Abstact

The Leibnizean cosmology, where both space and discrete objects (including particles) are assumed to be formed from continuous entities called infinitesimals, replaces the Euclidean Cosmology, where continuous space is assumed to be formed from discrete entities called points. Infinitesimals as well as infinitesimal numbers are defined in this monograph and an Arithmetic for non-standard analysis is developed by generalising the concept of real number to a wider class of numbers, called here the Cauchy numbers, and by rearranging Number Theory somewhat.

This is motivated in part one by showing that Cantor's famous diagonal proof, which is an algebraic formulation of Euclidean Cosmology, rests on the fallacy that an infinite decimal fraction can be identified by specifying its finite digits. The argument of this part includes a proof that the set of equivalence classes of Cauchy sequences is countable.

In part two the Leibnizean model for the infinite divisibility of space is developed by introducing the concepts of infinitesimal and infinitesimal number from the context of Calculus. An Arithmetic for Cauchy numbers is developed, including an interpretation of L'Hospital's rules and a description of the Real Continuum. The concept of cascades of infinitesimals as directed sets is introduced and the Fundamental Theorem of the Calculus is studied as a Net defined on such a cascade.

In part three the Leibnizean Cosmology is argued to be in line with the ideas of Parmenides and that space, in this cosmology, is 'fuzzy' - thus clarifying the paradox of the arrow. It is also pointed out that, with particles suitably defined as infinitesimals, Parmenides' ideas are vindicated because everything is one, and motion is only an illusion because nothing comes into being where it did not exist before. Furthermore, some intractable problems of Physics, like the particle/wave duality and action at a distance, are pointed out to be direct consequences of the Euclidean Cosmology and that they all but disappear in the Leibnizean Cosmology.

In part four it is pointed out that when the role of points as building blocks of space is discarded, points can be used comfortably in the Leibnizean model as indicators of locations in space. Thus Mathematics can once more become a canonical model, but without the parts that depend on the Euclidean properties of points; e.g. open and closed sets on the real line.

CANTOR'S FALLACY AND THE LEIBNIZEAN COSMOLOGY

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FOREWORD, SYNOPSIS AND ACKNOWLEDGEMENT

When a crime is committed the existence of a criminal is automatically known. It is then the function of the forces of justice to identify the criminal.

In Mathematics this distinction between “To exist” and “To be identified” is often vague or ignored.

Whenever a Cauchy sequence of points is specified, it is known that its limit exists because of the completeness of the real line. But it is also known that this limit cannot necessarily be identified.

In part one the consequences of this vagueness between existence and identification of real numbers are studied. The analysis starts with an example showing that the equality/inequality of two real numbers can only be decided if both are identified. It is then concluded that the similarity of the infinite decimal fraction of the example to the fraction that is constructed in the diagonal proof indicates, in a simplistic way, that the argument of the diagonal proof is itself simplistic.

Some relevant concepts are then discussed and new concepts are introduced to facilitate the analysis. This discussion culminates in a proof that the set of equivalence classes of Cauchy sequences is countable.

In the final section of part one it is shown that the use of the axiom of choice in the construction of a real number leads only to the existence of the number and not to its identification. This is done by showing that, if the Euclidean topology is taken into account, a contradiction results when it is assumed that the real number which is constructed in the diagonal proof, is identified. As a consequence of this it is concluded that the diagonal proof, in the presence of limits, is invalid.

In part two the conundrum that the real numbers are countable according to number theory but more than countable according to the real line, is addressed. This is done by looking at the infinite divisibility of space according to Euclid and according to Leibniz. It is concluded that these two approaches lead to two different models for Mathematics because the circumstances do not allow the order of sums and limits that occur to be inverted. These two models are then called the Euclidean- and the Leibnizean Model for Mathematics, and they would lead ultimately to two different cosmologies.

The concept of infinitesimal is introduced and then a re-interpretation of standard Number Theory is used to generalise the concept of number to what is called here the Cauchy numbers. It is shown that the Cauchy numbers consist of three classes of numbers, namely the infinitesimal numbers, the rated numbers and the infinite numbers and that these three classes describe the real continuum.

The relationship between these classes of numbers is then studied in the spirit of L'Hospital. The concept of differential is introduced. Finally cascades of infinitesimals are introduced and the fundamental Theorem of Calculus is studied in the Cauchy number context.

In part three the Leibnizean cosmology is introduced. It is pointed out how the Leibnizean cosmology fits in with the philosophy of Parmenides of Elea and Xeno. It is pointed out that the remark by Parmenides that there can be no motion (because everything is one) and the paradox of the arrow are explained by the Leibnizean cosmology. It is also pointed out that some of the intractable problems of Physics - like the particle/wave duality and action at a distance - are consequences of the Euclidean cosmology that become tractable in the Leibnizean cosmology.

In part four it is pointed out that once the Euclidean Cosmology is discarded, those results of the Euclidean model of Mathematics which are not related to the Euclidean cosmology, still form part of the Leibnizean model; thus creating once again a single model for Mathematics, but without the Euclidean Cosmology.

Finally, I should point out that the ideas presented in this document were never subjected to proper scientific scrutiny. This is because these ideas are completely contrary to current group-thinking in the scientific community and thus no willing peers could be found. In that aspect this document should be looked at as a discussion document with the objective of soliciting criticism from the scientific community at large. However, I would like to thank my good friend Dr Anneke Roux of the Department of Civil Engineering at the University of Pretoria for supporting me in this by always having been willing to listen to me; no matter how weird my ideas were.

PART ONE

THE FALLACY IN CANTOR’S DIAGONAL PROOF FOR REAL NUMBERS

SECTION 1: INTRODUCTION

Cantor's (well known) Theorem.
The real numbers are more than countable.

Proof
Assume that the real numbers are countable. Hence a list containing all the real numbers larger than or equal to zero and less than or equal to one can be made. Each such real number is an infinite decimal fraction, so that a list of all the infinite decimal fractions between zero and one inclusive can be made:

$0.a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15}\ a_{16}\ a_{17}\ a_{18}$ ....
$0.a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{25}\ a_{26}\ a_{27}\ a_{28}$, ...
$0.a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{35}\ a_{36}\ a_{37}\ a_{38}$ ....
$0.a_{41}\ a_{42}\ a_{43}\ a_{44}\ a_{45}\ a_{46}\ a_{47}\ a_{48}$ ....
$0.a_{51}\ a_{52}\ a_{53}\ a_{54}\ a_{55}\ a_{56}\ a_{57}\ a_{58}$ ....
$0.a_{61}\ a_{62}\ a_{63}\ a_{64}\ a_{65}\ a_{66}\ a_{67}\ a_{68}$ ....
.
.
.
Where each a_{ij} is one of the digits 0, 1, 2…, 9

Let b be that real number (infinite string of digits) $0,b_1b_2b_3b_4$.... which is such that

$b_i=2$ if $a_{ii} \neq 2$ and bi=5 if $a_{ii} = 2$. **[A]**

This real number b differs from any number in the list in at least one place. Thus b does not belong to the list. This contradicts the assumption that all real numbers larger than zero and less than one belong to the list. Thus all real numbers between 0 and 1 cannot be listed, and hence these numbers are more than countable. ■

The counterbalance to this theorem is the following example:

Example A

Let

$$a = 0.a_1a_2a_3a_4a_5....$$

be an irrational number between 0 and 1 as in Cantor's theorem.

Choose the digits of the infinite decimal fraction

$$a^* = 0.a^*_1 a^*_2 a^*_3 a^*_4 a^*_5....$$

as follows:

$a^*_i = a_i$ if the two strings of digits "$a_1a_2a_3a_4$ ….. a_i" and "$a_{(i+1)}a_{(i+2)}$ …..a_{2i}" are not identical

and

$a^*_i = 7$ if these strings are identical.

■

Note:

- The fractional part of an irrational number is a pseudo-random string of digits – i.e. even though the order of the digits is fixed, there may be no regularity in their occurrence. However, for any random string of digits of length 2n, the probability that the string consisting of the first n digits is the same as the string consisting of the last n digits is 10^{-n}. Thus, from a probabilistic point of view, the probability that a^*_n differs from a_n is 10^{-n}. Therefore the probability that the fraction a* differs from the fraction a diminishes rapidly with the length of the substrings under consideration. **But this probability never becomes zero.**
- In the light of this, it can be concluded that it is in principle possible to determine for a given infinite decimal fraction, a, whether a* is smaller or larger than a. This is done by generating the digits one after another until two is found that are not the same. But if no such digits appear, nothing at all about the equality/inequality of the numbers can be deduced. Hence it is not even in principle possible to determine whether a* is equal to a, nor is it possible to determine the value of a* should it differ from a.
- This example shows that the equality/inequality of two infinite decimal fractions can only be decided once they are identified, i.e. all their digits are known. Or, to put it milder, specifying the finite digits of an infinite decimal fraction does not provide enough information to decide the question of their equality/inequality.

This example reveals the simplistic nature of the argument in Cantor's diagonal proof. It shows that the information contained in the strings b and b* is not sufficient to determine whether b and the associated fraction b*, which by assumption belongs to the list, are equal or different. Consequently the argument in the proof becomes suspect[1]. This is looked at in formal detail in section five.

[1] This example is enough to destroy the logical structure of the proof. See the addendum at the back of the monograph.

A possible reason why this simplistic proof has been accepted without objection is because, according to the real line, the theorem is true.

SECTION 2: THE REAL LINE

As an addition to the existence of volumes, areas and lines that extend in three, two and one directions, Euclid defined a point as ‘That which has no extent’. This implies that a point is a thing (and consequently a piece of space) of which the volume, area and length are all zero. It is then required that a volume, area or line be formed by the combination of points. These assumptions will be called the **Euclidean Cosmology**.

For a line, the total length of a finite number of its points must be zero because it is a finite sum of zero’s. But then the total length of a countable number of its points must also be zero because this length is the limit of the total lengths of the partial sums which are all zero. However, the total length of all the points on the unit interval of the line must be the length of the line, and hence it must be one. Thus the number of points on a line of unit length cannot be either finite or countable, and therefore it is concluded that there must be **more** than countable many points in this interval.

The concept of the real line extends this property to real numbers:

The real line:

There is an order-preserving one-to-one mapping of the real numbers onto the points of a line. This mapping is a homeomorphism in the standard topologies of the real numbers and the line.

This one-to-one mapping then implies that the real numbers must be more than countable too, and this gave support to Cantor’s theorem.

SECTION 3: INFINITE DECIMAL FRACTIONS

3.1 Note about Notation

From the way the phrases ‘infinite decimal fraction’ and ‘diagonal’ are used in the proof of Cantor’s theorem, follows that the following phrases refer to infinite strings of digits with different other symbols interspersed, and thus are assumed to be equivalent:

- ‘Infinite decimal fraction’
- ‘Infinite (dimensional) vector of digits’
- ‘Infinite sequence of digits’

In the same context, an infinite matrix is considered to be an infinite list of infinite vectors.

3.2 A synopsis of the logical history of infinite decimal fractions

When the division algorithm is used to convert a rational number into decimal form, it is found that the resulting decimal fraction is either of finite length or becomes a repeating sequence of digits of which the length of the repeating string is less than or equal to the denominator of the fraction. Conversely, a decimal fraction of finite length or with a repeating sequence of digits can be transformed back into a proper fraction.

Certain numbers, e.g. the function value $\sqrt{2}$, can be shown to be not rational by using the properties of the function. In a case like this the properties of the function and a tool like Taylor's Theorem can be used to approximate the number to any desired accuracy by a decimal fraction. Because the given number is not rational, it cannot have the properties of the decimal representation of a rational number. Therefore this approximation cannot be either finite or have a repeating cycle of digits. Thus this approximation must be an infinite pseudo-random string of digits.

3.2.1 Notes

- In the above cases the digits of the resulting infinite decimal fraction are generated one-by-one. Thus the infinite decimal fraction $a = 0.a_1a_2a_3a_4\ldots.$ is generated as the Cauchy sequence of numbers $a = (0.a_1 ; 0.a_1a_2 ; 0.a_1a_2a_3 ; 0.a_1a_2a_3a_4 ; \ldots.)$. This Cauchy sequence is the form in which infinite decimal fractions are studied in Number Theory, and will be called the **Cauchy form** or **Cauchy representation** of the number.

- In both these cases the word 'infinite' refers to calculations that are performed repeatedly, and hence it means 'never ending'

3.2.2 The real line is introduced

When a line and a suitable scale is chosen as axis, there exists a simple geometrical procedure that allows the construction of a line of length equal to any given rational number. This allows any rational number to be associated with a point of the line. Thus the terms of the Cauchy representation of a number maps onto an open set of points of the line. Because the numbers form a Cauchy sequence, these points form a Cauchy sequence in the Euclidean topology of the line.

But according to the definition of a point the line is complete in the Euclidean topology, and therefore there **exists** a limit point for the Cauchy sequence of points. This validates the introduction of a new kind of number as a limit for the Cauchy sequence formed by the Cauchy representation of a number. This new number is called a **real number**. If these two limits are mapped onto each other, the **real line** is defined and there is a one-to-one order preserving homeomorphism of the real numbers onto the points of a line.

Because a real number is the limit of an ever longer string of digits, it is called an **infinite decimal fraction** or a string of digits of infinite length. Note that in this scenario the term 'infinite' means 'a number larger than any natural number'. This has to be so because, if the number denoting the length of the infinite string is not larger than any natural number, it has to be a natural number itself and consequently the number then refers to a term of the Cauchy sequence and not to its limit.

There exist many Cauchy sequences of points converging to any given point on the real line. For any given point on the real line these sequences form an equivalence class in the set of all Cauchy sequences of points. This equivalence class is uniquely associated with the given point. The real line then ensures that there is a unique equivalence class of Cauchy sequences of numbers that is associated with any real number.

Therefore a real number has three avatars: (1) It is a point of a line, (2) it is an infinite sequence of digits (with 'infinite' meaning 'larger than any natural number') and (3) It is an equivalence class of Cauchy sequences of numbers.

Remark

Even though all the limits of these Cauchy sequences exist, they are not necessarily **identified**, as illustrated by Example A of section one.

3.2.3 Conclusion

For two real numbers to be equal, they must both be identified and all three of their corresponding avatars must be identical.

Therefore, because an infinite decimal fraction is a single point on the real line, it is only identified if all its digits are known.

3.2.4 Note

A Cauchy sequence forms an open set on the real line; but a real number, being a single point, is a closed set.

3.3 The specification of numbers

As pointed out above, numbers are shown to be irrational by default: the value of a function at a given argument is shown to be not rational by using the properties of the function – like for $\sqrt{2}$. A real number, specified in this way, will be called

value specified. This is the number associated with the point on the real line onto which the argument of the function is mapped. Hence a real number specified in this way can also be said to be **value identified** because the point where it is located on the axis can be found by using the properties of the function. By implication both other avatars are then also value identified.

In Cantor's proof the constructed real number is described as an infinite decimal fraction by specifying the finite digits of its representation.

With the above note on notation in mind, a real number as well as an infinite matrix that is specified by stating their components, will be called **component specified**. Thus the real number constructed in Cantor's diagonal proof is an example of a component specified real number, as is the number constructed in Example A.

According to this example, although the irrational number √2 is both value specified and value identified, √2* is component specified but is not component identified. Hence the equality/inequality of these two numbers cannot be decided. This example emphasises that two real numbers cannot be compared unless both are identified.

Therefore the Cauchy form of a component-specified real number can be mapped onto an open set of points on the real line as a 'never-ending' sequence of points for which it is known that a limit exists, but for which the limit cannot necessarily be identified.

The same holds true for component specified infinite matrices.

SECTION 4: COMPONENT SPECIFIED REAL NUMBERS

4.1 Theorem

The set of component specified infinite decimal fractions is countable

Proof

(Tis proof mimics the proof that the set of rational numbers is countable)

Write the digits 4, 5, 6, 7, 8, 9, 0, 1, 2, 3 on 10 consecutive lines
Repeat this group another 9 times so that 100 lines are filled with ten of these groups of ten digits.
Add second digits to the lines: a 1 to the first group, and from 2 to 0 respectively to every consecutive group. All 100 possible permutations of two digits are now listed, the top one being the first two digits of the decimal part of √2.
Take this 100x2 array and repeat it 9 times so that 1000 lines now have two digits. Append the next digit of √2 to the first 100, and then proceed cyclically as above, adding the other nine digits to the next nine groups. All 1000 possible

permutations of three digits are now listed, the first line being the first three digits of √2.

Repeating this process, a component specified infinite two-dimensional array is constructed containing as rows all possible infinite permutations of the ten digits, the first row being the component specified fractional part of √2 i.e. the Cauchy form of √2.

Thus an infinite list of all possible component-specified infinite decimal fractions between zero and one is constructed, and the theorem is proved. ■

In the theory of numbers it is shown that any given Cauchy sequence can be converted to an equivalent infinite decimal fraction by using the definitions of equivalence and of Cauchy sequence. A number obtained in this way is a component specified real number.

Corollary
The set of equivalence classes of Cauchy sequences is countable.

Proof
Every equivalence class of Cauchy sequences contains at least one component specified infinite decimal fraction, as is pointed out above.

But transitivity of the equivalence relation prevents any two equivalence classes from sharing a component specified infinite decimal fraction. Thus there exists a one-to-many mapping of the set of equivalence classes into the set of component specified real numbers, and the latter is countable. ■

SECTION 5: INFINITE, NEVER-ENDING AND THE AXIOM OF CHOICE

5.1 Introduction

In the previous sections the attributes ‘infinite’ and ‘never ending’ were used in situations where the associated sets of points of the real line were respectively closed or open.

This is because, in the cases where these sets were closed, the term ‘infinite’ had properties of being a very large number – one that is ‘larger than any other number’. Two examples of such sets are relevant:

First, consider the closed interval [0,1]. This interval is a continuum. If a single point, a, is removed from this interval, the continuum is destroyed and two half open intervals [0,a) and (a,1] are formed. This implies that the continuum can only exist if ‘all’ its points are present.

Next, consider the irrational number a, and let a = $\{a_n\}$ be its Cauchy form. The number ‘a’ maps onto a single point of the real line. This point is the limit of the points onto which $\{a_n\}$ maps and, being a single point, is a closed set. If a is referred to as an infinite decimal fraction, then the word ‘infinite’ must refer to a ‘number larger that all natural numbers’ because any natural number of digits would refer to a term of the Cauchy representation and not the limit (as was pointed out before).

5.2 Existence and Identification

The example of section 1 shows that component-specified real numbers, and thus also the number ‘b’ of the diagonal proof, can not necessarily be identified in practice – i.e. although any one of the finite digits can be identified, all of its digits cannot be found.

When Zeno stated the paradox of Achilles and the tortoise, the response to the paradox was, in essence, that the real line is complete and that, although an infinite number of steps is required by Zeno’s argument, a limit exists and that limit fixes the point and time when Achilles would pass the tortoise.

But Zeno was intent on creating a paradox and thus used the expression ‘Achilles can never pass the tortoise’. If he had instead said ‘We can never know where Achilles would pass the tortoise’ the answer to his statement would have been much more difficult; because then he would have conceded the existence of the limit point, but now required its identification.

In modern times it became part of Mathematics to assume that an infinite number of choices can in principle be made; this is known as the ‘Axiom of Choice’. However, in the present situation, it is not clear whether application of the axiom of choice will result in ‘existence’ or in the more specific ‘identification’.

Note that the diagonal proof becomes valid if, in the construction of the number b, the axiom of choice should lead to ‘identification’ of the number and not merely to its ‘existence’. This is because then ‘all’ digits of the number b would in principle be known a posteriori. All the digits of all the numbers in the list are assumed to be known a priori because of the logical structure of the argument. Because all digits of all numbers are known they can be compared and the number b would differ from all numbers in the list if it should differ from any given one at one digit.

5.2.1 The Siegfried Lemma

A component-specified infinite decimal fraction is component-identified iff a component-specified two-dimensional array of digits is component-identified.

Proof

Consider the component-specified infinite decimal fraction

$0.a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9\ a_{10}\ a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15}\ a_{16}.....$

This component-specified decimal fraction can be split into infinitely many component-specified decimal fractions as follows:

$0.a_1\ a_2 \mid a_5\ a_6 \mid a_{11}\ a_{12} \mid a_{19}\ a_{20} \mid ......$
$0.a_3\ a_4 \mid a_9\ a_{10} \mid a_{17}\ a_{18} \mid ......$
$0.a_7\ a_8 \mid a_{15}\ a_{16}......$
$0.a_{13}\ a_{14}.......$

Thus an infinite vector of digits can be used to construct an infinite two-dimensional array of digits (or a component-specified infinite matrix).

Because an infinite decimal fraction, and hence an infinite vector, is formed by stringing together digits from the left, an infinite two dimensional array of digits is derived from the infinite vector by filling it from the top left corner in some zigzag way with digits from the fraction.

Conversely, if any component-specified infinite matrix of digits is set up by filling it from its top left corner with digits according to some scheme, this two dimensional array can be transformed back into a single component-specified decimal fraction by stringing together the digits in a zigzag fashion.

Hence component-specifying an infinite decimal fraction (infinite string of digits) and component-specifying an infinite matrix of digits are equivalent because one can be transformed into the other. Thus, if it is possible to component-identify one of them, it is possible to component-identify the other.

■

5.2.2 Theorem
It is not possible to component-identify an infinite decimal fraction.

Proof
Assume that it is possible to component-identify an infinite decimal fraction. The lemma above then implies that the previously constructed list of all possible component-specified real numbers is a component-identified array of digits. But if a matrix of digits is component-identified, then each of its rows is also component identified. Thus the matrix is a list of all possible component-identified real numbers, and hence these numbers are countable.

But the assumption that it is possible to component-identify a real number validates Cantor's diagonal proof. So that these numbers are then also more than countable by his proof.

The contradiction proves the theorem.

■

Thus it is shown that, in the case of component-specified real numbers, the axiom of choice leads only to the **existence.** Hence Cantor's diagonal proof is invalid and therefore there can be no component-identified irrational numbers.

Therefore there are now two types of real numbers that should be distinguished; component specified real numbers (or equivalence classes of Cauchy sequences) which are countable, and value identified real numbers (or points on the real line) which are more than countable. Therefore a real number no longer has three avatars.

PART TWO

THE INFINITE DIVISIBILITY OF SPACE

or

WHAT COMES FIRST: THE LIMIT OR THE SUM?

1.1 INTRODUCTION

Modern western civilisation is built on the philosophical foundations that were laid down in ancient Greece. The Greeks started what is now known as science – a deductive system of knowledge based on explicitly stated assumptions.

They settled on an atomic theory of matter. This meant that when a piece of matter is halved repeatedly, a piece of matter that cannot be divided again will be reached after a finite number of steps. This was called an 'atom' and continuous matter was assumed to be compounded of these discrete indivisible pieces of matter.

This is the explicit assumption that continuous matter is formed from discreet atoms, and is called the atomic theory.

But the Greeks accepted that space is infinitely divisible. Thus repeated division of a piece of space would lead to a never ending sequence of ever smaller pieces of space. For this sequence of ever smaller pieces of space they defined a discrete limit, called a point. The explicit assumption that continuous space is compounded of discrete points is called here the **Euclidean cosmology**

The ideas of the Euclidean cosmology was opposed by Parmenides of Elea and his eromenos Xeno. In their philosophy they concerned themselves with the concept of motion. Although they did not propose a cosmology of their own, their criticism of the Euclidean cosmology - and the resulting model of Mathematics - is today mostly known as Xeno's paradoxes. Looking at these paradoxes, it is quite clear that they considered motion to be in essence continuous and thus incompatible with Euclidean cosmology which is in essence discrete. Some of these paradoxes will be referred to in the paragraphs to come.

2.1 THE INFINITE DIVISIBILITY OF SPACE.

Strict rules apply today when limits and sums are to be interchanged in Mathematics. However, no such rules were followed when the infinite divisibility of space was considered at the time of laying down the foundations of Mathematics.

2.1.1(a) When the limit precedes the sum.

First the limit:

The Euclidean cosmology starts with a nested sequence of intervals of which the lengths of the intervals converge to zero. The limit of these intervals is then defined to be an entity of no extent, called a point.

Thus the length (or area or volume) of a point is zero.

Followed by the sum:

Next, the Euclidean cosmology states that a line of unit length is a string of points. This then requires that the length of the interval, which is non-zero, must be the sum of the lengths of all the constituent points, all of which are zero. This requires the line to be formed from more than countable many points, as was discussed in the first part.

The assumption that there are ‘more than countable’ many points is an ideological compromise that resolves the conflict that results when requiring that something which is continuous is to be formed by combining discrete entities - any discrete set can be counted one by one, but a continuum cannot be counted at all.

2.1.1(b) When the sum precedes the limit.

Two millennia later Leibniz studied motion, volumes, areas and lengths. These are all continuous entities.

Looking at the area under the curve y=1:

First the sum:

Consider the unit interval on the X-axis and let d(a;b)=b-a denote the length of the interval (a;b). Assume that the unit interval has been repeatedly subdivided in such a way that at each new subdivision all previous intervals are subdivided into three equal intervals. After n such partitions there are 3^n subintervals (parts), each of length 3^{-n} and

$$1 = \sum_{i=0}^{3^n - 1} d\left(\frac{i}{3^n} ; \frac{i+1}{3^n} \right)$$

After n steps the middle of interval i of these subintervals is at the point

$$x_{i+1} = \frac{2i - 1}{2 \cdot 3^n} : i = 0,1,..,3^n - 1$$

Notice that once a point is in the middle of a subinterval, it will be in the middle of a subinterval for all subsequent partitions.

Followed by the limit:

According to the theory of the Riemann integral:

$$1 = \int_0^1 1 \cdot dx = \lim_{\substack{n \to \infty \\ all\ \Delta x_i \to 0}} \sum_{i=0}^{3^{n-1}} 1 \cdot \Delta x_i$$

Where $\Delta x_i = 3^{-n}$ for all i. The parts of these partitions are intervals of which the lengths converge to zero as n becomes larger and larger and therefore any set of intervals with the same midpoint is a nested set that satisfies the requirements set out above for the definition of a point. Hence, according to the Euclidean cosmology, each set of nested intervals becomes a point in the limit. But only points that are in the middle of a subinterval for some value of n can be a limit point (because these intervals reduce symmetrically relative to their midpoints with each subsequent partition). Thus only rational points that are for some n and some i of the form

$$x_{n,i} = \frac{2i - 1}{2 \cdot 3^n}$$

qualify to be limit points. These points form a subset of the (countable) set of rational numbers.

Thus, if the sum precedes the limit, only countable many points, being the limits of monotonically decreasing nested intervals, are required to form the unit interval.

Remarks

- For any valid set of partitions these points are dense in the unit interval.

- In the Leibnizean approach to the infinite divisibility of space, the need for the existence of more than countable many points disappears because only countable many points are needed to cover an interval. Thus, in the Leibnizean approach, only countable many value specified real numbers need to exist in order to maintain the concept of the real line. Component specified real numbers and equivalence classes of Cauchy numbers have already been shown to be countable. Thus everything can become countable in a number system associated with the Leibniz approach.

2.1.2 Conclusion

These two different ways in which the limits and the sums are considered when studying the infinite divisibility of space, leads to two completely different sets of assumptions about the nature of space and therefore of numbers. The first, the traditional one, will be called the **Euclidean model of Mathematics** which is based on discrete points, while the second will be called the **Leibnizean model of Mathematics** which is based on continuous intervals. At this stage of the argument it looks as if these two models have contradicting properties and as such are not reconcilable.

Note that the order of the sums and the limits that occur here cannot be interchanged because the limit that occurs in the Euclidean model is zero.

3.1 NUMBER THEORY IN THE LEIBNIZEAN MODEL.

The study of Calculus inevitably ends in the use of infinitesimals. The word ‘infinitesimal’ is pidgin Latin that loosely translates into ‘that little thing at infinity’. Thus it is a valid question to ask whether points, as described in the Leibniz model for the infinite divisibility of space, are indeed the elusive ‘infinitesimals’; seeing that only countable many of them are required to form the unit interval. However, this is not so. In part four it will be shown that points, as defined in the Euclidean Model, can be re-introduced as utilitarian entities that are associated with infinitesimals. In what follows the word ‘point’ will mean ‘a place in space’ – like the endpoint of a line (or vector) or the intersection of two arcs. In the Leibnizean model a point will simply be a convenient word to describe the focus of a set of nested intervals.

In the Euclidean model the size of a point was defined as zero. In the Leibnizean model the size of an infinitesimal (a spatial entity) will be defined as an infinitesimal number (a numerical entity).

3.1.1 Cauchy Numbers

Number theory is based on interpreting an infinite decimal fraction as a Cauchy sequence, for instance

$$\sqrt{2} = (1. ; 1.4 ; 1.41 ; 1.414 ; 1.4142 ; \ldots\ldots)$$

In part one this representation of an infinite decimal fraction was called the **Cauchy Form** of the real number, and as such it is an element of some equivalence class of Cauchy sequences.

A Cauchy sequence that is associated with a positive infinite decimal fraction, like the one above, has non-decreasing components. To overcome this limitation the representation is generalised:

Definition
A Cauchy sequence of rational numbers is called a **Cauchy number**.

In general a Cauchy number will be specified as

$$a = (a_1 ; a_2 ; a_3 ; \ldots.)$$

The rational numbers a_n forming a Cauchy number are called its **components**.

The rules for the four basic operations on Cauchy numbers are those used for truncated decimal fractions:

Addition: $a+b = \{a_n\} + \{b_n\} = \{a_n + b_n\}$

Subtraction: $a - b = \{a_n\} - \{b_n\} = \{a_n - b_n\}$

Multiplication: $axb = \{a_n\}x\{b_n\} = \{a_n x b_n\}$

Division: $\frac{a}{b} = \frac{\{a_n\}}{\{b_n\}} = \left\{\frac{a_n}{b_n}\right\}$ provided none of the numbers $\{b_n\}$ is zero.

Function values: $f(a) = \{f(a_n)\}$

A Cauchy number that is associated with a positive number can now have decreasing terms, e.g.

$$2 - \sqrt{2} = (1. ; 0.6 ; 0.59 ; 0.586 \ldots.)$$

Thus a Cauchy number is but a different name for a Cauchy sequence of rational numbers as studied in number theory, but with an accompanying set of arithmetical operations defined. Thus all relevant results of Number Theory are Mutatis Mutandis applicable to Cauchy numbers. The principal results that are of interest here are the following:

- There is an equivalence relation defined between Cauchy numbers that causes the Cauchy numbers to be separated into equivalence classes. These equivalence classes are the **real numbers**.
- Each equivalence class contains at least one component described infinite decimal fraction, traditionally called the **main value**.
- Any two Cauchy numbers in the same equivalence class differ by a Cauchy number equivalent to zero.
- Zero is the Cauchy number (0 ; 0 ; 0 ; 0 …..).

Remark
It is necessary to extend the concept of Cauchy number further in order to make the Cauchy numbers closed under division by a Cauchy number that is equivalent to zero. This requires that the **infinite Cauchy numbers** be defined:

Definition
A sequence of numbers $\{a_n\}$ such that for any given number M there exists a number N such that $|a_n| \geq M$ for all $n \geq N$ is called an '**infinite Cauchy number**'.

Equivalent infinite Cauchy numbers differ by a finite Cauchy number.

3.1.2 Infinitesimal numbers

Definition:
The elements of the class of Cauchy numbers that are equivalent to zero are called the **infinitesimal numbers**.

Infinitesimal numbers will be indicated using Greek letters e.g.

$$\alpha = (\alpha_1 ; \alpha_2 ; \alpha_3 ; \ldots.)$$

Example:
Once more using subtraction, it is possible to change an increasing sequence into a decreasing sequence, and thus the infinitesimal number:

$$\{ 0.1 ; 0.01 ; 0.001 ; 0.0001 ; \ldots.\}$$

Is the Cauchy representation for

$$1-0.999999\ldots..$$

Where the periods at the end indicate that the '9' is a repeating digit.

The motivation for this definition is the observation that the lengths of the intervals of the partition of the unit interval, studied in section 2.1.1(b), form an infinitesimal number. But the reason becomes clearer when one notices that different infinitesimal numbers have different rates of convergence, even though they all converge to zero. This makes them suitable for the study of rates of change.

3.1.2.1 L'Hospital: Classes of Cauchy numbers

All four arithmetical operations can be performed as long as the Cauchy numbers involved do not have more than a finite number of zero components – i.e. from some point on they do not contain any zeroes.

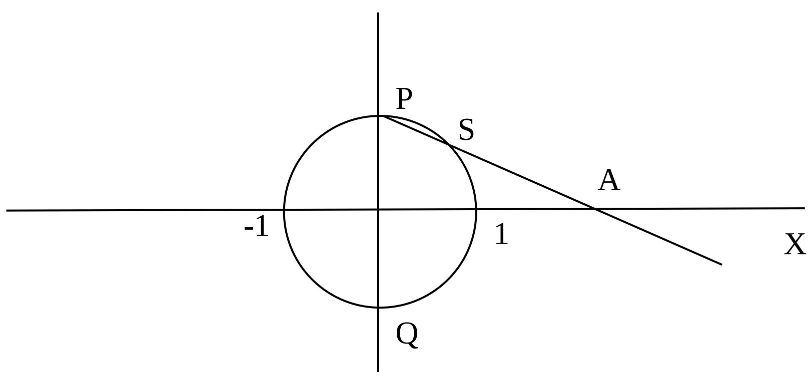
Three classes of Cauchy numbers have been defined:

- The **infinitesimal numbers**. These are Cauchy numbers equivalent to zero, indicated as the class A.
- The **infinite numbers**. These are sequences of numbers of which the magnitude of the terms increases without limit, indicated as the class B.
- The **rated numbers**. These are Cauchy numbers belonging to the other equivalence classes of Cauchy numbers, indicated as the class C.

Any sequence of numbers that does not belong to any of these classes will be called a **meandering** sequence.

The rule of L'Hospital indicates that when two Cauchy numbers are multiplied or divided the result can move from class to class. The transition most often used is when the quotient of two infinitesimal numbers becomes a rated number. This is traditionally referred to as differentiation.

The nature of the class B of infinite numbers introduced here becomes clearer when one emulates the compactification of the complex plane as was done by Lars Ahlfors when he introduced his 'point at infinity'.



The horizontal line in the figure is the real line and the circle is a unit circle centred at the origin. A line drawn from the topmost point P of the circle to any point A on the real line then maps that point onto a point S of the circle. The rightmost and leftmost points of the circle are the points +1 and -1 and they map onto themselves. The origin maps onto the lowermost point Q while the topmost point P corresponds to the ‘point at infinity’.

With a metric topology of ‘length of arc’ on the circle, the three classes of Cauchy numbers defined above correspond to classes of Cauchy sequences converging in this topology to points on this circle. Infinitesimals are Cauchy sequences that converge to Q, the image of zero. Rated Cauchy numbers are sequences that converge to all other points of the circle but the topmost, and infinite numbers correspond to Cauchy sequences converging to the topmost point P. In this sense the infinite numbers also form an equivalence class and this equivalence class will be called **infinity**[2]. This validates the term ‘Infinite Cauchy Numbers’, and allows ‘infinity’ to be considered as a real number.

The consequence of all this is that ‘infinity’ acquires a fine-structure and thus need not be avoided anymore (apart from division by the Cauchy number zero). For example, it will be shown later that the Dirac delta function (the derivative of the Heaviside function) is an ordinary piecewise function when using the Cauchy numbers and has an infinite number as value at the point of discontinuity.

Thus each point on this circle corresponds to an equivalence class of Cauchy numbers and is called, as is traditional, a real number. Thus the real line has been compacted to a real continuum.

3.1.3 Infinitesimals

Definition
A set of nested volumes, areas or lines of which the volumes, areas or lengths form infinitesimal numbers, is called an infinitesimal volume, -area or –line provided that the focus of the set is a point. (If the context is clear, the traditional way is to just simply call it an infinitesimal.)

The volumes, areas or lengths that form an infinitesimal are called its **parts.** An infinitesimal will be indicated using capital letters:

$$E = (E_1 ; E_2 ; E_3 ; \ldots.).$$

The partitions used in the example in section 2.1 consists of infinitesimals of which the lengths of the parts form the infinitesimal number
$\{3^{-n}; n= 0, 1, 2, 3\ldots\}$ from a value of n onwards.

The most used infinitesimals are the differentials:

[2] ‘Infinity’ as a ‘number larger than all other numbers’ is not required in this model because the concept of limit is not essential.

3.1.4 Differentials

Definition
Let d = {δ_n} be an infinitesimal number, and let a be any point on the X-axis. Let r be any number such that 0 ≤ r ≤ 1. Then a differential at the point a is the infinitesimal

$$D(d,a) = \{D_n\} = (a\text{-}rd\ ;\ a+[1-r]d) = \{(a - r\delta_n\ ;\ a + [1-r]\delta_n)\ ;\ n=1,2,3\ldots\}$$

Thus a differential at the point a is an infinitesimal focused at a.

It is called a left-differential when r=1 and a right-differential when r=0.

Let y = g(x). Then the Cauchy number

$$dy= g(a+[1-r]d) – g(a\text{-}rd)) = \{g(a_n + [1-r]\delta_n) – g(a_n - r\delta_n)\ ;\ n=1,2,3\ldots\}$$

can be formed. Traditionally the function g is called **differentiable at a** if dy is an infinitesimal number. The ratio $\frac{dy}{dx}$ is called the **derivative** of g at the point a.

For Cauchy numbers the derivative can exist even if dy is not an infinitesimal number.

In general:

$$\frac{dy}{dx} = \left\{ \frac{g(a_n + [1-r]\delta_n) - g(a_n - r\delta_n)}{\delta_n}\ ;\ n = 1,2,3.... \right\}$$

Consider the Heaviside function

$$H(a,x) = 0 \text{ if } x< a \text{ and } H(a,x) = 1 \text{ if } x \geq a$$

And let dx be an infinitesimal number. Choose r = 0.5 Then

$$dH(a,dx) = H(a+dx/2) – H(a\text{-}dx/2) = 1$$

Dividing by the infinitesimal number dx:

$$\frac{dH(a,dx)}{dx} = \begin{cases} 0 & \text{if } x \neq a \\ \frac{1}{dx} & \text{if } x=a \end{cases}$$

$= \delta(a)$

This is the Dirac-δ function of which the value at a is an infinite Cauchy number.

3.2 The infinite divisibility of space: Cascades of differentials

The sequence of partitions used in section 2.1 of this part has the following properties:

- Each new partition of the interval (0;1) is a refinement of the preceding partition.
- From one partition to the next there is a part of the new partition that has the same midpoint as the part of which it is a refinement.
- The lengths of all the parts of a partition are the same.
- The lengths of the parts from one partition to the next form an infinitesimal number

$$d^0 = \{3^{-n} ; n = 0; 1; 2; 3\ldots\}$$

Therefore the interval (0;1) is the first part of the differential $D(d^0, \frac{1}{2})$.

The second part of $D(d^0, \frac{1}{2})$ is the interval $(\frac{1}{3} ; \frac{2}{3})$. The intervals $(0; \frac{1}{3})$ and $(\frac{2}{3} ;1)$ are the first parts of the differentials $D(d^1, \frac{1}{6})$ and $D(d^1, \frac{5}{6})$ where $d^1 = \{3^{-n} ; n = 1; 2 ;3\ldots\}$.

This pattern is repeated for every following partition, so that in the end a cascade of differentials is obtained.

The properties of the partitions that are required to generate this cascade, are (1) that each new partition should be a refinement of the previous partition and (2) that the lengths of the parts of all the partitions should converge to zero. This is a restatement of the infinite divisibility of space from the Leibniz perspective.

A pre-order can be defined on any cascade using the two properties stated above:

Definition.
If all parts of the differential $D(d^r,a)$ are subsets of a part of the differential $D(d^s,b)$ then $D(d^s,b) \leq D(d^r,a)$.

This defines a pre-order on the cascade of differentials because it satisfies all the required properties of the ordering and has $D(d^0,c) \leq D(d^r,e)$ for all $r \geq 0$, which makes $D(d^0,c)$ the first element of the pre-order.

3.2.1 Functions on the cascade of differentials.

The first function of interest is the mapping $D(d^s, a_m) \rightarrow a_m$.

Although the numbers a_m in the example are all rational numbers, it is easy to construct a cascade of differentials for which a_m is a component specified Cauchy number. The mapping only implies existence and not identification of the focus points of the differentials. However, this is good enough to validate the conclusion, made at the beginning of this part, namely that the lengths of a countable number of points, in the Euclidean sense, can add up to the length of the unit interval in the Leibnizean model.

The second function of interest is the mapping $D(d^s, a_m) \rightarrow \frac{dF}{dx}(a_m)$ where F is a given function and $\frac{dF}{dx}(a_m)$ is the Cauchy number

$$\frac{dF}{dx}(a_m) = \left\{ \frac{F(a_m + 0.5\delta_r) - F(a_m - 0.5\delta_r)}{\delta_r} \; ; r = s, s+1, ... \right\}$$

Where, in the case of the above example, $\delta_r = 3^{-r}$.

Note that

$$\frac{dF}{dx}(a_m)\, dx = \left\{ \frac{F(a_m + 0.5\delta_r) - F(a_m - 0.5\delta_r)}{\delta_r} \delta_r \; ; r = s, s+1, ... \right\} \qquad [A]$$

The third function is

3.2.2 Nets: The Fundamental Theorem of Calculus

For the sake of simplicity and ease of notation, again consider the unit interval partitioned into $3^n = k$ intervals, all of the same length $\delta_n = 3^{-n}$ as in the example in section 2.1 of this part. Let F be a function defined on the unit interval.

Then

$F(1) - F(0) = F(3^{-n}) - F(0) + F(2x3^{-n}) - F(3^{-n}) + \ldots .+ F(1) - F([3^n - 1]x3^{-n})$

$$= \sum_{i=0}^{k-1} \left(F((i+1).\delta_n) - F(i\delta_n) \right)$$

$$= \sum_{i=0}^{k-1} \left(F\left((i+1).\delta_n\right) - F\left(i\delta_n\right)\right)\frac{\delta_n}{\delta_n}$$

$$= \sum_{i=0}^{k-1} \frac{F((i+1)\delta_n - F(i\delta_n))}{\delta_n}\delta_n \qquad \text{[B]}$$

The right hand side of [B] can be evaluated for each value of n, where n takes the values 0, 1 ,2, 3… Hence, if the right hand side of [B] is not a meandering sequence, it is a Cauchy number, and thus either belongs to an equivalence class of Cauchy sequences or is an infinite Cauchy number.

As was pointed out previously, when the right hand side of [B] is a finite Cauchy number, it follows from the theory of numbers that it is equivalent to a component specified infinite decimal fraction, called a main value of the equivalence class, plus an infinitesimal number.

If $\frac{dF}{dx}(x) = f(x)$ then this main value is called the "integral of f over the interval (0;1)" and is written as $\int_0^1 f(x)dx$. Thus

$$\text{F(1) – F(0)} = \int_0^1 f(x)dx + \alpha \qquad \text{[C]}$$

For any given function f this is a function from the cascade of directed differentials into the set of Cauchy numbers.

Remarks

1) In order for the right hand side of [C] to make sense, restrictions have to be placed on the properties of the function f that may appear in the integral. The fundamental restriction is mentioned above, namely that f should be such that the right hand side of [C] is not a meandering sequence. But interpretation of the integral as the area under the graph of the function f will require more drastic restrictions on the nature of f. Clearly, when f is the Dirac delta function, interpreting the integral as an area makes no sense.

2) There is nothing that prevents the right hand side of [C] to be an infinite Cauchy number. In that case α would be a finite Cauchy number.

3) Each infinitesimal in the cascade is component specified and therefore also the cascade as a whole. The right hand side of [C] is thus component specified too and the equivalence class to which it belongs cannot be

identified. Hence each component of the number α is at most an indication of the accuracy of the value of the integral at that value of n in the sum [B].

4) When F is a known function, the left hand side of [C] is value identified. This implies that the right hand side is also identified, i.e. the equivalence class of Cauchy sequences to which it belongs is fixed - even though it is component specified. In this case it is usual to assume that the integral, as the main value of that equivalence class, is also value identified. In this case α becomes the Cauchy number zero. Traditionally this is called 'taking the main value' and corresponds to taking limits in the Euclidean model.

5) The nets as well as the integral that was defined here, are functions on the directed cascade of differentials. Thus the cascade is the fundamental entity present, and it consists of a never ending selection of differentials which are themselves never ending sets of intervals.

6) Like with fractals, there is no obvious definable limit for the cascade. As the number of refinements of the partition increases, it is only the scale that changes but the pattern in the cascade remains the same.

PART 3: THE CASE FOR PARMENIDES

In the Euclidean model of the universe, that which is continuous (space) is compounded of that which is discrete (points). Parmenides and Xeno were critical of the concepts of Euclidean cosmology and, as mentioned before, their criticism found its way into history mainly in the form of Xeno's paradoxes.

These paradoxes are all about the consequences of describing the motion of a body – or particle - in terms of its position at specific points of space at specific instants of time.

Their concern about the cosmological implications of the Euclidean model is clearly stated by the paradox of the arrow. This paradox states that if at some instant of time every point (particle) of an arrow is at some point of space then the motion of the arrow is "frozen" and the arrow cannot move out of this position. This concern about describing the position of a moving particle has been echoed in the middle of the twentieth century by Heissenberg's uncertainty principle which states that when the position of a particle is identified then nothing can be known about its speed and vice versa.

Parmenides did not put forward a detailed alternative cosmological model of his own, but the aspects of his philosophy of interest here are (a) nothing can come into being that has not existed before and (b) one object cannot move relative to another because everything is one; therefore motion does not exist and what we see is an illusion.

In the Euclidean model particles are defined as points. In the Leibnizean model points are not spatial entities and hence a particle cannot be defined as a point but can only be defined as an infinitesimal with its focus at the position where the particle is perceived to be. This may be done as follows:

Assume that the universe is finite and has a radius R. Then a particle can be defined as an infinitesimal which is a set of nested spheres with radii $r_n = \frac{R}{n}$ (n=1, 2, 3, ….) and where the centre of each part of the infinitesimal has a suitable offset to focus the infinitesimal at the position of the particle.

This definition of a particle is completely in line with the requirements of the cosmology of Parmenides because:

(a) Everything exists throughout the universe and hence never come into being where it has not existed before.
(b) Everything is one because the infinitesimals of all particles share the whole universe as first part. Movement is an illusion because nothing moves – it is only the focus of the infinitesimal that shifts.

REMARKS

1. In the Leibnizean cosmology, space is modelled by infinitesimals which are component described. Although all infinitesimals are focussed at some place, the place where an infinitesimal is focused is in general not known. Therefore the description of space is fuzzy. Because of this, Heissenberg's uncertainty principle is to be expected in the Leibnizean cosmology. What is disconcerting is that the uncertainty principle prescribes a lower limit for this inaccuracy. One possible reason for this may be that a third cosmology, the Heissenberg cosmology, can be described where space, like energy, is quantised!

2. Civilisation today is but the Greek civilisation two and a half millennia on. Therefore, except for parts of Calculus, even today the Euclidean cosmology underpins the whole of science. But some intractable problems of Physics are direct consequences of the fact that particles are considered to be points and therefore localised. Action at a distance and the particle/wave duality comes immediately to mind. In the Leibnizean cosmology the nature of both these problems change and they all but disappear.

PART 4: CONCLUSION AND INCLUSION

The results of part one shows that the Euclidean cosmology leads to inconsistencies in the accompanying number system. It was therefore abandoned in favor of the Leibnizean cosmology which can accommodate a countable number system.

This countable number system was formed by a re-interpretation of existing number theory which led to an extension of the real numbers to the Cauchy numbers, which in turn form three classes of numbers, namely the infinitesimal numbers, the rated numbers and the infinite numbers. An arithmetic for these numbers was defined according to the everyday use of truncated numbers.

Some terms used in the Euclidean model had to be re-defined; the principal of which was that limits of numbers (infinite decimal fractions) were replaced by the concept of ‘main value’ and geometrical points lost their status of being spatial entities to become mere places. Many of the other terms remain valid, e.g. real numbers as equivalence classes of Cauchy numbers.

Other results of the Euclidean model had to be abandoned completely. These were results that follow directly from the Euclidean cosmology; the principal of these would most probably be the concept of open and closed sets in geometrical space. But results for non-geometrical spaces like function spaces, which are in essence discrete, should not be affected. Thus results from the Euclidean model should remain valid - perhaps with some adaptations - in such spaces.

The end result is that the Euclidean model, which is the smaller model of Mathematics, can easily - but with some alterations and omissions - become part of the Leibnizean model which is the larger model. Thus the structure of Mathematics as a canonical model is not lost.

ADDENDUM

Let

$$b = 0{,}b_1b_2b_3b_4....$$

be the number constructed in Cantor's diagonal proof.

There are infinitely many infinite decimal fractions in the list of which the first n digits are identical to $b_1b_2b_3b_4....b_n$, the first n digits of b. Select any one of them as the infinite decimal fraction c_n.

The sequence $\{c_n\}$ is a Cauchy sequence which is equivalent to b, and therefore its limit is equal to the real number b because both belong to the same equivalence class of Cauchy sequences.

Because of the real line, the assumption that all decimal fractions between zero and one (inclusive) belong to the list means that the list maps a priori onto the closed interval [0;1].

But c_n is in the list for all n and $\{ c_n \}$ maps onto a Cauchy sequence in [0;1]. Thus its limit, the infinite decimal fraction b, belongs to the closure of the list which is [0;1]. Hence b belongs to the list.

But Cantor's proof shows that b does not belong to the list.

Thus two independent contradictory results follow from the same assumptions of the theorem. This constitutes a proof by contradiction. But the logic of a proof by contradiction requires that there must be a **single** identifiable false assumption.

Here there are at least two possible false assumptions:

- That a list of all possible infinite decimal fractions between 0 and 1 can be made. The current consensus is that this is the relevant false assumption.

- That it is possible to identify an infinite decimal fraction by choosing its finite digits according to some rule. Example A constructed in part one shows this is the relevant false assumption.

The fact that more than one possible false assumption exist when the real line is taken into account destroys the logical structure of the proof and the diagonal argument is void.