In memoriam Hermann Minkowski,
who was born 150 years ago

Bridging the Gap between School Mathematics and the Mathematics of General Relativity

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Abstract

The Dirac operator is the general relativistic generalization of Minkowski's Lorentz operator and describes the geometric derivative in general relativity. But while the Dirac operator is discussed in length in books about Geometric Algebra, the discussion of the Lorentz operator usually is avoided. This leads to a huge didactical gap, as a discussion of the geometric derivative in special relativity becomes impossible without reciprocal frames.
These didactical problems are a consequence of three shocks special relativity suffered in its youth.

Keywords

Geometric Algebra, Clifford Algebra, Dirac Algebra, Special Relativity,
General Relativity, Lorentz Operator, Dirac Operator, Geometric Derivative

Contents

1 The tragic death of Hermann Minkowski

Hermann Minkowski untimely died on January 12, 1909, only 44 years of age [12]. His death was a tragedy for his family and colleagues. And it was a tragedy for mankind, for he died in the middle of a scientific revolution – the revolution of relativity.

Einstein had discovered special relativity in 1905. But he and his colleagues – although trying hard to understand its implications and its philosophy – never fully succeeded [20]. They needed the help of a mathematical genius like Minkowski with a convincing philosophical background and the ability to transform this background into a profound mathematical formulation. Nearly immediately after having heard about special relativity Minkowski grasped the philosophical consequences of this new way of thinking and started to re-formulate the basic equations of special relativity in a unifying, four-dimensional spacetime picture. But in the middle of this process, Minkowski's deadly appendicitis hit.

As Einstein shaped the infancy of relativity – he had invented it, – Minkowski shaped the adolescence [20, chap. 9] of relativity. Einstein's baby had become a young and still unsure boy in puberty with adolescent disorders, but firmly led by Minkowski into the right direction. And in the midst of this crucial age of puberty, Minkowski had to leave the scene. Relativity lost one of its most important guiding hands which was a first shock.

The second shock soon followed: Einstein invented general relativity in 1915. Within a blink of an eye, within an unbelievable short moment of scientific history, the pubescent relativity had become an adult. Relativity was now expected to be a serious, well-behaving grown-up. But it wasn't. It still was an unsure physical programme in an adolescent status looking only from far like a fully developed theory.

And soon again a third shock followed: Heisenberg & Pauli, Schrödinger & DeBroglie, Dirac and many others invented quantum mechanics. In the eyes of a psychologist, quantum mechanics was the younger, but more successful brother of relativity which soon outdid and surpassed relativity. Of course all this caused mental disorder and psychological fractions in the development of relativity. We still today see and feel these severe frictions in our textbooks of physics.

If relativity had been a human being we would have sent it to a psychiatrist to analyze its unhappy youth. The untimely death of Hermann Minkowski was followed by an untimely quick ageing of relativity.

2 Teaching general relativity before teaching calculus?

When we teach mathematics at school and high-school, calculus is an elementary and very central part of the mathematics curriculum. Usually the elementary rules of differentiating standard functions are taught at upper secondary level at schools in Berlin [21] and other states of Germany. And of course these lessons are taught before teaching the mathematical structures of curved space and general relativity. This makes sense: we are able to discuss derivatives in rectangular coordinates without problems.

Unfortunately textbooks about Geometric Algebra reflect the adolescent problems of special relativity in a very direct manner: Standard books like [6] – and many others – first introduce reciprocal frames with covariant and contravariant coordinates – e.g. section 4.3 at pages 100-103 of [6] – in much the same way as it is done in general relativity – e.g. page 65 of [7] or chapters 2 & 3 of [5] – with covariant and contravariant vectors.

They only later go on with introducing geometric derivatives. The reason for this order is simple: These standard textbooks need to explain reciprocal frames first because the ability to deal with them correctly is a necessary precondition for dealing with spacetime derivatives in the way they are presented in these textbooks – they all apply the Dirac operator.

But should we really follow a curriculum which necessarily first introduces the mathematics of oblique axes or curvilinear coordinates before introducing calculus? The didactical costs of this order are high: We would expect too much from many of our students. Therefore it makes sense to look for another route to general relativistic calculus by first discussing special relativistic calculus without the notion of reciprocal frames.

3 The mathematics of special relativity

With respect to physics, special relativity deals with and cares about the structure of space and time. With respect to mathematics, special relativity deals with and cares about signs – and how to hide them. Looking from a broader, more open perspective we of course recognize that both perspectives are more or less identical: Spatial or spacelike coordinates can be identified by their negative signs, timelike coordinates by their positive signs at the square of the special relativistic position vector

$$r^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{1}$$

or of its cosmological equivalent [3]

$$r'^2 = c^2t^2 - x^2 - y^2 - z^2 + \tau^2v^2 \tag{2}$$

But special relativity promised us the spirit of unification: “Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality” [18, p. 75]. Therefore Minkowski called for an “identical treatment of the four co-ordinates x, y, z, t” [18, p. 83].

Obviously this promise of an unequivocally identical treatment was broken: There are different signs in front of spacelike and timelike coordinate squares in eqs. (1) & (2). This is a huge mathematical disappointment. And as we cannot get rid of these different signs, the only other practical possibility is to hide them.

A very elegant strategy for hiding the signs of the special relativistic spacetime signatures (+, –, –, –) and (+, –, –, –, +) of eqs. (1) & (2) can be found in Clifford Algebra (or Spacetime Algebra): We hide all signs in spacetime base vectors γ_t, γ_x, γ_y, γ_z

$$r^2 = (ct\,\gamma_t)^2 + (x\,\gamma_x)^2 + (y\,\gamma_y)^2 + (z\,\gamma_z)^2 \tag{3}$$

or in cosmological base vectors γ_t, γ_x, γ_y, γ_z, γ_v [13]

$$r'^2 = (ct\,\gamma_t)^2 + (x\,\gamma_x)^2 + (y\,\gamma_y)^2 + (z\,\gamma_z)^2 + (\tau v\,\gamma_v)^2 \tag{4}$$

with spacelike and timelike base vector squares

$$\gamma_x{}^2 = \gamma_y{}^2 = \gamma_z{}^2 = -1 \qquad\qquad \gamma_t{}^2 = \gamma_v{}^2 = 1 \tag{5}$$

As all base vectors anticommute, the spacetime vector r and its cosmological counterpart r’ can be written as linear combination of these base vectors:

$$r = ct\,\gamma_t + x\,\gamma_x + y\,\gamma_y + z\,\gamma_z \tag{6}$$

$$r' = ct\,\gamma_t + x\,\gamma_x + y\,\gamma_y + z\,\gamma_z + \tau v\,\gamma_v \tag{7}$$

By the way: As Cartan already noticed, all these base vectors can be identified with the Dirac matrices γ_i [4, p. 133].

Unfortunately most books about Geometric Algebra or Clifford Algebra avoid a deep and profound mathematical discussion of eqs. (6) & (7) with respect to special relativistic calculus. They only present simple consequences of some algebraic properties, which surely are important: Special relativistic length contraction, time dilation, or spacetime rotations & Lorentz transformations offer a first glimpse about the extraordinary mathematical and physical concepts described by such a spacetime or cosmological picture given by the vectors of eqs. (6) & (7).

But the full chances and opportunities of this mathematical instrument and the physical ideas behind it will only be seen if we discuss it using the power of calculus, which "next to the creation of Euclidean geometry … has proved to be the most original and most fruitful concept in all of mathematics" [17, p. 363].

And at this central point there exists a huge and unbelievable gap in standard textbooks of Clifford Algebra. Instead of analyzing the consequences spacetime vectors offer they directly jump to the mathematics of general relativity using co- or contravariant spacetime vectors

$$r = \sum_{\mu} x_{\mu} e^{\mu} = \sum_{\mu} x^{\mu} e_{\mu} \tag{8}$$

This unexpected jump is a direct consequence of the three adolescent shocks experienced by special relativity: the early death of Hermann Minkowski, the invention of general relativity, and the invention and quantum mechanical misinterpretation of the Dirac operator.

And all this leads to a severe didactical problem: Should we really expect from our students first to learn the mathematics of general relativity before we teach them calculus? This is the strategy followed usually in Clifford Algebra textbooks – and from a didactical perspective, this is a weird strategy.

Of course calculus should be discussed in the much easier framework of special relativity first before implementing the conceptually more advanced and more complicated framework of co- and contravariant coordinates connected with the reciprocal frames of general relativity.

4 An early version of the Dirac operator

But in addition to this didactical gap, there is a basic structural gap, too: early versions of the Dirac operator had been written in different ways. One way to write this operator simply was

$$\gamma_t \frac{\partial}{c\,\partial t} + \gamma_x \frac{\partial}{\partial x} + \gamma_y \frac{\partial}{\partial y} + \gamma_z \frac{\partial}{\partial z} \quad \text{or} \quad \gamma_t \frac{\partial}{c\,\partial t} + \gamma_x \frac{\partial}{\partial x} + \gamma_y \frac{\partial}{\partial y} + \gamma_z \frac{\partial}{\partial z} + \gamma_v \frac{\partial}{\tau\,\partial v} \tag{9}$$

This version is only correct if we consider timelike coordinates as imaginary entities. Nevertheless this preliminary Dirac operator can be applied in the sense of a gradient operator to get a geometric derivative of the position vector squares r^2 and r'^2. Neglecting the imaginary nature of time (or cosmological velocity), this version of the Dirac operator results in a wrong expression:

$$\left(\gamma_t \frac{\partial}{c\,\partial t} + \gamma_x \frac{\partial}{\partial x} + \gamma_y \frac{\partial}{\partial y} + \gamma_z \frac{\partial}{\partial z}\right) r^2 = 2ct\,\gamma_t - 2x\,\gamma_x - 2y\,\gamma_y - 2z\,\gamma_z \neq 2r \tag{10}$$

$$\left(\gamma_t \frac{\partial}{c\,\partial t} + \gamma_x \frac{\partial}{\partial x} + \gamma_y \frac{\partial}{\partial y} + \gamma_z \frac{\partial}{\partial z} + \gamma_v \frac{\partial}{\tau\,\partial v}\right) r'^2 = 2ct\,\gamma_t - 2x\,\gamma_x - 2y\,\gamma_y - 2z\,\gamma_z + 2\tau v\,\gamma_v \neq 2r' \tag{11}$$

The geometric derivative of r^2 or r'^2 should be $2r$ or $2r'$ respectively.

Here we failed, because we ignored Minkowski. This early preliminary versions of the Dirac operators (9) do not describe geometric derivatives or geometric gradients in Clifford Algebra correctly. If we do not want to use reciprocal frames, we will need an operator designed by Minkowski shortly before his death and years before general relativistic ideas became relevant.

5 Minkowski's invention – The Lorentz operator

Dirac operators in modern versions, see e.g. [1, p. 112], are nothing else than an advanced version of the Lorentz operator, defined by Minkowski in 1910 and called ***lor*** [19, sec. 12, pp. 40 – 44]. The translation of this ***lor*** operator into Clifford Algebra can be written as:

$$\boldsymbol{lor} = \square = \frac{\partial}{\gamma_t c\,\partial t} + \frac{\partial}{\gamma_x \partial x} + \frac{\partial}{\gamma_y \partial y} + \frac{\partial}{\gamma_z \partial z} = \gamma_t \frac{\partial}{c\,\partial t} - \gamma_x \frac{\partial}{\partial x} - \gamma_y \frac{\partial}{\partial y} - \gamma_z \frac{\partial}{\partial z} \tag{12}$$

$$\boldsymbol{lor'} = \pentagon = \frac{\partial}{\gamma_t c\,\partial t} + \frac{\partial}{\gamma_x \partial x} + \frac{\partial}{\gamma_y \partial y} + \frac{\partial}{\gamma_z \partial z} + \frac{\partial}{\gamma_v \tau\,\partial v} = \gamma_t \frac{\partial}{c\,\partial t} - \gamma_x \frac{\partial}{\partial x} - \gamma_y \frac{\partial}{\partial y} - \gamma_z \frac{\partial}{\partial z} - \gamma_v \frac{\partial}{\tau\,\partial v} \tag{13}$$

It is extremely important to indicate the direction of the variables adequately, which of course are part of the denominator – and not of the numerator. Placing them into the numerator therefore necessitates an additional negative sign in case of spacelike coordinates. These Clifford Algebra equivalents of Minkowski's Lorentz operator then transform the wrong results of eqs. (10) & (11) into the following correct ones:

$$\boldsymbol{lor}\, r^2 = \square\, r^2 = 2\,(ct\,\gamma_t + x\,\gamma_x + y\,\gamma_y + z\,\gamma_z) = 2r \tag{14}$$

$$\boldsymbol{lor'}\, r'^2 = \pentagon\, r'^2 = 2\,(ct\,\gamma_t + x\,\gamma_x + y\,\gamma_y + z\,\gamma_z + \tau v\,\gamma_v) = 2r' \tag{15}$$

This time, the signature signs are taken into account according to eq. (5). Thus the minus signs hidden in the base vectors again reach daylight and make geometric derivation possible.

6 The geometric gradient of closed spacetime curves

The geometric derivatives of eqs. (14) or (15) can be visualized in a two-dimensional coordinate system, if we only look at two variables. We will have a pure spacelike picture, if $z = t = v = 0$. Then $r = r' = x\,\gamma_x + y\,\gamma_y$ and

$$\square\, r^2 = \pentagon\, r'^2 = \left(-\gamma_x \frac{\partial}{\partial x} - \gamma_y \frac{\partial}{\partial y}\right)(-x^2 - y^2) = 2x\,\gamma_x + 2y\,\gamma_y = 2r = 2r' \tag{16}$$

This is shown in figure 1a.

For a pure timelike picture we need two timelike coordinates. In standard special relativity there only is one timelike coordinate: the time which is measured by our clocks. Therefore we should think about another version of special relativity to model such a situation. Cosmological special relativity – invented by Carmeli [3] as a physical artifact of spinor theory [2, sec. 3.1.5, pp. 41 – 44] – offers two timelike coordinates. Carmeli suggests that velocity should be considered as a fifth independent, time-like coordinate. This is an intriguing and conceptually very interesting idea, explaining the Hubble expansion of the universe as a simple geometric consequence of a five-dimensional or even seven-dimensional [3, sec. 3.2, p. 32] spacetimevelocity world.

At present observational data does not support Carmeli's idea. For this reason cosmological relativity very probably will not represent physics correctly. Nevertheless it can be used as a toy model to discuss and to analyze consequences of an additional timelike dimension. We will get a pure timelike picture, if $x = y = z = 0$ and therefore $r' = ct\,\gamma_t + \tau v\,\gamma_v$ with

$$\pentagon\, r'^2 = \left(\gamma_t \frac{\partial}{c\,\partial t} + \gamma_v \frac{\partial}{\tau\,\partial v}\right)(c^2t^2 + \tau^2v^2) = 2ct\,\gamma_t + 2\tau v\,\gamma_v = 2r' \tag{17}$$

This is shown in figure 1c.

An even more interesting case – which indeed is highly relevant for the physics of special relativity – is a mixed spacetime picture, now modeled in two dimensions with $y = z = v = 0$ and $r = r' = ct\,\gamma_t + x\,\gamma_x$ as

$$\square\, r^2 = \pentagon\, r'^2 = \left(\gamma_t \frac{\partial}{c\,\partial t} - \gamma_x \frac{\partial}{\partial x}\right)(c^2t^2 - x^2) = 2ct\,\gamma_t + 2x\,\gamma_x = 2r = 2r' \tag{18}$$

This hyperbolic, pseudo-Euclidean picture is shown in figure 1b.

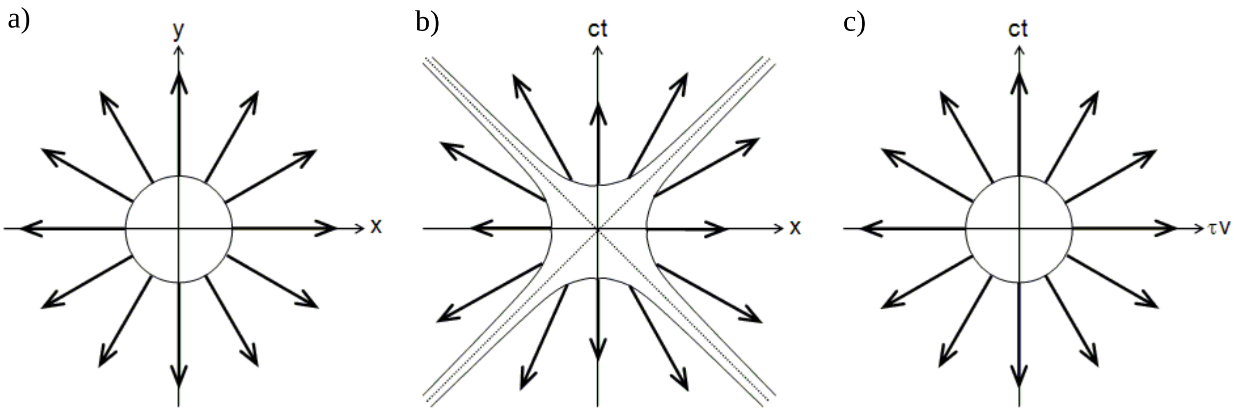

Fig. 1: Geometric derivatives of (a) pure spacelike, (b) mixed spacetime, and (c) pure timelike circles.

Please note, that all diagrams of figure 1 show circles: The distances of all points to the origin of the diagrams are identical. And all arrows representing spacetime vectors have equal spacetime length.

In string theory physicists deal with closed spacetime curves. Therefore, although somehow physically esoteric [6, p. 170], we should discuss simple manifestations of such curves with our students. A very simple closed spacetime curve s can be modeled by $s^2 = s'^2 = c^2t^2 + x^2$ with $s = ct\,\gamma_t + x\,\gamma_x\gamma_y\gamma_z$ or $s' = ct\,\gamma_t - x\,\gamma_t\gamma_y\gamma_z\gamma_v$. Then $y = z = v = 0$ and

$$\square\, s^2 = \pentagon\, s'^2 = \left(\gamma_t \frac{\partial}{c\,\partial t} - \gamma_x \frac{\partial}{\partial x}\right)(c^2t^2 + x^2) = 2ct\,\gamma_t - 2x\,\gamma_x \tag{19}$$

This closed spacetime curve and its gradient vectors (which are shortened to increase visual clarity) are shown in figure 2.

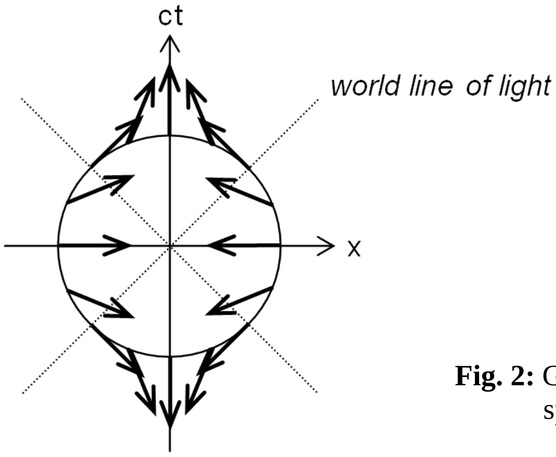

Fig. 2: Geometric derivative of the closed spacetime curve $c^2t^2 + x^2$.

It is an important feature of spacetime, that gradient vectors of closed spacetime curves deviate from radial directions due to the different signature signs of space- and timelike coordinates. At all points crossing the world line of light, the gradient vectors are perpendicular to the light cone and always point into the direction of the future light cone or the past light cone.

7 Closing the gap

After having discussed these geometric derivatives of different spacetime functions we are now in a position to look for a new mathematical perspective which describes these situations (e.g. spacetime circles and closed spacetime curves) using different mathematical tools.

This is a didactically rather convenient position: Instead of trying to find a new mathematical tool when confronted with a new physical situation, we are now able to find a new mathematical tool describing well-known situations. This makes it much easier to grasp a new mathematical perspective.

The central question of course again is: How can we hide the signs of the special relativistic spacetime signature $(\gamma_t^2, \gamma_x^2, \gamma_y^2, \gamma_z^2) = (+, -, -, -)$?

8 The philosophy of frames

And there is another central question: Where do we live? Do we live in a world which is situated in a right-handed frame? Or do we live in a world which is situated in a left-handed frame? Obviously we do not know the answer. But the mathematics of general relativity hints to the strange fact, that we might live in a world which is situated in a right-handed and a left-handed frame at the same time.

To distinguish between these frames, mathematicians and physicists use different base vectors. A right-handed frame is indicated by covariant base vectors e_μ, using suffixes in a subscript position – or downstairs suffixes as Dirac calls them [5, chap. 1, pp. 1–3]:

$$e_0 = \gamma_t \qquad e_1 = \gamma_x \qquad e_2 = \gamma_y \qquad e_3 = \gamma_z \tag{20}$$

Using the summation rule of Einstein, a vector r in this right-handed coordinate system then is given by

$$r = ct\,\gamma_t + x\,\gamma_x + y\,\gamma_y + z\,\gamma_z = \sum_{\mu=0}^{3} x^\mu e_\mu = x^\mu\, e_\mu \tag{21}$$

balancing upstairs and downstairs suffixes. The coordinates of a right-handed frame therefore are contravariant entities, using suffixes in a superscript or upstairs position with

$$x^0 = ct \qquad x^1 = x \qquad x^2 = y \qquad x^3 = z \tag{22}$$

This is an absolute conceptual mess for beginners! Right-handed frames are sometimes indicated by physical entities with upstairs and sometimes with downstairs suffixes.

A left-handed frame in four-dimensional spacetime has either a reversed time direction (which cannot be measured with the clocks mankind is able to produce till now) or reversed spatial directions (which our rulers will correctly show). Einstein once decided for coordinate systems with reversed time direction [7, eqs. 91 & 91a, p. 80], but today reversed spatial directions

$$e^0 = \gamma_t \qquad e^1 = -\gamma_x \qquad e^2 = -\gamma_y \qquad e^3 = -\gamma_z \tag{23}$$

usually are preferred by textbook authors. Thus the base vectors of a left-handed coordinate system are contravariant entities e^μ with upstairs suffixes, while the corresponding coordinates are covariant entities x_μ with downstairs suffixes.

$$x_0 = ct \qquad x_1 = -x \qquad x_2 = -y \qquad x_3 = -z \tag{24}$$

They again combine into the well-known special relativistic position vector r:

$$r = ct\,\gamma_t - x\,(-\gamma_x) - y\,(-\gamma_y) - z\,(-\gamma_z) = \sum_{\mu=0}^{3} x_\mu e^\mu = x_\mu\, e^\mu \tag{25}$$

And co- and contravariant entities are balanced again. Now let’s analyze the signs of the spacetime vector square

$$r^2 = r\,r = r \bullet r \qquad \text{as} \qquad r \wedge r = 0 \tag{26}$$

with respect to the handedness of spacetime coordinate systems using the inner product of Geometric Algebra [10] and Spacetime Algebra [11]:

$$\begin{aligned} r^2 &= (x^0 e_0) \bullet (x_0 e^0) + (x^1 e_1) \bullet (x_1 e^1) + (x^2 e_2) \bullet (x_2 e^2) + (x^3 e_3) \bullet (x_3 e^3) \\ &= (e_0 \bullet e^0)\, x^0 x_0 + (e_1 \bullet e^1)\, x^1 x_1 + (e_2 \bullet e^2)\, x^2 x_2 + (e_3 \bullet e^3)\, x^3 x_3 \\ &= \delta_0^0\, x^0 x_0 + \delta_1^1\, x^1 x_1 + \delta_2^2\, x^2 x_2 + \delta_3^3\, x^3 x_3 \end{aligned} \tag{27}$$

In this situation the Kronecker delta δ_ν^μ [8, p. 14] is called metric tensor g_ν^μ by relativists. It encodes the structure of the co- and contravariant special relativistic base vectors:

$$g_\nu^\mu = \delta_\nu^\mu = e_\nu \bullet e^\mu = e^\mu \bullet e_\nu \tag{28}$$

In this case the metric tensor equals the Kronecker delta and the square of the position vector can be written without any base vectors, using co- and contravariant coordinates only:

$$r^2 = \sum_{\mu=0}^{3} g_\nu^\mu\, x^\nu x_\mu = g_\nu^\mu\, x^\nu x_\mu = \delta_\nu^\mu\, x^\nu x_\mu = x^\mu x_\mu = x_\nu x^\nu \tag{29}$$

And where are the signature signs? To reproduce them, eq. (29) has to be evaluated:

$$r^2 = x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3 = ct\,ct - x\,x - y\,y - z\,z = c^2 t^2 - x^2 - y^2 - z^2 \tag{30}$$

Eq. (30) clearly shows that the signs are now hidden behind our different coordinate systems. The spacetime vector square r^2 is composed simultaneously by right-handed and left-handed coordinates at the same time [15]. Thus special relativity written in the language of general relativity tells us that we do not live in a left-handed or in a right-handed coordinate system alone. We simultaneously live in a left-handed and right-handed coordinate system – and theses coordinate systems are inextricably linked together. Or expressed in more glorious words: Henceforth every left-handed coordinate system by itself, and every right-handed coordinate system by itself, is doomed to fade away into a mere shadow, and only a kind of union of left- and right-handed coordinate systems will preserve an independent reality.

The intrinsically woven coexistence of both coordinate systems in the mathematics of general relativity might even be the place, where we should look for quantum mechanics. Obviously even nature does not know whether we live in a right-handed or in a left-handed coordinate system, only allowing for a probabilistic answer in some situations. Experiments might show that we live in a right-handed coordinate system with probability $\rho(1)$ and simultaneously in a left-handed coordinate system with probability $\rho(-1) = 1 - \rho(1)$, resulting in a physical world governed by quantum mechanics [14, sec. 10]. The result of such a measurement then represents the mean value of "quantities made from spin" [16, sec. 10]:

$$\langle U \rangle = (-1)\,\rho(1) + (1)\,\rho(1) \qquad [16, p. 73] \tag{31}$$

9 The mathematics of relativity

Relativists are orderly people. They do not like to work with different coordinate systems of different handedness at the same time. Therefore they prefer to model the spacetime vector square r^2 of eq. (1) by using a right-handed (or a left-handed) coordinate system only:

$$\begin{aligned}
r^2 &= (x^0 e_0) \bullet (x_0 e^0) + (x^1 e_1) \bullet (x_1 e^1) + (x^2 e_2) \bullet (x_2 e^2) + (x^3 e_3) \bullet (x_3 e^3) \quad \textit{mixed coordinate systems} \\
&= (x^0 e_0) \bullet (x^0 e_0) + (x^1 e_1) \bullet (x^1 e_1) + (x^2 e_2) \bullet (x^2 e_2) + (x^3 e_3) \bullet (x^3 e_3) \quad \textit{unmixed coordinate system} \\
&= (e_0 \bullet e_0)\, x^0 x^0 + (e_1 \bullet e_1)\, x^1 x^1 + (e_2 \bullet e_2)\, x^2 x^2 + (e_3 \bullet e_3)\, x^3 x^3 \\
&= g_{00}\, x^0 x^0 + g_{11}\, x^1 x^1 + g_{22}\, x^2 x^2 + g_{33}\, x^3 x^3 \\
&= \sum_{\mu,\nu=0}^{3} g_{\mu\nu}\, x^\mu x^\nu = g_{\mu\nu}\, x^\mu x^\nu
\end{aligned} \tag{32}$$

with the special relativistic metric tensor

$$g_{\mu\nu} = \begin{pmatrix} e_0 \bullet e_0 & e_0 \bullet e_1 & e_0 \bullet e_2 & e_0 \bullet e_3 \\ e_1 \bullet e_0 & e_1 \bullet e_1 & e_1 \bullet e_2 & e_1 \bullet e_3 \\ e_2 \bullet e_0 & e_2 \bullet e_1 & e_2 \bullet e_2 & e_2 \bullet e_3 \\ e_3 \bullet e_0 & e_3 \bullet e_1 & e_3 \bullet e_2 & e_3 \bullet e_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{33}$$

But this is a rather artificial description. This time we force all physical laws into a left-handed coordinate system. The signs, formerly hidden in the base vectors of different coordinate systems, are now buried in the metric tensor $g_{\mu\nu}$. And it is a messy description, as the position vector r now is a combination of right-handed coordinates x^μ and left-handed base vectors e^ν.

$$r = g_{\mu\nu}\, x^\mu e^\nu = ct\, \gamma_t - x\,(-\gamma_x) - y\,(-\gamma_y) - z\,(-\gamma_z) \tag{34}$$

But it is a description, which is useful for curvilinear coordinates and curved space of general relativity, as most textbooks avoid mixed metric tensors g^μ_ν and predominantly work with pure covariant metric tensors $g_{\mu\nu}$ or pure contravariant metric tensors $g^{\mu\nu}$.

10 Geometric calculus

The Dirac operator

$$\Box = \sum_{\mu=0}^{3} e_\mu \frac{\partial}{\partial x_\mu} = e_\mu \partial^\mu = \sum_{\mu=0}^{3} e^\mu \frac{\partial}{\partial x^\mu} = e^\mu \partial_\mu \tag{35}$$

is no quantum mechanical operator, but an operator which just delivers geometric derivatives [6, eq. 6.2, p. 168]. Inserting eqs. (20), (22), (23), and (24) recovers the Lorentz operator of eq. (12). But the didactical aim should be to enable students to apply the Dirac operator in a more systematic way using the Einstein summation rule. One way to reach this aim is to let them translate the given examples into the mathematical language of general relativity presented above.

For instance, the geometric derivative of the closed spacetime curve

$$s = g_{00}\, x^0 e^0 + g_{11}\, x^1 e^1 e^2 e^3 = x_0\, e^0 + x_1\, e^1 e^2 e^3 \qquad s^2 = x_0\, x_0 + x_1\, x_1 \tag{36}$$

will then be evaluated by

$$\begin{aligned}
\Box\, s^2 &= e_\lambda\, \partial^\lambda\, (x_0\, x_0) + e_\lambda\, \partial^\lambda\, (x_1\, x_1) \\
&= 2\, e_\lambda \frac{\partial x_0}{\partial x_\lambda}\, x_0 + 2\, e_\lambda \frac{\partial x_1}{\partial x_\lambda}\, x_1 \\
&= 2\, e_\lambda\, \delta_0^\lambda\, x_0 + 2\, e_\lambda\, \delta_1^\lambda\, x_1 \\
&= 2\, x_0\, e_0 + 2\, x_1\, e_1 \\
&= 2\, ct\, \gamma_t - 2\, x\, \gamma_x
\end{aligned} \tag{37}$$

And the geometric derivative of the position vector square (32) then reads

$$\begin{aligned}
\Box\, r^2 &= e^\lambda\, \partial_\lambda\, (g_{\mu\nu}\, x^\mu x^\nu) \\
&= e^\lambda\, g_{\mu\nu} \frac{\partial x^\mu}{\partial x^\lambda}\, x^\nu + e^\lambda\, g_{\mu\nu}\, x^\mu \frac{\partial x^\nu}{\partial x^\lambda} \\
&= e^\lambda\, g_{\mu\nu}\, \delta_\lambda^\mu\, x^\nu + e^\lambda\, g_{\mu\nu}\, x^\mu\, \delta_\lambda^\nu \\
&= g_{\mu\nu}\, x^\nu\, e^\mu + g_{\mu\nu}\, x^\mu\, e^\nu \\
&= x_\mu\, e^\mu + x_\nu\, e^\nu \\
&= 2\, x_\mu\, e^\mu = 2\, r
\end{aligned} \tag{38}$$

In this way students are able to go their first, still unsure steps within the mathematics of general relativity, guided by results they already know from a special relativistic perspective.

11 Outlook

We now have discussed how special relativistic spacetime can be described and discussed within the mathematical framework of Clifford Algebra, wandering from the mathematics of special relativity – Spacetime Algebra [11] – to the mathematical perspective of general relativity. Although having closed a great gap – the missing Lorentz operator Minkowski designed – we are still at the beginning of our discussion. Compared with Dirac's influential GTR book, this paper just covers the content of the first, short chapter [5, chap. 1].

The next natural steps might follow the didactical structure of this very dense book: a discussion of oblique axes [5, chap. 2] in the language of Geometric Algebra [6, sec. 4.3], and then the great step to curvilinear coordinates [5, chap. 3], again discussed by [6, sec. 6.2] from the geometric perspective of Clifford Algebra.

Only after having done this, we could call the dream of Grassmann [9] to have become true.

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13 SCED Reviews

This paper was submitted to the Springer journal *Science & Education: Contributions from History, Philosophy and Sociology of Science and Education* in summer 2014. Half a year later the editors informed me that the manuscript cannot be accepted for publication. And they send me the following reviews, indicating that the review process at Springer journals is a rather absurd procedure.

Reviewer #1:

The paper "Bridging the Gap between School Mathematics and the Mathematics of General Relativity" is submitted for publication in the journal Science Education. As can be seen already from the title of this paper, the author addresses teaching of mathematics and general relativity at high school. My first remark is that the title of this paper is absolutely misleading. The paper has nothing to do with high school mathematics, and the mathematical structure of general relativity is completely out of the scope of high school curricula anyway. What the paper, interesting as it is, is about is even at the university level only accessible to those students with special interest in theoretical physics.

My second point in general is that the motivation of the author for this paper seems to me rather week. It relays on the famous lecture by Hermann Minkowski at the meeting of the "Gesellschaft Deutscher Naturforscher und Ärzte" in Cologne 1908 where the unification of space and time to spacetime was introduced for the first time. Contrary to the author I never understood this lecture in such a way that Minkowski required the same sign in front of the spacial coordinates and the time coordinate. Instead, all aspects of spacetime can be seen in the formulation of the theory with an indefinite metric as well.

Another point is that the article heavily relies on the book by Moshe Carmeli "Cosmological Special Relativity". This book is far away from the mainstream on related publications, which, of course, says nothing about its quality. But the reader should now about this at least.

Nevertheless, the submitted paper is interesting and worth to be published. But I would recommend that the author explained a few notions like geometric derivative (page 1), Dirac operator (page 2) and its quantum mechanical misinterpretation (page 3) and cosmological velocity according to Carmeli (page 3). Furthermore, it would be helpful to explain to the reader how the spacetime vectors in formulas (6), (7) (page 2) can be vectors when the basis vector are identified with the Dirac matrices.

Reviewer #2:

Typographical error in Section 5, second sentence. The manuscript says "We will have a pure space-like picture, if $x = t = v = 0$," but it appears that the author's intention was to have $z = 0$, not $x = 0$.

Reviewer #3:

I read this article while I was in France (where I live part of the year) and I thought I sent in a report. Certainly my notes on the article are there and unaccessible now. Checking my laptop I do not have a typed report, so my handwritten draft is probably buried on my desk in France. I am truly sorry about the lapse.

I do know that I found the paper to be a bit bizarre and I did recommend rejection. It was not clear who was the target audience for this very off beat paper. The writing of the paper is jerky and colloquial, and with a perspective that I found to be slightly outside of the norm in the physics world. The subject matter, in my opinion, does not fit in well with education and general relativity.

For a blunt summary I would say that the subject matter is very much outside the realm of what your readers would find interesting. I think that the historical perspective on events does not reflect the present views of the relativity community (and the subset of those who worry about relativity education issues). The physics examples are abstract and of minor interest to most. I would advise rejecting this paper."