The Prime Number Formulas

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Abstract

There are many proposed partial prime number formulas, however, no formula can generate all prime numbers. Here we show three formulas which can obtain the entire prime numbers set from the positive integers, based on the Möbius function plus the "omega" function, or the Omega function, or the divisor function.

The history of searching for a prime number formula goes back to the ancient Egyptians. There have been many proposed partial prime number formulas (e.g., Euler: $\mathbf{P}(n) = n^2+n+41$), however, no formula can generate all prime numbers. Here we show $\mathbf{P}_k(n) \equiv n \cdot a^{\varpi(n)}$ to pick up the prime numbers from the positive integers, which is based on the prime number definition of $\mathbf{P} \equiv \mathbf{P} \cdot 1$ (i.e., a prime number can only be divided by 1 and itself). Let $\mathbf{P}, b \in \mathbb{Z}$, for a natural number $n \equiv \mathbf{P} \cdot b$, which is a prime number if $b = 1$ and a non-prime number if $b \neq 1$.

The Möbius function

The Möbius function is the sum of the primitive n-th roots of unity.[1]

$$\mu(n) = \sum_{\substack{1 \le k \le n \\ \gcd(k,n)=1}} e^{2\pi i k/n} \tag{1}$$

with values

$$\mu(n) = \begin{cases} (-1)^{\omega(n)} & \text{if } \Omega(n) = \omega(n) \\ 0 & \text{if } \Omega(n) > \omega(n) \end{cases} \tag{2}$$

where $\omega(n)$ is the number of distinct prime factors of n, $\Omega(n)$ is the number of factors (with repetition) of n, and n is square-free if and only if $\Omega(n) = \omega(n)$. Thus $\mu(1) = \mu(6 = 2 \times 3) = 1$, $\mu(2) = \mu(3) = \mu(5) = -1$, $\mu(4 = 2^2) = 0$, etc. In fact, only when $\omega(n) = 1, 3, 5, \ldots$ (an odd prime factor, e.g., Sphenic numbers: products of 3 distinct primes) make $\mu(\mathbf{P}) = \mu(\boldsymbol{pqr}) = \mu(\boldsymbol{pqrst}) = (-1)^{(2k+1)} = -1$. The variable term of the Euler identity $e^{i2\pi k/n}$ is involved in the Möbius function $\mu(n)$ sequence

1, -1, -1, 0, -1, 1, -1, 0, 0, 1, -1, 0, -1, 1, 1, 0, -1, 0, -1, 0, 1, 1, -1, 0, 0, 1, 0, 0, -1, -1, -1, 0, 1, 1, 1, 0, -1, 1, 1, ...(OEIS A008683)[2]

where $\mu(n) = -1$ gives the number sequence

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, **30**, 31, 37, 41, **42**, 43, 47, 53, 59, 61, **66**, 67, **70**, 71, 73, **78**, 79, 83, 89, 97, ...(OEIS A030059)[3]

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where **30**, **42**, **66**, **70**, **78** are from the sphenic number sequence

30, **42**, **66**, **70,** **78**, 102, 105, 110, 114, 130, ...(OEIS A007304)[4]

A set of sphenic numbers $J = \boldsymbol{p} \times \boldsymbol{q} \times \boldsymbol{r}$ has exactly eight divisors

$$\{1,\ \boldsymbol{p},\ \boldsymbol{q},\ \boldsymbol{r},\ pq,\ pr,\ qr,\ \boldsymbol{pqr}\} \tag{3}$$

For example $\{1,\ \mathbf{2},\ \mathbf{3},\ \mathbf{5},\ 6,\ 10,\ 15,\ \mathbf{30}\}$, where prime $\mu(\mathbf{2}) = \mu(\mathbf{3}) = \mu(\mathbf{5}) = -1$ and sphenic $\mu(\mathbf{30}) = (-1)^3 = -1$, while semiprime $\mu(6) = \mu(10) = \mu(15) = (-1)^2 = +1$.

Obviously, removing the sphenic number sequence (A007304) from the Möbius sequence (A030059) gives the prime number sequence $\mathbf{P}_k$ (<100)

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, ...(OEIS A000040)[5]

The "omega" function

The "omega" function $\omega(n)$ represents the number of *distinct prime factors* of $n = \mathbf{P}_1^{\rho_1}\mathbf{P}_2^{\rho_2}\cdots\mathbf{P}_k^{\rho_k}$ (OEIS A001221).[6] Since $\omega(\mathbf{P}) \equiv +1$ and $\mu(\mathbf{P}) \equiv -1 = e^{-i\pi}$, the prime numbers can be identified as $\varpi(n) = \mu(n) + \omega(n) \equiv 0$, which is equal to the Euler identity $e^{i\mathbf{P}_k\pi} + 1 = 0$. **Fig 1** shows that the prime numbers all have $\varpi(n) = \mu(n) + \omega(n) \equiv 0$, while all other non-prime numbers are $\varpi(n) \geq 1$.

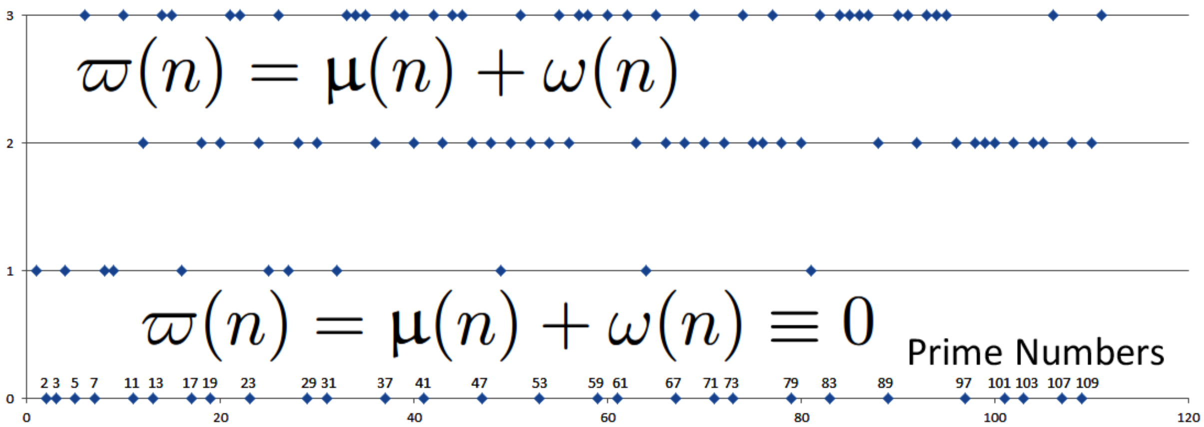

Fig. 1: The prime numbers can be identified as $\varpi(n) = \mu(n) + \omega(n) \equiv 0$, while other non-prime numbers are $\varpi(n) \geq 1$.

We define

$$\varpi(n) = \mu(n) + \omega(n) = \begin{cases} 0 & \text{prime numbers} \\ \geq 1 & \text{other numbers} \end{cases} \tag{4}$$

From $\varpi(n) \equiv 0$ for prime numbers, the entire prime numbers formula is

$$\mathbf{P}_k(n) \equiv n \cdot a^{\varpi(n)} \in \mathbb{N} \tag{5}$$

where $n = f(m,k) = 2^m(2k+1) = \mathbf{P}_1^{\alpha_1}\mathbf{P}_2^{\alpha_2}\cdots\mathbf{P}_k^{\alpha_k} = \prod_{i=1}^{\omega(n)} P_i^{\alpha_i} = 1, 2, 3...$ are positive integers $\mathbb{Z}$, $a \notin \mathbb{N}$ (e.g., an irrational number $a = \phi = 0.618033\ldots$ or sufficiently small $a = 1 \times 10^{-100}$), so only $a^0 \equiv 1$ makes $\mathbf{P}_k(n) \equiv n \cdot a^{\varpi(n)} \in \mathbb{N}$ if $\varpi(n) = 0$, and $n \cdot a^{\varpi(n)} \notin \mathbb{N}$ if $\varpi(n) \neq 0$.

$\mu(n) = -1 \doteq e^{i\mathbf{P}_k\pi}$ and $\omega(n) = +1$ get the Euler relationship $e^{i\mathbf{P}_k\pi} + 1 = 0$, where $\mathbf{P}_k \subset (2k+1)$ is an odd number. Therefore, production of prime numbers in the natural numbers $\mathbb{N}$ will occur if and only if $\varpi(n) \equiv 0$. In this way, the Euler identity regulates the Prime numbers.

Prime numbers already existed in the positive integers, we only needed to find a math formula (4) to lock-in on those primes within the natural numbers. All prime numbers can be found by solving the prime identical equation $\varpi(n) = \mu(n) + \omega(n) \equiv 0$ in (4).

Using Mathematica, $\varpi(n) = \mu(n) + \omega(n)$ gives the Prime number identification sequence (**Fig. 2**)

```
In[33]:= Table[MoebiusMu[k] + PrimeNu[k], {k, 200}]

Out[33]= {1, 0, 0, 1, 0, 3, 0, 1, 1, 3, 0, 2, 0, 3, 3, 1, 0, 2, 0, 2, 3, 3, 0, 2, 1, 3, 1, 2, 0,
  2, 0, 1, 3, 3, 3, 2, 0, 3, 3, 2, 0, 2, 0, 2, 2, 3, 0, 2, 1, 2, 3, 2, 0, 2, 3, 2, 3, 3,
  0, 3, 0, 3, 2, 1, 3, 2, 0, 2, 3, 2, 0, 2, 0, 3, 2, 2, 3, 2, 0, 2, 1, 3, 0, 3, 3, 3, 3,
  2, 0, 3, 3, 2, 3, 3, 3, 2, 0, 2, 2, 2, 0, 2, 0, 2, 2, 3, 0, 2, 0, 2, 3, 2, 0, 2, 3, 2,
  2, 3, 3, 3, 1, 3, 3, 2, 1, 3, 0, 1, 3, 2, 0, 3, 3, 3, 2, 2, 0, 2, 0, 3, 3, 3, 3, 2, 3,
  3, 2, 2, 0, 3, 0, 2, 2, 2, 3, 3, 0, 3, 3, 2, 3, 2, 0, 2, 2, 3, 0, 3, 1, 2, 2, 2, 0, 2,
  2, 2, 3, 3, 0, 3, 0, 2, 3, 2, 3, 2, 3, 2, 2, 2, 0, 2, 0, 3, 2, 2, 0, 3, 0, 2}
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Fig. 2: The $\varpi(n)$ table generated by Mathematica. The Prime numbers under 200 can be identified by $\varpi(n) = 0$.

The $\varpi(n)$ sequences are used in $\mathbf{P}_k(n) \equiv n \cdot 0.001^{\varpi(n)} \in \mathbb{N}$ to generate the prime number sequence, where only $\varpi(n) = 0$ yields prime numbers while all other non-zero terms in the above table yields 0 (**Fig. 3**) which can be easily filtered out (**Fig. 4**).

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In[32]:= IntegerPart[Table[k 0.001 ^ (MoebiusMu[k] + PrimeNu[k]), {k, 200}]]

Out[32]= {0, 2, 3, 0, 5, 0, 7, 0, 0, 0, 11, 0, 13, 0, 0, 0, 17, 0, 19, 0, 0, 0, 23, 0, 0, 0, 0,
  0, 29, 0, 31, 0, 0, 0, 0, 0, 37, 0, 0, 0, 41, 0, 43, 0, 0, 0, 47, 0, 0, 0, 0, 0, 53,
  0, 0, 0, 0, 0, 59, 0, 61, 0, 0, 0, 0, 0, 67, 0, 0, 0, 71, 0, 73, 0, 0, 0, 0, 0, 79,
  0, 0, 0, 83, 0, 0, 0, 0, 0, 89, 0, 0, 0, 0, 0, 0, 0, 97, 0, 0, 0, 101, 0, 103, 0, 0,
  0, 107, 0, 109, 0, 0, 0, 113, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 127, 0, 0, 0,
  131, 0, 0, 0, 0, 0, 137, 0, 139, 0, 0, 0, 0, 0, 0, 0, 0, 0, 149, 0, 151, 0, 0, 0, 0,
  0, 157, 0, 0, 0, 0, 0, 163, 0, 0, 0, 167, 0, 0, 0, 0, 0, 173, 0, 0, 0, 0, 0, 179, 0,
  181, 0, 0, 0, 0, 0, 0, 0, 0, 0, 191, 0, 193, 0, 0, 0, 197, 0, 199, 0}
```

Fig. 3: The Prime numbers under 200 can be identified by $\varpi(n) = 0$ which is generated by Mathematica.

```
In[34]:= Sort[IntegerPart[Table[k 0.001 ^ (MoebiusMu[k] + PrimeNu[k]), {k, 200}]]]

Out[34]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,|0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47,
  53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137,
  139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199}
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Fig. 4: Sorted table in **Fig. 3**, which is the prime number table if the zeros are filtered out.

All prime numbers can be found by solving the prime identity equation

$$\varpi(n) = \mu(n) + \omega(n) \equiv 0 \tag{6}$$

However, the prime number formula $\mathbf{P}_k(n) \equiv n \cdot a^{\varpi(n)} \in \mathbb{N}$ is used in Mathematica code as

Drop[Sort[IntegerPart[Table[k 0.0001^(MoebiusMu[k] + PrimeNu[k]), {k, 1000}]]], 832]

which yields the table of 168 prime numbers (< 1000). The prime number table can be generated, sorted and filter out of zeros from $\varpi(n) = \mu(n) + \omega(n) \equiv 0$ by Mathematica in **Fig. 5**.

```
In[31]:= Drop[Sort[IntegerPart[Table[k 0.0001^(MoebiusMu[k] + PrimeNu[k]), {k, 1000}]]], 832]

Out[31]= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83,
  89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179,
  181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271,
  277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379,
  383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479,
  487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599,
  601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701,
  709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823,
  827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941,
  947, 953, 967, 971, 977, 983, 991, 997}
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Fig. 5: The prime number table (< 1000) is generated, sorted and filtered out of zeros from $\varpi(n) = \mu(n) + \omega(n) \equiv 0$ by Mathematica.

The Omega function

The Omega function $\Omega(n) = \sum_{i=1}^{\omega(n)} \alpha_i$ is the number of prime factors (with repetition) of $n = \prod_{i=1}^{\omega(n)} P_i^{\alpha_i}$.

0, 1, 1, 2, 1, 2, 1, 3, 2, 2, 1, 3, 1, 2, 2, 4, 1, 3, 1, 3, 2, 2, 1, 4, 2, 2, 3, 3, 1, 3, 1, 5, 2, 2, 2, 4, ... (OEIS A001222)[7]

It has a similar property with the "omega" function as $\Omega(\mathbf{P}) = \omega(\mathbf{P}) = +1$,[8] while $\Omega(n) > \omega(n)$ are not for prime numbers. Therefore, it can also be used to define a new function for the prime number identification

$$\varpi'(n) = \mu(n) + \Omega(n) \equiv \begin{cases} 0 & \text{prime numbers} \\ \geq 1 & \text{other numbers} \end{cases} \tag{7}$$

For generating the table of 168 prime numbers (< 1000) in **Fig. 5**, $\mathbf{P}_k(n) \equiv n \cdot a^{\varpi'(n)} \in \mathbb{N}$ in Mathematica code is

Drop[Sort[IntegerPart[Table[k 0.01^(MoebiusMu[k] + PrimeOmega[k]), {k, 1000}]]], 832]

The divisor function

Divisor functions $\sigma_x(n) = \sum_{d|n} d^x$ were studied by Ramanujan,[9] where $\sigma_1(n)$ is given as the sequence

1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, 14, 24, 24, 31, 18, 39, 20, 42, 32, 36, 24, 60, 31, 42, 40, 56, 30, 72, 32, 63, ... (OEIS A000203)[10]

The prime numbers have

$$\mathbf{s}(\mathbf{P}_k) = \sigma_1(\mathbf{P}_k) - \mathbf{P}_k \equiv +1 \tag{8}$$

where $\mathbf{s}(n) = \sigma_1(n) - n$ involves much larger numbers than $\omega(n)$. For example, $\mathbf{s}(5) = (1+5) - 5 = 1$, while other non-prime numbers $\mathbf{s}(n) = \sigma_1(n) - n \geq 2$ (e.g., $\mathbf{s}(9) = (1+3+9) - 9 = 4$).

0, 1, 1, 3, 1, 6, 1, 7, 4, 8, 1, 16, 1, 10, 9, 15, 1, 21, 1, 22, 11, 14, 1, 36, 6, 16, 13, 28, 1, 42, 1, 31, 15, 20, 13, 55, ... (OEIS A001065)[11]

Therefore, $\mathbf{s}(n)$ can also be used to define a new function for the prime number identification (**Fig. 6**)

$$\varpi''(n) = \mu(n) + \mathbf{s}(n) = \begin{cases} 0 & \text{prime numbers} \\ \geq 1 & \text{other numbers} \end{cases} \tag{9}$$

```
In[16]:= Table[(MoebiusMu[k] + (DivisorSigma[1, k] - k)), {k, 100}]

Out[16]= {1, 0, 0, 3, 0, 7, 0, 7, 4, 9, 0, 16, 0, 11, 10, 15, 0, 21, 0, 22, 12, 15, 0, 36, 6, 17,
  13, 28, 0, 41, 0, 31, 16, 21, 14, 55, 0, 23, 18, 50, 0, 53, 0, 40, 33, 27, 0, 76, 8, 43,
  22, 46, 0, 66, 18, 64, 24, 33, 0, 108, 0, 35, 41, 63, 20, 77, 0, 58, 28, 73, 0, 123, 0,
  41, 49, 64, 20, 89, 0, 106, 40, 45, 0, 140, 24, 47, 34, 92, 0, 144, 22, 76, 36, 51, 26,
  156, 0, 73, 57, 117}
```

Fig. 6: The $\varpi''(n) = \mu(n) + \mathbf{s}(n)$ table generated by Mathematica. The Prime numbers can be identified by $\varpi''(n) = 0$.

For generating the table of 168 prime numbers (< 1000) in **Fig. 5**, $\mathbf{P}_k(n) \equiv n \cdot a^{\varpi''(n)} \in \mathbb{N}$ in Mathematica code is

```
Drop[Sort[IntegerPart[Table[k 0.1^(MoebiusMu[k] + (DivisorSigma[1, k] - k)),
{k, 1000} ]]], 832]
```

Since 2 is the only even prime number, for the odd primes, $\mathbf{P}_k(n) \equiv (2n+1) \cdot a^{\varpi(2n+1)} \in \mathbb{N}$ can be used to reduce the computation time.

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Competing financial interests
The authors declare no competing financial interests.

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[5] (OEIS A000040) https://oeis.org/A000040

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