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Point to Camera Problem

We consider the problem of determining how a point in 3 space is mapped to a pair of cameras. We begin by considering a single camera that is represented as a sphere at the origin. The vertical and horizontal angles from a point with cartesian coordinates:

$$P = (x, y, z)$$

Can be determined as

$$\theta_{vertical} = \tan^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right) = \sin^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right)$$

Through the use of simple trigonometry. Furthermore the horizontal angle can be determined as

$$\theta_{horizontal} = \tan^{-1}\left(\frac{y}{x}\right) = \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

We now label cartesian plot lines on the surface of the camera from the origin. Not that the vertical angle corresponds to the y axis and the horizontal angle to x axis. If the camera has radius r. Then it's clear that the coordinates are given as

$$\theta_{vertical} > 0 \rightarrow y = \frac{r}{4}\tan^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right), \theta_{vertical} < 0 \rightarrow y = -\frac{r}{4}\tan^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right)$$

Similarly for the horizontal

$$\theta_{horizontal} > 0 \rightarrow x = \frac{r}{4}\tan^{-1}\left(\frac{y}{x}\right), \theta_{horizontal} < 0 \rightarrow x = -\frac{r}{4}\tan^{-1}\left(\frac{y}{x}\right)$$

Assuming the CENTER of the image is 0,0

Now we consider the inverse question. Suppose that a point is located at (x,y) on our image, what angles does it correspond to in 3-space?

We have that

$$y = \frac{r}{4}\theta_{vertical} \rightarrow \frac{4y}{r} = \theta_{vertical}$$

And similarly

$$\frac{4x}{r} = \theta_{horizontal}$$

But this is all assuming that the entire sphere surface is mapped into the image. It may be that the image consists of a plane surface in which case behavior would be different. But this can be fairly easy to derive. We now consider the problem of two cameras placed horizontally registering the same point at

$$(x_1, x_2) : (x_3, x_4)$$

where the distance between them is d.

Again we can recover the angles of inclination for each point:

$$\left(\frac{4x_1}{r}, \frac{4x_2}{r}\right) : \left(\frac{4x_3}{r}, \frac{4x_4}{r}\right)$$

or

$$\left(\frac{4x_1}{r}, \frac{4x_2}{r}\right) : \left(-\frac{4x_3}{r}, \frac{4x_4}{r}\right)$$

Now the challenge is to recover the distance from this. It can be bashed aggressively with some trigonometry. From the law of sines we have that

$$\frac{d}{\sin\left(\pi - \frac{4x_1}{r} - \frac{4x_3}{r}\right)} = \frac{l_1}{\sin\left(\frac{4x_3}{r}\right)} = \frac{l_2}{\sin\left(\frac{4x_1}{r}\right)}$$

Thus:

$$l_1 = \frac{d \sin\left(\frac{4x_3}{r}\right)}{\sin\left(\pi - \frac{4x_1}{r} - \frac{4x_3}{r}\right)}, l_2 = \frac{d \sin\left(\frac{4x_1}{r}\right)}{\sin\left(\pi - \frac{4x_1}{r} - \frac{4x_3}{r}\right)}$$

where l1 are and l2 are the x-y plane projects of the vectors from the cameras to the point and the x-y plane in question is the plane within which the cameras travel as the vehicle they are contained in moves.

We can apply the same process now to the left and right triangles between the individual cameras and the point to derive that the exact distance between the cameras and the point in question (distance for camera 1 and camera 2 or l3 and l4 respectively)

$$\frac{l_3}{\sin\left(\frac{\pi}{2}\right)} = \frac{l_1}{\sin\left(\frac{\pi}{2} - \theta_2\right)} \rightarrow l_3 = \frac{l_1}{\sin\left(\frac{\pi}{2} - \theta_2\right)}$$

Similarly

$$l_4 = \frac{l_2}{\sin\left(\frac{\pi}{2} - \theta_4\right)}$$

Thus the distance from camera 1 to the point is

$$l_3 = \frac{d \sin\left(\frac{4x_3}{r}\right)}{\sin\left(\frac{\pi}{2} - \theta_2\right) \sin\left(\pi - \frac{4x_1}{r} - \frac{4x_3}{r}\right)}$$

The distance from camera 2 to the point is

$$l_4 = \frac{d \sin\left(\frac{4x_1}{r}\right)}{\sin\left(\frac{\pi}{2} - \theta_4\right) \sin\left(\pi - \frac{4x_1}{r} - \frac{4x_3}{r}\right)}$$

Now we seek to recover the coordinates in terms of each camera (and this can be easily adjusted to yield you coordinates from the center of the two cameras).

Start with the leftmost camera and recall that the horizontal angle from the point is given by

$$\frac{4x_1}{r}$$

Thus the slope of the outgoing line is

$$\tan^{-1}\left(\frac{4x_1}{r}\right)$$

And the distance of this line is given earlier as

$$l_1 = \frac{d \sin\left(\frac{4x_3}{r}\right)}{\sin\left(\pi - \frac{4x_1}{r} - \frac{4x_3}{r}\right)}$$

From here we note that the length of y will be proportional to x by the slope and therefore we can write the x coordinate as X and apply the pythagorean theorem to find

$$X^2 + \tan^{-1}\left(\frac{4x_1}{r}\right)^2 X^2 = \frac{d^2 \sin\left(\frac{4x_3}{r}\right)^2}{\sin\left(\pi - \frac{4x_1}{r} - \frac{4x_3}{r}\right)^2}$$

Which yields

$$X = \frac{1}{1+\tan^{-1}\left(\frac{4x_1}{r}\right)^2}\left(\frac{d\sin\left(\frac{4x_3}{r}\right)}{\sin\left(\pi-\frac{4x_1}{r}-\frac{4x_3}{r}\right)}\right)^2$$

And furthermore gives us a y coordinate of

$$Y = \frac{\tan^{-1}\left(\frac{4x_1}{r}\right)}{1+\tan^{-1}\left(\frac{4x_1}{r}\right)^2}\left(\frac{d\sin\left(\frac{4x_3}{r}\right)}{\sin\left(\pi-\frac{4x_1}{r}-\frac{4x_3}{r}\right)}\right)^2$$

Now the Z coordinate can be derive as satisfying

$$Z^2 + l_1^2 = l_3^2$$

Giving us

$$Z^2 + \left(\frac{d\sin\left(\frac{4x_3}{r}\right)}{\sin\left(\pi-\frac{4x_1}{r}-\frac{4x_3}{r}\right)}\right)^2 = \left(\frac{d\sin\left(\frac{4x_3}{r}\right)}{\sin\left(\frac{\pi}{2}-\theta_2\right)\sin\left(\pi-\frac{4x_1}{r}-\frac{4x_3}{r}\right)}\right)^2$$

We now simplify this is as

$$Z = \frac{d\sin\left(\frac{4x_3}{r}\right)}{\sin\left(\pi-\frac{4x_1}{r}-\frac{4x_3}{r}\right)}\sqrt{1-\frac{1}{\sin\left(\frac{\pi}{2}-\theta_2\right)}}$$

Thus the coordinates of our point are given as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\tan^{-1}\left(\frac{4x_1}{r}\right)^2}\left(\frac{d\sin\left(\frac{4x_3}{r}\right)}{\sin\left(\pi-\frac{4x_1}{r}-\frac{4x_3}{r}\right)}\right)^2 \\ \frac{\tan^{-1}\left(\frac{4x_1}{r}\right)}{1+\tan^{-1}\left(\frac{4x_1}{r}\right)^2}\left(\frac{d\sin\left(\frac{4x_3}{r}\right)}{\sin\left(\pi-\frac{4x_1}{r}-\frac{4x_3}{r}\right)}\right)^2 \\ \frac{d\sin\left(\frac{4x_3}{r}\right)}{\sin\left(\pi-\frac{4x_1}{r}-\frac{4x_3}{r}\right)}\sqrt{1-\frac{1}{\sin\left(\frac{\pi}{2}-\theta_2\right)}} \end{pmatrix}$$

Now the other camera and middle point coordiantes are trivial to derive.