Algebraic Partial equations

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We generalized the concept of hyperbolic
and trigonometric functions to the
third and the fourth case which
gives rise to the parametrization
of the orbifold defined by
$x^3 + y^3 + z^3 - 3xyz = 1$
in the third case.

Introduction

The trigonometric functions (cosinus and sinus) were historically discovered as real and imaginary parts of the complex exponential function which solves the first order partial equations. In addition, the hyperbolic functions are the results of the trigonometric function with imaginary arguments. Even if the trigonometric functions were find before the hyperbolic ones, try to imagine that this is the trigonometric which cames from the trigonometric ones.

Because the exponential function is defined as

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{0.1}$$

It seems to be natural to generalized the exponential function as

$$e_p(z) = \sum_{k=0}^{\infty} \frac{z^{pk}}{(pk)!} \tag{0.2}$$

that we have call the p-exponential function in [1]. Here we recognize the hyperbolic cosinus if we take $p = 2$. In comparaison with the hyperbolic functions, we remarks that the p-th exponential (0.2) can be written as

$$e_p(z) = \frac{1}{p} \left(\sum_{k=0}^{p-1} e^{{\omega_p}^k z} \right) \tag{0.3}$$

where ω_p is the p-th root of unity.

Hence the hyperbolic and the trigonometric functions each satisfie a partial equation

$$y^2 - \left(\frac{\partial y}{\partial z}\right)^2 = 1 \tag{0.4}$$

for the hyperbolic functions and the sames with a minus for the trigonometric ones.

By the way those functions are viewed as the unique solutions of two manifolds whom defining equations are

$$x^2 - y^2 = 1 \tag{0.5}$$

which defines a hyperbol for the hyperbolic functions and the sames with a minus which defines a circle for the trigonometric ones.

As a consequence, we searched for the partial equation solved by the third exponential.

To this end, we showed the equivalent addition relation for the case $p = 3$. In the same way we find (0.4) when we put $b = -a$ in the trigonometric addition formulae ($\cosh(a + b)$), we have put $b = ja, c = j^2 a$ in this equivalent trigonometric addition formulae ($\epsilon_3(a+b+c)$) and we have find the partial equation

$$\left(y(z)\right)^3 + \left(\frac{\partial y}{\partial z}\right)^3 + \left(\frac{\partial^2 y}{\partial z^2}\right)^3 - 3\left(y(z)\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial^2 y}{\partial z^2}\right) = 1 \tag{0.6}$$

Thereby, the third exponential is the unique parametric solution of the orbifold defined by the implicit equation

$$x^3 + y^3 + z^3 - 3xyz = 1 \tag{0.7}$$

The generalization of (0.6) leads to the equation

$$\prod_{q=0}^{p-1} \left(\sum_{k=0}^{p-1} \omega_p{}^{-kp} \; y^{(k)}(z) \right) \; = \; 1 \tag{0.8}$$

for each $p \in \mathbb{N}$ with $y(z) = e_p(z)$.

In a first time, we recall definitions of the third exponential and we give some expressions of this one. In a second time we give generalized hyperbolic parity and addition formulaes of this third exponential. In a third time, we arbitrarly put 0 in the argument of the generalized hyperbolic addition relations to obtain the defining equation of the emerging orbifold and the partial equation solved by the third exponential. In a fourth time, we give a defintion of the hypercomplex number which are a generalization of the complex ones. Finally, we completely solve the case of the fourth exponential and give the orbifold and the partial equation solved in the general case.

1 The p-exponential

We recall the definition of the p-exponential given in [1] by :

$$e_p(z) \;=\; \sum_{k=0}^{\infty} \frac{z^{pk}}{(pk)!} \;=\; \frac{1}{p}\left(\sum_{k=0}^{p-1} e^{{\omega_p}^k z}\right) \tag{1.1}$$

where ω_p is the p-th root of unity. We have shown in [1] that the p-exponential is solution of the partial equation

$$\frac{\partial^p e_p}{\partial z^p} \;=\; e_p(z) \tag{1.2}$$

and

$$\sum_{k=0}^{p-1} \frac{\partial^k e_p}{\partial z^k} \;=\; e^z \tag{1.3}$$

Example : For p=3, the set of 3-exponential are

$$e_3(z) \;=\; \sum_{k=0}^{\infty} \frac{z^{3k}}{(3k)!} \tag{1.4}$$

$$e'_3(z) \;=\; \sum_{k=1}^{\infty} \frac{z^{3k-1}}{(3k-1)!} \tag{1.5}$$

$$e''_3(z) \;=\; \sum_{k=1}^{\infty} \frac{z^{3k-2}}{(3k-2)!} \tag{1.6}$$

which can also be written

$$e_3(z) \;=\; \frac{e^z + e^{jz} + e^{j^2 z}}{3} \tag{1.7}$$

$$e'_3(z) \;=\; \frac{e^z + je^{jz} + j^2 e^{j^2 z}}{3} \tag{1.8}$$

$$e''_3(z) \;=\; \frac{e^z + j^2 e^{jz} + je^{j^2 z}}{3} \tag{1.9}$$

or

since $j = e^{\frac{2i\pi}{3}} = \frac{-1+i\sqrt{3}}{2}$ is the 3th root of unity

2 Trigonometric relations for the third exponential

For p=3, we generalized the parity relations for the third exponential

Proposition 1. *The parity relations are given by*

$$e_3(jz) = e_3(z) \qquad e_3(j^2 z) = e_3(z) \tag{2.10}$$

$$e'_3(jz) = j^2 e'_3(z) \qquad e'_3(j^2 z) = je'_3(z) \tag{2.11}$$

$$e''_3(jz) = je''_3(z) \qquad e''_3(jz) = je''_3(z) \tag{2.12}$$

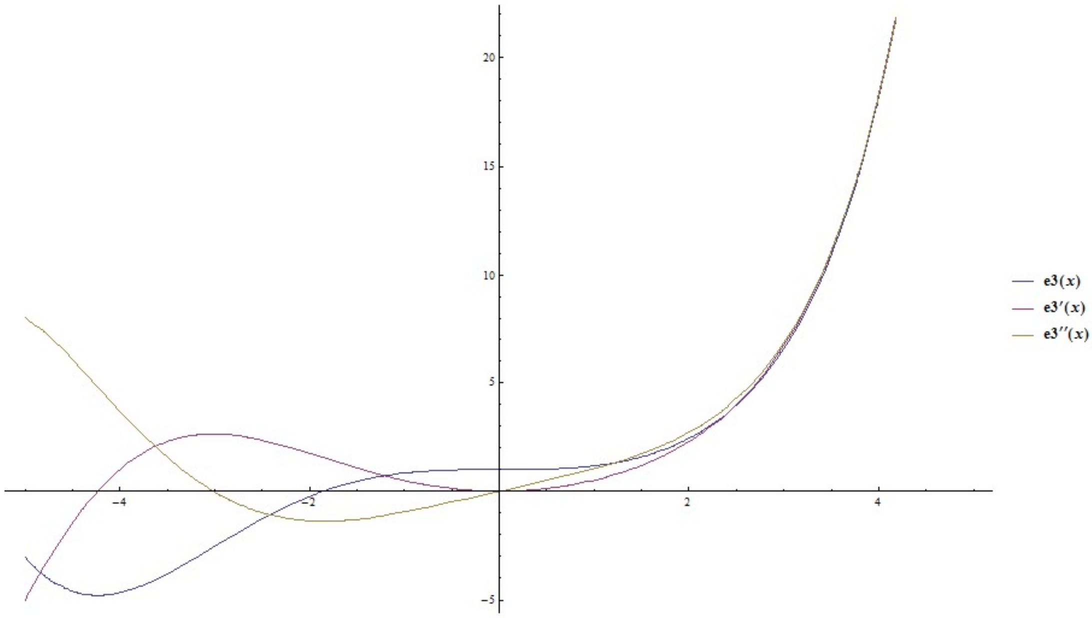

FIGURE 1 – The 3-exponential and its two derivatives

Proof :
From de definition (5.53), we obtain these relations.
For example, to show $e_3''(jz) = je_3''(z)$

$$\begin{aligned} 3e_3''(jz) &= e^{jz} + j^2 e^{j(jz)} + je^{j^2(jz)} \\ &= e^{jz} + j^2 e^{j^2 z} + je^{z} \\ &= j(j^2 e^{jz} + je^{j^2 z} + e^{z}) \\ 3e_3''(jz) &= 3je_3''(z) \end{aligned} \tag{2.13}$$

From (2.13), we obtain the other relation of (2.12) by successive derivations or by doing the same reasonment.

♦

Now, we generalized the addtion relations for the third exponential

Proposition 2. *The addition relations are given by*

$$e_3(a+b) = e_3(a)e_3(b) + e_3'(a)e_3''(b) + e_3''(a)e_3'(b) \tag{2.14}$$
$$e_3'(a+b) = e_3''(a)e_3''(b) + e_3'(a)e_3(b) + e_3(a)e_3'(b) \tag{2.15}$$
$$e_3''(a+b) = e_3'(a)e_3'(b) + e_3(a)e_3''(b) + e_3''(a)e_3(b) \tag{2.16}$$

Proof :

Developping the expression of e_3, e_3', e_3'' from their definition (5.53), we have the relations

$$\begin{aligned}
9e_3(a)e_3(b) &= e^{a+b}+e^{a+jb}+e^{a+j^2b}+e^{ja+b}+e^{ja+jb}+e^{ja+j^2b}+e^{j^2a+b}+e^{j^2a+jb}+e^{j^2a+j^2b}\\
9e_3'(a)e_3'(b) &= e^{a+b}+je^{a+jb}+j^2e^{a+j^2b}+je^{ja+b}+j^2e^{ja+jb}+e^{ja+j^2b}+j^2e^{j^2a+b}+e^{j^2a+jb}+je^{j^2a+j^2b}\\
9e_3''(a)e_3''(b) &= e^{a+b}+j^2e^{a+jb}+je^{a+j^2b}+j^2e^{ja+b}+je^{ja+jb}+e^{ja+j^2b}+je^{j^2a+b}+e^{j^2a+jb}+j^2e^{j^2a+j^2b}
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
9e_3(a)e_3'(b) &= e^{a+b}+je^{a+jb}+j^2e^{a+j^2b}+e^{ja+b}+je^{ja+jb}+j^2e^{ja+j^2b}+e^{j^2a+b}+je^{j^2a+jb}+j^2e^{j^2a+j^2b}\\
9e_3(a)e_3''(b) &= e^{a+b}+j^2e^{a+jb}+je^{a+j^2b}+e^{ja+b}+j^2e^{ja+jb}+je^{ja+j^2b}+e^{j^2a+b}+j^2e^{j^2a+jb}+je^{j^2a+j^2b}\\
9e_3'(a)e_3''(b) &= e^{a+b}+j^2e^{a+jb}+je^{a+j^2b}+je^{ja+b}+e^{ja+jb}+j^2e^{ja+j^2b}+j^2e^{j^2a+b}+je^{j^2a+jb}+e^{j^2a+j^2b}
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
9e_3'(a)e_3(b) &= e^{a+b}+e^{a+jb}+je^{a+j^2b}+je^{ja+b}+je^{ja+jb}+je^{ja+j^2b}+j^2e^{j^2a+b}+j^2e^{j^2a+jb}+j^2e^{j^2a+j^2b}\\
9e_3''(a)e_3(b) &= e^{a+b}+e^{a+jb}+e^{a+j^2b}+j^2e^{ja+b}+j^2e^{ja+jb}+j^2e^{ja+j^2b}+je^{j^2a+b}+je^{j^2a+jb}+je^{j^2a+j^2b}\\
9e_3''(a)e_3'(b) &= e^{a+b}+je^{a+jb}+j^2e^{a+j^2b}+j^2e^{ja+b}+e^{ja+jb}+je^{ja+j^2b}+je^{j^2a+b}+j^2e^{j^2a+jb}+e^{j^2a+j^2b}
\end{aligned} \tag{2.19}$$

Using (2.18) and (2.19), we have

$$\begin{aligned}
9(e_3'(a)e_3''(b)+e_3''(a)e_3'(b)) &= 2e^{a+b}-e^{a+jb}-e^{a+j^2b}-e^{ja+b}\\
&\quad +2e^{ja+jb}-e^{ja+j^2b}-e^{j^2a+b}-e^{j^2a+jb}+2e^{j^2a+j^2b}\\
&= 3(e^{a+b}+e^{ja+jb}+e^{j^2a+j^2b})-(e^{a+b}+e^{a+jb}+e^{a+j^2b}\\
&\quad +e^{ja+b}+e^{ja+jb}+e^{ja+j^2b}+e^{j^2a+b}+e^{j^2a+jb}+e^{j^2a+j^2b})\\
9(e_3'(a)e_3''(b)+e_3''(a)e_3'(b)) &= 9e_3(a+b)-9e_3(a)e_3(b)
\end{aligned}$$

which gives

$$e_3(a+b)=e_3(a)e_3(b)+e_3'(a)e_3''(b)+e_3''(a)e_3'(b) \tag{2.20}$$

which the first relation of (2.16).

In the same way, using (2.18) and (2.19), we have

$$\begin{aligned}
9(e_3(a)e_3''(b)+e_3''(a)e_3(b)) &= 2e^{a+b}-j^2e^{a+jb}-je^{a+j^2b}-j^2e^{ja+b}\\
&\quad +2je^{ja+jb}-e^{ja+j^2b}-je^{j^2a+b}-e^{j^2a+jb}+2j^2e^{j^2a+j^2b}\\
&= 3(e^{a+b}+je^{ja+jb}+j^2e^{j^2a+j^2b})-(e^{a+b}+j^2e^{a+jb}+\\
&\quad je^{a+j^2b}+j^2e^{ja+b}+je^{ja+jb}+e^{ja+j^2b}+je^{j^2a+b}+e^{j^2a+jb}+j^2e^{j^2a+j^2b})\\
9(e_3(a)e_3'(b)+e_3'(a)e_3(b)) &= 9e_3'(a+b)-9e_3''(a)e_3''(b)
\end{aligned}$$

which gives

$$e_3'(a+b)=e_3''(a)e_3''(b)+e_3'(a)e_3(b)+e_3(a)e_3'(b) \tag{2.21}$$

which the second relation of (2.16).

In the same way, using (2.18) and (2.19), we have

$$\begin{aligned}
9(e_3(a)e_3''(b)+e_3''(a)e_3(b)) &= 2e^{a+b}-je^{a+jb}-j^2e^{a+j^2b}-je^{ja+b} \\
&\quad +2j^2e^{ja+jb}-e^{ja+j^2b}-j^2e^{j^2a+b}-e^{j^2a+jb}+2je^{j^2a+j^2b} \\
&= 3(e^{a+b}+j^2e^{ja+jb}+je^{j^2a+j^2b})-(e^{a+b}+je^{a+jb}+j^2e^{a+j^2b}+je^{ja+b} \\
&\quad +j^2e^{ja+jb}+e^{ja+j^2b}+j^2e^{j^2a+b}+e^{j^2a+jb}+je^{j^2a+j^2b}) \\
9(e_3(a)e_3''(b)+e_3''(a)e_3(b)) &= 9e_3''(a+b)-9e_3'(a)e_3'(b)
\end{aligned}$$

which gives

$$e_3''(a+b)=e_3'(a)e_3'(b)+e_3(a)e_3''(b)+e_3''(a)e_3(b) \tag{2.22}$$

which the third relation of (2.16).

♦

3 Orbifold emerging of the trigonometric relations

Example :
For p=3, we have

Proposition 3. *The vector $(x,y,z)=(e_3,e_3',e_3'')$ is the solution of the equation such that*

$$x^3+y^3+z^3-3xyz=1 \tag{3.23}$$

The equation (3.23) defines an orbifold.

Proof :
From the addition relation, we have

$$\begin{aligned}
e_3(a+b+c) &= e_3(a)e_3(b+c)+e_3'(a)e_3''(b+c)+e_3''(a)e_3'(b+c) \\
&= e_3(a)\Big(e_3(b)e_3(c)+e_3'(b)e_3''(c)+e_3''(b)e_3'(c)\Big) \\
&\quad +e_3'(a)\Big(e_3'(b)e_3'(c)+e_3(b)e_3''(c)+e_3''(b)e_3(c)\Big) \\
&\quad +e_3''(a)\Big(e_3''(b)e_3''(c)+e_3'(b)e_3(c)+e_3(b)e_3'(c)\Big)
\end{aligned} \tag{3.24}$$

If we impose $b = ja$ and $c = j^2 a$ and using (2.12), we find

$$\begin{aligned} e_3(a + ja + j^2 a) &= e_3(a)\Big(e_3(a)e_3(a) + je'_3(a)e''_3(a) + j^2 e''_3(a)e'_3(a)\Big) \\ &\quad + e'_3(a)\Big(e'_3(a)e'_3(a) + j^2 e_3(a)e''_3(a) + je''_3(b)e_3(c)\Big) \\ &\quad + e''_3(a)\Big(e''_3(a)e''_3(a) + j^2 e'_3(a)e_3(a) + je_3(b)e'_3(c)\Big) \\ 1 &= \Big(e_3(a)\Big)^3 + \Big(e'_3(a)\Big)^3 + \Big(e''_3(a)\Big)^3 + (j + j^2 + j + j^2 + j + j^2)e_3(a)e'_3(a)e'_3(a) \end{aligned}$$

So we find

$$\Big(e_3(a)\Big)^3 + \Big(e'_3(a)\Big)^3 + \Big(e''_3(a)\Big)^3 - 3e_3(a)e'_3(a)e'_3(a) = 1 \tag{3.25}$$

So $y(z) = e_3(z)$ is the solution of the partial equation

$$\Big(y(z)\Big)^3 + \left(\frac{\partial y}{\partial z}\right)^3 + \left(\frac{\partial^2 y}{\partial z^2}\right)^3 - 3\Big(y(z)\Big)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial^2 y}{\partial z^2}\right) = 1 \tag{3.26}$$

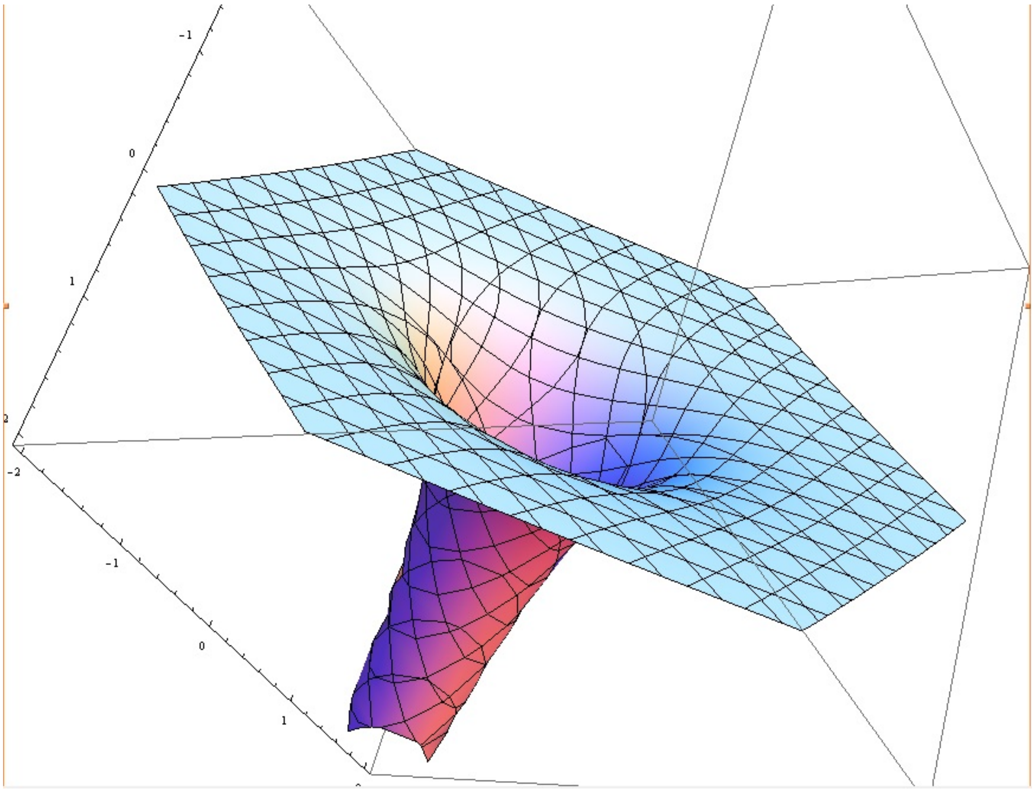

FIGURE 2 – Implicit plot of $x^3 + y^3 + z^3 - 3xyz = 1$

4 Hypercomplex numbers

From the definition of the orbifold emerging of the third exponential

$$x^3 + y^3 + z^3 - 3xyz = 1 \tag{4.27}$$

and because of the identity

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}\,(x+y+z)\,\Big[(x-y)^2 + (y-z)^2 + (z-x)^2\Big] \tag{4.28}$$

we obtain

$$\frac{1}{2}\,(x+y+z)\,\Big[(x-y)^2 + (y-z)^2 + (z-x)^2\Big] = 1 \tag{4.29}$$

Now we expand the second factor which gives

$$\begin{aligned}
(x+y+z)\,\Big[x^2 + y^2 + z^2 - xy - yz - xz\Big] &= 1 \\
(x+y+z)\,\Big[x^2 + y^2 + z^2 + (j+j^2)xy + (j+j^2)yz + (j+j^2)xz\Big] &= 1 \\
(x+y+z)\,\Big[x^2 + j^2\ xy + j\ xz + y^2 + j\ yx + j^2\ yz + z^2 + j^2\ zx + j\ zy\Big] &= 1 \\
(x+y+z)\,\Big[(x + j\ y + j^2\ z)(x + j^2\ y + j\ z)\Big] &= 1
\end{aligned} \tag{4.30}$$

So if we parametrize

$$\left\{\begin{array}{lcl} x &=& e_3(\theta) \\ y &=& e_3'(\theta) \\ z &=& e_3''(\theta) \end{array}\right. \tag{4.31}$$

We can write (4.30) as

$$e^{\theta}\ \ e^{j\theta}\ \ e^{j^2\theta} = 1 \tag{4.32}$$

since

$$e^{\theta} = e_3(\theta) + e_3'(\theta) + e_3''(\theta) \tag{4.33}$$

$$e^{j\theta} = e_3(\theta) + j^2\ \ e_3'(\theta) + j\ \ e_3''(\theta) \tag{4.34}$$

$$e^{j^2\theta} = e_3(\theta) + j\ \ e_3'(\theta) + j^2 e_3''(\theta) \tag{4.35}$$

$$\tag{4.36}$$

Then we can define the hypercomplex numbers

$$\mathbb{S} = \Big\{x + j\ y + j^2\ z \in \mathbb{C}\ \text{ with }\ x, y, z \in \mathbb{R}\Big\} \tag{4.37}$$

5 Generalization

Theorem 1. *The p-exponential $y(z) = e_p(z)$ is the solution the $p-1$ order partial equation*

$$\prod_{q=0}^{p-1} \left(\sum_{k=0}^{p-1} \ {\omega_p}^{-kp} \ y^{(k)}(z) \ \right) \ = \ 1 \tag{5.38}$$

This relation defines an orbifold $\mathcal{O}$:

$$\prod_{q=0}^{p-1} \left(\sum_{k=0}^{p-1} \ {\omega_p}^{-kp} \ x_k \ \right) \ = \ 1 \tag{5.39}$$

where $(x_0, ..., x_{n-1}) = \Big(e_p(z), e'_p(z), ..., e_p^{(n)}(z) \Big)$.

<u>Proof :</u> From the definition of the p-exponential, we have

$$\begin{aligned}
e_p(z) &= \frac{1}{p} \sum_{k=0}^{p-1} e^{{\omega_p}^k z} \\
e'_p(z) &= \frac{1}{p} \sum_{k=0}^{p-1} {\omega_p}^k \ e^{{\omega_p}^k z} \\
&\vdots \\
{e_p}^{(q)}(z) &= \frac{1}{p} \sum_{k=0}^{p-1} {\omega_p}^{qk} \ e^{{\omega_p}^k z} \\
&\vdots \\
{e_p}^{(n)}(z) &= \frac{1}{p} \sum_{k=0}^{p-1} {\omega_p}^{nk} \ e^{{\omega_p}^k z}
\end{aligned} \tag{5.40}$$

with ω_p the p-th root of unity. Now we deduce from (5.40) the generalized parity relation

$${e_p}^{(q)}(\omega_p z) = {\omega_p}^{q(p-1)} {e_p}^{(q)}(z) \tag{5.41}$$

or

$${e_p}^{(q)}(\omega_p z) = {\omega_p}^{-q} {e_p}^{(q)}(z) \tag{5.42}$$

From (5.42), we obtain

$$
\begin{aligned}
e^{z} &= \sum_{k=0}^{p-1} \; {e_p}^{(k)}(z) \\
e^{\omega_p z} &= \sum_{k=0}^{p-1} \; {\omega_p}^{-k} \; {e_p}^{(k)}(z) \\
\vdots & \\
e^{{\omega_p}^q z} &= \sum_{k=0}^{p-1} \; {\omega_p}^{-qk} \; {e_p}^{(k)}(z) \\
\vdots & \\
e^{{\omega_p}^n z} &= \sum_{k=0}^{p-1} \; {\omega_p}^{-nk} \; {e_p}^{(k)}(z)
\end{aligned}
\tag{5.43}
$$

Using (5.58) we obtain

$$
\begin{aligned}
e^{\left(\frac{1-{\omega_p}^p}{1-\omega_p}\right) z} &= 1 \\
e^{\,\sum_{q=0}^{p-1} \; {\omega_p}^q \; z} &= 1 \\
\prod_{q=0}^{p-1} \; e^{{\omega_p}^q \; z} &= 1 \\
\prod_{q=0}^{p-1} \; \left(\sum_{k=0}^{p-1} \; {\omega_p}^{-qk} \; {e_p}^{(k)}(z) \right) &= 1
\end{aligned}
\tag{5.44}
$$

which is the relation (5.38).

♦

Remark 1. *If we define the Armian of a vector* $\mathbf{e} = (e_1, ..., e_n) \in \mathbb{R}_n$ *:*

$$
\mathrm{Arm}(\mathbf{e}) \;=\; \prod_{q=1}^{n} \; e_q \tag{5.45}
$$

we can write (5.38) as

$$
\mathrm{Arm} \; (\, \mathrm{W} \;\; \mathbf{e_p} \,) \;=\; 1 \tag{5.46}
$$

with

$$\mathbf{e_p} = \begin{pmatrix} e_p(z) \\ e_p'(z) \\ \vdots \\ {e_p}^{(q)}(z) \\ \vdots \\ {e_p}^{(n)}(z) \end{pmatrix} \tag{5.47}$$

and $W = ({\omega_p}^{-(k-1)(q-1)})_{1\leq k,q\leq p}$ *or*

$$\mathrm{W} = \begin{pmatrix} 1 & \dots & 1 & \dots & 1 \\ 1 & \dots & {\omega_p}^{-(k-1)} & \dots & {\omega_p}^{-(p-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & {\omega_p}^{-(k-1)(q-1)} & \dots & {\omega_p}^{-(p-1)(q-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & {\omega_p}^{-(k-1)(p-1)} & \dots & {\omega_p}^{-(p-1)(p-1)} \end{pmatrix} \tag{5.48}$$

Using the Vondermonde determinant formula, we can check

$$\det(\mathrm{W}) = \prod_{1\leq i<j\leq p} \left({\omega_p}^{-(j-1)} - {\omega_p}^{-(i-1)} \right) \tag{5.49}$$

Example :

For p=4, the set of 4-exponential are

$$e_4(\theta) = \frac{e^{\theta} + e^{i\theta} + e^{-\theta} + e^{-i\theta}}{4} \tag{5.50}$$

$$e_4'(\theta) = \frac{e^{\theta} + ie^{i\theta} - e^{-\theta} - ie^{-i\theta}}{4} \tag{5.51}$$

$$e_4''(\theta) = \frac{e^{\theta} - e^{i\theta} + e^{-\theta} - e^{-i\theta}}{4} \tag{5.52}$$

$$e_4'''(\theta) = \frac{e^{\theta} - ie^{i\theta} - e^{-\theta} + ie^{-i\theta}}{4} \tag{5.53}$$

or

$$e_4(\theta) = \frac{1}{2} \left(\cos(\theta) + \cosh(\theta) \right) \tag{5.54}$$

$$e_4'(\theta) = \frac{1}{2} \left(-\sin(\theta) + \sinh(\theta) \right) \tag{5.55}$$

$$e_4''(\theta) = \frac{1}{2} \left(-\cos(\theta) + \cosh(\theta) \right) \tag{5.56}$$

$$e_4'''(\theta) = \frac{1}{2} \left(\sin(\theta) + \sinh(\theta) \right) \tag{5.57}$$

We have

$$
\begin{aligned}
e^{\theta} &= e_4(\theta) + e_4'(\theta) + e_4''(\theta) + e_4'''(\theta) \\
e^{i\theta} &= e_4(\theta) - ie_4'(\theta) - e_4''(\theta) + ie_4'''(\theta) \\
e^{-\theta} &= e_4(\theta) - e_4'(\theta) + e_4''(\theta) - e_4'''(\theta) \\
e^{-i\theta} &= e_4(\theta) + ie_4'(\theta) - e_4''(\theta) - ie_4'''(\theta)
\end{aligned}
\tag{5.58}
$$

Here we can see the W matrix

$$
\mathrm{W} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \tag{5.59}
$$

Using

$$
\begin{cases} x &= e_4(\theta) \\ y &= e_4'(\theta) \\ z &= e_4''(\theta) \\ t &= e_4'''(\theta) \end{cases} \tag{5.60}
$$

we can write

$$
\begin{aligned}
e^{\theta}e^{-\theta} &= 1 \\
(x+y+z+t)(x-y+z-t) &= 1 \\
(x+z)^2 - (y+t)^2 &= 1
\end{aligned}
\tag{5.61}
$$

and

$$
\begin{aligned}
e^{i\theta}e^{-i\theta} &= 1 \\
(x-iy+z+it)(x+iy+z-it) &= 1 \\
(x-z)^2 + (y-t)^2 &= 1
\end{aligned}
\tag{5.62}
$$

Combining (5.61) and (5.63), we obtain

$$
\begin{aligned}
e^{\theta}e^{-\theta}e^{i\theta}e^{-i\theta} &= 1 \\
\left[(x-z)^2 + (y-t)^2\right]\left[(x+z)^2 - (y+t)^2\right] &= 1
\end{aligned}
$$

the defining equation of the fourth exponential

$$
(x^2 - z^2)^2 + (y^2 - t^2)^2 + 4(xy - zt)(zy - xt) = 1 \tag{5.63}
$$

Conclusion

In fact there is a probleme with the parametrization of $x^3 + y^3 + z^3 - 3xyz = 1$ with the curve $(e_3(x), e_3'(x), e_3(x))$. It is because why I have written this paper using $z \in \mathbb{C}$ because $e_3(z)$ is two dimentional like $x^3 + y^3 + z^3 - 3xyz = 1$. At the time I am writting those lines, I do not have solved this problem.

The dimension of the emerging orbifold seems to depend on the independance of the p-th root of unity back the previous roots of unity. You can see it in the fourth exponential case where

$$\omega_4^2 = i^2 = -1 = \omega_2 \tag{5.64}$$

Hence we have two defining equations for the orbifold emerging of the fourth exponential whereas we find only one implicit equation for the orbifold emerging of the third exponential case. This gives us the good dimensional number since the generalized exponential are defined on $\mathbb{C}$ which is two dimensional.

Furthermore, I choose to give the generalized theorem for each $p \in \mathbb{N}$ to say that it is true for each p but I only formally prove it only for the third and the fourth exponential.

By the way, if somebody want to study the topology or the metric of the orbifold $x^3+y^3+z^3-3xyz = 1$, please contact me, I would be happy to work on it.

Références

[1] Arm B. N., The p-Arm Theory