# The Anomalous Nambu-Goldstone Theorem in Relativistic/Nonrelativistic Quantum Field Theory. II

Tadafumi Ohsaku

today

**Sehet, welch ein Mensch ( Ecce Homo )** ( Johannes, Kapitel 19 ).

## 1 Introduction

Theory of spontaneous symmetry breaking is one of the important results in modern theoretical physics, and it is expressed by the Nambu-Goldstone ( NG ) theorem [1,4,6,9,13,14,18,28,31,35,36,38,39,40,41,42,51,56,57,58,59]. The NG theorem is classified into the three categories. The NG theorem for a system where it has an exact continuous symmetry ( i.e., a Lie group defined over the real number field with a continuous topology ) from the beginning is the ordinary NG theorem [13,14,35]: It should be called as the normal Nambu-Goldstone ( NNG ) theorem [40,41]. All of the NG bosons must be massless in the NNG case. If the system contains an explicit symmetry breaking parameter and a VEV develops toward the same direction broken by the parameter, then the symmetry breaking phenomenon is described by the generalized Nambu-Goldstone ( GNG ) theorem [9,40]: In that case, the NG bosons associated with the symmetry breaking have finite masses. One of the important examples of such type of situations is the flavor symmetry which is explicitly broken by the current mass matrix of quarks. The anomalous NG ( ANG ) theorem is found in a Lorentz-symmetry-violating systems, such as a ferromagnet, or a relativistic model with a finite chemical potential [4,6,10,18,29,31,36,38,40,51,56-59]. In ANG, a subset of the

NG boson space acquire finite masses under a certain mechanism, while the complement of the subset gives massless bosons.

As the Part II of this paper ( we call Ref. [41] as Part I ), here, we investigate the structure of a low-energy effective theory for a situation of the ANG theorem. In this paper, we mainly consider the case where the topology of the base space is continuous, the same with $\mathbf{R}^n$ or $\mathbf{C}^n$. In principle, it is possible that a field theory is defined over a space with several topologies simultaneously.

The crucial point of the mechanism of the ANG case is found by Nambu himself, that an emergence of a Heisenberg algebra coming from a semisimple Lie algebra symmetry ( such as Lie($SU(N)$), Lie($SO(N)$), ... ) takes place [36]. In Part I of this paper, we have revealed that the statement of Nambu is generically observed in a diagonal breaking of $SU(N)$ ( where only the Cartan subalgebra remains unbroken in the Cartan decomposition ), and a (quasi-)Heisenberg algebra relation is obtained. It is also shown in the Part I that the (quasi-)Heisenberg algebra gives an uncertainty relation between a canonical conjugate pair obtained from a pair of Lie algebra generators in the Cartan decomposition ( so-called canonical basis ), and it is realized in the geometric structure of the one-loop effective potential of kaon condensation.

# 2 Theoretical Structure of the Ground States of the NNG/GNG/ANG Theorems

In this section, we discuss characteristic features of the ground states of the phenomena of broken symmetry. The infinite-order degeneracy of ground state of the NNG theorem is partially ( in general, but sometimes completely in a GNG case ) lifted in the GNG or the ANG theorems, by some mechanisms, revealed in Ref. [40] and the Part I ( see also, Ref. [9] ). The effective potential of the NNG case implicitly contains a coordinate system of the NG manifold, while that of the GNG and the ANG theorems have ( a part of ) them explicitly. The vacuum state of the NNG/GNG/ANG theorems is not uniquely defined over the coordinate system of a Lie group in general: The uniqueness of vacuum is usually discussed ( and, proved ) by an examination on whether the ground state eigenvalue have a degeneracy for the lowest energy eigenvalue. In a case of the NNG, the energy of the ground state of an effective potential is not changed under a Lie group operation ( usually given as an adjoint action to the order parameter ) of broken symmetry, while it is changed in a case of GNG at the tree level, and such a change is started from the one-loop level in a loop expansion in the ANG theorem ( shown by an example, in Ref. [41] ). Let us consider an effective potential of a loop expansion schematically given as follows:

$$V_{eff}(\Phi) \quad = \quad V^{(0)}_{eff}(\Phi) + V^{(1)}_{eff}(\Phi) + \cdots . \tag{1}$$

Here, $\Phi$ is an order parameter of non-vanishing VEV, (0) denotes the tree level, while (1) indicates the one-loop level. Then,

$$V^{(n)}_{eff}(\Phi) \quad = \quad V^{(n)}_{eff}(\mathrm{Ad}g(\Phi)), \quad \forall g \in G, \quad \forall n \in \mathbf{Z}_{\geq 0} \quad \text{(NNG)}, \tag{2}$$

$$V^{(n)}_{eff}(\Phi) \quad \neq \quad V^{(n)}_{eff}(\mathrm{Ad}g(\Phi)), \quad \exists g \in G, \quad n \geq 0 \quad \text{(GNG)}, \tag{3}$$

$$V^{(n)}_{eff}(\Phi) \quad \neq \quad V^{(n)}_{eff}(\mathrm{Ad}g(\Phi)), \quad \exists g \in G, \quad n \geq 1 \quad \text{(ANG)}. \tag{4}$$

Here, $G$ indicates a Lie group. Namely, the dependence of $V_{eff}$ on a group operation distinguishes the NNG, GNG, and ANG cases. It should be noticed that, not all of the broken generators can cause such a dependence in the GNG case, since there is a flat direction in an effective potential defined by a linear combination of the broken generators, in general ( see, Ref. [40] ). From the result of Part I, we know that the dependence of $V_{eff}$ on an adjoint action of broken generator in the ANG is caused by an emergent (quasi-)Heisenberg algebra at the one-loop level, while if the algebra is not generated at the one-loop level, it cannot be obtained in any higher-order contribution: Hence,

**Theorem**: *The dependence on an adjoint action of broken generator in the effective potential of the ANG is found at the one-loop level, namely, the first-order displacement given by the Lie group action on the order parameter. If the dependence is not caused at the one-loop level, it never takes place in the higher-order contributions of the expansion of the effective potential.*

In the frequently used discussion on the stability and uniqueness of the ground state of a theory, one introduces a Hamiltonian $H$ and a vacuum state $|0\rangle$, and argues such that the vacuum is unique up to a multiplication of a constant. Such a constant is restricted in a case of symmetry breaking, i.e., cases of NNG, GNG, and ANG theorems. In the NNG theorem, a group element $g = e^{iS} \in G$ can be regarded as a constant since it does not cause any change of the eigenvalue of $H$ against the vector $g|0\rangle$. On the contrary, such a group element cannot be regarded as a trivial multiplicative constant in the GNG and ANG, in general. Thus,

**Theorem**: *The geometric structure of an NG manifold defined over an effective potential restricts/distinguishes the type of multiplication constant for an eigenvector of the Hamiltonian of the system.*

Since a definition of a "trivial" multiplication constant determines a linear algebra of a Hilbert space, this theorem states that the Hilbert spaces of the NNG, GNG, ANG are somewhat different with each other. Moreover, this difference can be characterized by constructing principal bundles by using $G$ and $\Phi$ ( trivial bundles ), and the bundles may be classified by some operator algebra apparatus ( especially, $C^*$ and von Neumann algebras ). From this perspective, some works of Ojima are interesting for us [42], and we will be investigated this "simple but deep" theorem further in elsewhere.

It must be emphasized that the Nambu-Goldstone theorem gives not only a breakdown of a symmetry ( a reduction of symmetry ) but sometimes causes another new symmetry ( emergence of a symmetry ) in a physical theory. Such type of symmetries includes a Heisenberg group or a Galois group where they are not contained in the symmetry of the beginning of the theory [40,41]. This fact is not covered in the ordinary/naive formula of the NG theorem.

# 3 Nonlinear Sigma Models in the ANG Theorem

Now, we propose an effective Lagrangian which may capture the nature of the ANG theorem, for describing physics in the vicinity of the ground state. This Lagrangian models the low-energy excitations of the one-loop result of the kaon condensation model discussed in Part I of the ANG theorem, and we argue the model Lagrangian is also useful to understand a ferromagnet of $SU(2) \to U(1)$ since the nature of broken symmetries are quite similar in the both cases ( kaon condensation and ferromagnet ) [41]. With respect to the result of Part I, a nonlinear sigma model of $SU(2) \to U(1)$ ferromagnet, in which the NG bosons take their values on the unit sphere $S^2$, in the ANG theorem may be given as follows:

$$\mathcal{L}_{ANG}^{S^2} = \sum_{j=1}^{3}\Big(\partial_\nu\chi_j\Big)^2 + \lambda(\sum_{j=1}^{3}\chi_j^2 - 1) - \Big(m^2 e^{i\sqrt{\chi_1^2+\chi_2^2}} + c.c. + 2m^2\Big) \quad (5)$$

With respect to the result of Part I, we have chosen that $\sqrt{\chi_1^2+\chi_2^2} = \pi$ gives a stable point. Note that the $\chi_3$ direction, which the VEV of ferromagnetism develops, is special in the Lagrangian. Thus, we should say the space the sigma model takes its value is "topologically" the same with $S^2$. The shape of sphere may be deformed, which would be clarified by a Ricci flow of the one-loop level [44,45,46]. Since the model contains the mass parameter $m \in \mathbf{R}^1$ ( which has been introduced with respect to the result of Part I of the ANG theorem ) explicitly, the massive particles are relatively suppressed in its one-loop correction ( radiative correction ), the Ricci flow of the metric of the model cannot develop under keeping the "ideal" isotropy of a sphere even if the flow starts from an isotropic sphere of constant curvature: Such a Ricci flow does not have an absolute meaning, quite "relative." In other words, the mass parameter $m$ is not "geometric", causes a deviation from the target space geometry. Thus, we obtain an insight that the flow approaches to the unit sphere of isotropic $S^2$ at the UV region where $m$ can be neglected and the asymptotic freedom may be found ( especially in the case of two-dimensional spacetime ), while the $S^2$ will be "smashed"and a dimensional reduction in the target space takes place when the flow approaches to a low energy region, may give effectively a circle $S^1$. We can say an isotropic Ricci flow is ideal, only for the NNG case, and a Ricci flow becomes anisotropic in the GNG or ANG theorems. The flow may have a dependence on the starting point of a time evolution. After introducing the spherical coordinates,

$$\chi_1 = r\cos\theta\cos\phi, \quad \chi_2 = r\cos\theta\sin\phi, \quad \chi_3 = r\sin\theta, \quad (6)$$

one gets

$$\begin{aligned}\mathcal{L}_{ANG}^{S^2} &= \Big(\partial_\nu r\Big)^2 + r^2\Big[\Big(\partial_\nu\theta\Big)^2 + \cos^2\theta\Big(\partial_\nu\phi\Big)^2\Big] \\ &\quad -\Big(m^2 e^{ir\cos\theta} + c.c. + 2m^2\Big) + \lambda(r^2-1). \quad (7)\end{aligned}$$

In this Lagrangain, $r$ and $\theta$ modes have finite masses. These $r$ and $\theta$ have self-couplings in the Lagrangian, while the $\phi$ mode acquires a self-coupling through $r$ and $\theta$, relevant at the large-momentum region. The kinetic term of $\theta$ and $\phi$ will vanish at $r = 0$. In the case $\cos\theta = 0$, the kinetic term of $\phi$ vanishes, and the kinetic energy does not contribute to the partition function at $\theta = (n+1/2)\pi$ ( $n \in \mathbf{Z}$ ). From this form of the Lagrangian, it is apparent for us that the low-energy excitation defines a circle $S^1$ parametrized by $\phi$. We know from the result of Part I that $\chi_1^2 + \chi_2^2 = r^2\cos^2\theta = \mathrm{const}$ gives a set of stationary points of the system, while the phase of $\chi_1 + i\chi_2$ ( namely, the massless mode $\phi$ ) on a Gaussian plane is completely undetermined and equivalent ( $V_{eff}$ is flat ). In general, a diagonal breaking of $SU(N)$ in the ANG theorem gives $n_{NG} = [\dim G - \mathrm{rank} G]/2$ Heisenberg pairs, shown by the Lie algebra ( Cartan ) decomposition by $sl_2$-triples. It is shown in Part I that a Heisenberg pair gives an uncertainty relation given over a submanifold of the NG sector: One direction of the pair is "localized", while other direction is "delocalized" in the case of $SU(2)$ ferromagnet. ( More precisely, the pair $(S_1, S_2)$ forms the three-dimensional Heisenberg algebra, and linear combinations of them show such an uncertainty relation. ) This situation has some similarity with a metal-insulator transition [27], which takes place in the $r$-direction in the case given above. Then the uncertainty relation determines the form of a nonlinear sigma model of the NG sector: The low energy excitation of it gives a product of $n_{NG}$ circles. This means the relevant dynamical degrees of freedom is reduced at low energy region in the ANG theorem. Therefore, a Ricci flow of the nonlinear sigma model cannot give an isotropic sphere, the flow is deformed in general. From these observations, the Lagrangian at the low-energy low-temperature limit becomes,

$$\mathcal{L}_{ANG}^{S^2} \sim \langle r^2\cos^2\theta\rangle \Big(\partial_\nu\phi\Big)^2, \tag{8}$$

which is defined over an $S^1$. In the diagonal breaking of $SU(N)$, the sigma model is defined over

$$\underbrace{S^1 \otimes \cdots \otimes S^1}_{n_{NG}}, \tag{9}$$

and the effective sigma model becomes,

$$\mathcal{L}_{ANG}^{\prod_{n_{NG}} S^1} \sim \sum_{j=1}^{n_{NG}} (e_j^2 + f_j^2)\Big(\partial_\nu\phi_j\Big)^2. \tag{10}$$

Here, we denote a Cartan decomposition of the Lie algebra as $(h_i, e_j, f_k)$, where $h_j$ gives the Cartan subalgebra [12,16,22]. From the perspective of quasi-Heisenberg algebra we have revealed in Part I, this result implies that the symplectic vector space given by the Heisenberg pairs will be decomposed into a direct sum of two-dimensional symplectic subspaces at the low-energy limit. The localization takes place in $n_{NG}$-directions.

Since a Heisenberg group is a nilmanifold, its Killing form ( the kinetic term of the sigma model Lagrangian ) vanishes. Thus, it seems difficult to consider a Heisenberg group as a target space of a nonlinear sigma model. However, if we consider the following correspondence,

$$S_1 \to x, \quad S_2 \to p, \quad S_3 \to \hbar, \quad [x,p] = i\hbar, \tag{11}$$

then the following Lagrangian can be introduced:

$$\mathcal{L}_H \;=\; \Big(x \otimes \partial_\nu \phi_1\Big)^2 + \Big(p \otimes \partial_\nu \phi_2\Big)^2 + \Big(\hbar \otimes \partial_\nu \phi_3\Big)^2. \tag{12}$$

Where, the last term of $\mathcal{L}_H$ might be given as a result of a ( deformation ) quantization. This form of sigma model has an interesting aspect as its own right, since the quadratic part in terms of $x$ and $p$ can be diagonalized/solved by the similar method of a harmonic oscillator. Another possibility for candidates for an effective Lagrangian is given by using [26]

$$Z = \partial_z + i\bar{z}\partial_t, \quad \overline{Z} = \partial_{\bar{z}} - iz\partial_t, \quad T = \partial_t, \quad [Z, \overline{Z}] = -2iT, \tag{13}$$

then,

$$\mathcal{L}_H \;=\; \Big((\partial_z + i\bar{z}\partial_t) \otimes \partial_\nu \phi_1\Big)^2 + \Big((\partial_{\bar{z}} - iz\partial_t) \otimes \partial_\nu \phi_2\Big)^2 + 4\Big(T \otimes \partial_\nu \phi_3\Big)^2 \tag{14}$$

Note that this form is only meaningful at $T \neq 0$ due to the definition and relation of the generators of the Heisenberg algebra.

For example, let us consider the case of algebra of Galilei group,

$$\begin{aligned}
&[X_i, X_j] = [P_i, P_j] = 0, \quad [X_i, P_j] = i\hbar\delta_{ij}, \\
&[L_i, X_j] = i\hbar\epsilon_{ijk}X_k, \quad [L_i, P_j] = i\hbar\epsilon_{ijk}P_k, \\
&[L_i, L_j] = i\hbar\epsilon_{ijk}L_k, \quad [S_i, S_j] = i\hbar\epsilon_{ijk}S_k, \quad (i,j,k = 1,2,3).
\end{aligned} \tag{15}$$

If the VEVs of the spatial angular momenta become

$$\langle [L_1, L_2] \rangle = i\hbar \langle L_3 \rangle \neq 0, \quad \langle [L_2, L_3] \rangle = \langle [L_3, L_1] \rangle = 0, \tag{16}$$

then $(L_1, L_2)$ can be treated as a canonical conjugate pair. In this case, the Casimir element becomes

$$L_1^2 + L_2^2 + L_3^2 \sim X^2 + P^2 + \langle L_3 \rangle^2, \tag{17}$$

and it seems like a Hamiltonian of harmonic oscillator, with the square of the VEV $\langle L_3 \rangle^2$ as a part of the vacuum energy. Such a ( finite dimensional, finite degrees of freedom ) harmonic oscillator Hamiltonian generally arises in a diagonal breaking of $SU(N)$. Since it gives $n_{NG} = [\mathrm{dim}G - \mathrm{rank}G]/2$ Heisenberg pairs, the Casimir operator as an ensemble of harmonic oscillators is

$$H = \sum_{j=1}^{n_{NG}} (P_j^2 + X_j^2) + \mathrm{const.} \tag{18}$$

Here, $X_j$ and $P_j$ are coming from the broken generators of $SU(N)$, must not be confused with the Galilei algebra (15). However, after the conversion from Lie($SU(N)$) to the corresponding quasi-Heisenberg algebra via the ANG theorem, they ( $X_j$ and $P_j$ ) can be interpreted as the generators of Galilei algebra for $n_{NG}+1$-dimensional spacetime. Therefore, we can say such a Galilei group acts on the quasi-Heisenberg algebra, as a symmetry of the quasi-Heisenberg algebra:

Lie($SU(N)$) $\to$ quasi-Heisenberg algebra $\to$ Galilei algebra.

From the example $SU(2) \to U(1)$ of ferromagnet, we know the fact that $\mathrm{Arg}(S_1 + iS_2)$ is the massless mode while $|S_1 + iS_2|$ is massive and "localized" in the space of NG boson fields $(S_1, S_2)$. By using the radial coordinates $S_1 = X = r\cos\theta$ and $S_2 = P = r\sin\theta$, $S_1^2 + S_2^2 = X^2 + P^2 = r^2$ is obtained. Thus, the Hamiltonian of harmonic oscillators describes some degrees of freedom of Heisenberg pairs given as radial directions well localized by the uncertainty relations of pairs. A linear ( canonical ) transform of canonical variables $(X_i, P_j)$ corresponds to a transform between the generators of the Lie algebra. By employing

$$a_j = \sqrt{\frac{\omega_j}{2\hbar}} X_j + \sqrt{\frac{-1}{2\hbar\omega_j}} P_j, \tag{19}$$

$$N_j = a_j^\dagger a_j, \tag{20}$$

then the Casimir element is expressed as

$$H \sim 2\sum_{j=1}^{n_{NG}} \hbar\omega_j \Big(N_j + \frac{1}{2}\Big). \tag{21}$$

An interesting subject is the relation between the harmonic oscillators and the arithmetic number theory of quadratic forms of Weil [60]. They also may relate with a harmonic analysis of a Heisenberg group [20] via an integral,

$$\int e^{iH(X,P,\langle S_3\rangle)} f(X,P)dXdP. \tag{22}$$

Another interesting feature is found in our interpretation of the Bost-Connes-like model Hamiltonian ( a Riemann gas model of the Riemann zeta function ) [5] by our ANG theorem: The model Hamiltonian is a logarithm of our (18) of the bosonic oscillators. Thus, in our insight, the Bost-Connes Hamiltonian is subjected by operations of the Galilei algebra.

An interesting fact is that, due to the dynamical generation of a Heisenberg algebra Lie$(H_3)$ ( an algebra of Poisson-Lie group ) in the ferromagnet, one can consider a deformation quantization [3,48] in the nonlinear sigma model. The symplectic structure is defined by

$$\langle [S_1, S_2] \rangle = -\langle [S_2, S_1] \rangle. \tag{23}$$

( $S_1$ and $S_2$ are x and y components of spin variables, respectively. ) Here, Lie$(H_3)$ is decomposed as $(S_1, S_2) \oplus S_3 \simeq \mathbf{R}^2 \oplus \mathbf{R}^1$, and then the pair $(S_1, S_2)$ acquires an independent automorphism of the algebra which preserves the symplectic structure: This is an explicit realization of the broken isotropy of $SU(2)$. Thus, the symmetry conversions ( emergences ) are

$SU(2) \to$ Lie$(H_3) \supset$ symplectic symmetry.

$(S_1, S_2)$ form Darboux coordinates, and this means that we ( can ) choose a locally defined Euclidean space $\mathbf{R}^2$. Thus, $\nabla = d$ holds ( $\nabla$: connection ). A Weyl manifold can be defined by introducing a trivial bundle $M \times \mathcal{W}$, where $\mathcal{W}$ is a Weyl algebra coming from $(S_1, S_2, S_3)$ and $M$ denotes the two-dimensional symplectic manifold. Note that the quantization will be introduced in the algebra of generators $(S_1, S_2)$ of the symmetry $SU(2)$, namely an internal symmetry, and not into the scalar fields ( NG bosons ) themselves. Thus, it might be called as a quantization of gauge degree of freedom. Here, a reader must carefully consider. It may be called as a "quantum" and/or "Heisenberg" gauge. This gauge degree of freedom does not have its classical counterpart, and it arises by a quantization ( a "quantum symmetry" ). Since a geometric object maintains a symmetry, such a "quantum symmetry" might be realized in a "quantum geometry" with a "quantum algebra.".

The result of Kontsevich states that there is a deformation quantization in an arbitrary Poisson manifold ( a Poisson structure defined over a Euclidean space $\mathbf{R}^n$ ), a one-to-one correspondence exists [24,25]. Cattaneo and Felder re-express the result of Kontsevich by a path integration of Poisson sigma model which has a role in string theory [7,8]. The NG boson fields $(\chi_1, \chi_2)$ defines a Poisson ( precisely, symplectic ) structure, then they have a deformation quantization, and it will be rewritten in a path integral form of a Poisson sigma model. ( Now, a ferromagnet connects with string theory! ) In our case of the sigma model of a ferromagnet, the Poisson structure and its quantization depends on the VEV $\langle S_3 \rangle$, and thus we find a family of the deformation quantizations. For example, the following operator is introduced as a Moyal-Weyl star product [24,25,33]:

$$* = \exp\left[\frac{\langle S_3 \rangle}{2}\left(\overleftarrow{\partial_{S_1}}\overrightarrow{\partial_{S_2}} - \overleftarrow{\partial_{S_2}}\overrightarrow{\partial_{S_1}}\right)\right]. \tag{24}$$

Note that this operator is defined in the vicinity of the VEV $\langle S_3 \rangle$, and it lost the meaning at the vanishing VEV ( $S_3$; the z-component of spin variables ). Of particular importance/interesting here is the fact that there is a quantum fluctuation toward the direction of the VEV, and thus, we find a superposition of the *-products, caused by a set of displacements of the VEV such like $\langle S_3 \rangle + \delta\langle S_3 \rangle$ ( $\delta$ implies a quantum fluctuation ), after releasing the constraint of the unit sphere $S^2$ in the sigma model. So called sigma mode in the sigma model is given by $\delta\langle S_3 \rangle$. If a VEV such as $\langle S_3 \rangle$ is determined by a potential of $\lambda\phi^4$ type, $V \sim -m^2\phi^2 + \frac{\lambda}{2}\phi^4$, then, the VEV is parametrized as $\pm\sqrt{m^2/\lambda}$. Thus, the star product is ( or, may be ) singular at $\lambda = 0$: This fact is somewhat similar with the gap function of superconductivity which shows a non-perturbative effect. Moreover, when $S_3$ is coupled with an external ( magnetic ) field, the situation becomes the case of GNG theorem ( since the magnetic field acts as an explicit symmetry breaking parameter ), which should become an ANG case at the vanishing limit of the magnetic field. Such an external field can be stochastic, then the Heisenberg algebra $\langle [S_1, S_2] \rangle = i\langle S_3 \rangle$ also acquires the stochasticity, which is anisotropic in the $SU(2)$ space: The "Planck constant" is now stochastic [12,43,49]. Then, if $f(S_i)$ and $g(S_i)$ are functions of the spin variables, their quantization ( quantized algebra ) is obtained by

$$f(S_i) * g(S_i) \quad = \quad f(S_i)\exp\left[\frac{\langle S_3 \rangle}{2}\left(\overleftarrow{\partial_{S_1}}\overrightarrow{\partial_{S_2}} - \overleftarrow{\partial_{S_2}}\overrightarrow{\partial_{S_1}}\right)\right] g(S_i)$$

$$= \int \mathcal{D}X\mathcal{D}\eta e^{iS[X,\eta]} fg. \qquad (25)$$

Here, $S[X,\eta]$ indicates the classical action of Poisson sigma model [8]. It should be mentioned that the equivalence class of a deformation quantization ( defined by using a Hochschild cohomology, and the Deligne's relative class ) [24] can be found also the "quantization" prescription of our sigma model: Namely, it depends on (i) the value and direction of VEV ( vacuum alignment ), (ii) the equivalence class and Deligne's relative class of *-products. Since both $SU(2)$ and $H_3$ are embedded into $SU(2,1)$ [26], the deformation quantization discussed here acts as a functor between the origins ( tangent spaces ) of $SU(2)$ and $H_3$. Since the Heisenberg algebra is a central extension,

$$0 \to \mathbf{R}^1 \to \mathrm{Lie}(H_{2n+1}) \to \mathbf{R}^{2n} \to 0, \qquad (26)$$

such a path integral or a deformation quantization perturbatively gives the algebra extension to a commutative algebra. Due to the Stone-von Neumann theorem of quantum mechanics [50] of a finite degrees of freedom, an irreducible representation is uniquely determined up to unitary transformations ( a unitary equivalence ) by choosing the element of central extension ( now it is a VEV $\langle S_3 \rangle$ ). Those irreducible representations are defined on any point of the effective potential $V_{eff}$ parametrized by $\langle S_3 \rangle$. The transformation law ( if it exists ) between those irreducible representations of *different* VEVs may not be given as a simple/naive unitary transform. It is interesting for us to consider a theta function representation of a Heisenberg group in our ANG theorem [34]: It might be considered after globalizing the Heisenberg algebra to the Heisenberg group in our ANG theorem. This subject relates number theory, modular forms [17], upper half spaces, various types of theta functions ( especially, higher-dimensional analogues ), and ( generalized ) Heisenberg groups. In general, an equivalence class of deformation quantizations is defined by $T(f * g) = T(f) *' T(g)$ with $(C^\infty(M)[[\nu]], *) \simeq (C^\infty(M)[[\nu]], *')$, classified by the cohomology $H^2_{\mathrm{deRham}}(M, \mathbf{R})[[\nu]]$ ( $M$: a symplectic manifold ). Thus, it is defined by the same "$\nu$". In our context of this paper, this equivalence means a local aspect of a deformation, namely, it is locally defined over an NG manifold ( any point of the space of $V_{eff}$ given by a specified/fixed value of $\langle S_3 \rangle$ ), since the NG manifold ( now it is embedded in $V_{eff}$ ) has $S_j$ as its local coordinate system. By the result of Kontsevich, the deformation quantization will be uniquely defined if an NG manifold has a symplectic structure uniquely and globally.

From the perspective of deformation quantization, the Weyl representation of a Heisenberg group is interesting for us. Let $G$ ( $g \in G$ ) be a locally compact commutative group, $\widehat{G}$ ( $\gamma \in \widehat{G}$ ) be its dual. Let $\mathcal{H}$ be a Hilbert space, and let $U$ be a unitary representation of $G$ over $\mathcal{H}$, $V$ be a unitary representation of $\widehat{G}$. Then the Weyl representation of a Heisenberg group gives,

$$V(\gamma)U(g) = \gamma(g)U(g)V(\gamma). \tag{27}$$

This representation is remarkable since in the ferromagnet, effectively,

$$e^{i\alpha S_1}e^{i\beta S_2} = e^{-iS_3\alpha\beta}e^{i\beta S_2}e^{i\alpha S_1} \tag{28}$$

is satisfied. Namely, the Heisenberg pair $(S_1, S_2)$ is re-defined as a dual pair of locally compact commutative topological group.

In our ANG theorem, the ( quasi ) Heisenberg algebra emerged from a Lie algebra of internal symmetry is usually finite-dimesional. ( Namely, quantum mechanical, not quantum field theoretical. ) For example, in the case of $SU(2) \to U(1)$, a one-dimensional quantum mechanics $(X, P)$ arises. This fact indicates that the dynamical effect caused by the ( quasi ) Heisenberg algebra in a system of broken symmetry may be well described by a quantum mechanics, WKB, and classical mechanics of a point particle. Here, we must mention that, even though such a "finiteness" exists in a ( quasi ) Heisenberg algebra of an NG sector, the algebra of local field operators of the NG bosons does not show the von Neumann uniqueness theorem ( in fact, a unitary equivalence is violated ). In other words, one can say a finiteness and an infinity coexist in the NG sector of the ANG theorem by unitary (in)equivalence.

# 4 Around The Nambu Mechanics

From the symplectic structure of ANG theorem, it is interesting for us to enlarge our ANG theorem to the Nambu-Poisson manifold of the Nambu mechanics [37,54]. The Nambu mechanics is defined by several Hamilton functions ( conserved quantities ) to give a set of equations of motion of canonical variables. For example, we can prepare a usual Hamiltonian with a second-order Casimir invariant of a Lie algebra to construct a Nambu mechanics. ( An interesting fact is that a Casimir invariant of a Lie algebra is related with a Casimir energy via the mathematics of zeta functions and the Riemann hypothesis. ) The Nambu-Heisenberg equation and the Nambu bracket are defined by

$$i\frac{d}{dt}F = [F,H,G], \tag{29}$$

$$[A,B,C] = [A,B]C + [B,C]A + [C,A]B = \pm i\hbar, \quad \hbar = 1. \tag{30}$$

( We have chosen the unit $\hbar = 1$. ) Here, $H$ and $G$ denote Hamilton functions, and we assume them as Hermitian and lower-bounded. The set $(A,B,C)$ give the Nambu triplet of canonical variables. Of course, one can choose both of signs $\pm i\hbar$, and this implies the quantum mechanics ( both the ordinary Hamilton mechanics and the Nambu mechanics ) has a symmetry of Galois extension $Gal(\mathbf{C}/\mathbf{R})$. This formalism can be considered in the case of ferromagnet in our ANG theorem when we choose

$$\langle [S_1,S_2,S_3]\rangle = i, \quad \langle S_1^2\rangle = \langle S_2^2\rangle = 0, \quad \langle S_3^2\rangle \neq 0, \tag{31}$$

$$H = H(S_1,S_2,S_3), \quad G = G(S_1,S_2,S_3). \tag{32}$$

In the case of Lie($SU(2)$), $[S_1,S_2,S_3] = i(S_1^2+S_2^2+S_3^2)$ ( proportional to the second-order Casimir element ), and thus the above situation $\langle S_1^2\rangle = \langle S_2^2\rangle = 0$, $\langle S_3^2\rangle = 1$ ( $=\hbar$ ) is the same with the unit sphere condition of the nonlinear sigma model. Hence, under a rotation of the direction of the VEV $\langle S_3\rangle$ ( which may belong to $SL(3,\mathbf{R})$ or $SL(3,\mathbf{C})$ discussed by Nambu as Lie groups of canonical transformations ) does not cause any problem for the quantum Nambu bracket. Here, $H$ and $G$ take their value in Lie($SU(2)$), conserved quantities, and polynomials of the generators of $SU(2)$. In the frequently used formalism of the Nambu mechanics, $H$ is an ordinary Hamiltonian, while $G$ is a second-order Casimir invariant. However, we can say any type of analytic functions of $H$ and $G$, such like $F_1(H,G)$ and $F_2(H,G)$, are also conserved under satisfying a certain condition, utilized for our formalism of the Nambu mechanics: Namely,

$$F_1 = F_1(H_1,H_2,t), \quad F_2 = F_2(H_1,H_2,t), \quad i\frac{d}{dt}\Phi = [\Phi,F_1,F_2]. \tag{33}$$

Here, $H_1$ and $H_2$ are assumed as conserved quantities. We formally consider the classical counter part of the Nambu mechanics: The classical Nambu bracket of canonical triplet and the Nambu-Hamilton equation of motion are defined as follows:

$$\{F_1, F_2, F_3\} = \frac{\partial(F_1, F_2, F_3)}{\partial(S_1, S_2, S_3)}, \quad F_l = F_l(S_1, S_2, S_3), \quad (l = 1, 2, 3), \tag{34}$$

$$\frac{d}{dt}F = \{H_1, H_2, F\}. \tag{35}$$

The classical Nambu bracket $\{A, B, C\}$ is a generalization of a Poisson-Lie algebra: One may call it as "the Nambu-Poisson-Lie algebra" ( even though the operation is ternary ). The symplectic form $\omega$ and the volume form $v$ are given by the dual basis of canonical variables:

$$\omega = dS_1 \wedge dS_2, \quad v = dS_1 \wedge dS_2 \wedge dS_3. \tag{36}$$

Note that now $(S_1, S_2)$ gives Darboux coordinates. Hence, a conserved quantity defined over the Darboux coordinate system can be regarded as a Hamiltonian. For example,

$$H(S_1, S_2) \;\; = \;\; \frac{1}{2}\langle S_2\rangle^2 + V(\langle S_1\rangle). \tag{37}$$

Then the Hamilton vector field, the equation of motion, and the Poisson bracket are formally defined as follows [2]:

$$X_H \;\; = \;\; -\frac{\partial H}{\partial S_2}\partial_{S_1} + \frac{\partial H}{\partial S_1}\partial_{S_2}, \tag{38}$$

$$\frac{d}{dt}(S_1, S_2) \;\; = \;\; -X_H(S_1, S_2), \tag{39}$$

$$X_f g \;\; = \;\; \{f, g\}, \quad [X_f, X_g] = X_{\{f,g\}}. \tag{40}$$

The diagonal breaking of $SU(N)$ generally gives a (quasi-)Heisenberg algebra, and thus they will be embedded into the formalism of Nambu $n$-plet. For example, the set of differential operators $(Z, \overline{Z}, T)$ given above can also be utilized to define a quantum Nambu bracket $[Z, \overline{Z}, T]$ and Hamiltonians $(H(Z, \overline{Z}, T),\, G(Z, \overline{Z}, T))$. In this case, the Hamilton vector field is

$$X_H \;\; = \;\; -\frac{\partial H(Z, \overline{Z})}{\partial \overline{Z}}\partial_Z + \frac{\partial H(Z, \overline{Z})}{\partial Z}\partial_{\overline{Z}}. \tag{41}$$

This expression may have an interesting implication in geometry and theory of Riemann surfaces [26]. A possible generalization of the Nambu mechanics might be obtained by the Jacobian between two Haar measures:

$$\frac{d}{dt}g' \;\; = \;\; \frac{\partial(g')}{\partial(g)}, \quad \int_G f(g)dg \to \int_G f(g')dg'. \tag{42}$$

Some interesting aspects of the formalism of Nambu mechanics will be found ( or, enlarged ) when we introduce notions of functional analysis. Let us consider a quantum Nambu mechanical system of two Hamiltonians. For example, we consider a unitary operator ( propagator, wave operator, ... ),

$$U(t_1,t_2;0,0) = e^{-iH_1t_1-iH_2t_2}, \quad [H_1,H_2] = 0. \tag{43}$$

Here, we assume $t_1$ and $t_2$ can be treated independently with each other in our calculus ( beside the relation of Eq. (47) ). It is apparent that the unitary operator conserves the norm of a state vector. Then, we will obtain a rather simple solution of the quantum Nambu mechanics. ( An interesting subject of usage of such a wave operator is the Huyghens principle, in which a wave in an odd-dimensional spacetime is extinguished immediately. In a paper of conformal symmetry breaking, the anomalous behavior of the NG theorem is explained by transformation laws of a propagating wave [28]. More formal discussion on it is found in Ref. [19]. ) The equations of motion are defined by the following adjoint form:

$$F(t_1,t_2) = U(t_1,t_2;0,0)^{-1}F(0,0)U(t_1,t_2;0,0), \tag{44}$$

$$i\frac{d}{dt_1}F(t_1,t_2) = [F(t_1,t_2),H_1], \tag{45}$$

$$i\frac{d}{dt_2}F(t_1,t_2) = [F(t_1,t_2),H_2]. \tag{46}$$

Namely, the time evolutions toward $t_1$-direction is caused by $H_1$, while that of $t_2$-direction is given by $H_2$. For the consistency with the quantum Nambu-Heisenberg equation (29), we find

$$\frac{d}{dt}F = \Big(\frac{d}{dt_1}F\Big)H_2 - \Big(\frac{d}{dt_2}F\Big)H_1 \tag{47}$$

should be satisfied. Then the spectral representation is found to be

$$\langle\phi|e^{-iH_1t_1-iH_2t_2}|\psi\rangle = \sum_n \int e^{-i\lambda_1t_1}\int e^{-i\lambda_2t_2}d\langle\phi|E_{H_1}(\lambda_1)|n\rangle d\langle n|E_{H_2}(\lambda_2)|\psi\rangle. \tag{48}$$

In a typical case of Nambu mechanics, $H_1$ is the total energy while $H_2$ is the second-order Casimir invariant. Thus, if we prepare a simultaneous eigenstate of both $H_1$ and $H_2$,

$$H_1|n,m\rangle = \epsilon_n|n,m\rangle, \quad H_2|n,m\rangle = \epsilon_m|n,m\rangle, \tag{49}$$

then

$$\begin{aligned}\rho(\beta_1,\beta_2) &= e^{-\beta_1 H_1-\beta_2 H_2}, &(50)\\ \langle n',m'|\rho(\beta_1,\beta_2)|n,m\rangle &= \delta_{nn'}\delta_{mm'}e^{-\beta_1\epsilon_n-\beta_2\epsilon_m}. &(51)\end{aligned}$$

The functional form of $\rho$ is understood as a Wick rotated unitary operator $U$. Now we recognize the fact that $\rho(\beta_1,\beta_2)$ gives a Schrödinger semigroup:

$$\rho(\beta_1,\beta_2)\rho(\beta_1',\beta_2') = \rho(\beta_1',\beta_2')\rho(\beta_1,\beta_2) = \rho(\beta_1+\beta_1',\beta_2+\beta_2'). \quad (52)$$

After introducing the state vector of the system in the following manner,

$$|\beta_1,\beta_2\rangle = e^{-\beta_1 H_1-\beta_2 H_2}|0,0\rangle, \quad (53)$$

we find the following equations of motion:

$$\begin{aligned}\frac{d}{d\beta_1}|\beta_1,\beta_2\rangle &= -H_1|\beta_1,\beta_2\rangle, &(54)\\ \frac{d}{d\beta_2}|\beta_1,\beta_2\rangle &= -H_2|\beta_1,\beta_2\rangle. &(55)\end{aligned}$$

Since there is an inequivalence of time evolutions caused by $H_1$ and $H_2$ in the statistical factor $\rho$, diffusion equations obtained from hydrodynamic limits of the Boltzmann equation of $\rho$ become

$$\partial_{t_1}\rho = D_1\nabla^2\rho, \quad \partial_{t_2}\rho = D_2\nabla^2\rho, \quad (56)$$

and $D_1 \neq D_2$ in general. The vacuum energy can be defined as follows:

$$E_0(H_1,H_2) = -\lim_{\beta_1\to\infty}\frac{1}{\beta_1}\langle\psi e^{-\beta_1 H_1}\psi\rangle - \lim_{\beta_2\to\infty}\frac{1}{\beta_2}\langle\psi e^{-\beta_2 H_2}\psi\rangle. \quad (57)$$

The partition function is defined by the following two-inverse-temperature form:

$$\mathcal{Z}(\beta_1,\beta_2) = \mathrm{Tr} e^{-\beta_1 H_1-\beta_2 H_2}, \quad (58)$$

and then the entropy of the system is given by

$$\begin{aligned}S &= -\mathrm{Tr}\tilde{\rho}\ln\tilde{\rho}, &(59)\\ \tilde{\rho} &= \mathcal{Z}^{-1}(\beta_1,\beta_2)\rho(\beta_1,\beta_2). &(60)\end{aligned}$$

The density matrix $\tilde{\rho}$ is re-expressed as follows:

$$\tilde{\rho} \;=\; (\mathcal{Z}(\beta_1,\beta_2))^{-1}\sum |n,m\rangle e^{-\beta_1 E_n-\beta_2 E_m}\langle n,m|. \tag{61}$$

A statistical average of an operator $\mathcal{O}$ is given by

$$\overline{\mathcal{O}} \;=\; \mathrm{Tr}\mathcal{O}\tilde{\rho}. \tag{62}$$

The average value of energy is

$$\overline{E} \;=\; -\Big(\frac{\partial}{\partial\beta_1}+\frac{\partial}{\partial\beta_1}\Big)\ln\mathrm{Tr}\rho. \tag{63}$$

Here, we regard the inverse temperatures $\beta_1$ and $\beta_2$ are independent with each other.

# 5 Physical Implications and Perspectives

In this section, we discuss some physical implications and perspectives of our ANG theorem.

For example, a spin density of ferromagnet gives a similar role with a charge density of a relativistic model with a finite chemical potential, and both of the examples show the ANG behavior [41]. In the case of the breaking scheme $SU(2)\to U(1)$ of a ferromagnet, a nonlinear sigma model of it keeps the spin density ( a uniformly conserved quantity ), namely the second-order Casimir element of Lie($SU(2)$), since the sigma model describes a low-energy excitation of a Heisenberg ferromagnet of localized spin system. On the contrary, a spin density is modulated in an itinerant electron ferromagnetism [32,53], thus a more complicated situation takes place in the NG sector of such a system when we consider the ANG theorem from a local point of view. The Casimir element is not a conserved quantity in theory of itinerant electron ferromagnetism, and a deformation quantization or the Heisenberg algebra $\langle[S^1,S^2]\rangle = i\langle S^3\rangle$ itself depends on spatial coordinates in the case of itinerant electron ferromagnetism due to the modulation. In that case, one cannot apply naively the framework of the Nambu mechanics we have dicussed. It was emphasized in theory of itinerant electron ferromagnetism that a careful consideration on both quantum and thermal fluctuations of spins are important for obtaining a qualitatively/quantitatively correct description of thermodynamics of a ferromagnet. Since a statistical fluctuation is defined by a displacement from a statistical average,

$$\delta X \quad = \quad X - \langle X \rangle, \quad \langle X \rangle = \frac{1}{T}\int_0^T dt X(t), \tag{64}$$

it is apparent for us that it has a deep connection with an ergodic theory in the sense of dynamical system in mathematics. ( Usually, such a statistical average in statistical physics is taken by a Hamiltonian in phase space, though, as we have observed in the Nambu mechanics, group operations of a Lie group on an NG sector is not far from it. ) Namely, the NG modes of ferromagnet give quantum/thermal fluctuation, which may be interpreted by an ergodic theory, theory of dynamical systems especially by connections with Lie groups and number theory [11,30,47].

In the anomalous NG theorem of a spin system, we have observed the commutator $[S_+, S_-]$ ( of course, exactly equivalent with $[S_1, S_2]$ ) has a crucially important role. We can find a case where such a commutator is generated radiatively/perturbatively. A typical example is the famous calculation of Kondo [23,62], the second-order perturbation of the Kondo effect of a local spin fluctuation. Moreover, a single-cite Kondo effect can be understood as a symmetry breaking: A generation of a local spin moment gives a broken state ( but not a phase ), and the essence of the Kondo effect is physics of fluctuation of spins. Therefore, a theory which bridges between a single-cite spin model and an infinitely extended Heisenberg-type model may provide some interesting features for our ANG theorem. A supersolid is studied by a Bose-Hubbard model, which is transformed into an anisotropic Heisenberg model [27]. Thus, an NG boson in a supersolid is a "magnon" with an explicit symmetry breaking of $SU(2)$. A spin liquid state is usually considered in a Mott insulator, not a metal, and the formalism and the model Hamiltonian of Wen can be examined by the framework of the anomalous NG theorem [61]. His model is a Heisenberg Hamiltonian which is derived from the half-filled Hubbard model with strong on-site Coulomb repulsion ( the strong-coupling limit ), and then it is converted as a fermion system interacting with an $SU(2)$ lattice gauge theory: That is a composite boson model ( the NG bosons are given as composites of fermions ).

Since a relatively large fluctuation will develop toward the direction of massless NG boson or a relatively small mass mode, which we have examined, it will dominate the fluctuation-dissipation theorem ( expressed directly by the Keldysh Green's function [21], though a massless mode may have an IR divergence ) in a ferromagnet. Another interesting issue is the Casimir effect of zero point ( quantum fluctuation ) energy of a ferromagnet. This quantity is affected from a geometry of a system, and it is expressed by the Riemann zeta function [52]. A thermal spectrum of the Hawking-Unruh effect may obey the fluctuation-dissipation theorem, and thus a system of spontaneous symmetry breaking and its restoration which show the ANG behavior would be found [15,39,55].

If a nuclear matter of kaon condensation has a surface, the density of nucleons changes drastically at the vicinity of surface: In such a situation, a "metamorphose" from the NNG to the ANG theorems might be taken place. Since the mass of NG bosons will change from the NNG to the ANG situations, the thermodynamical nature ( low-energy excitation ) may change at the vicinity of the surface. If the difference of the numbers of up-spin and down-spin fermions is finite, which can be settled by two chamical potentials $\mu_{up}$ and $\mu_{down}$: A symmetry breaking in such a case also be subjected by our ANG theorem. One of interesting problems for us from the context of our ANG theorem is the case where the chemical potential $\mu$ depend on time, such that a situation where the symmertry of matter/anti-matter is broken by an underlying mechanism ( for example, CP violation ).

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