)$ as a funtion of time across a modified path of size 43 which contains stars of size 8. We can see that the highest peak is roughly around $P(t) \approx 0.95$.

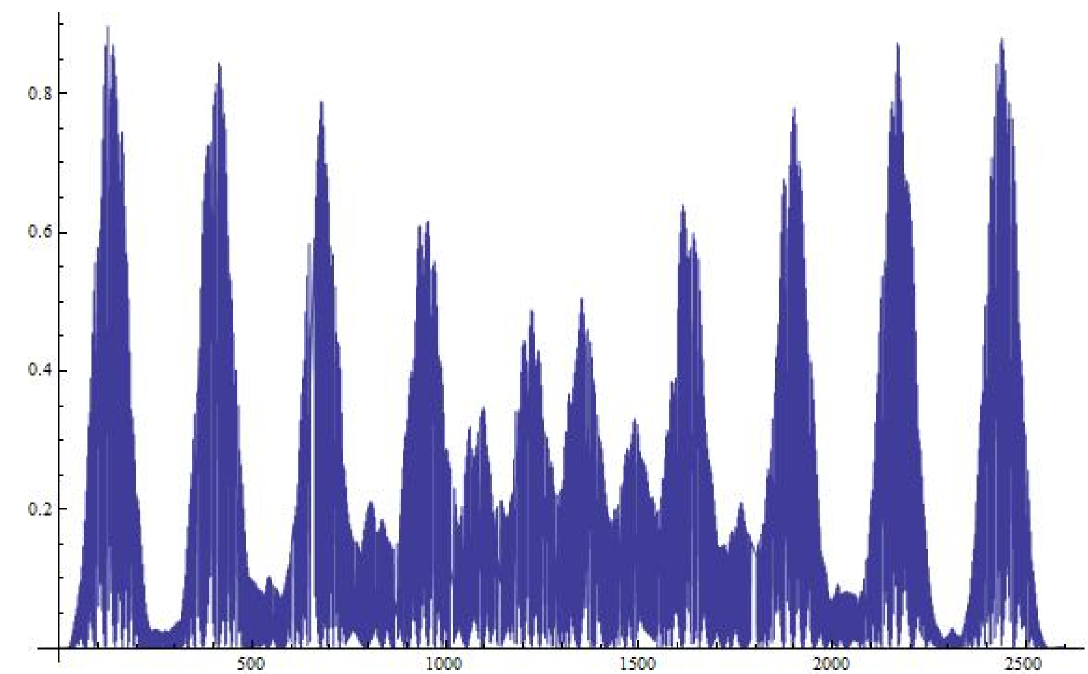


Figure 8: This figure also shows the output probability as a function of time for the same modified P43 network, but for a longer period of time. We can see that it is indeed periodic.

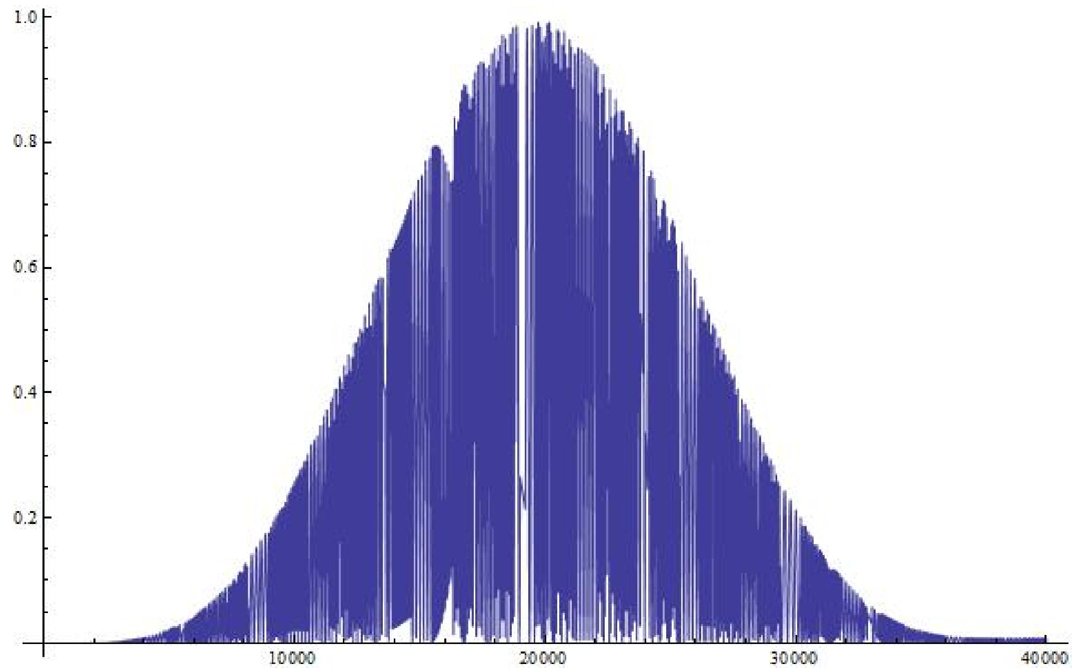


Figure 9: This figure shows $P(t)$ as a funtion of time across a modified path of size 1471 with stars of size 200. We can see that the highest peak is roughly around $P(t) \approx 0.99$.

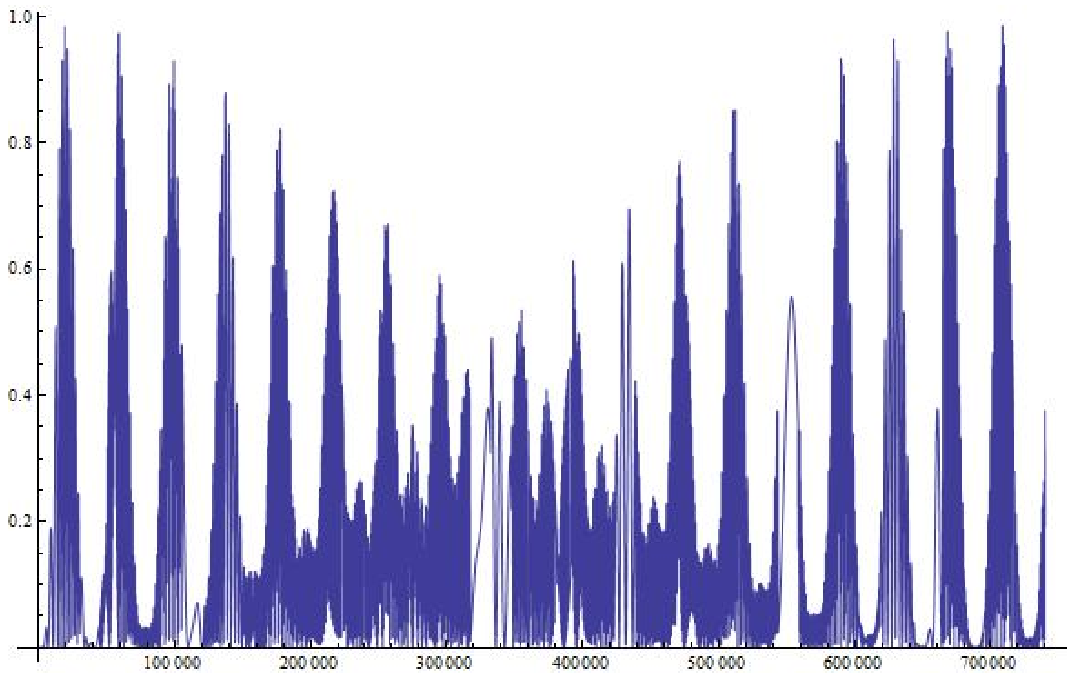


Figure 10: This figure also shows the output probability as a function of time for the same modified P1741 network, but for a longer period of time. We can see that it is also periodic.

It true that for all tested cases of these networks, as the diameter increases significantly there is a clear rise in peak output probability.

There are a few more interesting properties of these networks that may prove to be very advantageous experimentally:

- The frequency of oscillation of $P(t)$ is not increasing, so the time window to retrieve a state should stay fixed as the size of a network increases, while the output probability increases.

- The added star figures can be collapsed to a single weighted edge that is weighted as the square root of the star size. For example, a network consisting of a diameter of 740 has added stars of size 100. This network behaves identically if single branches of weight 10 are added. This relationship is necessarily true for all cases. Therefore experimentally, one can add only two single weighted edges to any size path graph and achieve PGST.

- Not only does PGST occur between the end vertices of these networks, but it also occurs between the second and second last vertices as well. That is, the two vertices that are adjacent to the end vertices. The output probability between these two vertices also increases as the network grows in size. This implies that there are four available sites where PGST can occur.

This family of networks, which have been dubbed ”Mereau Networks”, allow for PGST over arbitrary distances in short time. It has been shown numerically, that the output probability increases as the network size increases, and that the star size and expected time corresponding to a given diameter is easily predicted. Furthermore, these networks are easily constructed in two dimensions so they may be engineered in practice. Since the graphs are undirected, they may allow for PGST with minimal external intervention, as the state transfer may be realized by the underpinning dynamics of the networks themselves.