A Manifestation toward the Nambu-Goldstone Geometry

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today

Ich bete an die Macht der Liebe
Die sich in Jesu offenbart
Ich gebe mich hin dem freien Triebe
mich dem ich treu geliebet ward
Ich will anstatt an mich zu denken
Ins Meer der Liebe mich versenken

The Fundamental Human Rights:
Dignity, Freedoms, Equality, Solidarity, Citizens' Rights, Justice
(a booklet of european union agency for fundamental human rights, 2012).

Abstract:

Various geometric aspects of the Nambu-Goldstone (NG) type symmetry breakings (normal, generalized, and anomalous NG theorems) are summarized, and their relations are discussed. By the viewpoint of Riemannian geometry, Laplacian, curvature and geodesics are examined. Theory of Ricci flow is investigated in complex geometry of the NG-type theorems, and its diffusion and stochastic forms are derived. In our anomalous NG theorems, the structure of symplectic geometry is emphasized, Lagrangian submanifolds and mirror duality are noticed. Possible relations between the Langlands correspondence, the Riemann hypothesis and the geometric nature of NG-type theorems are given.

Keywords: Nambu-Goldstone theorem, Lie algebras and groups, Riemannian geometry, complex geometry, symplectic geometry, algebraic geometry, number theory, deformation quantization, the mirror duality, the Langlands correspondence, the Riemann hypothesis.

1 Introduction

Needless to say, a consideration on symmetry and its breakdown in a physical system provides an important way to approach a problem of modern physics. The Nambu-Goldstone (NG) theorem on spontaneous symmetry breaking gives one of main "principles" in theoretical physics [29,46,47,75,88,90,99,100] [107,108,109,110,111,112,113,114,116,117,118,119,120,140,143,147,160,170,172], and it may be regarded as a theory of phase transitions.

Recently, several schema of symmetry breakings in the NG-type theorem was classified into three categories. A spontaneous symmetry breaking of a system which has an exact continuous symmetry given by a Lie group from the beginning is considered by the ordinary NG theorem [46,47,107,108]: It should be called as the normal Nambu-Goldstone (NNG) theorem [116,117,118,119]. It is a well-known fact that all of the NG bosons generated in an NNG case is massless. If the system contains an explicit symmetry breaking parameter which breaks a symmetry of a Lie group, and simultaneously a VEV dynamically develops toward the same direction broken by the parameter, then the symmetry breaking phenomenon is described by the generalized Nambu-Goldstone (GNG) theorem [30,116,122]: Due to the explicit symmetry breaking parameter, the degeneracy of vacua in the NG boson space is lifted, and an NG boson associated with the symmetry breaking has a finite mass. A typical example is the flavor symmetry which is explicitly broken by the current mass matrix of quarks, and the constituent mass matrix develops toward the same direction. The anomalous NG (ANG) theorem is found in a Lorentz-symmetry-violating systems (a ferromagnet as a nonrelativistic system, a relativistic model with a finite chemical potential, so on) [3,14,20,49,60,76,98,109,117,118,119,127,136,146,153,154,155,156,157]. In an ANG case, a subset of the NG boson space acquire finite masses under a certain mechanism, while the complement of the subset gives massless bosons. (See also, Refs. [61,91]) In the symmetry-breaking mechanism of the ANG theorem, an emergence of a (quasi-)Heisenberg algebra coming from a set of VEVs of a semisimple Lie algebra symmetry is found, and the (quasi-

)Heisenberg algebra plays the crucial role in a realization of characteristic aspect of the ANG theorem [117,118,119]. It was shown in Ref. [117] that the (quasi-)Heisenberg algebra gives an uncertainty relation in a canonical conjugate symplectic pair obtained from a pair of Lie algebra generators in the Cartan decomposition, and the uncertainty relation determines the global geometric structure of the one-loop effective potential of a kaon condensation model.

In Ref. [116], the mathematical structure of the GNG theorem is studied in detail (Lie algebras/groups, topology, differential geometry, algebraic geometry, and number theory). In Ref. [117], the mathematical structure of the ANG theorem is revealed, and several important facts we use here are obtained. In this paper, by utilizing those previous results, we discuss several differential geometric aspects of the NG-type theorems (NNG/GNG/ANG). Especially, Riemannian, complex (Hermitian, Kähler, and their generalizations), and sympletic geometry in the NNG/GNG/ANG theorems will be studied.

This paper is organized as follows. After giving some general situations of the NG-type theorems useful for us, Riemannian, complex, and symplectic geometry in the NG-type theorems are studied in Sec. 2. In Sec. 3, we discuss mathematical structure of phase of a matrix in theoretical physics, since the NG-type theorems also consider phases of matrices. In Sec. 4, algebraic geometry of renormalization groups is discussed, since the method of renormalization groups is frequently combined with an evaluation of physical quantity under a symmetry breaking in the NG theorem. A summary and a perspective of this work is presented in Sec. 5.

2 Geometric Nature of the NG-type Symmetry Breakings

Modern framework of differential geometry [77] have mainly Riemannian, (almost) complex, and symplectic structures as the fundamental structures. In this section, we discuss several aspects of Riemannian, complex, and symplectic geometry in our NNG/GNG/ANG theorems.

2.1 General Situation

In this subsection, we discuss a general situation it is valid in various breaking schema of the NG-type symmetry breakings.

To determine what we will consider for "geometry" in this paper, we will give the following general discussion. Any case of the NG-type theorems considers the following triple: (X, F_X, G), where X is a (topological, Hausdorff) space with a certain type of measure, possibly has a definition of distance, F_X is a space of mathematical objects defined over X such as functions (for example, an effective potential), distributions, hyperfunctions, sheaves, bundles, sections, or operator algebras, and G is a (Lie/topological) group, or a set of mappings of a dynamical system, acts on X and/or F_X. Frequently, X is given as $X = G - H$ in a breaking scheme $\pi : G \to H$. Thus, we have two classes of mathematical objects, X and F_X, to study their geometry. While X can be studied in theory of Lie groups/algebras [58], an investigation on F_X provides us some special and characteristic aspects of the NG-type theorems: Thus, we mainly focus on geometric nature of F_X, which is typically expressed in an effective potential V_{eff} of a quantum field theoretical example, by employing some apparatus of theory of X. If G is a Lie group, it is locally homeomorphic with a Euclidean space. A symmetry and its breaking in the NG-type theorems is usually assumed G as a continuous group, i.e., a Lie group. A Hausdorff nature is important to obtain a stationary point of a theory by a variational calculus. For example, the so-called classical-quantum correspondence can be expressed for a function $f \in F_X$ such that

$$f = f^{(0)} + f^{(1)}\hbar + f^{(2)}\hbar^2 + \cdots. \quad (1)$$

Each term of the infinite-order series is subjected by a (Lie) group action. If such a function f defines a space (especially, a manifold), then a sequence of quantum correction possibly changes the topological nature of the space, and then a similar notion with "so-called" quantum cohomology might be concerned. An effective potential as a tool to investigate a symmetry breaking belongs to such a function space f. Since an effective potential generally depends on a gauge choice, the gauge degree of freedom should carefully be treated. After the gauge degree of freedom (or high-frequency/momentum components) is integrated out, sometimes a certain type of topological invariant is obtained, and a partition function or an effective potential contains it (given by it). If an effective action is expanded as a series given above, then the geometry and topology of it (geometry given by the effective potential itself) are interpreted as expansions in terms of $\hbar$, and sometimes the nature of geometry of an effective action is altered by a choice of truncation of the expansion. This is a kind of "quantization" of a geometry, and a non-perturbative evaluation for an expansion of geometry is crucial to understand the Nambu-Goldstone-type symmetry breaking. Such a non-perturbative evaluation usually employs a mean field approximation. For example, a Hartree-Fock nonperturbative approximation is "integrable", and an expansion of the Hartree-Fock potential by quantum fluctuations is an expansion of geometry, makes the integrable model non-integrable (a quantum fluctuation is interpreted as a displacement from an integrable system). In other words, a mean-field approximation frequently utilized for investigating a symmetry breaking is a fixing of geometry with a physical assumption (Ansatz), especially via a variation principle. A path integral is defined by using a measure $\mu(X)$ such that $\int_X d\mu(X) f$, and a large part of information of the NG-type theorems is derived by examinations of the path integral. A steepest decent is frequently employed for an estimation of a path-integral, by a justification of a mean-field theory: Hence, a steepest decent implies a fixing of geometry. If we use a Gaussian measure for the path integration, then a random geometry will be obtained.

Another important aspect of the NG-type theorems is that a space of our consideration will be divided into two parts, "symmetric" and "broken" spaces. A mixing of two parts does not take place after a breaking scheme is chosen, and then the dimension of NG manifold is kept fixed. In a viewpoint of Lie group, it implies a subgroup and its orthogonal complement. While a submanifold/subvariety will be a subject of our consideration, from the viewpoint of manifold, variety and geometry. Thus, even if geometric natures of symmetric and broken spaces are changed by a quantum correction, the dimensions of them should be kept, and a morphism which changes the dimension of a manifold/variety is not found. This fact gives a restriction for usage of several morphisms studied in algebraic geometry.

If we consider a "continuation" from NNG to GNG, sometimes we will meet with an example where the cardinality of stationary points of $V_{eff} \in F_X$ changes. For example, a broken $U(1)$ symmetry of the NNG case to that of the GNG case changes the cardinality of set of stationary points:

$$\pi_{GNG} : \mathrm{Card}(A) \to \mathrm{Card}(B), \quad \mathrm{Card}(A) > \mathrm{Card}(B). \tag{2}$$

Here, A and B denote the sets of stationary points of NNG and GNG, respectively. Later, we will discuss Galois groups and the Langlands correspondence [53,148] in our NG-type theorems. Such a change of cardinality relates with an *explicit* realization of Galois group in NG-type theorems. At first glance, the topological nature of the set of stationary points seems to be changed from an NNG case to a GNG case, but a naive consideration on it is in fact dangerous, since an NG subspace frequently has a flat direction on which an effective potential does not depend: An example was found and discussed in Ref. [117], where an $SU(2)$ model was examined, and it was shown that a set of stationary points in that model becomes a two-dimensional plane.

Our effective potential V_{eff} is a section of a sheaf $\mathcal{O}$ on the base space M. For example, we frequently meet with a flag manifold such as $G/T = U(N)/\mathbf{T}^N$ as a base space. In such a case, V_{eff} acquires the induced topology which is coming from a Lie group G/T. A Lie group G acts on the sections of $\mathcal{O}$ as G-equivariant manner. By this set up, we can utilize the method of geometric representation.

The following statement is a well-known in Lie groups and representation theory. Let G be a compact connected Lie group, and let T be a maximal torus of G. Let $X = G/T$ (i.e., a flag manifold) be a space which G acts. By an embedding of G/T to $(\mathrm{Lie}(G))^*$ (a moment map), G/T is found as a symplectic manifold (so-called Hamiltonian manifold). Since a diagonal breaking of NNG, GNG or ANG gives G/T, this fact implies that a symplectic space (in the base space) naturally arises in NNG, GNG, and ANG theorems. Therefore, generically, Riemannian, complex, and symplectic geometry occupy our main concerning on geometry of the NG-type theorems. Usually, one employs a method of cohomology to examine a topological nature of a manifold. Such a method of course reduces the total information of a manifold. It is a well-known fact that a continuous function over a compact group G is almost periodic. Thus, if V_{eff} defined over a compact group G is continuous, then it must be almost periodic.

Needless to say, a symmetry breaking considers a (Lie) group and its subgroup. Thus, a lot of examples of symmetry breaking schema consider homogeneous spaces and their geometric nature. The first step for such a consideration is provided by the method of differential forms: A p-form over the homogeneous space G/H is given by a section of

$$G \times_H \bigwedge^p (\mathrm{Lie}(G/H)^\vee) \to G/H. \tag{3}$$

Similarly, a p-form over a manifold M is given by a C^∞-class section of

$$\bigwedge^p (T^*M) \to M. \tag{4}$$

Here, T^*M is a cotangent bundle. Namely, it defines a total space of p-form $\mathcal{E}^p(M) = \Gamma^\infty(M, \bigwedge^p(T^*M))$. A Maurer-Cartan 1-form which is frequently used in the method of Cartan geometry (for example, nonlinear sigma models) $g^{-1}dg$ (and gdg^{-1}) belongs to this space. A projection $\pi : G \to G/H$ is globally defined by a symmetry breaking, and it induces a morphism between p-forms defined over G and G/H.

The nonlinear Lagrangian of our ANG theorem may be given by

$$\mathcal{L} = \frac{1}{2} G_{\mu\nu} H_{\alpha\beta} D_\mu X^\alpha D_\nu X^\beta, \quad D_\nu = \partial_\nu - iA_\nu. \tag{5}$$

A chemical potential will be introduced as a constant gauge field $A_\nu = (\mu, 0, 0, 0)$. Later, we frequently utilize some results of Ref. [117] of our ANG theorem, especially an $SU(2)$ kaon condensation model. The low-energy excitation of the model is described by

$$\begin{aligned}
\tilde{\partial}^\dagger_\nu \Phi^\dagger \tilde{\partial}^\nu \Phi &= -\Phi^\dagger_0 (g^{-1}\tilde{\partial}_\nu g)(g^{-1}\tilde{\partial}^\nu g)\Phi_0 \\
&= -\Phi^\dagger_0 \Big(g^{-1}(\partial_\nu \partial^\nu - 2i\mu\partial_0 - \mu^2) g \Big) \Phi_0, \qquad (6) \\
\Phi &= g\Phi_0, \qquad g = e^{iQ^A\chi^A} \in G, \qquad (7) \\
\tilde{\partial}_\nu &= \partial_\nu - i\mu\delta_{0\nu}. \qquad (8)
\end{aligned}$$

Here, Φ is a bosonic field, and χ^A give NG bosons. After some manipulation (expansion of g in terms of the Lie algebra generators $\{Q_A\}$), we get the following expression which is proportional to μ:

$$2i\mu\Phi^\dagger_0 g^{-1}\partial_0 g\Phi_0 = 2\mu \sum \sum_{A>B} \Big\{ \chi^A(\partial_0\chi^B) - \chi^B(\partial_0\chi^A) \Big\} \Phi^\dagger_0 [Q^A, Q^B] \Phi_0 + \cdots. \tag{9}$$

This term gives a Berry phase [119,127,157]. The VEVs $\Phi^\dagger_0[Q^A, Q^B]\Phi_0$ form a (quasi-)Heisenberg algebra [117].

2.2 Riemannian Geometry

The geometric aspects of NG-type theorems may firstly be characterized by the notions of Riemannian manifolds, if we consider the case the theory is defined over a usual topological space with a Euclidean norm (an induced topology by a Lie group). A Riemannian manifold is characterized by the following notions: Metric and tangent spaces, submanifolds and covering spaces, connections and bundles, several curvatures, geodesics, Laplacians, exponential mappings, isometry, holonomy groups. In this section, we mainly discuss on curvature, Laplacian, and geodesic curves in geometry of our NG-type theorems. Several important transformation groups act on a Riemannian manifold M are summarized such that an isometry group $I(M)$, an affine transformation group $A(M)$, a projective transformation group $P(M)$, and a conformal transformation group $C(M)$. Their relations are known as $I(M) \subset A(M) \subset P(M)$ and $I(M) \subset C(M)$. As we will see, a local structure of NG manifold found in $V_{eff} \in F_X$ is not simple compared with X, we choose other tools for our study.

An interesting and important tool to characterize a compact Riemannian manifold is provided by a Laplacian $\triangle$. We introduce the following known theorem: Let (M, g) be a C^{∞}-class *compact* Riemannian manifold, and let $\mathcal{D}^p(M)$ be a space of p-forms defined over the manifold. Then the eigenvalues of $\triangle$ acts on $\mathcal{D}^p(M)$ are distributed on $[0, \infty)$ discretely, and the degeneracy of any eigenvalue is finite. If (M, g) and (N, h) are isometric, then they share the same spectrum. A spectrum can judge whether any geodesic of (M, g) is periodic (Duistermaat-Guillemin). In our problem of the NG-type theorems, we will meet with mainly three types of Laplacians: The first one is defined by a second-order Casimir operator $\mathcal{C} = \rho_{ij}X^iX^j \sim g_{ij}D_iD_j$ (ρ is a Killing form, D_i is a differential operators, $X^i \in \mathrm{Lie}(G)$ at the origin) of universal enveloping algebra of the Lie algebra. The second example is coming from an inner product of a tangent space of V_{eff} (the hypersurface of V_{eff}) defined by a Riemannian metric: To give a Laplacian defined over the hypersurface of V_{eff} globally, one needs to consider a moving frame on the hypersurface. While, we are interested in the second-order derivative of V_{eff} (which gives the curvature of V_{eff}, while generally the second-order part of a Taylor series of a Riemannian metric also defines a curvature) as the third example:

$$\theta_\alpha \frac{\partial^2 V_{eff}}{\partial\theta_\alpha \partial\theta_\beta} \theta_\beta = \theta_\alpha (\lambda)_{\alpha\beta} \theta_\beta. \tag{10}$$

($\{\theta_\alpha\}$; a local coordinate system of a subspace of Lie group.) Note that the Laplacian of the third example arising from the second-order derivative is defined over a Euclidean space $ds^2 = g_{\alpha\beta} d\theta_\alpha d\theta_\beta$ with $g_{\alpha\beta} = \mathrm{diag}(1, \cdots, 1)$, since the field-theoretical Taylor expansion of V_{eff} is given over the Euclidean coordinate system. Note that a Lie group is locally Riemannian symmetric, isomorphic with a Euclidean space $\mathbf{R}^n$. Then the Laplacian of the equation obtained from this,

$$\frac{\partial^2}{\partial\theta_\alpha \partial\theta_\beta} u = \widetilde{\lambda} u, \quad \widetilde{\lambda} = \frac{\lambda}{V_{eff}}, \tag{11}$$

is especially important for us since its eigenvalues correspond to mass eigenvalues of NG bosons: Needless to say, a massless NG boson $\lambda = 0$ defines a harmonic function, while the case $\lambda > 0$ is a pseudo-NG bosons, and $\lambda < 0$ corresponds to a tachyon (in the case where the "scaling factor" V_{eff} is positive). Moreover, the action of Laplacian to u defines a metric of Finsler geometry. Since a second-order Casimir element expressed by the differential operators of the local coordinate system of a Lie group is a Laplacian, the set of eigenvalues $\widetilde{\lambda}$ (namely, the mass spectrum of of NG modes) of the Euclidean Laplacian $\frac{\partial^2}{\partial\theta_\alpha \partial\theta_\beta}$ will obtain a relation (some correspondence) with the Casimir element via an appropriate coordinate transformation. In other words, the mass spectrum of NG bosons is determined by the Casimir element of the universal enveloping algebra of a Lie group: Thus, representation theory of a Lie algebra (a famous example is theory of D-modules) deeply relates with the mass spectrum of an NG manifold.

A topological nature (for example, the fundamental group) might be defined for any point by a naive application of some methods *formally*, though, when we observe a global structure of V_{eff} (by specifying its orientation), it contains a huge number of components of Riemannian manifolds of positive, zero, and negative (locally defined) curvatures (a superposition of various manifolds). Hence a definition for global/topological nature of V_{eff} is not familiar with modern theory of geometric topology. Though, if V_{eff} can be regarded as a Morse-Bott function at a critical point, a topological nature of it may reflects that of the base space. Moreover, we have an example where a curvature becomes $\pm\infty$ under the definition given above (an example: a conical intersection). Usually, a curvature given as a Hessian of V_{eff} is finite, though sometimes it will diverge in a case of conical intersection (Jahn-Teller type potential sometimes has a conical intersection of two potential surfaces): Such a case is not fall into the class of Hermitian/Kähler geometry even if we employ a complexification (later, we will discuss complex geometry). A conical intersection of two potential energy surface can give an example where the curvature of ground state is always positive. We do not consider a possibility of conical intersection of two surfaces in NG-type theorems, thus, a curvature is always finite. Then, we will introduce the following lemma.

Lemma: *In any case of NG-type theorem, an effective potential under a broken phase must have a point of inflection.*

Then we argue that a "topological nature" defined on a curvature is drastically changed at the point of inflection. For example, a wine-bottle-like potential of second-order phase transition has a point of inflection in the region between the origin (the point the order parameters vanish) and the stationary point, and it exists in the path of phase transition dynamics. A case of first-order phase transition also be given by a potential with inflection points. Usually, a Riemannian manifold with positive curvature has only closed/periodic geodesics, and a chaotic behavior of dynamical system is observed at the region of negative curvature. Those generic nature of an effective potential of NG-type theorem (globally, a curvature with local definition can have both positive and negative values) reflects in a mass spectrum of NG bosons via the Laplacian discussed above. It must be mentioned that the choice of orientation of the surface of V_{eff} is crucial to determine whether a curvature is positive or negative in our prescription, and also to give a definition of distance (especially to consider a geodesic) between two points, for an appropriate setting for our NNG/GNG/ANG theorems. Since the negative curvature part of V_{eff} is not isolated, connects with the positive curvature part of it, it is subtle (difficult, impossible) to apply several results of global Riemannian geometry to our NNG/GNG/ANG theorems. One of the reason of this difficulty is coming from the fact that V_{eff} contains the amplitude mode of an order parameter $|\Phi|$, which takes its value as $0 \le \Phi < +\infty$, thus the geometry of $V_{eff} \in F_X$ is essentially non-compact. Moreover, such an amplitude mode can couple with an NG mode inside a path-integral evaluation of V_{eff}, and thus one cannot treat the amplitude mode separately from the beginning [117]. While, in a case of base space X, to utilize the results on a global nature of a compact Riemannian manifold, one may use the fact that a compact Riemannian symmetric space is dual with a noncompact Riemannian symmetric space, some geometric results of X.

A closed geodesic in a Riemannian manifold closely relates with its covering group and the fundamental group. Closed geodesics and its number are evaluated in a compact Riemannian manifold with a constant curvature such as a sphere, while it will become more subtle issue if the base space is different from a sphere, for example in an ellipsoid. In our cases of NG-type theorems, such an attempt to examine geodesics is not very meaningful, especially in a Riemannian geometry defined over the surface space of V_{eff}, since the geometric structure of it is too complicated for this purpose. However, an S^1-circle as a set of stationary points in a wine-bottle-like potential of NNG theorem is a geodesic. If we consider a nonlinear sigma model Lagrangian $L(x(t),\dot{x}(t))$, which may have an absolutely continuous closed geodesic on its target space, and if its classical dynamics is periodic in "time" t (here, one can consider any type of a real parameter t, does not have to coincide with a physical time variable), then a closed geodesic γ may be obtained as the set of critical points by a variational calculus of the following action functional:

$$F(\gamma) \;=\; \int_\gamma L(x,\dot{x})dt, \quad x : S^1 \to M, \quad F(\gamma) < +\infty. \tag{12}$$

Usually, L is bilinear in terms of a bosonic field x such as $L = \langle \dot{x}, \dot{x} \rangle$, namely a length squared, and thus L is defined in the form of an inner product: L gives a Hilbert manifold HM. Then a geodesic is a locally shortest curve in a family of curves, uniquely defined after giving two end points ("start" and "goal") of the curves. The shortest curve is generated by an exponential mapping. Needless to say, a geodesic is defined by the Levi-Civita connection (a torsion-free flat connection, $\nabla g = 0$, g; metric),

$$\nabla_{\dot{x}}\dot{x} = 0. \tag{13}$$

Thus, an S^1-circle as a set of stationary points in NNG or ANG cases gives a geodesic: Namely, a set of stationary points given from a gap equation $\frac{\partial V_{eff}(x)}{\partial x} = 0$ coincides with a closed geodesic as the solution of $\nabla_{\dot{x}}\dot{x} = 0$ starts from a point of the S^1 circle. This coincidence is lost if the degeneracy of the ground state of V_{eff} is completely lifted, as a case of our GNG theorem (which contains an explicit symmetry breaking parameter) of a broken $U(1)$ symmetry [116], and then the sigma-model description becomes not very meaningful. In a case of broken $SU(2)$, V_{eff} contains a flat direction in the NG space, and the stationary point of the system does not become a point but a two-dimensional plane, and thus its situation is more complicated [116].

It is well-known that this type of L given above always has a $U(1) = SO(2)$ map which is homotopical with the identity map. Since $HM/SO(2)$ is an orbifold, it is singular. Physically, such a closed geodesic corresponds to a periodic motion of an order parameter in the target space: A motion of spin vector x of $SO(3)$ starts from the north pole of S^2 and coming back to the north pole along with a great circle is a typical example. Such a motion is called "prime" because it cannot be given by an iteration of another closed geodesic, and it is called "simple" since it has no self-intersection [133,145]. In the case of $SO(3)$, such types of periodic geodesics exists densely over the target space. This type of a "motion" (but not physical in general) may be found also in the effective potential V_{eff} if it has a flat direction (degeneracy) in the NG manifold (for example, a $U(1)$ circle): A symmetry breaking of $U(1)$ in the NNG case and the $SU(2)$ kaon condensation model of the ANG case give the examples of such a situation, since effective potentials of both of them have the $U(1)$ "symmetry". A gradient flow of such an effective potential vanishes toward the direction along with an S^1-circle (tangent of S^1). While if the degeneracy is lifted in V_{eff}, then the "time"-dependent motion becomes more complicated: Such a situation will be found in a GNG case. As we will discuss later, the term of Berry phase in the low-energy effective Lagrangian of an ANG case vanishes on any point of an S^1-circle. Thus, we can say the Lagrangian submanifold of the symplectic subspace defined by a set of symplectic pairs in a diagonal symmetry breaking of ANG case gives a closed geodesic of the target manifold of the breaking scheme. Such a geodesic is always "closed" due to the periodicity of V_{eff} in the local coordinates of a Lie group, forms a lattice group. Thus, a Hilbert orbifold is generically found in the Lagrangian submanifold of ANG situations. Needless to say, a periodic geodesic is not found/observed in a physical system (for example, a nonlinear sigma model), at least at the classical level of the theory or in the vicinity of the ground state of it. The ergodicity of a motion of a representation point on a target manifold can also be considered.

It is a known fact that any Riemannian manifold (S^2, g) has an infinite number of geometrically distinguished closed geodesics, and $CF(l) > al/\ln l$ is satisfied, where $CF(l) = \sharp\{\gamma : \text{prime closed geodesic with its length } \le l\}$ is the so-called counting function, and a is a positive constant [62,63]. The function $l/\ln l$ is a well-known quantity in the prime number theorem. Of course, our V_{eff} is topologically different from S^2, though a sigma model description of low-energy excitations which is usually defined on a compact space as its target will relate with such a geometric property of S^2 (S^n; Riemannian symmetric spaces).

2.3 Complex Geometry

A complex manifold, especially a Kähler geometry frequently appears to the NG-type theorem due to the fact that a symmetry of a theory is expressed by a Lie group. It is a known fact that a complex submanifold of a Kähler manifold becomes Kähler by an induced metric. This fact becomes important when both G and H are Kähler in a breaking scheme $f : G \to H$. Especially, an important theorem is the Kähler immersion/imbedding: A complex analytic isometric imbedding of (M, J, g) into $(\widetilde{M}, \widetilde{J}, \widetilde{g})$. In that case, M is a Kähler and minimal submanifold of $\widetilde{M}$. By using the GAGA principle of Serre [135], a complex manifold can be studied as an algebraic variety, namely any problem of complex manifold becomes an algebraic problem expressed by a meromorphic function. From those facts, we can consider a mirror duality of a symplectic subspace and its Lagrangian subspace in an NG manifold by algebro-geometric setting.

Another occasion we meet a complex geometry in our NG-type theorems is a complexification of a Lie group, $G_{\mathbf{R}} \to G_{\mathbf{C}}$ [116]. Such a complexification may relate with the following correspondences: The Harish-Chandra correspondence between representations of $G_{\mathbf{R}}$ and Harish-Chandra modules. Beilinson-Bernstein correspondence between Harish-Chandra modules and $(D_X, K_{\mathbf{C}})$-modules [10]. The Riemann-Hilbert correspondence between $(D_X, K_{\mathbf{C}})$-modules and $K_{\mathbf{C}}$-equivariant sheaves. The Matsuki correspondence between G_R-equivariant sheaves and $K_{\mathbf{C}}$-equivariant sheaves. Here, $G_{\mathbf{R}}$ is a connected real semisimple Lie group, $K_{\mathbf{R}}$ is its maximal compact subgroup, $K_{\mathbf{C}}$ is the complexification of $K_{\mathbf{R}}$.

Later, we will examine symplectic geometry in our NG-type theorems, especially in our ANG theorem. A symplectic structure is compatible with an almost complex structure, and thus, we will find complex geometry simultaneously with symplectic geometry in the NG-type theorems.

Let us consider the Hessian matrix of V_{eff} obtained as a second-order derivative of local coordinates of Lie group, discussed in the previous section. After a complexification of the Lie group, one may obtain

$$\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} u, \quad z_\alpha \in \mathbf{C}, \quad \bar{z}_\beta \in \overline{\mathbf{C}}. \tag{14}$$

This form of the Hessian relates with a subharmonic function of complex analysis of several variables, appears in the problem of Einstein-Kähler manifolds (the complex Monge-Ampere equation). Note that the complexification is physically natural (at least not "strange") when we consider a complex mass (order parameter) of a theory [116]: For example, $\mathcal{M} = re^{i\theta} \to z$, so forth. A complexification of a Lie group, which frequently utilized in representation theory, gives an example for it. It is a known fact that a complex manifold with a Kähler metric $\hat{g}$ has the holonomy of a unitary group, and has a C^∞-class real function F (namely, a Kähler potential), which gives the Kähler metric via $g_{\alpha\bar{\beta}} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} F$. In the case $F = V_{eff} \in \mathbf{R}^1$, the most important part of it is the critical point: $\frac{\partial}{\partial \theta_\alpha} V_{eff} = 0$ ($\forall \alpha$), and if the Hessian $H = \frac{\partial^2 V_{eff}}{\partial \theta_\alpha \partial \theta_\beta}$ is regular (i.e, $\det H \neq 0$), then the critical point is isolated and non-degenerate. If F is complex (sometimes V_{eff} is complex), then it cannot be regarded as a Morse-Bott function from the context of variational calculus, since the Morse-Bott function must be real: In modern framework of Morse theory, a global characterization of base manifold is achieved by a *real* valued function defined over the base manifold, it must admit a local expression of usual quadratic form $x_1^2 + \cdots + x_j^2 - x_{j+1}^2 - \cdots - x_n^2$, and thus a complex-valued function is the outside of the scope of this modern framework. This issue might also relate (but, not the same!) with the problem of how to "complexify" the Morse-Bott function, and it might derive modifications of notions of Morse polynomials and Poincare polynomials: Since the Morse-Bott theory is a theory for de Rham cohomology, this issue might relate with Dolbeault cohomology and Hodge decompositions in variational calculi.

For example, a Lorentz violation in quantum field theory can be achieved by (i) tachyonic mode, (ii) indefinite-metric Hilbert space with a complex mass term [106], (iii) vector-type condensation. It is a known fact that a presence of tachyonic mode causes an effective potential complex. The usefulness and natural appearance of complex mass parameter for NG-type theorems is noticed above. Our consideration of a complex effective potential may be relevant for a spontaneous Lorentz violation with a tachyon.

The Hessian $H = \partial_i \partial_j F$ breaks the non-degenerate condition det$H \neq 0$ at any point of a critical submanifold in the NNG/GNG/ANG cases and a critical point of V_{eff} of the NNG/GNG/ANG theorems is not isolated in general: Very generically, a broken symmetry of $U(N)$ or $O(N)$ contains a Lie group action g which gives $V_{eff}(g\Phi) = V_{eff}(\Phi)$ or $V_{eff}(\mathrm{Ad}(g)\Phi) = V_{eff}(\Phi)$ (Φ; a field) even if g belongs to the space of a broken symmetry (in NNG, GNG, or ANG), and it causes no energy difference in V_{eff}. In other words, such a Lie group action cannot give any effective variation in a variational calculus, and usually one has to examine the global structure of V_{eff} to find a critical submanifold. An invariant function (and, a class function) which is important for representation theory (for example, a character) is a key for considering a Morse (Morse-Bott) theory from our context. If F ($\in F_X \ni V_{eff}$) is real, and the Hessian of normal directions of a critical submanifold is non-degenerate in the sense of Morse-Bott, then the critical submanifold is called as a nondegenerate critical submanifold, and F can be called as a Morse-Bott function. If $F : M \to \mathbf{R}^1$ is a Morse-Bott function on a compact manifold M, then the famous relation $MB(t) = P(t) + (1+t)Q(t)$ holds, where $MB(t)$ is a Morse-Bott polynomial, $P(t)$ is a Poincare polynomial, and $Q(t) \geq 0$. The Morse-Bott polynomial is given by indices of critical points, while the Poincare polynomial is defined by the set of Betti numbers of M. In our case, a small "tilting" of a wine-bottle-type NNG potential to obtain the case of GNG potential gives a conversion from a nondegenerate critical submanifold (an S^1-loop) to a nondegenerate critical point (a variationally determined point of S^1) in the sense of Morse-Bott, while the global topological nature of the base space must be conserved under the tilting. Later, we will find the fact that an S^1-loop defined on the hypersurface of V_{eff} in the ANG case is a Lagrangian submanifold, gives a nondegenerate critical submanifold. A Morse-Bott function defined over a generalized flag manifold $G/T = G_{\mathbf{C}}/B$ (T; the maximal torus of G, B; the Borel subgroup of $G_{\mathbf{C}}$) was examined by Bott [16]: Such a flag manifold is frequently obtained in a diagonal breaking of our NG-type theorems. A torus action (namely, maximal torus of $G_{\mathbf{C}}$) on the flag manifold was considered in the work of Bott, and it was found that the decomposition of $G_{\mathbf{C}}/B$ into stable manifolds (here, "stable" is the notion of similar sense in dynamical systems) gives the Bruhat decomposition $G = \bigcup_{\omega \in W} B\omega B$ (W; Weyl group, B; Borel subgroup). An effective potential V_{eff} of the breaking scheme $G \to T$ (a diagonal breaking) is defined on the flag manifold, and thus V_{eff} itself also acquires the Bruhat decomposition by a set of group actions,

$V_{eff}(y) = V_{eff}(\mathrm{Ad}(G)x)$ or $V_{eff}(y) = V_{eff}(Gx)$.

As we have stated many times, after the complexification of V_{eff}, it deviates from the naive notion of a Morse-Bott function: It acquires a symmetry $V_{eff}(z,\bar{z}) \to V_{eff}(z,\bar{z}) + \mathcal{F}(z) + \mathcal{G}(\bar{z})$ modulo a Kähler metric. The Ricci tensor is defined in a Kähler manifold by

$$R_{i\bar{j}} \;=\; -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \ln \det \hat{g}. \tag{15}$$

If V_{eff} (it belongs to a set of smooth sections of the bundle $\pi : \mathcal{E} \to M$) has a non-vanishing imaginary part (such as a chiral anomaly, a Chern-Simons term generated by a fluctuation of mean fields, a radiative correction which contains a tachyonic mode, a theory of complex mass parameter with indefinite metric Hilbert space) as a phase factor of a matrix, then which destroys the Hermiticity of the Kähler metric and the Ricci tensor, and their eigenvalues are complex in general: Therefore, we have found the possibility, via a field-theoretical apparatus, to construct a complex geometry which has a non-Hermitian metric with complex curvatures (our notion may be called as "non-Hermitian complex geometry" coming from the fact that our "Kähler potential" V_{eff} can become complex). The theory of stratified Morse theory of Goresky and MacPherson [50] also usually considers a C^∞ real function $f : X \to \mathbf{R}$ with X a smooth algebraic variety and S a Whitney stratification of X, and thus a naive application of it for our problem might be impossible. An interesting fact for us is that a "non-Hermitian" model admits a symplectic structure (for example, in the ANG case) while it can have a non-Hermitian metric. On the other hand, $g(Jv, Jw) \neq g(v,w)$ and $\omega(v,w) \neq -\omega(w,v)$ where g is a metric on a tangent space, ω is a symplectic structure defined by $\omega(v,w) = g(Jv,w)$, and J is an almost complex structure [105]. We recognize that this case is simply coming from the fact that a symplectic structure in our ANG theorem is found in an algebra given by a set of VEVs of Lie algebra generators, independent from the metric g obtained from a Hessian of V_{eff}. For example, $V_{eff} \propto \cos(r)$ ($\chi_1 + i\chi_2 = re^{i\phi}$) is obtained in the kaon condensation model which shows an ANG situation, where (χ_1, χ_2) form the local coordinates of a symplectic pair [117]: In this case, the symplectic structure is not obvious in V_{eff}. The Hermiticity of a metric $g(Jv, Jw) = g(v,w)$ arises from a traditional Riemannian metric is lost if $F \in \mathbf{C}^1$ or $V_{eff} \in \mathbf{C}^1$, and the "Kähler" metric defined by $g_{\alpha\bar{\beta}} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} F$ under $F \in \mathbf{C}^1$ indicates we already have a deviation from Riemannian geometry. A Ricci tensor closely relates with the volume form:

$$\mathrm{Vol}^n \;=\; \frac{i^n}{n!}(\det\hat{g})dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n. \qquad (16)$$

Thus, our non-Hermitian case gives a complex volume form in general.

Usually, the first Chern class is given by the Ricci form, $c_1(M)=\frac{i}{2\pi}R_{i\bar{j}}dz^i\wedge d\bar{z}^j$ (frequently used to define a Calabi-Yau manifold or an Einstein manifold), then the notion of Chern class might also be enlarged in our non-Hermitian complex geometry. A phase of path-integral weight (i.e., a topological term defined over a compact space) of a Yang-Mills theory is separately handled in the classical action $S=\frac{1}{2e^2}\int_M\Omega\wedge*\Omega+\frac{i\theta}{2}\int\frac{\Omega}{2\pi}\wedge\frac{\Omega}{2\pi}$ (the four-dimensional case) [166]. Our approach given above unifies the "pure-phase" part and another part. When V_{eff} is complex $V_{eff}:\mathbf{C}^n\otimes\overline{\mathbf{C}}^n\to\mathbf{C}^1$ ($2n$; the dimension of NG space) and has an imaginary part, V_{eff} is chiral in the sense of $V_{eff}\in\mathbf{C}^1$ and $\overline{V_{eff}}\in\overline{\mathbf{C}}^1$: Thus, our non-Hermitian complex geometry obtained from an effective potential contains the notion of chirality. It should not be confused that this fact does not imply the holomorphicity $\partial_{\bar{z}}V_{eff}=0$, while it is possible that $\partial_z V_{eff}=h_1\neq 0$, $\partial_{\bar{z}}V_{eff}=h_2\neq 0$ and $h_1\neq h_2^*$.

In the traditional framework of theory of Ricci flow [24,25,26,124,125,126,151], which is of course defined locally, several functionals defined by integrals over the whole part of a manifold will be considered for studying the global nature of a geometry of the manifold. At first sight, an effective potential seems to be defined over a non-compact non-closed space due to the existence of the amplitude mode (sigma mode σ in a Ginzburg-Landau model or a nonlinear sigma model) which is the variable essential to determine a stationary point of the theory (needless to say, here we indicates the sigma mode σ of the gap equation $\frac{\partial V_{eff}}{\partial\sigma}=0$ with $\sigma\in\mathbf{R}^1$), and thus we might need a compactification for considering a geometric problem (this issue of a non-compact space is still investigated in literature). We can say this problem of the modern framework of Ricci flow is a typical example of local-global relation (as we have observed in a problem of curvature of Riemannian geometry in previous section), where sometimes a global nature of a space cannot describe enough in detail of a local structure of the space, since a global nature is obtained after reducing an information of the manifold (hence we speculate that such a modern framework of geometry contains an information entropy). If we omit the sigma mode (amplitude of an order parameter) and restrict ourselves on the NG manifold $X=G-H$ where G is the Lie group of the beginning and

H as the remaining symmetry, then it is a priori determined that whether X is a compact manifold or not. If X is compact, then the framework of Ricci flow can be adopted to study a global nature of our problem. It should be noteworthy to mention that an omission of a sigma (amplitude) mode of an order parameter sometimes causes a serious short comings since a sigma mode can couple with an NG mode via a radiative correction [117]. For another approach, we consider a polydisc $X = \prod D_j$ where a disc is given as a domain of a symplectic pair (explicitly appears in the ANG case) of a diagonal breaking, and set a unit disc with its boundary $\partial D_j = S^1$ (S. K. Donaldson considered such a polydisc in Ref. [31]). Then one can consider our non-Hermitian complex Monge-Ampere equation defined over the polydisc as a boundary value problem to settle a Ricci flow in our theory. Over the domain where our complex Monge-Ampere equation is defined, we can consider a cotangent bundle T^*X with a complex symplectic structure. Then a 1-form of T^*X is expressed as $\zeta_i dz_i$ (z_i, ζ_j; the local coordinate system of T^*X). By the decomposition $z_i = x_i + iy_i$, $\zeta_i = \theta_i + i\eta_i$, then we yield the expression of the real part of the 1-form as $\Omega^1 = \theta_i dx_i - \eta_i dy_i$. This expression gives the Berry phase, a holonomy of the symplectic manifold we consider in Ref. [119] of a paper of our ANG theorem.

Moreover, one can (in principle) evaluate quantum corrections of several orders for V_{eff}, and we know some examples give drastic changes of global natures of effective potentials: One of such examples will be found in the kaon condensation model of the ANG theorem [117]. In that case, the tree level shows no dependence on a local coordinate system of a Lie group, while the one loop correction for V_{eff} acquires a periodicity. Thus, a Riemannian or a Kähler metrics are modified under the quantum correction of a "Kähler"-like potential. It is interesting for us if such a geometric modification caused by a quantum correction gives a change in a topological nature of the system (our interest is somewhat similar with the so-called quantum cohomology [44,64,128,134]). A Kodaira-Spencer deformation in a complex mass matrix gives a series of the deformation parameter t [80,54,55]. The tangent bundle of the deformation parameter space gives a cohomology group. Thus, the expansion of a path integral of a theory by the complex mass parameter gives an infinite-order series of deformation parameters and cohomology groups. This implies that the path integral is a summation of various geometric spaces/objects, and the deformation parameter may be regarded as a variational parameter: A complex structure inside the path integral is variationally determined. A fixed point of Ricci flow is called as a Ricci soliton. A Ricci curvature form gives a first Chern class, and thus the curvature of a manifold, which is given from V_{eff} in our case, is expressed by a first Chern class. (There are several works on classifications on relations between Chern classes, Kähler metrics, Ricci curvatures, and scalar curvatures in various types of manifolds.) If $\mathrm{Ric} = \lambda\hat{g}$, then the manifold is Einstein-Kähler. From our context, an interesting subject is a possibility on quantum modification of such an Einstein-Kähler metric such that

$$\hat{g} \;=\; \hat{g}^{(0)} + \hbar\hat{g}^{(1)} + \hbar^2\hat{g}^{(2)} + \cdots. \tag{17}$$

Then, for example, a dynamical equation of Ricci flow $\frac{\partial}{\partial\tau}\hat{g} = -2\mathrm{Ric}$ also acquires such a sequence of quantum corrections (quantum Ricci flow), similar to the case of nonlinear sigma model [39].

Let us consider the case $V_{eff} = A\cos(r)$, $z = \chi_1 + i\chi_2 = re^{i\phi}$, and $A \in \mathbf{R}^1$. This type of effective potential, a real form, is found in an ANG case [117]. It should be noteworthy to mention that this V_{eff} does not depend on the sigma mode, it is only written by the local coordinates of the NG manifold $X = G - H$. By putting $r = \sqrt{z\bar{z}}$, we obtain our Kähler potential $V_{eff} = A\cos(\sqrt{z\bar{z}})$, and the Kähler metric is evaluated as

$$g_{z\bar{z}} = g_{\bar{z}z} = -\frac{A}{4r}\Big(\sin r - r\cos r\Big). \tag{18}$$

Namely, the metric shows the dependence only on the radial direction r, as the result of Heisenberg uncertainty relation between χ_1 and χ_2 observed in Ref. [117]. If the effective potential acquires an imaginary part such as

$$V_{eff} = A\cos(\sqrt{z\bar{z}}) + i\Gamma(z,\bar{z}), \tag{19}$$

then the metric becomes

$$\tilde{g}_{z\bar{z}} \;=\; g_{z\bar{z}} + i\frac{\partial^2}{\partial z\partial\bar{z}}\Gamma(z,\bar{z}). \tag{20}$$

In ordinary Kähler geometry, a Ricci curvature is a real closed (1,1)-form, while it is complex in our case. A Ricci flow equation will be given by

$$\frac{\partial}{\partial\tau}\tilde{g} = -2\mathrm{Ric}(\tilde{g}), \tag{21}$$

or, the form of Kähler-Ricci flow,

$$\frac{\partial}{\partial \tau}\tilde{g} = -2\mathrm{Ric}(\tilde{g}) + \tilde{g}. \tag{22}$$

We assume τ-dependences of the metric and Ricci tensor, might be obtained from, for example, a renormalization-group prescription for V_{eff}. Note that these dynamical equations are defined locally, in the sense that we do not consider a space of constant curvature defined globally over a manifold. Hamilton showed in Ref. [52] that a Ricci flow equation has a unique solution for a short time evolution of an arbitrary smooth metric of a closed manifold. In various cases of quantum field theory, one has to consider a non-compact non-closed manifold as a base space. It is known fact that if the starting metric is Kähler, then a Ricci flow is always Kähler. In the case of complex effective potential, it is possible that the imaginary part Γ vanishes or becomes to give no result on the metric in a time evolution, and then the metric restores the Kähler nature. On the contrary, the opposite case where a time evolution of Kähler gives a non-Kähler is impossible.

Let us consider a fluctuation of the metric $\hat{g}(\tau) = \hat{g}^{(0)} + \delta\hat{g}(\tau)$ where, the τ-dependence is found only in the fluctuating part $\delta\hat{g}(\tau)$. Such a fluctuation might be caused in a dynamics of the quantum field theoretical background which might depend on the "true" time t or on the spacetime coordinates. Then,

$$\begin{aligned} \ln\det\hat{g}(\tau) &= \ln\det(\hat{g}^{(0)} + \delta\hat{g}(\tau)) \\ &= \ln\det\hat{g}^{(0)} + \ln\det(1 + \frac{\delta\hat{g}(\tau)}{\hat{g}^{(0)}}) \\ &= \ln\det\hat{g}^{(0)} + \mathrm{tr}\ln(1 + \frac{\delta\hat{g}(\tau)}{\hat{g}^{(0)}}) \\ &= \ln\det\hat{g}^{(0)} - \mathrm{tr}\sum_{n=1}^{\infty}(-\frac{\delta\hat{g}(\tau)}{\hat{g}^{(0)}})^n. \end{aligned} \tag{23}$$

Here, we assume $\hat{g}^{(0)}$ is invertible. Thus, after the linearization, our Ricci flow equation becomes

$$\frac{\partial}{\partial \tau}\delta\hat{g}(\tau) = \frac{\partial^2}{\partial z^i \partial \bar{z}^j}\frac{\delta\hat{g}(\tau)}{\hat{g}^{(0)}}. \tag{24}$$

Namely, we obtain a complex diffusion equation for the fluctuating metric $\delta\hat{g}$, with the diffusion coefficient $\frac{1}{\hat{g}^{(0)}}$. As a result, we consider a Brownian motion of the fluctuating metric defined over a complex domain. We emphasize that a representation point of a physical system usually fluctuates in the vicinity of stationary point, and the effective potential may contain such a fluctuation. Now we acquire a connection with number theory and modular forms since an elliptic theta function satisfies a real diffusion equation as a special solution [33]. (An important issue is when the fluctuating metric becomes a theta function, a condition to give a theta function. It is also an important problem to investigate its geometric implication in our complex geometry.) Moreover, since the diffusion equation is an example of parabolic partial differential equation, it is suitable/possible for us to extend it to the Fokker-Planck equation [131], a typical stochastic differential equation [121]:

$$\begin{aligned} \frac{\partial}{\partial\tau}\delta\hat{g} &= a_{i\bar{j}}(z,\bar{z},\tau)\frac{\partial^2}{\partial z^i\partial\bar{z}^j}\delta\hat{g} + b_i(z,\tau)\frac{\partial}{\partial z^i}\delta\hat{g} + b_i^\dagger(\bar{z},\tau)\frac{\partial}{\partial\bar{z}^i}\delta\hat{g} \\ &\quad + c(z,\bar{z},\tau)\delta\hat{g} + f. \end{aligned} \tag{25}$$

Here, we do not assume the Hermiticity of the equation. We call it as a "complex Fokker-Planck equation." One may find another choice of such an extension. Thus, now we meet a "stochastic complex geometry." Such a situation is easily obtained via a quantum field theoretical model which contains a stochastic external force (for example, a sigma model with a stochastic random force). Hence, our Ricci flow equation will connect with the fluctuation-dissipation theorem. (Beside our context of this paper, these extension of diffusion and Fokker-Planck equations to complex domains might find an interesting application to the method of diffusion MRI (magnetic resonance imaging).)

A Ricci flow study on three-dimensional Lie groups, such as the Heisenberg group H_3 or $SU(2)$, is found in Refs. [45,123]: They defined a Riemannian metric via an inner product of local coordinates of a Lie group, and then construct a flow equation. Needless to say, an NG mode is just a local coordinate of a Lie group. While, a classification of three-dimensional Poisson-Lie group is given in Ref. [7]. The work of Drinfeld considers symplectic/Poisson homogeneous spaces [32]. In the next subsection, we discuss symplectic geometry in the NG-type theorems. It is interesting for us to study the relation between our NG cases with their works, and also with theory of dynamical systems.

2.4 Symplectic Geometry

A diagonal breaking of a Lie group G in the ANG theorem gives $n_{NG} = (\mathrm{dim}G - \mathrm{rank}G)/2$ symplectic pairs [117], and $\otimes^{n_{NG}} Sp(1)$ naturally acts on it: It may noteworthy to mention that the set of symplectic pairs *does not* give a skew-symmetric bilinear form. The pairs are coming from VEVs (more precisely, expectation values not restricted at a vacuum state) of the Lie algebra which is spontaneously broken, such as $\langle [Q_i, Q_j] \rangle = if_{ijk}\langle Q_k \rangle$, and the VEVs of the right-hand side of these equations (given by VEVs of the Cartan subalgebra) do not give the same value. In a Cartan decomposition $\mathbf{g} = \mathbf{h} \oplus \mathbf{g}_\alpha$, where $\mathbf{g} = \mathrm{Lie}(G)$ and $\mathbf{h}$ as the Cartan subalgebra, the Lie algebra basis $\mathbf{g}_\alpha$ (they form the set of broken generators) are algebraically equivalent with each other, though the equivalence is generally broken after taking their VEVs (naively, unclear), and thus, their equivalence is also broken in V_{eff} which is a functional of those VEVs. Simultaneously, the algebraic equivalence between the generators of Cartan subalgebra also be broken by their VEVs, even though a moment map of G/T (the base space of the problem) gives a symplectic manifold. The equivalence of symplectic pairs are broken in general, even in the Lagrangian level (see Eq.(9) or, Ref. [117]). (Of course, it is possible that the equivalence of the broken generators are restored at a special point of V_{eff}, for example, over a stationary point which is a subspace of the NG space.) Therefore, one can consider a group action of $Sp(n_{NG})$ on the space of set of symplectic pairs in principle, though the symmetry of $Sp(n_{NG})$ is broken, and thus not natural. In a diagonal breaking of NG-type theorems $G \to H$ where H is the Cartan subgroup of G, the space of the group manifold is divided into two parts, H and $M = G - H$ (the orthogonal subspace for H in G). In the ANG theorem, M has generically a symplectic structure, and a holonomy group is defined on M: The holonomy group is a product of $Sp(1)$. While, H consists with a product of $U(1)$. Thus, a characterization of the global nature of the NG space in V_{eff} of any diagonal breaking of our ANG theorem is given by $\mathcal{G}_G = \mathcal{G}_H \otimes \mathcal{G}_M$ where $\mathcal{G}_H = \prod^{\mathrm{rank}G} U(1)$ and $\mathcal{G}_M = \prod^{(\mathrm{dim}G - \mathrm{rank}G)/2} Sp(1)$. It should be noticed that $\mathcal{G}_H = H$, namely the remaining symmetry in any case of diagonal breakings of the NG-type theorems, while $\mathcal{G}_M$ is *not* the symmetry of the ground state of the NG-type theorems. Each factor of $Sp(1)$ defines a symplectic manifold, and a geometric invariant of it is examined by the corresponding almost-Kähler manifold.

The double covering group of a symplectic group is called as a metaplectic group, and its unitary representation is the Weil (oscillator) representation. Thus, each symplectic pair of a diagonal breaking of our ANG theorem acquires such a representation. Moreover, it is a known fact that the Harish-Chandra module of a Weil representation [159] of metaplectic group is isomorphic with a polynomial ring $\mathbf{C}[a_1, \cdot, a_n]$, where a_j and $a_j^\dagger$ are creation and annihilation operators defined by a symplectic pair via its corresponding Weyl algebra, respectively [6]. Hence, a system of diagonal breaking of our ANG theorem consists with a finite number of harmonic oscillators. The categorical equivalences between Harish-Chandra modules and D-modules (Beilinson-Bernstein correspondence), the Riemann-Hilbert correspondence, the Matsuki correspondence, are also well-known in literature [71].

For example, in the kaon condensation model of $SU(2)$, we obtain the three-dimensional Heisenberg algebra $\langle [S^1, S^2] \rangle = i\langle S^3 \rangle$ [117]. The corresponding Weyl algebra is $[x, -i\partial_x] = i$, $[x, x] = [\partial_x, \partial_x] = 0$, and its Weil representation is also well known. Needless to say, this Weyl algebra has an $SL(2, \mathbf{R})$ symmetry, such as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ \partial_x \end{pmatrix} = \begin{pmatrix} ax + b\partial_x \\ cx + d\partial_x \end{pmatrix},$$
$$[ax + b\partial_x, cx + d\partial_x] = [x, \partial_x] \quad \text{if } ad - bc = 1 \tag{26}$$

This symmety, $SL(2, \mathbf{R})$, acts on the symplectic pair (S^1, S^2) as a canonical transformation.

As we have stated above, in any type of diagonal breaking where only the Cartan subalgebra acquires non-vanishing VEVs, it defines a set of two-dimensional symplectic spaces, and they are explicitly realized in an ANG case. Since a two-dimensional space with a specified orientation defines a Riemann surface, the diagonal breaking implicitly gives a set of mappings from each of symplectic spaces to the corresponding Riemann surfaces. (This fact has a similarity with a topological sigma model with quantum cohomology.) An orientation of a Riemann surface in such a case is naturally defined by an effective potential itself, namely by the direction in which energy of the system increases. For example, a diagonal breaking of $G = SU(N)$ gives $n_{sp} = (\mathrm{dim}G - \mathrm{rank}G)/2$ symplectic pairs and then we have a polydisc consists with n_{sp} discs, $D_{poly} = \prod_{j=1}^{n_{sp}} D_j$. Then any point of V_{eff} has the map $\pi : D_{poly} \to V_{eff}$. A Riemann surface is regarded as a curve, and then we meet a problem of how many biholomorphically equivalent curves can exist over such a two-dimensional symplectic space. Since a line bundle = a vector bundle of rank 1 = locally free sheave = an invertible sheave = a curve = a Riemann surface, one can utilize an algebro-geometric method of sheaves and schemes for our theory [54,55,96]. A physical quantity may contain a part of the local coordinate system of a Lie group, and thus the quantity is given over an NG manifold $X = G - H$. Such a quantity may be expressed (for example, linearly) by a basis over X. Since X is decomposed into a direct product of symplectic spaces in a case of diagonal breaking, then the basis also be defined over the space of direct product. As we will see, the nature of a basis on each symplectic space relates with the Gromov-Witten invariant of the symplectic space.

Let us consider the result of kaon condensation model discussed in Ref. [117] as a typical example of our ANG theorem. Each symplectic pair (χ_j, χ_{j+1}) in our ANG theorem defines a two-dimensional torus $\mathbf{T}^2$, a genus 1 Riemann surface, due to the uncertainty relation of the symplectic pair realized in the effective potential V_{eff} of the theory [117]:

$$\chi_j + i\chi_{j+1} = r_j e^{i\phi_j}, \quad 0 \le j \le n_{NG}, \tag{27}$$

$$\begin{aligned} V_{eff}(\{\chi_j\}) &= V_{eff}(r_j, \phi_j) = V_{eff}(r_j + 2\pi, \phi_j) \\ &= V_{eff}(r_j, \phi + 2\pi) = V_{eff}(r_j + 2\pi, \phi + 2\pi). \end{aligned} \tag{28}$$

The periodicity in the r_j-direction is coming from the uncertainty relation of χ_j and χ_{j+1} explicitly realized in V_{eff}, and also relates with a holonomy (Berry phase) in the symplectic space (χ_j, χ_{j+1}) [119]. Thus, the total space of the NG manifold defines a product of tori $\otimes_{l=1}^{n_{NG}} \mathbf{T}_l^2$, which is isomorphic with a product of elliptic functions. V_{eff} is also defined over the product of tori, and behaves similar to an elliptic function. Since an elliptic function, an elliptic integral, an elliptic curve ($y^2 = x^3 + ax + b$, $4a^3 + 27b^2 \neq 0$, a torus when it is defined over $\mathbf{C}$) and V_{eff} relate with each other via their doubly periodic nature, then the arithmetic of elliptic curve (and a modular form) in number theory will be introduced in V_{eff} of the ANG situation. Thus, an elliptic modular group $SL(2, \mathbf{Z})$ as a discrete subgroup of $SL(2, \mathbf{R})$ naturally acts on a Riemann surface of (χ_j, χ_{j+1}). A torus $\mathbf{C}/\Gamma$ is biholomorphically equivalent with an elliptic curve, and then one may consider an elliptic cohomology [92] in our ANG theorem: In a diagonal breaking of $G = SU(N)$ in our ANG theorem, totally $n_{NG} = (\mathrm{dim}G - \mathrm{rank}G)/2$ elliptic curves are obtained, and it gives a Hilbert modular form. A theta function deeply relates with elliptic functions, and it gives a representation of Heisenberg group [67,72,86,130], a globalization of Heisenberg algebra [102]. Thus, the double periodicity realized in V_{eff} of the ANG theorem is the result of a (quasi-)Heisenberg algebra. Thus, a symplectic pair of our ANG theorem which obeys a Heisenberg algebra naturally acquires its globalization, a Heisenberg group, via the double periodicity. As shown in the kaon condensation model, the set of stationary points in V_{eff} becomes an S^1-loop, and it a Lagrange submanifold (we will discuss on it later) may found in each torus. While, an S^1 can be embedded in an S^2 as a closed geodesic of it. Sometimes S^2 is chosen as a target space of a nonlinear sigma model.

The fact that a symplectic pair always appears in a diagonal breaking in any case of NNG, GNG, and ANG theorems has already been mentioned in Ref. [119]. In a diagonal breaking of ANG case, the total NG space is decomposed into a set of symplectic pairs, each of them is oriented and two dimensional, namely a Riemann surface. A Riemann surface is Kähler and Einstein, and the symplectic structure gives a Kähler form. A low-energy effective Lagrangian of our ANG theorem contains a Berry-phase (holonomy) term given by a symplectic pair [119,127,157]. Since any complex p-form has a Hodge decomposition:

$$H^p(X)^{\mathbf{C}} \;=\; \bigoplus_{r+s=p} H^{r,s}(X), \tag{29}$$

the 1-form of the Berry phase (gives a holonomy of $Sp(1)$) acquires the decomposition $\Omega^1 = H^{1,0} \oplus H^{0,1}$, while the Berry curvature is a 2-form $\Omega^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$. It is noteworthy to mention that, for example in simply connected irreducible 4-manifolds, the class of complex manifolds is included in symplectic manifolds as its subset: $\{\text{complex}\} \subset \{\text{symplectic}\}$. Those Hodge decompositions might give period maps/domains [33]. Since $z = \chi_1 + i\chi_2 = re^{i\phi}$ and the set of stationary points is contained in a circle $\sqrt{z\bar{z}} = r \sim S^1$, it satisfies the condition of a Lagrangian submanifold, where the symplectic structure vanishes. Thus, we can say the 1-form of Berry phase and the 2-form of Berry curvature given by the symplectic structure also vanish over an S^1 loop. The massless NG mode will be found in the direction of the S^1 loop of V_{eff} [117]. Therefore, the mass-generating term (i.e., the Berry phase term) of a Lagrangian of NG modes does not work for the NG mode along with an S^1-loop in a two-dimensional symplectic space, and the NG mode becomes massless, while any vertical (namely, orthogonal, transversal, radial) direction r of the S^1-loop remains massive: This is the mechanism to generate a finite mass to an NG mode of a symplectic pair in the ANG theorem. An interesting fact is that this mechanism deeply relates with both the Morse-Bott variational calculus and the uncertainty relation of a (quasi-)Heisenberg algebra generated by a diagonal breaking scheme of ANG theorem: The Heisenberg uncertainty relation between two NG coordinates as a symplectic pair realizes in a Morse-Bott-type variational problem. The mechanism we have found here will be re-expressed by theory of symplectic homogeneous spaces [2,19,28,142]. For example, an S^1 loop is regarded as a classical solution, and a quantum fluctuation given as NG modes exists along with (in the vicinity of) the Lagrange submanifold S^1 according to the (quasi-)Heisenberg uncertainty relation of NG modes. The direction of quantum fluctuation is found in the direction vertical to the Lagrangian submanifold, and then it coincides with the normal direction of Morse-Bott nondegenerate critical submanifold. Such a variationally determined Lagrangian submanifold is embedded into a torus $\mathbf{T}^2$, and thus it belongs to a family of lines of $y = ax + b$ defined over $\mathbf{T}^2$ where $a \in \mathbf{Q}$: The S^1-curves do not give a foliation of $\mathbf{T}^2$. Since the symplectic structure $\sum_j d\chi_j \wedge d\chi_{j+1}$ vanishes at the Lagrange submanifold, the Liouville theorem is automatically satisfied [4].

An S^1-loop defined in a two-dimensional subspace of an NG sector in our ANG theorem (for example, $\chi_1^2/a^2 + \chi_2^2/b^2 = 1$, $a, b \in \mathbf{R}$) can satisfy the condition of Lagrangian submanifold. It is noteworthy to mention that not only a closed loop S^1 but parabolic and hyperbolic curves can also give Lagrangian submanifold in our case (usually they are not considered in topological quantum field theory). Sometimes we find those S^1-loops cross with the circle $\sqrt{z\bar{z}} = r$ (namely, $\chi_1^2 + \chi_2^2 = r^2$) of V_{eff}, give intersection points: Such intersection points may be studied by the framework of the Lagrangian intersection Floer homology [41,42,43], while the physically relevant Lagrangian submanifold among them is uniquely determined for a V_{eff}, from an Euler-Lagrange variational calculus. As we have mentioned above, any S^1 loop defined over V_{eff} has its natural orientation coming from V_{eff} itself: Thus, one can consider an orientation preserving functor $f : V_{eff} \to S^1$. The effective potential on S^1-loops may belong to a sheaf $\mathcal{O}$ (a sheaf of germs of continuous regular functions, a structural sheaf of S^1). We can always find a point of an S^1-loop of the circle where $\partial_\phi V_{eff} = 0$ is satisfied (in the NNG, GNG, and ANG cases), and thus it gives a harmonic map $\triangle_\phi V_{eff} = 0$, which can be rewritten by a Maurer-Cartan form [152]. The holonomy group $Sp(1)$

of symplectic manifold may have its counterpart in the mirror pair of Kähler manifold. The subject we consider here may relate with some notions of topological quantum field theory [5,162,163], such as the Jones polynomial, Frobenius algebras, so forth.

In our cases, especially in an ANG cases, a symplectic manifold is realized in a vacuum state of a theory via its effective potential. Thus, symplectic topology [48] and holonomy [138], Morse theory [17] in a symplectic (homogeneous) space, Floer homology [64,128,134], homological mirror symmetry [42,82], Hodge theory and Kähler geometry [33], deformation theory of complex structures [80], homogeneous dynamics [74,95,129], and some notions of Poisson geometry [161] (also, Poisson homogeneous spaces [32]) may be related with each other in our theory [116,117,118,119]. Here, we will discuss some of them. In our context, their mathematical structures should be derived from a unified viewpoint of theory of geometry of symplectic homogeneous spaces (which can be viewed as a phase space of classical particle). We give the definition of a symplectic homogeneous space [2,19,28,142]: Let G be a semisimple connected Lie group, and let H be a connected closed subgroup of G, let O be a G-invariant symplectic form. A symplectic homogeneous space is given by the triple (G,H,O). After choosing Z from an element of a Cartan subalgebra of Lie(G), then such a G-invariant symplectic form O is constructed by the Killing form $O_Z(G) = -\mathrm{tr}\langle Z,[X,Y]\rangle = -\mathrm{tr}(Z\wedge X\wedge Y)$ ($X,Y,Z\in \mathrm{Lie}(G)$) for a fixed Z. Then, a symplectic homogeneous space $(G,C_G(Z),O_Z(G))$ is obtained as a coset $G/C_G(Z)$, where $C_G(Z)=\{g\in G|\mathrm{Ad}(g)Z=Z\}$ is an adjoint orbit. (Such an adjoint orbit becomes an elliptic orbit if G is compact.) Since we consider G as a compact Lie group, the Clifford-Klein form $\Gamma\backslash G/C_G(Z)$ with a uniform lattice Γ may be found, due to the known theorem [78,79]. Thus, we can say a diagonal breaking of a Lie group in our NG-type theorems (for example, $SU(2)\to U(1)$, $SU(3)\to U(1)\times U(1)$, ..., in NNG/GNG/ANG cases) generally defines an associated symplectic homogeneous space: In a diagonal breaking, Z belongs to the space of symmetric generators. Since a diagonal breaking of G of the ANG theorem gives n_{NG} symplectic submanifolds (n_{NG} Riemann surfaces) in the NG boson space, and as we have stated that they give a Hilbert modular form can be defined over the NG boson space, then the arithmetic of Hilbert modular form will enter into the symplectic homogeneous space given above.

The classically main theorems of symplecic geometry were given by (i)

Darboux (local triviality), (ii) Moser (deformation is unobstructed), and (iii) Gromov (incompressible). It is a well-known fact that the Gromov-Witten invariant is the invariant of a symplectic structure [42], depends only on the symplectic structure (M, ω), namely, it is invariant under a continuous deformation of the symplectic structure ω. The homological mirror symmetry [42,82] states that there is an equivalence between the category of Lagrangian submanifold of a symplectic manifold and the category of coherent sheaves of the corresponding complex manifold (the mirror dual). Moreover, Seidel and Thomas discussed Braid group actions on derived categories of coherent sheaves [139], related with a context of mirror symmetry. Needless to say, a Braid group deeply relates with a Chern-Simons theory.

Fukaya discussed [41,42,43] that the torus $M = (\mathbf{R}^2/\mathbf{Z}^2, dx \wedge dy)$ as a symplectic space has its mirror counterpart $(M^\dagger, J_\tau) = \mathbf{C}/(\mathbf{Z} \oplus \tau\mathbf{Z})$. This type of mirror duality naturally holds also in a higher-dimensional torus. These situations is just the case in a diagonal breaking of our ANG theorem, where (x_j, y_j) as NG bosonic coordinates of a set of symplectic pairs obtained after taking VEVs of the Lie algebra generators form a product of tori $\otimes \mathbf{T}_j^2$. The discussion of Fukaya on the homological mirror symmetry contains the Floer homology of Lagrange submanifolds (maximal totally isotropic subspace of a symplectic space) as its key, and a Lagrange submanifold is determined by a *variational* calculus in our case (explicitly demonstrated in an $SU(2)$ kaon condensation model [117]). Thus, one can say a topological property (mirror duality) of a total manifold is subtracted by its submanifold, a nature of submanifold ("local" in some sense) reflects that of the total (global) manifold: This case is somewhat similar with the spirit of Morse theory.

The following combination of the Berry phase term (given in the last paragraph of Subsection 2.1) and the symplectic structure of the $SU(2)$ symmetry breaking in the ANG cases gives a Chern-Simons theory, which can live only in a three-dimensional space, defined over the "parameter space" $X_{ANG} = (t, \chi_1, \chi_2)$:

$$\Omega^1 \ni \chi_2 \partial_0 \chi_1 - \chi_1 \partial_0 \chi_2 \to A_\mu \epsilon^{\mu\nu\rho} \partial_\nu A_\rho \to A \wedge dA, \tag{30}$$

$$\Omega^2 \ni O_Z(G) = -\mathrm{tr}\langle Z, [X, Y]\rangle \chi_3 \chi_1 \chi_2 \to \epsilon^{\mu\nu\rho} A_\mu A_\nu A_\rho \to A \wedge A \wedge A, \tag{31}$$

$$\mathcal{S}_{CS} = \frac{1}{4\pi} \int_{X_{ANG}} \mathrm{tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big), \tag{32}$$

$$\mathcal{Z}(k) = \int \mathcal{D}A \exp\Big[ik\mathcal{S}_{CS}\Big]. \quad (33)$$

$A = g^{-1}dg$ of Cartan geometry may give us a familiar expression of this theory in the community of physics, defined on a G-bundle $E \to X_{ANG}$. By the Euler-Lagrange equation, A is found as a flat connection. Here, $\mathcal{S}_{CS}$ is gauge invariant modulo $2\pi\mathbf{Z}$. The classical Chern-Simons invariant is an obstruction of conformal immersion of three-dimensional manifold into a Euclidean manifold, while the quantum Chern-Simons invariant Witten considered gives a topological invariant of the base manifold since the gauge field was already integrated out. One can assume the two-dimensional space (χ_1, χ_2) as a torus $\mathbf{T}^2$ in the case of ANG theorem. Thus, this model describes a "time evolution" of a gauge field defined over a torus as the base space. By an appropriate restriction from the Chern-Simons model (isotropic) to our ANG case (anisotropic), the first term of the integrant of a symplectic structure vanishes on the Lagrange submanifold $\chi_1 + i\chi_2 = re^{i\phi}$, and the second term gives a winding number at a fixed r. Therefore, we find that a topological field theory is naturally defined on a symplectic submanifold of the NG boson space. Since a diagonal breaking of G of the ANG case gives the number of symplectic pairs over a vacuum of the theory as

$$n_{NG} = (\mathrm{dim}G - \mathrm{rank}G)/2 = \mathrm{dim}(X = G - H) - \mathrm{rank}\frac{\partial^2 V_{eff}}{\partial X_i \partial X_j}, \quad (34)$$

we will yield n_{NG}-replicated Chern-Simons theories in the diagonal breaking. Each of those Chern-Simons theory must be defined over a three-dimensional base space given by a "time" and a symplectic pair, because a Chern-Simons form can live only in a three-dimensional manifold. Since a symplectic pair gives a (quasi-)Heisenberg algebra via its VEV (the Cartan subalgebra is fixed), the corresponding Chern-Simons theory implicitly contains a VEV of Cartan subalgebra as a parameter. Moreover, the Berry-phase term (gives a holonomy group) of the NG boson Lagrangian in an $SU(2)$ model of the ANG theorem is the result of realization of three-dimensional Heisenberg algebra, and thus the Chern-Simons theory we have constructed is a result of the Heisenberg algebra. The stationary point of a Chern-Simons action is given by the flat connection $F = d_A^2 = dA + A \wedge A = 0 \in \Omega_h^2(P) \otimes \mathrm{Lie}(G)$ (h; horizontal components, P; principal G-bundle,), which will be determined by the Berry phase term in our case. The condition of a flat connection implies that the connection is locally "constant" (including the case the connection vanishes, namely, zero). Actually, the Berry phase term of the NG boson Lagrangian vanishes at any point of an S^1-loop as the set of stationary points of V_{eff} (this results a massless NG boson along with the S^1 loop), the stationary condition of the Chern-Simons functional coincides with the S^1-loop of the ANG case. This fact indicates the consistency of the description of topological nature of the S^1-loop (the set of vacua of the ANG case) by the Chern-Simons theory. Our Chern-Simons theory has a correspondence with quantum fluctuations of the parameter space in the vicinity of a set of stationary points, an S^1-loop. As we have mentioned, such quantum fluctuations are given in the normal direction of the nondegenerate critical submanifold (i.e., S^1) of V_{eff} in the sense of Morse-Bott. In fact, a diagonal breaking of $SU(N)$ in the ANG case causes anisotropies, inequivalence between the Cartan subalgebra and others. Thus, such an anisotropy gives an inequivalence of quantum fluctuations. For example, in the $SU(2)$ kaon condensation model, we take $\chi_1 + i\chi_2 = re^{i\phi}$, and the quantum fluctuations are given as $(\delta r, \delta\phi)$. Our Chern-Simons model is defined to be isotropic in any direction of the NG boson space, thus it is made so as to neglect those anisotropy. Note that the Chern-Simons theory breaks parity, which is now defined in the parameter space (i.e., the NG coordinates (χ_j, χ_{j+1})).

Witten showed that Chern-Simons gauge theory gives the Jones polynomial, links and knot theory, braid groups (thus, quantum groups also) by an estimation of a path-integral quantum expectation value of Wilson lines [164]. Hence, the three-dimensional Heisenberg algebra and class field theory [158,97] is bridged via the Chern-Simons model in our theory. The Chern-Simons theory also has a close relation with rational conformal field theory (which also has a deep connection with number theory) and an affine Lie algebra, and also a Hecke algebra. In literature, it was shown that the partition function of Chern-Simons theory $\mathcal{Z}_k = \int \mathcal{D}A(\exp(i\mathcal{S}_{CS}))^k$ ($k \in \mathbf{Z}$) takes its value in the cyclotomic field (an algebraic number field) [85]. From the Kronecker-Weber theorem (any Abelian extension of $\mathbf{Q}$ is a cyclotomic extension), the Chern-Simons partition function relates with class field theory. After the complexification of $SU(2)$ gauge group into $SL(2,\mathbf{C})$, we yield a complex Chern-Simons theory [168]. From the viewpoint of symplectic homogeneous space, probably, an equivariant Cherm-Simons theory defined over G/H or $G_{\mathbf{C}}/H_{\mathbf{C}}$ (for example, symplectic homogeneous space) can be constructed. There are criteria for a double coset $\Gamma\backslash G/H$ to be a compact Hausdorff quotient by a discrete subgroup Γ of G (existence problem of a uniform lattice) [78,79]. It is desirable that Γ acts on G/H proper (

compact) discontinuously and freely: In that case, $\Gamma\backslash G/H$ has a Hausdorff quotient topology. Such criteria are crucial for us to consider a gauge group.

As we have mentioned many times, a diagonal breaking scheme of ANG gives a set of symplectic manifolds, constructed by the local coordinates of broken generators of a Lie group. In this case, a set of stationary points gives sometimes S^1 in a two-dimensional surface, as we have found in a kaon condensation model in Ref. [117], since the effective potential is obtained in the form $V_{eff} \sim \cos(\sqrt{z\bar{z}})$ ($z = \chi_1 + i\chi_2$, (χ_1, χ_2); the local coordinates of $SU(2)$) which was also discussed in the previous section on complex geometry. Thus, the S^1-loop gives an example of a Lagrangian submanifold of a symplectic manifold, and we can define a Fukaya category. For example, the NG space which are given as a direct product of two-dimensional symplectic spaces has a well-defined Lagrangian subspace in the ANG theorem since the dimension of its massless NG boson space is $n_{NG} = (\mathrm{dim}G - \mathrm{rank}G)/2$, just the half of the number of broken generators. It is explicitly realized in a theory if the set of stationary point of V_{eff} acquires the dimension n_{NG}. Note that such an S^1 in which a set of stationary points (vacua) of NG-type theorem lies is similar to a brane of string theory. Namely, a Ginzburg-Landau-type wine-bottle potential corresponds to a world sheet of a closed string. Hence, we can introduce several concepts/notions of string theory (such as the A-model, B-model, ...) in our theory. A morphism between two S^1 loops (for example, caused by a generic variation of the model) determines a Floer morphism $\mathrm{Hom}(L_1, L_2) = FC(L_1, L_2)$. Such a morphism is generated by an action of $SL(2, \mathbf{R})$ to a two-dimensional symplectic subspace of the NG manifold. More generally, a diagonal breaking scheme gives the NG space as a symplectic manifold, a symplectic Floer homology arises associated with a symplectomorphism of the NG space. Due to the conjecture of Kontsevich on the homological mirror symmetry (a derived Morita equivalence), such a Fukaya category may be equivalent with the derived category of coherent sheaves over an algebraic (complex) manifold. It was proved by Fukaya, Polischuk-Zaslow that the homological mirror symmetry is correct in a two-dimensional torus $\mathbf{T}^2; dz \wedge d\bar{z}$. Since a symplectic submanifold is embedded into a (quasi-)Heisenberg algebra in an ANG case [117], one can say such a mirror duality exists in a (quasi-)Heisenberg algebra and a Heisenberg group. Furthermore, a symplectic structure is generically found in a nonlinear sigma model Lagrangian in the ANG case (as a Berry phase), the symplectic Floer homology should be considered combined with the symplectic holonomy of the NG space.

Let M be a manifold, and let us consider a loop space $l : S^1 \to M$. A loop is parametrized such as $(s,t) \to l(s,t)$ ($0 \le s < 2\pi$, $-\infty < t < +\infty$). Then we yield a map $l(s,t) : S^1 \times \mathbf{R} \to M$. By $(t,s) \to e^{t+is}$, the space $S^1 \times \mathbf{R} \sim \mathbf{C}\backslash\{0\}$ acquires a complex structure. This is just the traditional preparation for a construction of a string theory. Thus, the gap equation $\partial_z V_{eff} = 0$ ($z = \chi_1 + i\chi_2 = re^{i\phi}$) naturally obtains a string-theoretical (in a broad sense) interpretation [162,163] when V_{eff} is expanded by a complex mass parameter. Since $\partial_z V_{eff} = 0$ gives a constant function, we need a source term for this equation, $\partial_z V_{eff} = f$, to generate a dynamics of $V_{eff}(t,s)$ over the set of stationary points (S^1 loop). V_{eff} must satisfy the single-value condition $V_{eff}(\phi + 2\pi) = V_{eff}(\phi)$. An NG mode wavefunction can live on a loop l.

A Donaldson invariant is evaluated as a scattering amplitude of three Lagrangian submanifold (pseudoholomorphic curves) in a topological quantum field theory: Namely, a scattering between two closed strings. Thus, these strings do not "cross" with each other. Such an amplitude gives a so-called pants diagram, which ends at three S^1 loops (Lagrangian submanifolds). It is an interesting fact for us that we can obtain a geometric object topologically the same with a pants diagram, by cutting a V_{eff} of wine-bottle-type at three non-crossed non-contact loops (biholomorphically the same with a unit disc with two punctuates). Thus, a V_{eff} expanded by a complex mass parameter may admit a string-theoretical description.

Now, we summarize some mathematical definitions to approach the Gromov-Witten invariant and homological mirror symmetry in our NG-type theorems. Let (M,ω) be a symplectic manifold, where a 2-form ω is a symplectic structure $\omega = \sum dx^j \wedge dy^j$, $d\omega = 0$, and $\omega^n \neq 0$ ($2n = \mathrm{dim}M$). The 2-form ω belongs to the class of symplectic structures, $[\omega] \in H^2_{deRham}(X) \simeq H^2(X,\mathbf{R})$. A complex coordinate is given by $z^j = x^j + iy^j$. An almost complex structure J consistent with the symplectic structure of (M,ω) satisfies, (i) $J : TM \to TM$, (ii) $J^1 = -1$, (iii) $\omega(X,JX) \ge 0$ (the "taming" condition, where $\omega(X,JX) = 0$ is only satisfied in the case $X = 0$), (iv) $\omega(JX,JY) = \omega(X,Y)$ (compatibility of ω and J). For example, M is a Riemann surface. With implementing some conditions with this definition, one obtains the so-called special Kähler manifold which is used for studying a harmonic mapping. (Harmonic forms and a harmonic mapping of a compact Kähler manifold gives a Hodge decomposition and a variation of Hodge structure for its mirror dual, a symplectic manifold.) Now, we can introduce a Gromov-Witten invariant for a symplectic manifold acquired in a diagonal breaking of NNG, GNG, ANG cases. The following is the definition of it given by Fukaya [42]. Let Σ be a Riemann surface, and its complex structure given as $j_\Sigma : T\Sigma \to T\Sigma$. A map $\phi : \Sigma \to M$ is pseudoholomorphic if $d_p\phi : T_p\Sigma \to T_{\phi(p)}M$ (p; a point) satisfies $d_p\phi(j_\Sigma V) = J(d_p\phi(V))$. Let $g, m (\geq 0)$ be integers, $\beta \in H_2(M, \mathbf{Z})$, and $2g + m \geq 3$, $\beta \neq 0$. Let Σ be a Riemann surface with its genus g, let $\mathbf{p} = (p_1, \cdots, p_m)$ a set of m-distinguished and ordered (i.e., their numbering is performed) points of Σ, and let $\phi : \Sigma \to M$ be a pseudoholomorphic map with $\phi_*([\Sigma]) = \beta$. Let $\psi : \Sigma \to \tilde{\Sigma}$ be a biholomorphic map between Riemann surfaces. An isomorphism between $(\Sigma, \mathbf{p}, \phi)$ and $(\tilde{\Sigma}, \tilde{\mathbf{p}}, \tilde{\phi})$ is defined by $\psi(p_j) = \tilde{p}_j$, $\tilde{\phi} \circ \psi = \phi$. The total set of the isomorphism is denoted by $\mathcal{M}_{g,m}(M, \omega, J, \beta)$. The virtual fundamental class of it is called as a Gromov-Witten invariant. The Gromov-Witten potential can also be defined formally in our case. Simultaneously, the theory of Seiberg-Witten invariant can be introduced to symplectic manifolds, and used to find a deep relation between Seiberg-Witten invariant and pseudoholomorphic curves [94]. The nice feature of the definition of Fukaya for us (namely, for a usage of it for our problems in the NNG, GNG, and ANG theorems) is that the Gromov-Witten invariant is determined only by the symplectic manifold (M, ω), does not depend on the almost complex structure J, and it is invariant under a continuous deformation of ω. Since the set of stationary points will be found as a Lagrangian subspace of the symplectic manifold, one may consider a homological mirror symmetry, an isomorphic pair between the category of Lagrangian submanifolds (the Fukaya category) and the category of complex manifolds as their counterparts [82].

A set of stationary points will be found in a circle S^1 defined over V_{eff}, and such a circle is a periodic orbit in the sense of theory of dynamical systems. Then, theory of dynamical systems may also join with those subjects via our NNG, GNG, and ANG theorems. For example, our insight is that a dynamical zeta [132], a Selberg zeta, and the Riemann zeta function might be related with each other in our NG-type theorems. Note that a Selberg trace formula bridges between the dynamical zeta function and the Riemann zeta function, while Frenkel and Ngo implemented the Langlands correspondence to the trace formula of Selberg type by a very generic manner [36]. Thus, we found three dualities, the Pontrjagin-Tannaka-Klein duality, the mirror duality, and the Langlands duality in our theory. Fukaya also discussed a Galois group $\widehat{\mathbf{Z}} = \lim_{\leftarrow} \mathbf{Z}/N\mathbf{Z} \simeq \prod_{l:prime} \mathbf{Z}_l$, which acts on a Novikov ring (defined over $\mathbf{Q}$) appeared in a quantum cohomology. A Lagrange submanifold has a Novikov ring (defined over $\mathbf{R}$) as the coefficient ring of the Floer homology. We think, via the well-known isomorphism $\widehat{\mathbf{Z}} \to G_{\mathbf{F}_p}$, $1 \to Fr_p$ (Fr_p: a geometric Frobenius, an inverse image of the absolute Galois group $G_{\mathbf{F}_p} = Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$), one can find a Galois representation, can consider Galois cohomology theory [59] in those geometric objects. The notion of rational Lagrange submanifold L, where $\pi_1(L) \to U(1)$ gives a finite group [41], is also realized in our ANG theorem [117,118,119]. The symplectic structure ω defines a so-called $U(1)$ prequantum bundle E and its connection ∇, $(E, \nabla) \to M$ where M is a manifold. The rational Lagrange submanifold L is defined that the image of a monodromy representation $\pi_1(L) \to U(1)$ of the restriction of (E, ∇) to L gives a finite group [41].

Hausel and Thaddeus discussed the relation between homological mirror symmetry, Hitchin Higgs bundles, non-Abelian Hodge theory, and the Langlands duality [56,57]. (Hence, the works of Simpson on non-Abelian Hodge theory [141] also be introduced in our theory.) The Langlands duality [103,167] is a key of the Langlands correspondence, namely, theory of Galois representations in the noncommutative class field theory [34]: Any compact Lie group G has its dual $G^\vee$: For example, $SU(N)$ has its Langlands dual $PSU(N) = SU(N)/(\mathbf{Z}_N)^\times$. It is noteworthy to mention on the following extension

$$1 \to (\mathbf{Z}/N\mathbf{Z})^\times \to SU(N) \to PSU(N) \to 1, \tag{35}$$

$$(\mathbf{Z}/N\mathbf{Z})^\times \simeq Gal(\mathbf{Q}(\zeta_N)/\mathbf{Q}). \tag{36}$$

This type of group extension is obtained in a diagonal breaking of $SU(N)$ in our GNG theorem [116]. (It should be mentioned that it is not easy for us to incorporate such a discrete symmetry in a nonlinear sigma model approach, as it was stated in the paper of general theory of ANG theorem [117]). Thus, a diagonal breaking of $SU(N)$ in our GNG theorem naturally contains its Langlands dual group $PSU(N)$. A diagonal breaking is defined that the Cartan subalgebra remains unbroken while other Lie algebra generators are all broken. Thus, it is defined by a root system of the Lie algebra which is explicitly determined by the Cartan decomposition. While, the Langlands correspondence indicates that the root lattice of G becomes the coroot lattice of $G^\vee$, vice versa. Therefore, the root lattice of broken generators of a diagonal breaking of G of our NG-type theorems is the coroot lattice of $G^{\vee}$ which may certainly realizes (is expressed) in the path integral and effective potential of our theory. Especially, it is a well-known fact that the coroot system is given by the element of Cartan subalgebras in a semisimple Lie algebra. Hence, the role of Cartan subalgbera in a diagonal breaking provides a key for us to understand/find the Langlands dual group in our theory. In a diagonal breaking, usually we choose the VEVs of order parameters (namely, Higgs fields) toward the directions of elements of Cartan subalgbera to make a Lie bracket between any order parameter and Cartan subalgebra commutative. Let us introduce a Cartan decomposition of a Lie algebra $\mathbf{g} = \mathbf{h} + \mathbf{e} + \mathbf{f}$. For example, in our ANG theorem, a set of VEVs $\langle [e, f] \rangle$ contains information of the root system, and thus the deviation from the skew-symmetric bilinear structure (as stated in the beginning of this section) reflects the structure of root system, which should realize in the global structure of V_{eff} of our ANG theorem. There is a study on zeta functions of root systems as generalizations of the Witten zeta function [81,165,171]. The so-called Witten zeta function, which was studied in partition functions (the volume of a moduli space) of two-dimensional quantum Yang-Mills theory and three-dimensional Chern-Simons theory [165], is regarded that it deeply relates with the Riemann hypothesis of the ordinary Riemann zeta function ζ (one of candidates which provide the way toward the proof of the Riemann hypothesis). For example, $\zeta(k) = \mathbf{Q} \times \pi^k$ holds at some special values. The definition of Witten zeta function given by Zagier is that $\zeta_{W,\mathrm{Lie}(G)}(s) = \sum_{\phi} (\mathrm{dim}\phi)^{-s}$ ($s \in \mathbf{C}$) where ϕ runs over all finite-dimensional irreducible representations of Lie(G) which may explicitly evaluated via the Weyl dimension formula, and it satisfies $\zeta_{W,\mathrm{Lie}(G)}(s) \in \mathbf{Q} \times \pi^{rs}$ (r; the number of positive roots of Lie(G), $s = 2, 4, 6, \cdots$) [171].

From the discussion of Frenkel given in his paper on a mirror symmetry and the Langlands duality [35] (see also, Ref. [37]) (he discusses on the works of Kapustin and Witten [70], also of Nadler and Zaslow [104]), we consider the categorical correspondences between a symplectic subspace of NG manifold, its homological mirror pair, and a Langlands correspondence can be introduced in our theory. The Langlands correspondence may be understood as a phenomenon of mirror symmetry, a categorical equivalence between symplectic and complex geometry [70]. In the Langlands correspondence, the modularity theorem, theory of modular forms and Galois representations play the crucial roles. The modularity theorem states that there is a correspondence between an automorphic representation and a Galois representation, formulated by a correspondence between a modular form and an L-function (zeta function) defined over an algebraic variety. It was also shown that a three-dimensional quantum gravity gives a modular form explicitly as its partition function [93]: The paper also discusses on the Lee-Yang zeroes, they relate with the Riemann zeta function. Probably, the way toward the solution of the Riemann hypothesis might be found in the relations of discussions of those subjects. (A summary of several relations between physics and the Riemann hypothesis is found in [137].) A famous approach toward the Riemann hypothesis is provided by the Bost-Connes model [18], which shows a close relation between (commutative) class field theory and the Riemann hypothesis. We argue that those several approaches toward the Riemann hypothesis, class field theory and the Langlands correspondence, are contained in our framework of the NG-type theorems (NNG/GNG/ANG) from our several observations presented here: Thus, our framework gives a unified viewpoint of those approaches. Usually in representation theory of Lie groups, it handles a finite-dimensional (especially, unitary) representation. From those contexts, it is interesting to investigate a mirror duality between an infinite-dimensional symplectic space, an infinite number of Lagrangian submanifolds contained in it (quantum field theory is an example of it), and an infinite-dimensional complex geometry. Such an infinite-dimensional mirror duality might give us an infinite number of Langlands pairs, and then they give a superposition of an infinite number of L-functions via the (possibly, generalized) modularity theorem, and it might corresponds to the Riemann zeta function. Thus, via the Hilbert-Poliya approach of a determinantal representation of the Riemann zeta function (it closely relates with the Selberg zeta function and the dynamical zeta function) which will be expressed by a path-integral of quantum fields (an infinite-dimensional symplectic/Poisson space and its quantization) with an appropriate measure modulo a moduli space, and an infinite-dimensional mirror duality might be found in the path integration. The unitary inequivalence in quantum field theory might have a crucial role in the mechanism of phenomena of the Riemann hypothesis defined over a Gaussian plane, which is clearly realized in a non-perturbative symmetry breaking dynamics, which can bridge between an infinite-dimensional geometry and an infinite-dimensional algebra (several important problems of modern mathematics, such as quantum gravity, noncommutative class field theory, and the Riemann hypothesis). One of the keys toward the Riemann hypothesis might be found in an application of method of conformal filed theory in two-dimensional spacetime to quantum field theory in four-dimensional spacetime. One of candidates for universal framework on these issues might be provided from the notion of Poisson-Lie groupoids [144], especially its algebro-geometric extension. Since an affine Lie algebra [38] as an infinite-dimensional Lie algebra can be interpreted as a non-Abelian version of a Heisenberg algebra [68,69], several notions (theta functions, modular invariance, Hecke operators, ...) may also enter into a generalization of our NG-type theorems, and the close relation with number theory may become more obvious. The character of such a Heisenberg algebra is a modular function [68]. There is an attempt to an algebro-geometric approach to the Weil representation of Heisenberg algebra in literature [51]. There is also a study on representation theory of the infinite-dimensional Heisenberg group [65]. A non-Abelian version of the infinite-dimensional Heisenberg group might be needed for our purpose. The relation between the Weyl algebra and D-modules is investigated in Ref. [6]. Beilinson studied on the Heisenberg algebra in the context of the Langlands correspondence [9]. To consider an infinite-dimensional problem, one may have to introduce an infinite-dimensional flag variety, D-modules and perverse sheaves over an infinite-dimensional space. Kashiwara introduced an infinite-dimensional flag manifold as a completion of infinite-order sequence of finite-dimensional flag manifolds [66]. In our ANG theorem, a set of stationary points gives a Lagrangian subspace of a symplectic space. It is known fact that they have a Witt basis. After introducing an appropriately defined quadratic form, a Witt group and a Witt ring may be introduced to our theory. Since a crystalline cohomology is defined by a Witt ring [12,15], this cohomology might be useful for studying our ANG theorem. Those considerations presented here will be summarized into the following scheme:

Riemann ζ $\to$ quantum field theory $\to$ infinite-dimensional Heisenberg algebra $\to$ Weyl algebra, deformation theory, D-modules, $\to$ modular forms, Weil representation, arithmetic of quadratic forms (local, global) $\to$ several cohomology theories,
quantum field theory $\to$ infinite-dimensional Lie groups $\to$ infinite-dimensional mirror duality (dualities) $\to$ infinite-dimensional Langlands $\to$ L-functions and modular forms $\to$ infinite-dimensional algebraic variety.

It is known fact that any symplectic manifold admits a symplectic connection [13], and thus we can define symplectic connection and curvature, and its Yang-Mills-type functional in our ANG theorem. An interesting issue is how we can define a topological term in the symplectic Yang-Mills theory. Another interesting issue is to construct theory for all of the mathematical objects discussed above by theory of Riemann-Finsler geometry (and symplectic-Finsler geometry), since a variational calculus plays a crucial role in our NG-type theorems. From our context, a conformal invariance of path integral of 4D gauge field action (Witten), quadratic forms, Siegel modular forms, and arithmetic are interesting [169]. Witten discussed that a path integral of 4D gauge theory gives a Siegel theta series, and it has a modular invariance [166]. The gauge field action contains both the Yang-Mills part $|F^{+}|^2 + |F^{-}|^2$ and the topological part $|F^{+}|^2 - |F^{-}|^2$. Witten discussed the Langlands duals in such type of gauge field actions. Thus, the Langlands dual (electromagnetic duality) may deeply relate with the phase of a matrix.

As we have stated many times in this paper, a diagonal breaking of our ANG theorem gives a sympletic pair (χ_j, χ_{j+1}), which is a special case of a Poisson structure $\pi \in \Gamma(\wedge^2 TM)$ of bilinear forms of bivector fields of T^*M, and π satisfies the relation $[\pi, \pi] = 0$ of the Schouten-Nijenhuis bracket. This bracket has an important role for considering a cohomology group of a Poisson manifold and its deformation quantization [8,22,23,83,84,101,115]. Drinfeld theorem states that a Poisson-Lie group has a natural bialgebra structure [32]. He discussed a Poisson structure ξ of a Poisson homogeneous space which is given by $\xi(x) \in \wedge^2(T_x M) = \wedge^2(\mathbf{g}/\mathbf{h}_x)$, where, M is a homogeneous G-space, $\mathbf{g} = \mathrm{Lie}(G)$, and $\mathbf{h}_x$ is the Lie algebra of a stabilizer H at $x \in M$. He also considered on a Lagrangian subspace of ξ, and shows a bijective correspondence between pairs (H, ξ) (ξ; a Poisson structure on G/H) and Lagrangian subalgebras. In our case, especially in a diagonal breaking of our ANG theorem, such a Poisson structure and Lagrangian subalgebra can be defined in a set of symplectic pairs [117,118,119]. We think $-\langle Z, [X, Y]\rangle XY \sim X \wedge Y$ in the construction of a symplectic homogeneous space discussed above has the bivector structure. The author discussed a deformation quantization of Poisson (symplectic) spaces of our ANG theorem [118]. It is an important issue that how such a bialgebra plays a role in a mirror symmetry, and also in a Hodge decomposition and deformation theory, especially in a mirror symmetry of a symplectic homogeneous space from our context of this paper. Since Kontsevich discussed the mirror duality by using some algebras used in deformation quantization of a Poisson manifold (differential graded algebra, Hochschild cohomology, A_∞-algebra, ...) [21,82,83,84], while Kontsevich also proved that any symplectic (more generally, Poisson) manifold admits a unique deformation quantization [83], a bialgbera structure will enter into a deformation quantization of symplectic space. This case is important for us to consider a Heisenberg algebra (and Weyl algebra) given from a symplectic pair in a diagonal breaking of our ANG theorem. Those several notions and concepts of deformation quantization in a symplectic space should be explained by the words of complex manifolds. Since the Langlands correspondence may be understood as a mirror duality, the bialgebra structure might find its counter part in the Langlands dual pair. Moreover, we might find a "quantization" over the Langlands correspondence. This might indicate the quantized modularity theorem, quantum theory of numbers, and noncommutative quantum class field theory.

It should be mentioned that, various statements on mirror dualities and the Langlands correspondences are given in the framework of $\mathcal{N}=4$ supersymmetric Yang-Mills theory [70], while we consider some similar phenomena of them by our NG-type theorems in four dimensional spacetime. Probably, as a Morse function in Morse theory to investigate a topological nature of base manifold, those quantum field theoretical models might be interpreted as "Morse functions", could be used for some topological nature of the base spaces. Hence we speculate that those (geometric, number-theoretical, ...) mathematical subjects (mirror, Langlands, ...) may deeply relate with some topological natures of our world (but here, we do not restrict the meaning of our world as a spacetime or the physical Universe!). Hence, it may be desirable to find relations (or, a classification) between mathematical structures of several quantum field theories as "Morse functions", where they give similar (the same) mathematical nature of the common base space (Euclidian/Lorentzian spacetimes, a group manifold, ...). It might give us a new criterion and a perspective on a possible quantum field theory. We have discussed that the relation between the root system of G and the coroot system of $G^\vee$ should appear in a diagonal breaking of NG-type theorems. On the other hand, our nature sometimes shows successive symmetry breakings. Thus, one can consider the case where G and its Langlands dual $G^\vee$ are the result of symmetry after a spontaneous symmetry breaking took place. Hence, we should consider a hierarchy of Langlands correspondences in quantum field theory.

Before closing this section, we will discuss a helical spin ordering and its symplectic structure in the ANG theorem as a physical example. It is a well-known fact that a helical ordering of an $SU(2)$ Heisenberg model takes place in the two-dimensional space (X_1, X_2) where X_3 is the direction of magnetization. Thus, a helical ordering provides a nice field to observe a role of physics of broken generators of the NG theorem. This situation is easily generalized in our ANG case, namely, a helical ordering can be found in any symplectic pair of the quasi-Heisenberg algebra of a diagonal breaking of $SU(N)$, and the number of helical orderings (counting both of right/left helicities) is maximally $\dim G - \mathrm{rank} G$ where $G = SU(N)$. We can apply this result to another type of Lie group/algebra, $SO(4) \simeq SU(2) \times SU(2)$ (Lie($SO(4)$)=Lie($SU(2)$)$\oplus$Lie($SU(2)$)), and which can give a double helical ordering. The case Lie($SO(4)$)=Lie($SU(2)$)$\otimes$Lie($SU(2)$) which frequently used in a spin-orbital model of orbital ordering in condensed matter [73] is an interesting subject for the application of our ANG theorem.

3 The Phase of a Matrix

Needless to say, an NG boson is a phase degree of freedom of a matrix. There are several examples relate with mathematics coming from a phase of matrix in theoretical physics. The strong-CP phase (axion) [27], the $U(1)_A$-problem, instanton, the Fujikawa method of anomaly, 't Hooft interaction, color confinement, gauge fixing and the FP ghosts are all related with the phase of a matrix, defined by $\ln\det M$. Not all of them are included in (related with) the NG-type theorems, though here we will discuss some mathematical aspects of them.

Let M be a matrix, and let $g \in G$ be an element of Lie group. Let us consider an adjoint action $\widetilde{M} = \mathrm{Ad}(g)M$ (we denote $\mathrm{Ad}(g)M = g^{-1}Mg$, $\mathrm{Ad}(g^{-1})M = gMg^{-1}$). The group element g will have several Lie group (Lie algebra) decompositions, and they result several definition of a phase of M. Then, the phase of M is also decomposed according to the Lie group decomposition.

Let O be a Hermitian operator. A matrix representation of O in an indefinite-metric vector (Hilbert) space does not give a Hermitian matrix in general [106]. Then, a non-vanishing phase of will be obtained. As we have mentioned in the previous section, a quantum field theoretical model with complex mass parameter defined over an indefinite-metric Hilbert space makes its effective potential complex.

A logarithm of determinant (or, also a Pfaffian) of a matrix $\ln\det M$ subtracts the $U(1)$ phase of M. A determinant (Pfaffian) gives a section of a determinant (Pfaffian) line bundle. Such a phase can, for example, exist on an S^1-loop as the set of stationary points of V_{eff} in our ANG theorem: A phase as a function is distributed along with the Lagrange submanifold. The definition of the real and imaginary parts of $\det M$ are given as follows:

$$\Re\ln\det M = \frac{1}{2}\ln\det M^\dagger M, \tag{37}$$

$$\Im\ln\det M = \frac{1}{2i}\ln\det(M/M^\dagger). \tag{38}$$

For example, the logarithm of the 't Hooft matrix [149],

$$T_D = \kappa_D\Big[e^{i\theta_D}\det\bar{\psi}P_+\psi + e^{-i\theta_D}\det\bar{\psi}P_+\psi\Big], \quad P_\pm = \frac{1}{2}(1\pm\gamma_5) \tag{39}$$

(it is known that this type of 't Hooft matrix is derived from a Nambu−Jona-Lasinio (NJL) four-fermion interaction model [75]) or, more generally,

$$T_D = \kappa_D\Big[e^{i\theta_D}\det\bar{\psi}P_+\Phi\psi + e^{-i\theta_D}\det\bar{\psi}P_-\Phi\psi\Big], \tag{40}$$

(Φ: a matrix bosonic field) give phases: It sbtracts and fixes a $U(1)$-phase of the matrix of $U(N_f)_R\times U(N_f)_L$ symmetry. Namely, the fixing of phase of the 't Hooft matrix is mathematically equivalent with the fixing of the phase of the Dirac mass term,

$$m\bar{\psi}_R\psi_L + m^*\bar{\psi}_L\psi_R. \tag{41}$$

Needless to say, an explicit symmetry breaking fixes a phase, and a GNG case considers a quantum fluctuation in the vicinity of the fixed phase [116]. The 't Hooft matrix can fix the $U(1)_A$ phase under the manner of explicit symmetry breaking. Such type of fixings of mass matrices can also be considered for Majorana mass terms:

$$T_R = \kappa_R\Big[e^{i\theta_R}\det\psi^T CP_+\psi + e^{-i\theta_R}\det\bar{\psi}P_+C\bar{\psi}^T\Big], \tag{42}$$

$$T_L = \kappa_L\Big[e^{i\theta_L}\det\psi^T CP_-\psi + e^{-i\theta_L}\det\bar{\psi}P_-C\bar{\psi}^T\Big]. \tag{43}$$

Due to the redundancy, we need the condition $\theta_D + \theta_R + \theta_L = \text{const.}$ to fix $U(1)_A$ and $U(1)_V$. One can consider a field-theoretical model such as the NJL model with those matrices T_j ($j = D, R, L$). For example,

$$\kappa_B \text{tr}\Big(m\Phi + \Phi^\dagger m^\dagger\Big) \tag{44}$$

can fix the phase of Φ. A typical example is found in the following Lagrangian:

$$\begin{aligned} \mathcal{L}_{ch} &= \frac{f^2}{4}\text{tr}(\partial_\nu U^\dagger \partial^\nu U) - \frac{1}{2}\Sigma \text{tr}(e^{i\theta/N_f} M U^\dagger + e^{-i\theta/N_f} U M^\dagger) \\ &\quad -\frac{1}{2}\chi(\ln \det U)^2, \end{aligned} \tag{45}$$

$$\Sigma = -\langle\bar{\psi}\psi\rangle / N_f, \tag{46}$$

$$\chi = m\Sigma / N_f. \tag{47}$$

For example, the Yukawa interaction term,

$$g_Y(\bar{\psi}P_+\Phi^\dagger\psi + \bar{\psi}P_-\Phi\psi), \quad \Phi = e^{i\theta}\Phi', \tag{48}$$

fixes the "phase" of the Yukawa coupling constant. For example, a quark-axion model is given by the following Lagrangian:

$$\begin{aligned} \mathcal{L} &= \frac{f^2}{2}\partial_\nu\theta\partial^\nu\theta - \frac{1}{4g^2}(G^a_{\mu\nu})^2 + \bar{\psi}i\gamma^\nu D_\nu \\ &\quad + c_1\bar{\psi}\gamma^\nu\gamma_5\psi \cdot \partial_\nu\theta - m(e^{ic_2\theta}\bar{\psi}P_+\psi + e^{-ic_2\theta}\bar{\psi}P_-\psi) \\ &\quad + c_3\frac{\theta}{32\pi^2}G^a_{\mu\nu}\widetilde{G}^a_{\mu\nu}. \end{aligned} \tag{49}$$

From the form of derivative coupling term, θ must be a pseudo-scalar for the parity invariance of the theory. (Sometimes we meet an axion coupling of the form $\bar{\psi}\gamma^\mu(a + b\gamma_5)\psi \cdot \partial\phi$ in literature. The axion field ϕ which has this type of coupling might be assumed as a mixing, or a linear combination of scalar and pseudo-scalar components.) The mass term of quarks fixes the $U(1)$-symmetry of the θ direction. This type of model belongs to a class of explicit symmetry breakings (here, a $U(1)$-symmetry).

In the case of color confinement, the following terms will be the objects of main consideration [87]:

$$\mathcal{L}_{GF} = B\partial^\nu A_\nu + \frac{\alpha}{2}B^2, \tag{50}$$

$$\mathcal{L}_{FB} = i\bar{c}\partial^\nu D_\nu c. \tag{51}$$

Namely, the gauge fixing part and the Fadeev-Popov ghost part, respectively. B is the Nakanishi B-field. An interesting fact of $\mathcal{L}_{GF}$ is that it has a similarity with a model of topological quantum field theory [162,163]. The physics of confinement is mainly coming from these parts of Lagrangian of non-Abelian gauge symmetry. Needless to say, $\mathcal{L}_{GF}$ fixes the phase of gauge degree of freedom, while $\mathcal{L}_{FB}$ absorbs the fixed phase, via

$$\ln(\det M_F/\det M_B) \sim \ln|M_F/M_B|e^{i\phi}. \tag{52}$$

(M_F and M_B are fermion and boson matrices, respectively.) Namely, a subtraction of the relative phase between M_B and M_F. This BRST prescription introduces a Lagrange multiplier (i.e., a constraint) to lift a degeneracy in the Poisson structure of the phase space which exists in the classical Lagrangian: This results a restriction of a path integration in the gauge degree of freedom, and thus the BRST prescription restricts a phase degree of freedom of a matrix. The gauge invariance/dependence of the theory will be found in the effective action (V_{eff} depends on a gauge parameter), or the propagator of the theory. Therefore, if we can control the particle statistics of fermions/bosons, then the relation between numerator and denominator also may be changed, and we could achieve another way of gauge fixing and confinement.

The choice of 't Hooft monopole of $SU(2)$ is frequently given by [150]

$$\left(\begin{array}{cc} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{array}\right). \tag{53}$$

This form is a special choice of a matrix of three-dimensional linear space. The phase φ play the crucial role in the monopole physics. Of course, it depends on the representation of the Lie group $SU(2)$, especially σ_3 of the Pauli matrices of Lie($SU(2)$). Then the fixing of the form of monopole determines a symplectic vector space given by the direction of (σ_1, σ_2): It defines a symplectic homogeneous space, and a Gromov-Witten invariant (and also, its mirror counterpart) will be given for the symplectic structure. The symplectic structure defined by a Killing form $B = \langle Z, [X, Y]\rangle$ of Lie algebra ($X, Y, Z \in$Lie(G)) depends on the choice of Z but it is uniquely determined after the choice.

It is a well-established fact that a chiral transform $\psi \to e^{i\alpha\gamma_5}\psi$ and $\bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_5}$ ($\alpha \in \mathbf{R}^1$) gives

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\nu \exp\Big[\int \bar{\psi}(i\gamma^\nu D_\nu - m)\psi d^4x + S_{Maxwell}\Big]$$

$$\to \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\nu \exp\Bigg[\int d^4x\Big(\bar{\psi}(i\gamma^\nu D_\nu - m)\psi$$
$$+\alpha(x)[\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) - 2im\bar{\psi}\gamma_5\psi]$$
$$-2i\alpha(x)\frac{e^2}{32\pi^2}e^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\Big) + S_{Maxwell}\Bigg], \qquad (54)$$

in the case of QED [40]. Namely, it contains the famous chiral anomaly term in the Lagrangian, arising as an infinite-dimensional Jacobian of the transformation of the integration variables of ψ and $\bar{\psi}$. Thus, the chiral anomaly arises from a "volume element" of the quantized fields, gives a volume form. By using the classical solution (for example, instanton), one yields the following expression (by the Laidlaw-DeWitt-Schulman theorem):

$$\mathcal{Z} \sim \sum_\nu \int \mathcal{D}\mu_\nu e^{i\nu\theta + S_E}, \quad -\infty < \nu < +\infty, \quad \nu \in \mathbf{Z}, \qquad (55)$$

where, ν is a winding number of the fundamental group estimated by an instanton solution of a compact manifold (ν corresponds to the index of the Dirac operator $\gamma^\nu D_\nu$ by using the method of heat kernel, and a Mellin transform of a heat kernel gives the Riemann zeta function [89]), and S_E is a Euclidian action. In general, a winding number is evaluated by a self-dual part R_+ of a curvature 2-form, which is defined (subtracted from R) by the Hodge *-operator in a case of 4-dimensional spacetime, (or an anti-self dual part R_-) of a curvature 2-form $R = R_+ \oplus R_- \in P \times_{Ad} \mathrm{Lie}(G)$ (P; a principal bundle, G; a structure group) via a Chern class. A Donaldson invariant closely relates with a counting of instanton. Under the chiral transform of the Dirac fields defined above, the integration measure is transformed as $\mathcal{D}\mu_\nu \to \mathcal{D}\mu_\nu e^{-2i\nu\alpha}$, we get

$$Z_\theta \sim \sum_\nu \int \mathcal{D}\mu_\nu e^{i\nu\theta + S_E} \to \sum_\nu \int \mathcal{D}\mu_\nu e^{i\nu(\theta - 2\alpha) + S'_E}. \qquad (56)$$

Namely, in the phase factor $e^{i\nu\theta}$ which is apparently not self-adjoint, $\nu \in \mathbf{N}$ is defined by a configuration of gauge field which is called a winding number, while $\theta \in \mathbf{R}^1$ gives the angle of chiral transform. In other words, under a chiral transform, the integration measure of the fermion sector acquires a scaling proportional to the winding number which is determined by the gauge sector. Note that a measure (length) is changed (scaled). As we have mentioned, the chiral anomaly is caused from the "volume element" defined by a Jacobian in the sense of the Fujikawa method, and it might be interpreted rigorously by geometric measure theory in mathematics [1]. In other words, the configuration and winding number of gauge fields, and also the number of zero modes of the Dirac operator (corresponds to the condition of "flat", "constant", "integrable"), determines the volume element of a geometric measure. Let us turn our attention to the NG-type theorems. In a case of GNG theorem of chiral symmetry breaking, θ takes its value in a stationary point, and then the main contribution of integrant to the path integral is coming from the stationary point: In a case of broken chiral symmetry in our GNG theorem, θ corresponds to the coordinate of explicitly broken chiral symmetry of a theory, and θ will takes a VEV obtained from the corresponding stationary point of the theory. Even though the phase factor $e^{i\nu\theta}$ is not self-adjoint, it does not directly imply that the theory is not Hermitian, since the theory is given by the total sum of ν. An important aspect is that, even though the gauge field can have its value over a non-Abelian group, the phase factor $e^{i\nu\theta}$ is essentially Abelian. We can say this situation more generally. Let M be a base space which is a topological space, and let $\Psi(x)$ and $A(x)$ ($x \in M$) be a quantum field and a connection, and they obey certain types of algebras. Then, a (complex) scaling of integration measure $\mathcal{D}\Psi \to \lambda\mathcal{D}\Psi$ is determined by a characteristic class evaluated by a curvature 2-form derived from $A(x)$. Since this general statement is given logically, we consider it is the case in several types of manifolds/varieties (complex manifolds, algebraic varieties, arithmetic varieties, ...). A path integral is decomposed as a sum indexed by winding numbers, the phase factor $\exp(iN\Theta)$ may satisfy both the Goldbach conjecture (any positive even number $n_e \geq 6$ is a sum of two prime numbers, any positive odd number $n_o \geq 9$ is a sum of three prime numbers) and the statement of Fermat (any natural number is at most given by a sum of m polygonal numbers of order m).

In summary, from our observation given above, we find that our GNG theorem contains two types of winding numbers ν and n, where ν is coming from a configuration of gauge sector, evaluated by instanton calculus, characteristic class, and the index theorem of Dirac operator (for example, the Fujikawa method of anomaly), while n is given by a chiral transform of fermion field.

4 Renormalization Groups and Algebraic Geometry

The method of renormalization groups is an important technique to evaluate a physical quantity accurately [11,172]. Since an effective potential or a mass parameter should be estimated under a renormalization-group (RG) invariant manner, the method deeply relates with our NG-type theorems. In other words, several geometric aspects we have found should not only qualitatively, but also quantitatively valid, must be inert from a choice of RG prescription.

The axiom/condition of RG procedure may be summarized as (i) any physical observable is invariant under a scaling (but it is not always true), (ii) the unitarity of the system is conserved. An RG flow can be put on a manifold for our examination of it. Then, it is desirable to extend the notion of attractor (equilibrium, fixed point) to include not only geometric but also some types of Milnor attractors. An RG equation is a dynamical system, and it takes generically in the following form:

$$\frac{dH}{dt} = F(H,t), \tag{57}$$

(H: a Hamiltonian). After a linearization of the right-hand side, we find a Jacobian, and its eigenvalues λ_j define whether it is relevant ($\lambda_j > 1$) or irrelevant ($\lambda_j < 1$), similar to the definition of Lyapunov exponent of hyperbolic dynamics. Due to a hyperbolicity, a chaotic behavior of an RG flow is possible, though such a solution may regard as a pathological one. The Gaussian fixed point corresponds to the Gaussian model (depends on the number of dimensions of spacetime, it sometimes becomes an attractor of Milnor type), which is based on the realization of central limiting theorem in a quantum field theoretical model.

The prescription of RG consists with a scaling,

$$F(b^{\alpha_1}x_1, b^{\alpha_2}x_2, \cdots, b^{\alpha_n}x_n) = b^{\gamma} f(x_1, x_2, \cdots, x_n). \tag{58}$$

It is a well-known fact that a partial differential equation,

$$\left(\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}\right) u = ku, \tag{59}$$

has its characteristic curves such as

$$x_j(t) = e^t x_j(0), \quad u(t) = e^{kt}u(0), \quad \forall j \in (1, \cdots, n), \tag{60}$$

and the solution satisfies the following scaling law:

$$u(\lambda x_1, \cdots, \lambda x_n) = \lambda^k u(x_1, \cdots, x_n). \tag{61}$$

A scalling law is determined by a long-distance asymptotic behavior of a correlation function. A fractal dimension D is defined by $f(x/b) = af(x)$ with $b^D = a$. For example, a magnetization of Ising model is invariant under a scaling. Let us consider a dynamical generation of Dirac mass in a Nambu$-$Joba-Lasinio model with an explicit symmetry breaking parameter (even though it is non-renormalizable, we use it as a simplest example for what we consider here) [116]. The massive Dirac operator we consider is given by $i\gamma^\nu D_\nu - \widetilde{\mathcal{M}}$ with $\widetilde{\mathcal{M}} = |\mathcal{M}^{(0)}| + \mathcal{M}_{dyn}P_+ + \mathcal{M}^\dagger_{dyn}P_-$, where $\mathcal{M}^{(0)}$ is an explicit chiral symmetry breaking parameter, while $\mathcal{M}_{dyn}$ is a dynamically generated mass and $P_\pm$ are the chiral projectors. A scaling of mass function in our GNG case under a complex gauge transformation [116] is given by

$$\widetilde{\mathcal{M}} = |\widetilde{\mathcal{M}}|e^{i\Re\theta - \Im\theta}, \quad \theta \in \mathbf{C}, \tag{62}$$

namely, a complexification of a Lie group action on the mass function contains a "pure" scalling factor (a dilatation) $e^{-\Im\theta}$ (the Callan-Symanzik equation shows a response of a system under a scaling of mass parameter). Thus, a complex chiral/gauge transformation gives a chiral/gauge transformation and a scaling simultaneously, which means an enlargement of gauge transformation space: This fact may suggest us the meaning and implication of a field theoretical model with a complex gauge. While, this type of a chiral transformation gives also a chiral anomaly in a theory: Hence, such a complex chiral (gauge) transformation gives a chiral anomaly and a scaling simultaneously, where the anomaly term is scaling invariant/free. If we absorb the phase $e^{i\theta}$ ($\theta \in \mathbf{C}^1$) of the mass function $\mathcal{M}_{dyn}$ in $\widetilde{\mathcal{M}}$ by a fermion field redefinition, then the effect of redefinition reflects in the bare mass parameter $|\mathcal{M}^{(0)}|$. Such a transformation generically contains a scaling effect on the mass parameters, and it may observe the "perturbation" $|\mathcal{M}^{(0)}|$ to the corresponding massless theory. In fact, a Wilsonian block-spin transform in momentum space $\phi'(k) = L^{-\theta}\phi(p)$ ($k = Lp$) is a special case of a complex gauge transform. A Wilsonian block-spin transform seems to have the assumption that we have a "uniform" Euclidean space, and a discrete lattice is put perfectly uniformly in the space (a uniform lattice), and one can define the same phase (or the phase rotation) of a variable of any point of the lattice. (Thus, very abstractly, one might consider a "block-spin" prescription suitable for a lattice of Γ of a Riemannian space G, namely G/Γ, generated by a Lie group action of a homogeneous space in which an RG-type equation or a field theoretical model are defined. This viewpoint may enlarge the application of RG methods, for example, geometry of numbers, number theory.) More generally, a scaling of RG defines an affine transform $z \to \alpha z + \beta$ (z: a field), and it gives $F(z) \to F(\alpha z + \beta) = \alpha^m F(z)$ (a special case of conformal group $SL(2, \mathbf{C})$), and an RG invariance defines a projective hyperplane. Since the classical massless QED gives a scaling invariance,

$$\int L(x^\nu, \psi(x), A_\mu(x))dx = \int L(e^{-\alpha}x^\nu, e^{\frac{3}{2}\alpha}\psi(x), e^{\alpha}A_\mu(x))dx, \quad (63)$$

it defines a special case of conformal group defined over a projective space. Needless to say, a mass parameter for the QED model breaks the scaling invariance.

Obviously, the condition of a scaling invariance in an RG prescription contains a projective space in which the point (x_0, x_1, x_2) coincides with $(\lambda x_0, \lambda x_1, \lambda x_2)$ of a homogeneous coordinate system: The scaling of an RG gives the special case of a projective space. The scaling relation of massless QED given above gives a nice example of a projective space, and the breakdown of conformal invariance of $SL(2, \mathbf{C})$ is understood by a projective geometry. More generally, one considers a linear transformation

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}. \quad (64)$$

Since we take a homogeneous coordinate system, A and ρA ($A = \mathrm{Mat}(a_{ij})$, $\rho \in \mathbf{R}^1$ or $\rho \in \mathbf{C}^1$) give the same transform. For example, a zero of m-th order polynomial $F(x_0, x_1, x_2) = F(\lambda x_0, \lambda x_1, \lambda x_2) = 0$ defined over a projective space satisfies

$$F(\lambda x_0, \lambda x_1, \lambda x_2) = \lambda^m F(x_0, x_1, x_2), \quad \lambda \neq 0, \quad (65)$$

This condition is satisfied by an m-th order homogeneous curve, for example,

$$F(x_0, x_1, x_2) = \sum_{i_0+i_1+i_2=m} a_{i_0 i_1 i_2} x_0^{i_0} x_1^{i_1} x_2^{i_2}. \quad (66)$$

Since a scaling of RG does not give such a simple scaling relation generally, a scaling procedure of RG is understood that it contains a deviation from the homogeneous curve in a quantum field theoretical model such that:

$$\begin{aligned}\widetilde{F}(x_0,x_1,x_2) &= \sum_{i_0+i_1+i_2=m} a_{i_0i_1i_2}x_0^{i_0}x_1^{i_1}x_2^{i_2} \\ &+ \sum_{i_0+i_1+i_2=m\pm1} b_{i_0i_1i_2}x_0^{i_0}x_1^{i_1}x_2^{i_2} \\ &+ \sum_{i_0+i_1+i_2=m\pm2} c_{i_0i_1i_2}x_0^{i_0}x_1^{i_1}x_2^{i_2} + \cdots. \end{aligned} \tag{67}$$

It should be mentioned that the convergence property (convergence radius) of this expansion is not guaranteed, since the increasing rates of coefficients $\{b_{i_0i_1i_2}\}$, $\{c_{i_0i_1i_2}\}$, ..., and the variables (x_0,x_1,x_1) may depends on coupling constants and cutoff in a quantum field theoretical model, may be determined by a combinatorial calculation in a field theory, and which may sometimes rapidly increase. Note that an m-th order polynomial $f(x,y)=0$ defined over an usual plane can always be transformed into a projective space by

$$F(x_0,x_1,x_2) = x_0^m f\left(\frac{x_1}{x_0},\frac{x_2}{x_0}\right). \tag{68}$$

Since a set of projective curves and the images of their projective transformations defined over the three-dimensional space (x_0,x_1,x_2) at $x_2=0$ is naturally regarded as a Riemann sphere, a projective curve (and then, a projective variety) defines a homogeneous space.

In summary, we argue that (i) a scaling invariance of RG defines a projective variety $\mathbf{CP}^n$, (ii) the deviation form an ideal scaling low of RG corresponds to a deviation from the projective variety. In other words, we meet a concept which enlarges the usual/ordinary notion of projective transform of a homogeneous coordinate system, for understanding an RG prescription. We can consider a weighted (complex) projective space $W\mathbf{CP}^n$:

$$(z_1,\cdots,z_{n+1}) \simeq (\lambda^{k_1}z_1,\cdots,\lambda^{k_{n+1}}z_{n+1}), \quad \lambda\in\mathbf{C}^\times, \quad k_l\in\mathbf{N} \quad (\forall l). \tag{69}$$

This definition of a weighted projective space is an orbifold, namely $W\mathbf{CP}^n \simeq \mathbf{CP}^n/(\mathbf{Z}_{k_1}\times\cdots\times\mathbf{Z}_{k_{n+1}})$. It is interesting for us that if a complex manifold (Calabi-Yau, K3 surface, so on) can be embedded (or a foliation, imbedded) into such a weighted projective space, then we might consider a mirror pair of it in a scaling relation of RG: Namely, a role of symplectic manifold in RG would be found.

5 Summary and Perspective

In this paper, we have discussed on differential geometric nature of the NNG, GNG, and ANG theorems. From the viewpoint of Riemannian geometry, the Laplacian, the curvature, and the geodesics have been examined in detail. After an analytic continuation of the Riemannian geometry of an effective potential in the NG-type theorems, we have studied the complex geometry, and the Ricci flow equation has been obtained. From several viewpoints, symplectic geometry has been introduced in the NG-type theorems, and it has been examined especially in the setting of problem of the ANG theorem. The mirror duality, the Langlands correspondence, and some number theoretical aspects have been discussed. In our consideration on Riemannian and complex geometry, their methods/tools are not quite useful, while we have found that several notions of symplectic geometry are more relevant for describing the geometric nature of the NG-type theorems. Some mathematics of phases of matrices in theoretical physics, and the algebraic geometry in renormalization groups have also been discussed in our context of the NG-type theorems.

We have mainly considered some generic mathematical structure of the NG-type theorems in this paper. Needless to say, the reason why the NG-type theorems is universal and powerful is that they can explain the physical nature very well, by a simple methodology. The next step we should investigate is how the mathematical structure we have revealed here realizes in the physical nature more concrete manner, not only theoretically but also experimentally. In our results presented here, it seems the case that it is crucially important for us to reveal some infinite-dimensional groups/algebras/geometry in our NG-type theorems to understand them more deeply. For this purpose of our further investigations, we obtain an insight that several techniques, notions, and concepts of conformal field theory might be useful for us.

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