# Phase Velocity And Group Velocity For Beginners

*In the first section of this paper I derive the formulas for the phase velocity and group velocity as a function of the total relativistic energy and the momentum of a particle. In the second section I derive similar formulas as a function of the de Broglie and the Compton wavelengths of the particle. In the third section, I derive similar formulas as a function of the angular frequency and the wave number. Finally, two additional meanings of the Compton wavelength are derived. The first one is derived from the equation of the group velocity in terms of the de Broglie and the Compton wavelengths and the second one from the range of the weak nuclear force.*

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## Contents



## 1. Phase Velocity and Group Velocity as a Function of the Total Relativistic Energy and the Relativistic Momentum of a Particle

### 1.1. Phase Velocity

Let us consider the following three laws from Einstein's special theory of relativity:

a) The formula of equivalence of mass and energy

$$E = m c^2 \tag{1.1-1}$$

b) The formula of the relativistic momentum

$$p = m v_g \tag{1.1-2}$$

c) The formula of the relativistic mass

$$m = \frac{m_0}{\sqrt{1 - \frac{v_g^2}{c^2}}} \tag{1.1-3}$$

where

$E =$ total relativistic energy of a particle
$m =$ relativistic mass of a particle
$m_0 =$ rest mass of a particle
$p =$ relativistic momentum of a particle
$v_g =$ group velocity of a particle
$c =$ speed of light in vacuum

Let us begin by dividing equation (1.1-1) by equation (1.1-2)

$$\frac{E}{p} = \frac{m c^2}{m v_g} \tag{1.1-4}$$

Simplifying we get

$$\frac{E}{p} = \frac{c^2}{v_g} \tag{1.1-5}$$

The dimensions of equation (1.1-5) tells us that the ratio $s = E/p$ is a velocity. Furthermore, because the group velocity is always smaller than the speed of light, this velocity, $s$, must be greater than the speed of light, $c$, Therefore the ratio $s$ must be the phase velocity, $v_f$ . Thus, we can write

$$v_f = \frac{E}{p} \tag{1.1-6}$$

Thus we can draw the following conclusion

**Phase Velocity**

*The phase velocity of any particle (massive or massless) is equal to its total relativistic energy divided by its momentum.*

Finally from equations (1.1-5) and (1.1-6) we get

$$v_f v_g = c^2 \tag{1.1-7}$$

Which can be translated into words as follows

**The Product of the Phase Velocity and The Group Velocity**

*The product between the phase velocity and the group velocity of any particle (massive or massless) equals the square of the speed of light in vacuum.*

## 1.2. Group Velocity

Let us consider the Einstein's total relativistic energy formula

$$E^2 = p^2 c^2 + m_0^2 c^4 \tag{1.2-1}$$

Now we derive both sides of this equation with respect to $p$

$$\frac{d}{dp}\left(E^2\right) = \frac{d}{dp}\left(p^2 c^2 + m_0^2 c^4\right) \tag{1.2-2}$$

Observing that both $c$ and $m_0$ are constants we get

$$2E\frac{dE}{dp} = 2c^2 p\frac{dp}{dp} + 0 \tag{1.2-3}$$

After simple mathematics steps we get

$$\frac{dE}{dp} = \frac{p c^2}{E} \tag{1.2-4}$$

Substituting the denominator of the second side, $E$, with the second side of equation (1.1-1) we get

$$\frac{dE}{dp} = \frac{p c^2}{m c^2} = \frac{p}{m} \tag{1.2-5}$$

Substituting the numerator of the second side, *p,* with the second side of equation (1.1-2) we get

$$\frac{dE}{dp} = \frac{m\, v_g}{m} = v_g \tag{1.2-6}$$

Finally we swap sides to get the formula for the group velocity

$$v_g = \frac{dE}{dp} \tag{1.2-7}$$

Thus we can draw the following conclusion

**Group Velocity**

*The group velocity of any particle (massive or massless) is equal to the derivative of its total relativistic energy with respect to its relativistic momentum.*

# 2.
# Phase Velocity and Group Velocity as a Function of the de Broglie and the Compton Wavelengths of a Particle

## 2.1. Phase Velocity

In this section we shall derive the expression of the phase velocity of a particle as a function of its de Broglie wavelength and its Compton wavelength. To do that I will consider equation (1.1-6)

$$v_f = \frac{E}{p} \tag{2.1-1}$$

and the de Broglie law

$$p = \frac{h}{\lambda} \tag{2.1-2}$$

where

$h =$ Planck's constant
$\lambda =$ de Broglie wavelength

Now I shall define the Compton momentum, $p_C$, as follows

$$p_C \equiv m_0\, c \tag{2.1-3}$$

The Compton momentum is, as far as I know, not normally defined anywhere in the literature. However, since the Compton momentum is a very important concept it is convenient to introduce it here. This definition will allow us to write the Einstein's equation (1.2-1) for the total relativistic energy of a particle, as follows

$$E^2 = p^2 c^2 + p_C^2 c^2 \tag{2.1-4}$$

Taking $c^2$ as a common factor we can write

$$E^2 = c^2 \left( p^2 + p_C^2 \right) \tag{2.1-5}$$

Before we continue I shall define the Compton wavelength of a particle of rest mass $m_0$ , as

$$\lambda_C \equiv \frac{h}{p_C} = \frac{h}{m_0 c} \tag{2.1-6}$$

The Compton wavelength was introduced by the American physicist Arthur Compton to explain the scattering of photons by electrons. The Compton wavelength of a particle is the wavelength of a photon whose energy is equal to the rest energy, $E_0$ , of the particle ( $E_0 = m_0 c^2$ ). Now let us substitute the relativistic momentum *p* with the second side of equation (2.1-2) and the Compton momentum with the Compton wavelength given by equation (2.1-6). This gives

$$E^2 = c^2 \left( \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda_C^2} \right) \tag{2.1-7}$$

Taking $h^2$ as a common factor we have

$$E^2 = h^2 c^2 \left( \frac{1}{\lambda^2} + \frac{1}{\lambda_C^2} \right) \tag{2.1-8}$$

This equation can be written as

$$E^2 = \frac{h^2 c^2}{\lambda^2} \left( 1 + \frac{\lambda^2}{\lambda_C^2} \right) \tag{2.1-9}$$

Taking the square root on both sides

$$E = \frac{h\,c}{\lambda} \sqrt{1 + \frac{\lambda^2}{\lambda_C^2}} \tag{2.1-10}$$

Now we use equation for the phase velocity: (1.1-6), where we substitute *E* with the second side of the above equation, and *p* with the second side of equation (2.1-2). These substitutions yield

$$v_f = \frac{E}{p} = \left( \frac{h\,c}{\lambda} \sqrt{1 + \frac{\lambda^2}{\lambda_C^2}} \right) \frac{\lambda}{h} \qquad (2.1\text{-}11)$$

Finally, simplifying, we get the formula for the phase velocity in terms of the de Broglie wavelength, $\lambda$ , and the Compton wavelength, $\lambda_C$

$$v_f = c \sqrt{1 + \frac{\lambda^2}{\lambda_C^2}} \qquad (2.1\text{-}12)$$

**Particular case of the phase velocity for photons**

For the particular case of photons, the rest mass is zero, mathematically

$$m_0 = 0 \qquad (2.1\text{-}13)$$

therefore

$$\lambda_C = \frac{h}{m_0 c} = \frac{1}{0} = \infty \qquad (2.1\text{-}14)$$

$$v_f = c \sqrt{1 + 0} \qquad (2.1\text{-}15)$$

$$v_f = c \qquad (2.1\text{-}16)$$

Thus the phase velocity for photons equals the speed of light.

## 2.2. Group Velocity

We begin from equation (1.1-7)

$$v_f\, v_g = c^2 \qquad (2.2\text{-}1)$$

Solving for $v_g$ we get

$$v_g = \frac{c^2}{v_f} \qquad (2.2\text{-}2)$$

Now we substitute the denominator, $v_f$ , with the second side of equation (2.1-12) and after simplifying we get the formula for the group velocity in terms of the de Broglie wavelength, $\lambda$ , and the Compton wavelength, $\lambda_C$

$$v_g = \frac{c}{\sqrt{1 + \frac{\lambda^2}{\lambda_C^2}}} \qquad (2.2\text{-}3)$$

**Particular case of the group velocity for photons**

For the particular case of photons, the rest mass is zero, mathematically this means that

$$m_0 = 0 \qquad (2.2\text{-}4)$$

therefore

$$v_g = \frac{c}{\sqrt{1+0}} \qquad (2.2\text{-}5)$$

$$v_g = c \qquad (2.2\text{-}6)$$

Thus the group velocity for photons also equals the speed of light

# 3.
# Phase Velocity and Group Velocity as a Function of the Angular Frequency and the Wave Number

## 3.1. Phase Velocity

Let us multiply equation (2.1-1) by $\hbar/\hbar$ :

$$v_f = \frac{\hbar}{\hbar}\frac{E}{p} = \frac{E}{\hbar}\frac{\hbar}{p} \qquad (3.1\text{-}1)$$

But we know that the total energy of a photon is proportional to its frequency

$$E = \hbar\,\omega \qquad (3.1\text{-}2)$$

and the momentum of a photon is proportional to its wave number

$$p = \hbar\, k \qquad (3.1\text{-}3)$$

Thus equations (3.2-2) and (3.2-3) allow us to rewrite equation (3.1-1) as

$$v_f = \frac{\omega}{k} \qquad (3.1\text{-}4)$$

This result may be expressed as follows

**Phase Velocity**

The phase velocity is the ratio of the angular frequency to the wave number.

### 3.2. Group Velocity

Let us multiply equation (1.2-7) by $\hbar/\hbar$

$$v_g = \frac{\hbar}{\hbar}\frac{dE}{dp} = \frac{d(E/\hbar)}{d(p/\hbar)} \tag{3.2-1}$$

Using equations (3.1-2) and (3.1-3) from the previous subsection; the above equation transforms into

$$v_g = \frac{d\,\omega}{dk} \tag{3.2-2}$$

This result may be expressed as follows

**Group Velocity**

The group velocity is the derivative of the angular frequency with respect to the wave number.

## 4. Physical Meanings of the Compton Wavelength

I have pointed out that the Compton wavelength of a particle is the wavelength of a photon whose energy is equal to the rest energy of the particle. There are, however, many other "equivalent definitions". We can take equation (2.2-3), for example, and make the de Broglie wavelength of the particle the same as its Compton wavelength. Mathematically this means that we substitute $\lambda$ with $\lambda_C$ . This yields

$$v_g = \frac{c}{\sqrt{1+\frac{\lambda_C^2}{\lambda_C^2}}} \tag{4-1}$$

$$v_g = \frac{c}{\sqrt{2}} \tag{4-2}$$

Thus, the Compton wavelength of a particle is the wavelength associated with a particle that is moving at a speed equal to $\sqrt{2}/2$ times the speed of light.

Another interpretation of the Compton wavelength is related to the weak nuclear force and to the electromagnetic force (both forces have been unified under the name of: *the electroweak force*). The range of the weak interaction may be expressed as follows

$$R_{weak} = \frac{\hbar}{m c} = \frac{h}{2\,\pi\,m c} = \frac{\lambda_C}{2\,\pi} \tag{4-3}$$

Thus, the range of the weak force is the Compton wavelength for a given intermediary particle (the *W*+ boson, the *W*- boson or the *Z*° boson), divided by 2*π*. The Compton wavelength for the force carrier (boson) of the electroweak force is

$$\lambda_C \equiv \frac{h}{m\,c} \tag{4-4}$$

(a) For the *W*+ or the *W*- bosons the range of the weak force is

$$R_{weak}\left(W\right) = \frac{h}{2\,\pi\,m_W\,c} \tag{4-5}$$

where $m_W = m_{W^+} = m_{W^-}$ is the mass of either the *W*+ boson or the *W*- boson (both bosons have the same mass).

(b) For the *Z*° boson the range of the weak force turns out to be

$$R_{weak}\left(Z°\right) = \frac{h}{2\,\pi\,m_Z\,c} \tag{4-6}$$

This means that the weak nuclear force has two ranges, $R_{weak}\left(W\right)$ and $R_{weak}\left(Z°\right)$ . The reason of having a dual range is because the mass of the *Z*° boson is different from the mass of the other two *W's* bosons. The values of these masses are

$$m_W \approx 143.1\times10^{-27}\,Kg$$
$$m_Z \approx 162.3\times10^{-27}\,Kg$$

Introducing these values into equations (4-5) and (4-6), respectively, we get

Range of weak force for the *W* boson $\qquad R_{weak}\left(W\right) = 2.5\times10^{-18}\;m$

Range of weak force for the *Z*° boson $\qquad R_{weak}\left(Z°\right) = 2.2\times10^{-18}\;m$

It is interesting to compare the range of the weak force with the diameter of the proton. The observed value of the proton radius is

Measured radius $\qquad r_{p\,exp} = 0.84087\left(39\right) fm \approx 8.4\times10^{-16}\;m$

Thus, the ratio of the range of the weak force for the $Z°$ boson to the diameter of the proton is

Comparison $$\frac{R_{weak}(Z°)}{2\,r_{p\,exp}} \approx \frac{2.2\times 10^{-18}\ m}{2\times 8.4\times 10^{-16}\ m} \approx 0.001$$

Thus, the range of the weak force is about 0.1 % of the diameter of the proton.
It is also interesting to compare the masses of these bosons with the mass of the proton

$$\frac{m_W}{m_p} \approx \frac{143.1\times 10^{-27}\,Kg}{1.172\,621\,777\times 10^{-27}\,Kg} = 86$$
$$\frac{m_Z}{m_p} \approx \frac{162.3\times 10^{-27}\,Kg}{1.172\,621\,777\times 10^{-27}\,Kg} = 97$$

Thus, the mass of the $W$ bosons is about 86 times the mass of the proton while the mass of the $Z°$ boson is about 97 times the mass of the proton.

(c) If the force is the electromagnetic force, the boson that transmits the force is the photon. Because the rest mass of the photon is zero (within the experimental errors of the measurement), it turns out that the range of the electromagnetic force is infinite. Mathematically

$$R_{electromagnetic} = \frac{h}{2\,\pi\, m_{\gamma\,0}\, c} = \frac{h}{2\,\pi\times 0\times c} = \infty \tag{4-7}$$

Thus the Compton wavelength for the photon is infinite.

(see next page)

# Summary

The two following tables summarises the above results.

| | MASSIVE PARTICLES (e.g. electrons) | MASSLESS PARTICLES (e.g. photons) |
|---|---|---|
| PHASE VELOCITY (Special Relativity) (See section 1) | $v_f = \frac{E}{p}$ | $c$ |
| PHASE VELOCITY (De Broglie) (See section 2) | $v_f = c\sqrt{1 + \frac{\lambda^2}{\lambda_C^2}}$ where $\lambda_C \equiv \frac{h}{m_0 c}$ is the Compton wavelength | $c$ |
| PHASE VELOCITY (Angular frequency and wave number) (See section 3) | $v_f = \frac{\omega}{k}$ | $c$ |

**Table 1:** Phase velocity formulas. Both massive particles and photons obey the same equation. However for the latter the Compton wavelength is infinite. This means that, for photons, the phase velocity is equal to the speed of light in vacuum.

| | MASSIVE PARTICLES (e.g. electrons) | MASSLESS PARTICLES (e.g. photons) |
|---|---|---|
| GROUP VELOCITY (Special Relativity) (See section 1) | $v_g = \frac{dE}{dp}$ | $c$ |
| GROUP VELOCITY (De Broglie) (See section 2) | $v_g = \frac{c}{\sqrt{1 + \frac{\lambda^2}{\lambda_C^2}}}$ where $\lambda_C \equiv \frac{h}{m_0 c}$ is the Compton wavelength | $c$ |
| GROUP VELOCITY (Angular frequency and wave number) (See section 3) | $v_g = \frac{d\omega}{dk}$ | $c$ |

**Table 2:** Group velocity formulas. Both massive particles and photons obey the same equation. However for the latter the Compton wavelength is infinite. This means that, for photons, the group velocity is equal to the speed of light in vacuum.

Three of the physical meanings of the Compton wavelength are

**Physical Meanings of the Compton Wavelength**
(See Section 4)

*The Compton wavelength of a particle is*

*1) the wavelength of a photon whose energy is equal to the rest energy,* $E_0$ *, of the particle (* $E_0 = m_0 c^2$ *).*

*2) the wavelength of a particle that is moving at a speed equal to* $1/\sqrt{2} \approx 0.707$ *times the speed of light. In other words*

$$\text{If} \quad v_g = \frac{c}{\sqrt{2}} = \frac{\sqrt{2}}{2} c \quad \text{then} \quad \lambda = \lambda_C$$

*3)* $2\pi$ *times the range of the weak force:* $\lambda_C = 2\pi R_{weak} = \dfrac{h}{mc}$ *.*

*For the weak nuclear force,* $m$*, is the mass of any of the intermediate-vector bosons:* $m = m_{W^+} = m_{W^-}$ *or* $m = m_{Z^0}$ *. Because these masses are relatively large, the range of the weak force is, as we have seen, relatively very small. On the other hand, for the electromagnetic force, the force carrier is the photon. Because the rest mass of the photon is zero,* $m = m_{\gamma 0} = 0$ *, the range of the electromagnetic force turns out to be infinite. Mathematically this is usually expressed as* $R_{electromagnetic} = \infty$ *. In fact the above formula may be written as:* $\lambda_C = 2\pi R = \dfrac{h}{mc}$ *; where* $R$ *is the range of any force of nature and* $m$ *is the mass of any of its force carriers.*