Prior to my retirement from the service I had been teaching mathematics at the graduate level for 40 odd years, in Rajaram College Kolhapur, which is run by the Govt. of Maharashtra.

My specialization is **Pure Mathematics**. Naturally, the branches which I use to teach were; calculus, advance real analysis, topology, modern algebra and set theory.

It is well known that, we learn more while teaching a subject rather than while studying it. I seldom took anything for granted simply because it is printed in a standard book.

Almost all books, which I use to refer during the tenure of my service have categorized the **"brotherhood relation"** as non-reflexive, non-symmetric but transitive. I couldn't agree with this. I have penned down my thoughts in this regard in the following pages which the reader will find in my own handwriting.

I propose to upload this on many other sites also as I believe that, **whenever any new thought stickers a mind of some human being it should become a property of all mankind.**

Thank you.

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Introduction:

In the discussion that follows it is expected that the reader is familliar with elementary concepts of Set Theory such as sets, elements of a set, the curly bracket notation, belonging relation, subsets of a set, equality of two sets, union & intersection operations, complement of a set.

With these elementary ideas we ~~set~~ now start the discussion of a topic that will take us to study "Binary Relations"

Order Pair

In a set the only important information is 'which are the elements of it?' and nothing else. The order of elements in a set is not important.

eg. The sets $\{3,4\}$ and $\{4,3\}$ are equal or in other words they are 'same' sets.

But in many instances we require the elements taken in specific order. eg. In an army parade when the soldiers are standing in rows, the 3rd person in the 4th row is not same as the 4th person in the 3rd row.

In order to achieve this we introduce the following definition:

Defn. If a and b are two elements then the set defined as

$\{\{a\},\{a,b\}\}$ is called as

the ordered pair of a and b with the understanding that 'a' is the first element and 'b' is the second. This will be denoted as (a, b).

Thus

$$(a, b) = \{\{a\}, \{a, b\}\}$$

It is easy to see and to prove that this concept certainly enjoys the following two properties

(i) $(a, b) \neq (b, a)$ unless $a = b$

and (ii) $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

As (i) says that if $a \neq b$ then $(a, b) \neq (b, a)$ it justifies the name given to it as "an ordered pair"

Cartessian Product of two Sets.

Defn' If A and B are two non-empty sets then the Cartessian Product of A with B is defined as

$$\{(x, y) / x \in A \text{ and } y \in B\}$$

and is denoted as $A \times B$

Thus

$$A \times B = \{(x, y) / x \in A \text{ and } y \in B\}$$

It can be verified that

$A \times B \neq B \times A$ unless of course $A = B$

Also if A has m number of element and B has n number of element

the $A \times B$ has mn elements in it.
But the nature of elements of $A \times B$ is quite different from those of A or B.

eg. If $A = \{x, y\}$ and $B = \{1, 2, 3\}$
then $A \times B = \{(x,1), (x,2), (x,3), (y,1), (y,2), (y,3)\}$
and $B \times A = \{(1,x), (1,y), (2,x), (2,y), (3,x), (3,y)\}$.

Binary Relations.

In everyday life we use statements of the following type

Arjun 'is the father of' Abhimanyu.
or Laxman 'is a brother of' Rama
or Usha 'is taller than' Rahul. etc.

In Mathematics we use simillar statements,
eg. 4 'is greater than' 3
or L_1 'is parallel to' L_2
or ΔABC "is simillar to" ΔPQR etc.

In all the above examples the phrase put in the inverted commas is called a relation which relates the element before the phrase with the element that follows the ~~phase~~ phrase.

eg. In the statement "Arjun 'is the father of' Abhimanyu" the relation considered is "is the father of" and it relates Arjun to Abhimanyu.
Simillarly in the statement "4 'is greater than' 3" the relation considered is

'is greater than' and it relates 4 with 3.

Please see that when 4 is related to 3 under the ~~rest~~ relation 'is greater than'; 3 is not related to 4 although 3 ~~is~~ may be considered related to 4 under different relation 'is less than' but certainly not under "is greater than".

Consider the following example.

Let $S = \{2, 3, 4, 6, 8\}$ and the relation being considered is 'is a factor of'

The set of true statements formed in this case are.

2 is a factor of 2
2 is a factor of 4
2 is a factor of 6
2 is a factor of 8
3 is a factor of 3
3 is a factor of 6
4 is a factor of 4
4 is a factor of 8
6 is a factor of 6
8 is a factor of 8

To avoid repeatedly writting the phrase "is a factor of" we can say that let us denote this complete phrase by R (the first letter in the word relation) So now in this context R always mean "is a factor of".

The above set of true statements can now be reproduced as

$2R2,\ 2R4,\ 2R6,\ 2R8,\ 3R3,$
$3R6,\ 4R4,\ 4R8,\ 6R6,\ 8R8.$

Here is another way to look at the problem. Every statement of relation requires two elements. One is the element which is related and the other is the element to which ~~it~~ the first is related. Hence such relations are called 'the binary ~~set~~ relations'.

It is interesting to see that not only every statement incorporates a pair of elements but the order in which ~~this~~ these elements occure is also important. For, in the above example $2R4$ but $4 \not R 2$.

Hence the statement in a binary relation not only involves 'a pair' but it involves 'an ordered pair.'

If it is stipulated that in the topic of binary relation (a, b) will mean

a is related to b
i.e. in other words the related element will be written in the first place and then the above statement will now appear as

(2, 2) (2, 4) (2, 6) (2, 8)
(3, 3) (3, 6) (4, 4) (4, 8) (6, 6) (8, 8).

When this set of true statements is enclosed with the set bracket the above relation will now look like

{(2,2), (2,4) (2,6) (2,8) (3,3) (3,6) (4,4) (4,8), (6,6), (8,8)}

This set will be called the relation set R

Thus here

$R = \{(2,2), (2,4), (2,6), (2,8), (3,3), (3,6), (4,4), (4,8), (6,6), (8,8)\}$

It can be easily seen that

$R \subset A \times A$ (Although here $A \times A \neq R$)

Here the relation is first introduced with meaningful words like 'is a factor of' and is developed to its abstract form as a subset of Cartessian Product. This development suggests the definition of a binary relation as follows —

Def^n: If A ~~[illegible]~~ is a non-empty set then a non-empty subset R of $A \times A$ is called a binary relation in A.

Thus $A \neq \emptyset$, $R \subset A \times A$ and $R \neq \emptyset$

If $x, y \in A$ such that $(x, y) \in R$ we shall say that x is related to y in A by R. This is same thing as writing xRy.

Thus $(x, y) \in R \equiv xRy$

In the earlier example we have seen that 2 is a factor of 4 but 4 is not a factor of 2.

i.e. $2R4$ by $4 \not R 2$

or $(2,4) \in R$ but $(4,2) \notin R$

Thus the elements here can not be interchanged without affecting the truth value of the statements. We see that the elements 2, 4 are not symetrically placed on both sides of R and indeed we are going to admit that this relation "is a factor of" is not symetric.

Considering this and such other ~~possibilit~~ possibilities we define certain

Types of Binary Relations

(1) Reflexive Relations.

Def^n: A binary relation R in a non-empty set A is said to be a reflexive relation if xRx for every $x \in A$. i.e. if every element of A is related to itself by R. The property that

$x \in A \Rightarrow xRx$

is called the reflexivity.

eg (i) / is a factor of relation in any subset of integers is reflexive as every integer is a factor of itself.

(ii) If parallelism of lines is defined as 'two lines L_1 & L_2 are said to be parallel if they are co-planar and $L_1 \cap L_2 = \phi$'

then parallelism can not be reflexive. Because for any line L although L & L are co-planar $L \cap L = L$ and it can not be empty.

(iii) But if parallelism is defined as two lines L_1 & L_2 in a plane are said to be parallel if their slopes are equal then for any line L

slope of L = slope of L and we shall have to accept that a line is always parallel with itself. Hence the relation will be reflexive.

The above example will show that whether a relation is reflexive or not depends strictly on the definition of that relation. i.e. it depends upon the meaning we atribute to the relation. The reader should keep this point in his mind to understand the further discussion.

(iv) In many books on Set Theory, Modern Algebra the relation 'is a brother of' is considered in the set of humans. As the statement

x is a brother x

is found unacceptable to many authors the brotherhood relation is declared as non-reflexive in these books.

The reflexivity is discussed in detail because the main purpose of this writting is to establish the reflexivity of brotherhood relation.

(2) Symetric relation.

Defn. A binary relation R in A is said to be a symetric relation if

$$xRy \Rightarrow yRx \quad \text{for all } x, y \in A.$$

As is seen the elements are symetrically placed on both the sides of R and their interchange does not affect the truth value.

Ex.

(i) The relation "is a factor of" is certainly not symmetric as we have seen that 2 is a factor of 4 but 4 is not a factor of 2 i.e. $2R4$ but $4 \not R 2$

(ii) In the set of all triangles in a plane the congruency of triangles is symmetric. Because
$\Delta ABC \cong \Delta PQR \Rightarrow \Delta PQR \cong \Delta ABC$.

(iii) The brotherhood relation is not reflexive
In almost all books the brotherhood relation is declared to be non-symmetric in the set of all human beings. This is so because the statement that x is a brother of y does not necessarily mean that y is a brother of x as in some instances y may be a sister of x. But this imediately suggests that in the set of all male human beings the brotherhood relation will be symetric. We will turn to this relation in detail later.

(3) Transitive Relations

A binary relation R in A is said to be a transitive relation if
xRy and $yRz \Rightarrow xRz$
for all $x, y, z \in A$

The property is called the transitivity. It can be described in the words that if there are two statements of relation taken in specific order such that the second statement begins where the first ends then the relation is transited. In xRy and yRz, the intermediate element y is sometimes looked upon as the median through which R gets transited from x to z.

Ex. (i) The relation 'is a factor of' is transitive.

If xRy and yRz then in this context it will mean that

x is a factor of y i.e. $y = mx$ for some integer m

and y is a factor z i.e $z = ny$ for some integer n

By substitution $z = n(mx)$

i.e. $z = (nm)x$ and

nm is certainly an integer we get that x is a factor of z i.e. xRz

(ii) In a book shop if we say that for two books x & y xRy if the ~~pr~~ difference in their prices is less that 10 Rs., then this relation will be reflexive, symetric but not transitive. For,

(a) $|\text{price of } x - \text{price of } x| = 0 < 10\,Rs.$

$\therefore\ xRx$ (Reflexivity

(b) If xRy then

$|\text{price of } x - \text{price of } y| < 10\,Rs.$

But

$|\text{price of } x - \text{price of } y| = |\text{price of } y - \text{price of } x|$

$\therefore\ |\text{price of } y - \text{price of } x| < 10\,Rs.$

$\therefore\ yRx$ (Symetry

(ii) But we can have a case of the following type

x, y, z are books s.t.

price of x = 100 Rs.

price of y = 108 Rs.

price of z = 116 Rs.

Clearly here xRy and yRz.

But $|\text{price of } x - \text{price of } z| = 16 \not< 10$

$\therefore\ x \not R z$ ✡.

(iii) In almost all books, the brotherhood relation is considered transitive probably because the following statements is felt to be obvious

If x is a brother of y and y is a brother of z then x is a brother of z

Now here is the main important analysis of this problem.
In the definition of transitivity the condition xRy & $yRz \Rightarrow xRz$ for all $x, y, z \in A$ ~~be~~ it is not at all mentioned that x, y, z should ~~be~~ be different from each other. The condition must be satisfied for all x, y, z whether equal or unequal. Hence a situation can arise of the following type. Suppose that x and y are brothers of each other. Then we will have

xRy and yRx.

The first statement ends on y where the second begins. y, therefore, becomes the median. Now if we are to agree that R is transitive then xRy and yRx must give us xRx i.e. we will have to accept that a person is brother of himself i.e. brotherhood is reflexive (at least among males). If we donot agree about the reflexivity then because of the above situation we have to say that the relation is not transitive.

Thus brotherhood relation can be either reflexive and transitive or it can be non-reflexive and non-transitive.

Truly speaking there is no ~~ra~~ reason why brotherhood should not be accepted as reflexive. The main trouble lies in the fact that ~~but~~ the term brotherhood is not defined here and it is taken as if a known thing. If at all we decide to define the term then our definition will obviously be of the following type.

Defn: x is said to be a brother of y iff x is a male human being and both x and y have same parents.

Once this definition is given, whenever ~~we~~ consider a male human being x then the statement that both x & x have ~~the~~ same parents becomes a true statement and we get xRx. i.e. 'x is a brother of x' and the brotherhood relation becomes reflexive at least in the set of all male humans (and therefore transitive also).

We now consider this problem from another angle in order to again establish the reflexivity of this brotherhood relation.

Equivalence Relations

A binary relation R in A is said to be an equivalence relation if R is

i) reflexive

ii) symmetric

and iii) transitive

Before establishing various properties of equivalence relation we will first show that in the above defn. all the three conditions are essential and no two of them can imply the third. This becomes clear from the following examples.

(i) Example 4 on page 12 shows us that a relation can be reflexive, symetric but not transitive. Thus

reflexivity & symetry $\not\Rightarrow$ transitivity

(ii) If we consider the relation 'is a factor of' then it becomes reflexive, transitive but not symetric. Thus

reflexivity & transitivity $\not\Rightarrow$ symetry

(iii) The above two cases are readily accepted but many students (and some teachers/authors) feel that the condition of reflexivity is a redundancy, it is really not necessary, it follows from symetry & transitivity together, that it is only mentioned in the defn. of equivalence relation as a formality just as the closure property is mentioned in the defn. of a group. This thinking is based on the following logic.

Let R be a symetric & transitive relation

in a set A.
Now $xRy \Rightarrow yRx$ (by symetry
We then get xRy and yRx.
As the relation is transitive we get
xRx and reflexivity.
But this arguement is wrong. It
only shows that in a symetric & transitive
relation if an element is related
to some element of the set then it
is related to itself. But it does
not show that if an element is
not related to any other element then
will it be related to itself? The
answer is not necessarily. This
suggests the construction of the following
example.

Let $A = \{x, y, z\}$

Let $R = \{(x,x), (x,y), (y,x), (y,y)\}$

The reader is asked to verify that
this relation is symetric & transitive.
But is it reflexive? No. Because
$z \in A$ and yet $(z,z) \notin R$.
Thus symetry & transitivity
$\not\Rightarrow$ reflexivity.
These examples show that all the three
conditions i.e. reflexivity, symetry and
transitivity are required in the defn of

the equivalence relation.

The concept of equivalence relation has its roots in the concept of equality. Irrespective of the nature of objects we certainly have in our mind that

i) $x = x$ for any x

ii) $x = y \Rightarrow y = x$ for all x, y

and iii) $x = y$ and $y = z \Rightarrow x = z$ for all x, y, z.

These properties are generalised and are named as reflexive, symetric and transitive respectively. This clearly indicates that equality is indeed an example of equivalence relation.

Some other examples

i) Let I denotes the set of all integers i.e. $I = \{\ldots -3, -2, -1, 0, 1, 2, 3, 4 \ldots\}$

We fix some positive integer, say 5. Once fixed this will not be changed now. In I we define a relation as follows — For $x, y \in I$ we say that xRy if $5 | y - x$. This is read as "five divides $y-x$" meaning that $y-x$ is divisible by 5 i.e. $y-x$ has a factor 5 or $y - x = 5m$ for some $m \in I$

We will first show that this is an equivalence relation in I

(a) Let $x \in I$.

Then $x - x = 0 = 0 \cdot 5$

$\therefore 5 | x - x$ and xRx which means R is reflexive

(b) Let $x, y \in I$ with xRy

$\therefore 5 \mid y-x$

$\therefore y-x = 5 \cdot m$ for some $m \in I$

$\therefore x-y = 5 \cdot (-m)$ where $-m \in I$

$\therefore 5 \mid x-y$

$\therefore yRx$

$\therefore R$ is symetric

(c) Let $x, y, z \in I$ with xRy and yRz

$5 \mid y-x$ and $5 \mid z-y$

$\therefore y-x = 5m$ and $z-y = 5n$

where $m, n \in I$

Adding these two eqns:

$(\cancel{y}-x) + (z-\cancel{y}) = 5m + 5n$

$\therefore z-x = 5(m+n)$

Now $m+n \in I$ when $m \,\&\, n \in I$

$\therefore 5 \mid z-x$

$\therefore xRz$

$\therefore R$ is transitive

$\therefore R$ is an equivalence relation.

(ii) We are very much (and mainly) interested in the brotherhood relation. We investigate this in detail.

For the time being we restrict ourselve to the set of all male human beings.

Let $M = \{\text{all male humans}\}$

Let R mean "is a brother of"

Since we are analysing this relation now strictly Mathematically it is necessary for us to define the term 'brother' formally. As mentioned earlier we define

Defn x is said to be a brother of y if x is male and both x & y have same parents.

(a) Let $x \in M$

x is male and

x and x have same parents

$\therefore xRx$ and R is reflexive

(b) Let $x, y \in M$ s.t. xRy

Now x & y are males and x and y have same parents

Hence y is a male and y and x have same parents.

$\therefore yRx$ and R is symetric

(c) Let $x, y, z \in M$ s.t.

xRy and yRz. Here x, y, z are all males.

(x and y have same parents) and

(y and z have same parents)

$\therefore$ x and z have same parent

$\therefore xRz$ and R is transitive.

Thus R is an equivalence relation.

09/08/10

Of course, for the sake of arguement, one may say that if in the definition of 'brother' we introduce a condition that $x \neq y$ then reflexivity ~~can~~ will be impossible. i.e. If we say that x is a brother of y if x is a male, $x \neq y$ and both x & y have same parents then brotherhood will not be reflexive (and of course it will not be transitive also)

But I think it will be more natural to accept the reflexivity than trying to remove it by putting such additional condition. The reason for my thinking on these lines lies in the effect of equivalence relation on a set and the effect of a partition in a set. We proceed to study this in detail.

Equivalence classes of a set.

Defn Let R be an equivalence relation in a set S. Let $a \in S$.

Then the set of all elements of S which are related to 'a' is called as the equivalence class of a in S under R. It is denoted as $[a]$ (read as class a) Thus

$$[a] = \{x \in S / xRa\}$$

or $[a] = \{x \in S / aRx\}$ (Because of symmetry $xRa \Rightarrow aRx$)

clearly $[a] \subset S$

In the example $\dot{X}$ on page 17 if we consider $0 \in I$ then

$$[0] = \{x \in I \,/\, xR0\}$$
$$= \{x \in I \,/\, 5 \mid (0-x)\}$$
$$= \{x \in I \,/\, 5 \text{ is a factor of } -x\}$$
$$= \{x \in I \,/\, -x \text{ (hence } x) \text{ is a multiple of}$$
$$= \{\ldots -15, -10, -5, 0, 5, 10, 15, 20 \ldots\}$$

Similarly

$$[1] = \{\ldots -9, -4, 1, 6, 11, 16 \ldots\}$$

The reader may try to find other equivalence classes and see how many he get.

Here are some important properties of equivalence classes.

<u>Property I</u> An equivalence class is never empty.

This follows almost immediately. As R is an equivalence relation, it is reflexive. Hence for any $a \in S$ we must have aRa.

$$\therefore a \in [a] \quad [a] \neq \phi$$

Thus the element always belongs to its equivalence class. $[a]$ may or may not have other elements but it will certainly have 'a'.

<u>Property 2</u> Any two equivalence classes are either disjoint or equal.

<u>Proof:</u> - Let R be an equivalence relation in S.

Let $a, b \in S$ and consider $[a]$ & $[b]$

obviously $[a] \cap [b] = \phi$ or $[a] \cap [b] \neq \phi$

<u>Case i)</u> If $[a] \cap [b] = \phi$ then the two classes are disjoint and there is nothing to prove.

<u>case ii)</u> If $[a] \cap [b] \neq \phi$ then there must exist at least one $x \in S$ s.t. $x \in [a] \cap [b]$

$\therefore x \in [a]$ and $x \in [b]$

$\therefore xRa$ and xRb (by defn of classes

$\therefore aRx$ and xRb (By symetry

$\therefore aRb$ (by transitivity

Now for any $y \in [a]$

we have yRa

But since aRb is already established

we get yRa & aRb

$\therefore yRb$ (by transitivity

$\therefore y \in [b]$

$\therefore y \in [a] \Longrightarrow y \in [b]$

or $[a] \subset [b]$ ——— (i)

This result is desired from the statement

aRb

But by symetry, aRb also gives us bRa.

Hence we can show that

$$[b] \subset [a]$$

If it is needed to show this explicitely then

$y \in [b] \Rightarrow yRb$

But bRa is given

$\therefore$ yRb and bRa

$\therefore$ yRa (by transitivity)

$\therefore$ $y \in [a]$

i.e. $y \in [b] \Rightarrow y \in [a]$.

$\therefore$ $[b] \subset [a]$ ——— (ii)

From (i) and (ii) we get

$$[a] = [b]$$

Thus for any two equivalence classes $[a]$ and $[b]$ either they are disjoint or equal.

This can be described ~~as~~ also as

~~any~~ two equivalence classes either have nothing in common or have everything in common.

For an equivalence relation R in S we may contruct the set of all distincts equivalence classes of S under R and ~~call~~ denote such a set as ~~R~~ P

Thus $\mathcal{P} = \{[a] \,/\, a \in S$ and $[a]$ occurs only once.$\}$.

Then no two members of $\mathcal{P}$ are equal hence as seen in property two they must be disjoint.

Consider

$$\bigcup_{[a] \in \mathcal{P}} [a]$$

$$x \in \bigcup_{[a] \in \mathcal{P}} [a] \Rightarrow x \in [a] \text{ for some } [a] \in \mathcal{P}$$

$$\Rightarrow x \in S \text{ as } [a] \subset S.$$

Equally

$$x \in S \Rightarrow [x] \in \mathcal{P}$$

$$\Rightarrow x \in \bigcup_{[a] \in \mathcal{P}} [a]$$

$$\therefore \boxed{S = \bigcup_{[a] \in \mathcal{P}} [a]}$$

Thus S is split into different subsets (as classes) such that any two are disjoint and all together give us S. To use a favourite language among ~~most~~ Mathematicians these classes are mutually disjoint and collectively exhaustive. But such a splitting of a set in different mutually disjoint subsets is called a partition of S. Here is the defn

Defn: Let S be a non-empty set.

A ~~set~~ set $\mathcal{P}$ of subsets of S is said to be a partition ~~in~~ of S if

i) $A \in \mathcal{P}$ and $B \in \mathcal{P}$
$$\Rightarrow A \cap B = \phi$$

and ii) $\bigcup_{A \in \mathcal{P}} A = S$.

The elements of $\mathcal{P}$ are called as the members of partition.

With this defn. and the earlier discussion we conclude that an equalence relation in S introduces ~~a~~ partition in S wherein the equivalence classes are the members of the partition.

induces

In example 8 on page 17 the reader must have found ~~that~~ here we get only 5 distinct classes namely

$$[0] = \{ \ldots, -15, -10, -5, 0, 5, 10, 15 \ldots \}$$

$$[1] = \{ \ldots -14, -9, -4, 1, 6, 11, 16 \ldots \}$$

$$[2] = \{ \ldots -13, -8, -3, 2, 7, 12, 17 \ldots \}$$

$$[3] = \{ \ldots -12, -7, -2, 3, 8, 13, 18 \ldots \}$$

and $[4] = \{ \ldots -11, -6, -1, 4, 9, 14, 19 \ldots \}$

If we continue further we will get same classes again and again as $[5] = [0]$ $[6] = [1]$ and so on.

Hence the partition introduced in I is

induced

$\mathcal{P} = \{ [0], [1], [2], [3], [4] \}$

Just as an equivalence relation induces a partition it is equally true that a partition induces an equivalence relation in such a manner that the members of the partition become the equivalence classes. Here is the proof of this.

Let $\mathcal{P}$ be a partition of a non-empty set S.

Hence S is split into different mutually disjoint subsets which are members of $\mathcal{P}$.

Define a relation R in S as follows.

For $x, y \in S$ we say that

$x R y$ iff ~~both~~ x & y belong to same member of the partition.

We then get

(a) for any $x \in S$,

since $S = \bigcup_{A \in \mathcal{P}} A$

$x \in A$ for some $A \in \mathcal{P}$.

But then $x \in A$ and $x \in A$ ($p \Rightarrow p \wedge p$)

$\therefore$ x & x belong to same member A of $\mathcal{P}$.

$\therefore x R x$

$\therefore$ R is reflexive.

(b) Let $x, y \in S$ such that $x R y$.

By the defn of R this means ~~that~~ there

is some $A \in \mathcal{P}$ such that both
$x \& y \in A$
i.e. $x \in A$ and $y \in A$
$\therefore$ $y \in A$ and $x \in A$ ($p \wedge q \Leftrightarrow q \wedge p$)
$\therefore$ y & x belong to same member
namely A of $\mathcal{P}$
$\therefore$ $y R x$ by defn.
$\therefore$ R is symetric

ⓒ Let $x, y, z \in S$ such that xRy & yRz
$xRy \Rightarrow \exists A \in \mathcal{P}$ such that
$x \in A$ and $y \in A$

$yRz \Rightarrow \exists B \in \mathcal{P}$ such that
$y \in B$ and $z \in B$.
$\therefore$ $A, B \in \mathcal{P}$ and $\mathcal{P}$ is a partition
$A = B$ or $A \cap B = \phi$
But $y \in A$ and $y \in B$
$\therefore y \in A \cap B$
$\therefore A \cap B \neq \phi$
$\therefore A = B$.
Calling B as A now we can write
$x \in A$ and $y \in A$ and $y \in A$ and $z \in A$
i.e. $x, y, z \in A$
In particular
x & z belong to same
member A.
$\therefore xRz$
$\therefore$ R is transitive

R is thus an equivalence relation in S induced by the partition $\mathcal{P}$.

If we consider any element a of S then as $S = \bigcup_{A \in \mathcal{P}} A$ we will get some A in $\mathcal{P}$ s.t. $a \in A$.

Now by $[a]$ mean the equivalence class of a in S under R.

$$[a] = \{x \in S / xRa\}$$

As $a \in A$, xRa will mean $x \in A$.

Hence

$$[a] = \{x \in S / x \in A\}$$

i.e. $[a] = A$.

Thus this equivalence relation induced by the partition $\mathcal{P}$ is such that it's equivalence classes are exactly the members of the partition.

We have seen now that with an equivalence relation there corresponds a partition and with a partition there corresponds an equivalence relation such that equivalence classes are same as the members of the partition. Hence we can say that an equivalence relation and partition are but two sides of the same coin.

We turn our attention once again to the set of male humanbeings. i.e. $M = \{\text{all male humans}\}$

12/08/10 Supose we introduce a split in this set by saying "all brothers should form their groups." The set M will be broken into different subsets such that each subset will comprise of men who are brothers of each other. If a certain male humanbeing is the only son of his parents then he will form a singleton subset in this split. These different subsets of M would be mutually disjoint and if the union of all of them is taken the we will get our M back. But this means that such a split forms a partition of M. As we have seen earlier a partition indnces an equivalence relation. So this partition of M will ~~also~~ give in an equivalence relation in M such that the equivalence classes will be the members of the partition. In this equivalence relation an equivalence class will be comprised of men who are brothers of each other.

Since the members of an equivalence class are all related to each other, this relation that we get is nothing but 'the brotherhood relation.' The very fact that we are getting this as an equivalence relation ~~again~~ indicates that it is reflexive. If a person x is the only son of his parents (thus forming a singleton set in this partition) then also we get xRx.

In view of all that is said so far about equivalence relation and partition it is obvious that the brotherhood relation should be accepted as reflexive.

Extension to all humanbeings.

~~[illegible]~~

The difference between the words 'brother' and 'sister' is only of gender. The remaining part of the condition of 'having same parents' is common to both of them. Hence it is sugested that this gender difference should be abondoned. In many instances we do this. For example, when we talk about 'international brotherhood' or when we say that 'the peaceful progress of mankind is possible if we behave like brothers irrespective of our race, religion or or nationality' do we exclude all female humanbeings from our consideration? Certainly not.

Neglecting this gender difference we can identify the words 'brother' and 'sister' to mean samething. When this is done it is no longer necessary for us to consider only male humanbeings. We can extend the ~~complete~~ entire argument made so far to all humanbeings.

Thus if x is any humanbeing and R means 'is a brother of' then we conclude

$x R x$

or x is a brother of x

or R is reflexive.

14/08/10

Now there is no intention whatsoever of connecting Mathematics & Meta-Physics Nor I am trying to analyse anything said in Srimad Bhagwadgeeta as such an attempt will be beyond my capacity. Yet when in the concluding part of this epilouge I mentioned my humble opinion that brotherhood relation should be accepted as reflexive, I was reminded of a simillar statement in Srimad Bhagwadgeeta somewhere. I searched and found that in Chapter 6 Stanza 5 Lord Krishna has said to Arjuna that

आत्मैव ह्यात्मनो बंधुः।

meaning that 'we (and only we) are brother of ourselves'.

Because of the word एव comming in आत्मैव (this is a combination of आत्मा एव)

the statement can be symbolically written as follows.

If R means 'x is a brother of y'

then for ~~many~~ men x & y

$(xRx$ and $(x \neq y \Rightarrow x \not R y))$

Since $p \wedge q \Rightarrow p$ ~~and~~ always the above statement certainly gives us

xRx.

Let me make this clear that the context of the above statement is quite different. It is cited here only because the similarity was felt interesting.

॥ श्रीकृष्णार्पणमस्तु ॥

(Shrikrishnarpanmastu)

Meaning that whatever is here written and done be offered to Lord Krishna.

———x———