

# THE CLASSICAL MAXWELL ELECTRODYNAMICS AND THE ELECTRON INERTIA PROBLEM WITHIN THE FEYNMAN PROPER TIME PARADIGM

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ABSTRACT. The Maxwell electromagnetic and the Lorentz type force equations are derived in the framework of the R. Feynman proper time paradigm and the related vacuum field theory approach. The electron inertia problem is analyzed within the Lagrangian and Hamiltonian formalisms and the related pressure-energy compensation principle. The modified Abraham-Lorentz damping radiation force is derived, the electromagnetic electron mass origin is argued.

## 1. INTRODUCTION

The elementary point charged particle, like electron, mass problem was inspiring many physicists [30] from the past as J. J. Thompson, G.G. Stokes, H.A. Lorentz, E. Mach, M. Abraham, P.A. M. Dirac, G.A. Schott and others. Nonetheless, their studies have not given rise to a clear explanation of this phenomenon that stimulated new researchers to tackle it from different approaches based on new ideas stemming both from the classical Maxwell-Lorentz electromagnetic theory, as in [11, 20, 21, 47, 18, 19, 24, 25, 31, 32, 38, 39, 41, 42, 43, 45, 54, 53, 55, 56, 59], and modern quantum field theories of Yang-Mills and Higgs type, as in [3, 26, 27, 58] and others, whose recent and extensive review is done in [57].

In the present work I will mostly concentrate on detail analysis and consequences of the Feynman proper time paradigm [18, 19, 13, 14] subject to deriving the electromagnetic Maxwell equations and the related Lorentz like force expression considered from the vacuum field theory approach, developed in works [8, 49, 10, 6], and further, on its applications to the electromagnetic mass origin problem. Our treatment of this and related problems, based on the least action principle within the Feynman proper time paradigm [18], has allowed to construct the respectively modified Lorentz type equation for a moving in space and radiating energy charged point particle. Our analysis also elucidates, in particular, the computations of the self-interacting electron mass term in [38], where there was proposed a not proper solution to the well known classical Abraham-Lorentz [1, 35, 36, 37] and Dirac [12] electron electromagnetic "4/3-electron mass" problem. As a result of our scrutinized studying the classical electromagnetic mass problem we have stated that it can be satisfactory solved within the classical H. Lorentz and M. Abraham reasonings augmented with the additional electron stability condition, which was not taken before into account yet appeared to be very important for balancing the related electromagnetic field and mechanical electron momenta. The latter, following recent enough works [54, 41], devoted to analyzing the electron charged shell model, can be realized within there suggested *pressure-energy compensation principle*, suitably applied to the ambient electromagnetic energy fluctuations and the own electrostatic Coulomb electron energy.

## 2. THE LORENTZ TYPE FORCE ANALYSIS WITHIN THE FEYNMAN PROPER TIME PARADIGM

As it was reported by F. Dyson [13], the original Feynman approach to derivation of the electromagnetic Maxwell equations was based on *a priori* general form of the classical Newton type force, acting on a charged point particle moving in three-dimensional space  $\mathbb{R}^3$  endowed with the canonical Poisson brackets on the phase variables, defined on the associated tangent space  $T(\mathbb{R}^3)$ . As a result of this approach there was derived only the first part of the Maxwell equations, as the second part, owing to F. Dyson [13], is related with the charged matter nature, which appeared

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to be hidden. Trying to complete this Feynman approach to derivation of the electromagnetic Maxwell equations more systematically we have observed [49] that the original Feynman's calculations, based on the Poisson brackets analysis, were performed on the *tangent space*  $T(\mathbb{R}^3)$  which is, subject to the problem posed, physically not proper, as the true Poisson brackets can be correctly defined only on the *coadjoint phase space*  $T^*(\mathbb{R}^3)$ , as it follows from the classical Lagrangian equations and the related Legendre type transformation [2, 4, 23, 5] from  $T(\mathbb{R}^3)$  to  $T^*(\mathbb{R}^3)$ . Moreover, within this observation, the corresponding dynamical Lorentz type equation for a charged point particle should be written for the particle momentum, not for the particle velocity, whose value is strongly defined only with respect to the proper relativistic reference frame, associated with the charged point particle owing to the fact that the searched for Maxwell equations are Lorentz invariant.

Thus, from the very beginning, we will reanalyze the structure of the Lorentz type force exerted on a moving charged point particle with a charge  $\xi \in \mathbb{R}$  by other point charged particle with a charge  $\xi_f \in \mathbb{R}$ , making use of the classical Lagrangian approach, and rederive the corresponding electromagnetic Maxwell equations. The latter appears to be strongly related with the charged point mass structure of the electromagnetic origin as it was suggested before by R. Feynman and F. Dyson.

Consider a motion of a charged point particle  $\xi \in \mathbb{R}$  (we will further identify a point charged particle with its charge value) located at point  $r \in \mathbb{R}^3$  at time  $t \in \mathbb{R}$  and which is under influence of a smooth vector magnetic potential  $A \in T^*(\mathbb{R}^3)$ , measured with respect to a reference frame. Taking into account the gauge structure of the magnetic potential, it is natural [2, 10, 23, 33, 52] to consider the extended phase space  $\mathcal{M} := \mathbb{R}^3 \times G$  with an abelian gauge group  $\exp \mathcal{G}$ , where  $\mathcal{G}$  is the linear space  $\mathbb{R}$  acting on  $\mathbb{R}^3$  by means of smooth vector fields on  $\mathbb{R}^3$  and next to define the related fiber bundle  $\pi : \mathcal{M} \rightarrow \mathbb{R}^3$  with the structure group  $G$ . Having assumed that the gauge group  $G$  action is naturally lifted on the cotangent space  $T^*(\mathcal{M})$  equivariantly subject to the natural  $G$ -invariant canonical symplectic structure

$$(2.1) \quad \begin{aligned} \omega^{(2)}(r, p; g, y) & : = d p r^* \alpha^{(1)}(r, p; g, y) = \langle dp, \wedge dr \rangle + \\ & + \langle dy, \wedge g^{-1} dg \rangle_{\mathcal{G}} + \langle y dg^{-1}, \wedge dg \rangle_{\mathcal{G}} \end{aligned}$$

for all  $(r, p; g, y) \in T^*(\mathcal{M})$ , where  $\alpha^{(1)}(r, p; g, y) := \langle p, dr \rangle + \langle y, g^{-1} dg \rangle_{\mathcal{G}}$  is the corresponding Liouville form on  $\mathcal{M}$ , one can define on the fibered manifold  $\mathcal{M}$  a related connection differential form

$$(2.2) \quad \sigma^{(1)}(r; g) := g^{-1} \langle \xi A(r), dr \rangle_{\mathcal{G}} + g^{-1} dg,$$

with  $\xi \in \mathcal{G}^*$ ,  $(r, g) \in \mathbb{R}^3 \times G$ ,  $\langle \cdot, \cdot \rangle$  being the scalar product in  $\mathbb{E}^3$ . Its curvature 2-form  $\Sigma^{(2)} \in \Lambda^2(\mathbb{R}^3) \otimes \mathcal{G}$  equals, by definition,

$$(2.3) \quad \Sigma^{(2)}(r) := d\sigma^{(1)}(r; g) + \sigma^{(1)}(r; g) \wedge \sigma^{(1)}(r; g) = \xi \sum_{i,j=1}^3 F_{ij}(r) dr^i \wedge dr^j,$$

where

$$(2.4) \quad F_{ij}(r) := \frac{\partial A_j}{\partial r^i} - \frac{\partial A_i}{\partial r^j}$$

for  $i, j = \overline{1, 3}$  is the so-called electromagnetic field tensor with respect to the reference frame  $\mathcal{K}(t; r)$ .

As an element  $\xi \in \mathcal{G}^*$  is still not fixed, it is natural to apply the standard [2, 4, 5] invariant Marsden-Weinstein-Meyer reduction to the orbit factor space  $\tilde{P}_{\xi} := P_{\xi}/G_{\xi}$  subject to the related momentum mapping  $l : T^*(\mathcal{M}) \rightarrow \mathcal{G}^*$ , constructed with respect to the suitably reduced canonical Poisson bracket

$$(2.5) \quad \begin{aligned} \{r^i, r^j\} & = 0 = \{p_i, p_j\}, \\ \{p_i, r^j\} & = \delta_i^j, \end{aligned}$$

on  $T^*(\mathbb{R}^3)$  for  $i, j = \overline{1, 3}$ , where, by definition,  $\xi \in \mathcal{G}^*$  is constant,  $P_{\xi} := l^{-1}(\xi) \subset T^*(\mathcal{M})$  and  $G_{\xi} = \{g \in G : Ad_{G^*}^* \xi\}$  is the isotropy group of the element  $\xi \in \mathcal{G}^*$ . As a result of this Marsden-Weinstein reduction one obtains that  $G_{\xi} \simeq G$ , the factor-space  $\tilde{P}_{\xi} \simeq T^*(\mathbb{R}^3)$  and the Poisson

brackets (2.5) on  $T^*(\mathbb{R}^3)$  becomes on the reduced manifold  $\tilde{P}_\xi$  as follows:

$$(2.6) \quad \begin{aligned} \{r^i, r^j\}_\xi &= 0, \quad \{p_j, r^i\}_\xi = \delta_j^i, \\ \{p_i, p_j\}_\xi &= \xi F_{ij}(r) \end{aligned}$$

for  $i, j = \overline{1, 3}$ , considered with respect to the reference frame  $\mathcal{K}(t; r)$ . It is now worth to observe that the Poisson brackets (2.6) can be equivalently rewritten as

$$(2.7) \quad \begin{aligned} \{r^i, r^j\}_\xi &= 0 = \{\tilde{\pi}_{p,i}, \tilde{\pi}_{p,j}\}, \\ \{\tilde{\pi}_{p,i}, r^j\} &= \delta_i^j \end{aligned}$$

for  $i, j = \overline{1, 3}$ , if to define a vector of new momentum coordinates

$$(2.8) \quad \tilde{\pi}_p := p + \xi A(r)$$

at any point  $(r, p)$  of the cotangent space  $T^*(\mathbb{R}^3)$ .

Proceed now to a dynamical description of the interaction between two moving charged point particles  $\xi$  and  $\xi_f$ , moving respectively, with the velocities  $u := dr/dt$  and  $u_f := dr_f/df$  with respect to the reference frame  $\mathcal{K}(t; r)$ . To the regret there is a fundamental problem how to write down correctly a physically suitable action functional and to formulate the related least action condition. Namely, there exist there evident possibilities:

$$(2.9) \quad S_p^{(t)} := \int_{t_1}^{t_2} dt \mathcal{L}_p^{(t)}(r; dr/dt)$$

on a temporal interval  $[t_1, t_2] \subset \mathbb{R}$  with respect to the laboratory reference frame  $\mathcal{K}(t; r)$ ,

$$(2.10) \quad S_p^{(t')} := \int_{t'_1}^{t'_2} dt' \mathcal{L}_p^{(t')}(r; dr/dt')$$

on a temporal interval  $[t'_1, t'_2] \subset \mathbb{R}$  with respect to the moving reference frame  $\mathcal{K}(t'; r - r_f)$

$$(2.11) \quad S_p^{(\tau)} := \int_{\tau_1}^{\tau_2} dt \mathcal{L}_p^{(\tau)}(r; dr/d\tau)$$

on a temporal interval  $[\tau_1, \tau_2] \subset \mathbb{R}$  with respect to the proper time reference frame  $\mathcal{K}(\tau; r - r_f)$ , naturally related with the moving charged point particle  $\xi$ .

It was first observed by A. Poincare and by H. Minkowski [44, 20, 21] that the temporal differentials  $dt$  and  $dt'$  are not closed differential one-forms, physically meaning that a particle can traverse many different paths in space  $\mathbb{R}^3$  during any given proper time interval  $d\tau$ , naturally related with its motion. This fact was stressed [15, 16, 40, 44, 46] by A. Einstein, H. Minkowski and A. Poincare, and later deeply analyzed by R. Feynman, who argued [18] that the dynamical equation of a moving point charged particle is physical sensible only with respect to the proper time reference frame of this particle. This Feynman's proper time reference frame paradigm was recently further elaborated and applied both to the electromagnetic Maxwell equations in [20, 21] and to the Lorentz type equation for a moving charged point particle under external electromagnetic field in [49, 10, 6, 48]. As it was there argued from a physical point of view, the least action principle should be applied only to the expression (2.11) written with respect to the proper time reference frame  $\mathcal{K}(\tau; r - r_f)$ , whose temporal parameter  $\tau \in \mathbb{R}$  is independent of an observer and is a closed differential one-form. Owing to these properties this action functional is also mathematically sensible, what on part reflects the A. Poincare and H. Minkowski observation that the infinitesimal quadratic interval

$$(2.12) \quad d\tau^2 = (dt')^2 - |dr - dr_f|^2,$$

relating to each other the reference frames  $\mathcal{K}(t'; r - r_f)$  and  $\mathcal{K}(\tau; r - r_f)$ , can be invariantly used for the four-dimensional relativistic geometry. As the most natural way to contend with this problem is to consider, as first, the quasi-relativistic dynamics of the charged point particle  $\xi$  with respect to the moving reference frame  $\mathcal{K}(t'; r - r_f)$  subject to which the charged point particle  $\xi_f$  is at rest. The latter makes it possible to write down a suitable up to  $O(1/c^2)$  as the light velocity  $c \rightarrow \infty$  action functional (2.10), where the Lagrangian function  $\mathcal{L}_p^{(t')}(r; dr/dt')$  is naturally chosen as

$$(2.13) \quad \mathcal{L}_p^{(t')}(r; dr/dt') := m'(r) |dr/dt' - dr_f/dt'|^2 / 2 - \xi \varphi'(r).$$

with  $m'(r) \in \mathbb{R}_+$ , being the charged particle  $\xi$  mass parameter and  $\varphi'(r)$ , being the potential function generated by the charged particle  $\xi_f$  at a point  $r \in \mathbb{R}^3$ .

Since the standard temporal relationships between reference frames  $\mathcal{K}(t; r)$  and  $\mathcal{K}(t'; r - r_f)$  :

$$(2.14) \quad dt' = dt(1 - |dr_f/dt'|^2)^{1/2},$$

as well as between the reference frames  $\mathcal{K}(t'; r - r_f)$  and  $\mathcal{K}(\tau; r - r_f)$  :

$$(2.15) \quad dt = dt'(1 - |dr/dt' - dr_f/dt'|^2)^{1/2},$$

give rise up to  $O(1/c^2)$  as  $c \rightarrow \infty$ , to  $dt' \simeq dt$  and  $d\tau \simeq dt'$ , respectively, it is easy to obtain that the least action condition  $\delta S_p^{(t')} = 0$  is equivalent to the dynamical equation

$$(2.16) \quad d\pi_p/dt = \nabla \mathcal{L}_p^{(t')}(r; dr/dt) = \nabla m(\frac{1}{2}|dr/dt - dr_f/dt|^2) - \xi \nabla \varphi(r),$$

where we have defined the generalized momentum as

$$(2.17) \quad \pi_p := \frac{\partial \mathcal{L}_p^{(t')}(r; dr/dt)}{\partial (dr/dt)} = m(dr/dt - dr_f/dt),$$

having dropped the dash signs and denoted by "∇" the usual gradient operator in  $\mathbb{E}^3$ . Having equated the momentum expression (2.17) to that of (2.8) one obtains that

$$(2.18) \quad m = -\xi \varphi(r),$$

since there holds the relationship

$$(2.19) \quad \varphi(r) dr_f/dt = A(r).$$

The latter relationship is well known in the classical electromagnetic theory under the Lorentz condition

$$(2.20) \quad \partial \varphi(r)/\partial t + \langle \nabla, A(r) \rangle = 0$$

yet the expression (2.18) is very nontrivial and relates the mass of the charged point particle  $\xi$  with the electric potential, generated by the charged point particle  $\xi_f$ . As it was argued in articles [8, 49, 48], the mentioned above mass phenomenon is closely related with the classical electromagnetic mass problem and reflects from a physical point view its deep relationship with the matter.

Before proceeding to further analyzing the completely relativistic the charge  $\xi$  motion under regard, we preliminarily substitute the mass expression (2.18) into the quasi-relativistic action functional (2.10) with the Lagrangian (2.13). As a result we obtain two possible action functional expressions, taking into account two main temporal parameters choices:

$$(2.21) \quad S_p^{(t')} = - \int_{t'_1}^{t'_2} \xi \varphi'(r) (1 + \frac{1}{2} |dr/dt' - dr_f/dt'|^2) dt'$$

on an interval  $[t'_1, t'_2] \subset \mathbb{R}$ , or

$$(2.22) \quad S_p^{(\tau)} = - \int_{\tau_1}^{\tau_2} \xi \varphi'(r) (1 + \frac{1}{2} |dr/d\tau - dr_f/d\tau|^2) d\tau$$

on an  $[\tau_1, \tau_2] \subset \mathbb{R}$ . It is easy to observe that the first expression (2.16) fails to satisfy upon transforming it to the proper time relativistic representation form the suitable quasi-relativistic limit for the Lagrangian function (2.13). In contrary to that above the direct relativistic generalization of (2.22) follows:

$$(2.23) \quad \begin{aligned} S_p^{(\tau)} &= - \int_{\tau_1}^{\tau_2} \xi \varphi'(r) (1 + \frac{1}{2} |dr/d\tau - dr_f/d\tau|^2) d\tau \simeq \\ &\simeq - \int_{\tau_1}^{\tau_2} \xi \varphi'(r) (1 + |dr/d\tau - dr_f/d\tau|^2)^{1/2} d\tau = \\ &= - \int_{\tau_1}^{\tau_2} \xi \varphi'(r) (1 - \frac{1}{2} |dr/dt' - dr_f/dt'|^2)^{-1/2} d\tau = - \int_{t'_1}^{t'_2} \xi \varphi'(r) dt', \end{aligned}$$

giving rise exactly to the correct, from physical point of view, relativistic action functional form (2.10), suitably transformed before to the proper time reference frame representation (2.11) within

the mentioned already Feynman proper time paradigm. Thus, we have stated that the true action functional procedure consists in physically motivated choosing either the action functional expression form (2.8) or that (2.10), its next transformation to the proper time action functional representation form (2.11) within the Feynman paradigm, and applying to it the standard least action principle.

Concerning the discussed above problem of description the motion of a charged point particle  $\xi$  under the electromagnetic field generated by a moving in space other charged point particle  $\xi_f$  we need to mention that we have chosen the quasi-relativistic functional expression (2.13) in the form (2.10) with respect to the moving reference frame  $\mathcal{K}(t'; r - r_f)$ , because its form has from physical point of view a natural and approved argumentation, taking into account that the charged point particle  $\xi_f$  is then at rest.

Based now on the found above relativistic action functional expression

$$(2.24) \quad S_p^{(\tau)} := - \int_{\tau_1}^{\tau_2} \xi \varphi'(r) (1 + |dr/d\tau - dr_f/d\tau|^2)^{1/2} d\tau$$

written with respect to the proper reference frame  $\mathcal{K}(\tau; r - r_f)$ , one finds the following evolution equation:

$$(2.25) \quad d\pi_p/d\tau = -\xi \nabla \varphi'(r) (1 + |dr/d\tau - dr_f/d\tau|^2)^{1/2},$$

where the generalized momentum is given by the relationship (2.17):

$$(2.26) \quad \pi_p = m(dr/dt - dr_f/dt).$$

Making use of the relativistic transformation (2.14) and the next one (2.15), the equation (2.25) easily transforms to

$$(2.27) \quad \frac{d}{dt}(p + \xi A) = -\nabla \varphi(r) (1 - |u_f|^2),$$

where we took into account the definitions: (2.18) for the charged particle  $\xi$  mass, (2.19) for the magnetic vector potential and  $\varphi(r) = \varphi'(r)/(1 - |u_f|^2)^{1/2}$  for the scalar electric potential with respect to the laboratory reference frame  $\mathcal{K}(t; r)$ . The equation (2.27) can be further transformed, using the elementary vector algebra, to the classical Lorentz type form:

$$(2.28) \quad dp/dt = \xi E + \xi u \times B - \xi \nabla \langle u - u_f, A \rangle,$$

where

$$(2.29) \quad E := -\partial A/\partial t - \nabla \varphi$$

is the related electric field and

$$(2.30) \quad B := \nabla \times A$$

is the related magnetic field, exerted by the moving charged point particle  $\xi_f$  on the charged point particle  $\xi$  with respect to the laboratory reference frame  $\mathcal{K}(t; r)$ . The Lorentz type force equation (2.28) was before obtained in [49, 10], written with respect to the moving reference frame  $\mathcal{K}(t'; r - r_f)$ , and recently reanalyzed in [6, 47], being derived in part from the classical Ampere's reasonings [50, 51] subject to constructing the magnetic force between two neutral conductors with stationary currents.

Concerning the Lorentz type force equation (2.28) it is a natural problem to analyze its form in the case of many external charged point particles  $\xi_j \in \mathbb{R}$ ,  $j \in \mathbb{Z}_+$ , moving with velocities  $dr_j/dt$ ,  $j \in \mathbb{Z}_+$ , with respect to the laboratory reference frame  $\mathcal{K}(t; r)$ . As in this case there is no possibility to choose a common moving reference frame  $\mathcal{K}(t'; r - r_f)$  with respect to which all of the charged particles  $\xi_j$ ,  $j \in \mathbb{Z}_+$ , would be in rest, none the less we are endowed with the unique proper time parameter  $\tau \in \mathbb{R}$ , related to each charged point particle  $\xi_j$ ,  $j \in \mathbb{Z}_+$ , via the infinitesimal relativistic transformation expressions

$$(2.31) \quad dt'_j = d\tau (1 - |dr/dt'_j - dr_f/dt'_j|^2)^{-1/2}$$

to the moving reference frames  $\mathcal{K}(t'_j; r - r_j)$ ,  $j \in \mathbb{Z}_+$ , fixing the  $\tau$ -clock for all charged particles under regard. Thus, making use of the same scheme as demonstrated above, we can now write

down jointly with the superposition principle, the net Lorentz type force expression for the charged point particle  $\xi$  as

$$(2.32) \quad dp/dt = \xi \bar{E} + \xi u \times \bar{B} - \xi \nabla \sum_{j \in \mathbb{Z}_+} \langle u - u_j, A_j \rangle,$$

where, by definition, we put

$$(2.33) \quad \bar{E} := \sum_{j \in \mathbb{Z}_+} E_j, \quad \bar{B} = \sum_{j \in \mathbb{Z}_+} B_j,$$

and  $A_j \in T^*(\mathbb{R}^3)$ ,  $j \in \mathbb{Z}_+$ , are magnetic vector potentials, generated by the set of distant charged point particles  $\xi_j$ ,  $j \in \mathbb{Z}_+$ . If to take into account that this system of external charges is in average neutral, that is  $\sum_{j \in \mathbb{Z}_+} \xi_j \simeq 0$ , and their spatial distribution is, in average, symmetric subject to the charges signs and velocities, one obtains from (2.32) that

$$(2.34) \quad dp/dt = \xi \bar{E} + \xi u \times \bar{B},$$

coinciding exactly with the classical Lorentz type expression for the charged point particle  $\xi$  moving under the influence of external electromagnetic field with respect to the laboratory reference frame  $\mathcal{K}(t; r)$ .

The obtained equation (2.34) can be physically naturally interpreted as the Lorentz type force, exerted by a virtual net charge  $\bar{\xi}$  at rest and located at the common charges system centrum with respect to the laboratory reference frame  $\mathcal{K}(t; r)$ . The latter makes it possible to write down the corresponding effective relativistic invariant action functional expression

$$(2.35) \quad \bar{S}_p^{(t)} := \int_{t_1}^{t_2} dt (m_\xi + \langle \bar{A}, dr/dt \rangle - \xi \bar{\varphi})$$

on an interval  $[t_1, t_2] \subset \mathbb{R}$  with respect to the laboratory reference frame  $\mathcal{K}(t; r)$ . Here we denoted by  $m_\xi \in \mathbb{R}$  a possible internal charged particle mass energy value and as before,  $\bar{\varphi} := \sum_{j \in \mathbb{Z}_+} \varphi_j$ ,  $\bar{A} := \sum_{j \in \mathbb{Z}_+} A_j$ , and took additionally into account the suitable relativistic electric potentials transformations from the moving reference frames  $\mathcal{K}(t'_j; r - r_j)$ ,  $j \in \mathbb{Z}_+$ , to the laboratory reference frame  $\mathcal{K}(t; r)$  with respect to which the averaged set of charges  $\bar{\xi}$  is assumed to be virtually at rest:

$$(2.36) \quad -\varphi'_j dt'_j = \varphi_j dt + \langle A_j, dr \rangle$$

holding for all  $j \in \mathbb{Z}_+$  and giving rise, upon summing over  $j \in \mathbb{Z}_+$ , to the expression

$$(2.37) \quad -\sum_{j \in \mathbb{Z}_+} \varphi'_j dt'_j = -\bar{\varphi} dt + \langle \bar{A}, dr \rangle,$$

used for construction of the action functional (2.35). As the latter is already considered to be written for the averaged set of charges  $\bar{\xi}$ , whose virtual location is now assumed to be at rest, we can apply to this action functional (2.35) the Feynman proper time paradigm and construct the corresponding physically reasonable action functional

$$(2.38) \quad \bar{S}_p^{(\tau)} = \int_{\tau_1}^{\tau_2} d\tau (-\xi \bar{\varphi} + \xi \langle \bar{A}, dr/d\tau \rangle) (1 + |dr/d\tau|^2)^{1/2},$$

defined on an independent time interval  $[\tau_1, \tau_2] \subset \mathbb{R}$  with respect to the proper time reference frame  $\mathcal{K}(t; r)$ , whose time parameter  $\tau \in \mathbb{R}$  is infinitesimally related to the laboratory time parameter  $t \in \mathbb{R}$  as

$$(2.39) \quad d\tau = dt (1 - |dr/dt|^2)^{-1/2}.$$

As a result of the least action principle applied to the functional (2.38) one easily obtains the evolution equation

$$(2.40) \quad \frac{d}{dt} (p + \xi \bar{A}) = -\xi \nabla \bar{\varphi} + \xi \nabla \langle \bar{A}, u \rangle,$$

where, as before, the charged particle  $\xi$  momentum is defined classically as

$$(2.41) \quad p := m dr/dt,$$

and its mass parameter is defined, respectively, as

$$(2.42) \quad m := -\xi \bar{\varphi}(r).$$

As the four-vector potentials  $(\varphi_j, A_j) \in T^*(M^4), j \in \mathbb{Z}_+$ , where  $M^4 := \mathbb{R} \times \mathbb{E}^3$  is the standard Minkowski pseudo-Euclidean metric space, satisfy the Lorentz conditions

$$(2.43) \quad \partial\varphi_j/\partial t + \langle \nabla, A_j \rangle = 0$$

for any  $j \in \mathbb{Z}_+$ , it is evident that the same condition

$$(2.44) \quad \partial \bar{\varphi}/\partial t + \langle \nabla, \bar{A} \rangle = 0$$

holds also for the averaged potentials  $(\bar{\varphi}, \bar{A}) \in T^*(M^4)$ . The further standard calculations, applied to the expression (2.40), bring about exactly the same as (2.34) Lorentz force equation

$$(2.45) \quad dp/dt = \xi \bar{E} + \xi u \times \bar{B},$$

thereby demonstrating the mathematical agreement between two physically different approaches to its derivation, based on the classical averaging procedure and the superposition principle. In particular, the presented above results are in part supported by reasonings from [7], where the averaged nature of the related Maxwell equations was argued taking into account their quantum electrodynamic quantization problems satisfactory solved only for the suitably averaged electromagnetic potentials.

### 3. THE ELECTROMAGNETIC MAXWELL AND LORENTZ FORCE EQUATIONS ANALYSIS

**3.1. The Maxwell equations derivation.** As a moving charged particle  $\xi_f$  generates the suitable electric field (2.29) and magnetic field (2.30) via their electromagnetic potential  $(\varphi, A) \in T^*(M^4)$  with respect a laboratory reference frame  $\mathcal{K}(t; r)$ , we will supplement them naturally by means of the external material equations, describing the relativistic charge conservation law:

$$(3.1) \quad \partial\rho/\partial t + \langle \nabla, J \rangle = 0$$

where  $(\rho, J) \in T^*(M^4)$  is a related four-vector for the charge and current distribution in the space  $\mathbb{R}^3$ . Moreover on can augment the equation (3.1) with the experimentally well established the Gauss law

$$(3.2) \quad \langle \nabla, E \rangle = \rho$$

and to calculate the quantity  $\Delta\varphi := \langle \nabla, \nabla\varphi \rangle$  from the expression (2.29):

$$(3.3) \quad \Delta\varphi = -\frac{\partial}{\partial t} \langle \nabla, A \rangle - \langle \nabla, E \rangle.$$

Having taken into account the relativistic Lorentz condition (2.20) and the expression (3.2) one easily derives that the wave equation

$$(3.4) \quad \partial^2\varphi/\partial t^2 - \Delta\varphi = \rho$$

holds with respect to the laboratory reference frame  $\mathcal{K}(t; r)$ . If to apply the operation rotor-" $\nabla \times$ " to the expression (2.29) we obtain, owing to the expression (2.30) that

$$(3.5) \quad \nabla \times E + \partial B/\partial t = 0,$$

giving rise, together with (3.2), to the first pair of the classical Maxwell equations. To obtain, respectively, the second pair of the Maxwell equations, it is first, necessary to apply the rotation " $\nabla \times$ " to the expression (2.30):

$$(3.6) \quad \nabla \times B = \partial E/\partial t + (\partial^2 A/\partial t^2 - \Delta A)$$

and, second, apply to the wave equation (3.4) the first derivative  $-\partial/\partial t$ :

$$(3.7) \quad \begin{aligned} & -\frac{\partial^2}{\partial t^2} \left( \frac{\partial\varphi}{\partial t} \right) + \langle \nabla, \nabla \frac{\partial\varphi}{\partial t} \rangle = \frac{\partial^2}{\partial t^2} \langle \nabla, A \rangle - \\ & - \langle \nabla, \nabla \langle \nabla, A \rangle \rangle = \langle \nabla, \frac{\partial^2 A}{\partial t^2} - \nabla \times (\nabla \times A) - \Delta A \rangle = \\ & = \langle \nabla, \frac{\partial^2 A}{\partial t^2} - \Delta A \rangle = \langle \nabla, J \rangle. \end{aligned}$$

The result (3.7) strictly entails the relationship

$$(3.8) \quad \partial^2 A/\partial t^2 - \Delta A = J,$$

if to take into account that both the vector potential  $A \in \mathbb{E}^3$  and the vector of current  $J \in \mathbb{E}$  are determined up to a rot-vector expression  $\nabla \times S$  for some smooth vector-function  $S : M^4 \rightarrow \mathbb{E}^3$ .

Having inserted the relationship (3.8) into (3.6), we obtain jointly with (3.5) the second pair of the Maxwell equations:

$$(3.9) \quad \nabla \times B = \partial E / \partial t + J, \quad \nabla \times E = \partial B / \partial t.$$

It is important that the system of equations (3.9) can be represented by means of the least action principle  $\delta S_{f-p}^{(t)} = 0$ , where the action functional

$$(3.10) \quad S_{f-p}^{(t)} := \int_{t_1}^{t_2} dt \mathcal{L}_{f-p}^{(t)}$$

is defined on an interval  $[t_1, t_2] \subset \mathbb{R}$  by the Lagrangian function

$$(3.11) \quad \mathcal{L}_{f-p}^{(t)} = \int_{\mathbb{R}^3} d^3r \left( \frac{|E|^2 - |B|^2}{2} + \langle J, A \rangle - \rho\varphi \right)$$

with respect to the laboratory reference frame  $\mathcal{K}(t; r)$ . From (3.11) we derive that the generalized field momentum

$$(3.12) \quad \pi_f := \partial \mathcal{L}_{f-p}^{(t)} / \partial (\partial A / \partial t) = -E$$

and the related its evolution is given as

$$(3.13) \quad \partial \pi_f / \partial t := \delta \mathcal{L}_{f-p}^{(t)} / \delta A = J - \nabla \times B,$$

being equivalent to the first Maxwell equation of (3.9). As the Maxwell equations allow the least action representation, the latter entails [2, 4, 10, 48] their dual Hamiltonian formulation with the Hamiltonian function

$$(3.14) \quad H_{f-p} := \int_{\mathbb{R}^3} d^3r \langle \pi_f, \partial A / \partial t \rangle - \mathcal{L}_{f-p}^{(t)} = \int_{\mathbb{R}^3} d^3r \left( \frac{|E|^2 + |B|^2}{2} - \langle J, A \rangle \right),$$

satisfying the invariant condition

$$(3.15) \quad dH_{f-p} / dt = 0$$

for all  $t \in \mathbb{R}$ .

It is worthy to note here that the Maxwell equations were derived above under the important condition imposed on the charged system: the charged system  $(\rho, J) \in T(M^4)$  exerts no influence on the ambient electromagnetic field potentials  $(\varphi, A) \in T^*(M^4)$ . As it is not the case owing to the existence the damping radiation reaction, acting on accelerated charged particles, one can try to describe this self-interacting influence by means of the modified least action principle, making use of the Lagrangian expression (3.11) in the case of a separate charged particle  $\xi$ . Following the well known approach from [34] this Lagrangian can be rewritten (in the Gauss units) as

$$(3.16) \quad \begin{aligned} \mathcal{L}_{f-p}^{(t)} &= \int_{\mathbb{R}^3} d^3r \left( \langle -\nabla\varphi - \frac{1}{c}\partial A/\partial t, -\nabla\varphi - \frac{1}{c}\partial A/\partial t \rangle - \langle \nabla \times (\nabla \times A), A \rangle \right) + \\ &\quad + \int_{\mathbb{R}^3} d^3r \left( \frac{1}{c} \langle J, A \rangle - \rho\varphi \right) - \langle k(t), dr/dt \rangle = \\ &= \int_{\mathbb{R}^3} d^3r \left( \frac{1}{2} \langle -\nabla\varphi, E \rangle - \frac{1}{2c} \langle \partial A/\partial t, E \rangle - \frac{1}{2} \langle A, \nabla \times B \rangle + \right. \\ &\quad \left. + \frac{1}{c} \langle J, A \rangle - \rho\varphi \right) + \langle k(t), dr/dt \rangle = \\ &= \int_{\mathbb{R}^3} d^3r \left( \frac{1}{2} \varphi \langle \nabla, E \rangle + \frac{1}{2c} \langle A, \partial E/\partial t \rangle - \frac{1}{2c} \langle A, J + \partial E/\partial t \rangle + \frac{1}{c} \langle J, A \rangle - \rho\varphi \right) - \\ &\quad - \frac{1}{2c} \frac{d}{dt} \int_{\mathbb{R}^3} d^3r \langle A, E \rangle - \frac{1}{2} \lim_{r \rightarrow \infty} \int_{\mathbb{S}_r^2} \langle \varphi E + \langle A \times B, dS_r^2 \rangle - \langle k(t), dr/dt \rangle = \\ &= \frac{1}{2} \int_{\mathbb{R}^3} d^3r \left( \frac{1}{c} \langle J, A \rangle - \rho\varphi \right) - \frac{1}{2c} \frac{d}{dt} \int_{\mathbb{R}^3} d^3r \langle A, E \rangle - \\ &\quad - \frac{1}{2} \lim_{r \rightarrow \infty} \int_{\mathbb{S}_r^2} \langle \varphi E + \langle A \times B, dS_r^2 \rangle - \langle k(t), dr/dt \rangle - , \end{aligned}$$

where we have introduced still not determined both an internal charged particle stability energy impact  $m_\xi c^2$  and a radiation damping force  $k(t) \in \mathbb{E}^3$ , as well as we denoted by  $\mathbb{S}_r^2$  a two-dimensional sphere of radius  $r \rightarrow \infty$ . Having additionally assumed that the radiated charged particle energy at infinity is negligible, the Lagrangian function (3.16) becomes equivalent to

$$(3.17) \quad \mathcal{L}_{f-p}^{(t)} = \frac{1}{2} \int_{\mathbb{R}^3} d^3r \left( \frac{1}{c} \langle J, A \rangle - \rho\varphi \right) - \langle k(t), dr/dt \rangle,$$



which we now need to calculate taking into account that the electromagnetic potentials  $(\varphi, A) \in T^*(M^4)$  are retarded and given as  $1/c \rightarrow 0$  in the following Lienard-Wiechert form:

$$\begin{aligned}
\varphi &= \int_{\mathbb{R}^3} d^3r' \frac{\rho(t', r')}{|r - r'|} \Big|_{t'=t-|r-r'|/c} = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^3} d^3r' \frac{\rho(t - \varepsilon, r')}{|r - r'|} + \\
&+ \frac{1}{2c^2} \int_{\mathbb{R}^3} d^3r' |r - r'| \partial^2 \rho(t, r') / \partial t^2 + \frac{1}{6c^3} \int_{\mathbb{R}^3} d^3r' |r - r'|^2 \partial \rho(t, r') / \partial t + O(1/c^4), \\
(3.18) \quad A &= \frac{1}{c} \int_{\mathbb{R}^3} d^3r' \frac{J(t', r')}{|r - r'|} \Big|_{t'=t-|r-r'|/c} = \lim_{\varepsilon \downarrow 0} \frac{1}{c} \int_{\mathbb{R}^3} d^3r' \frac{J(t - \varepsilon, r')}{|r - r'|} - \\
&- \frac{1}{c^2} \int_{\mathbb{R}^3} d^3r' \partial J(t, r') / \partial t + \frac{1}{2c^3} \int_{\mathbb{R}^3} d^3r' |r - r'| \partial^2 J(t, r') / \partial t^2 + O(1/c^4).
\end{aligned}$$

Here the current density  $J(t, r) = \rho(t, r) dr(t) / dt$  for all  $t \in \mathbb{R}$  and  $r \in \Omega(\xi) := \text{supp } \rho(t; r) \subset \mathbb{R}^3$ , being the compact support of the charged particle density distribution, and the limit as  $\varepsilon \rightarrow +0$  takes into account that the potentials (3.18) are both retarded and singular at the charged particle  $\xi$  center, moving in space with the velocity  $u \in T(\mathbb{R}^3)$  subject to the laboratory reference frame  $\mathcal{K}(t; r)$ . As a result of simple enough calculations like in [29] and the suitable regularization procedure one obtains up to  $O(1/c^4)$  that the electric potential integral, entering (3.17), equals:

$$\begin{aligned}
(3.19) \quad & \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^3} d^3r \rho(t, r) \int_{\mathbb{R}^3} d^3r' \frac{\rho(t - \varepsilon, r')}{|r - r'|} = \\
&= \int_{\mathbb{R}^3} d^3r \rho(t, r) \int_{\mathbb{R}^3} d^3r' \frac{\rho(t, r')}{|r' - r|} - \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^3} d^3r \rho(t, r) \int_{\mathbb{R}^3} d^3r' \frac{\varepsilon \partial \rho(t, r') / \partial t}{|r - r'|} = \\
&= \int_{\mathbb{R}^3} d^3r \rho(t, r) \int_{\mathbb{R}^3} d^3r' \frac{\rho(t, r')}{|r - r'|} + \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^3} d^3r \rho(t, r) \int_{\mathbb{R}^3} d^3r' \frac{\varepsilon \langle \nabla', J(t; r') \rangle}{|r' - r|} = \int_{\mathbb{R}^3} d^3r \rho(t, r) \int_{\mathbb{R}^3} d^3r' \frac{\rho(t, r')}{|r - r'|} \\
&- \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^3} d^3r \rho(t, r) \int_{\mathbb{R}^3} d^3r' \langle \frac{\varepsilon u}{|r' - r|}, \frac{r' - r}{|r' - r|^2} \rangle \rho(t; r') := 2\mathcal{E}_{es} + m_\xi |u|^2,
\end{aligned}$$

where we denoted the averaged as  $\varepsilon \downarrow 0$  limiting integral expression

$$(3.20) \quad \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^3} d^3r \rho(t, r) \int_{\mathbb{R}^3} d^3r' \langle \frac{\varepsilon u}{|r' - r|}, \frac{r' - r}{|r - r'|^2} \rangle \rho(t; r') := m_\xi |u|^2,$$

strongly depending on the internal electron structure, ensuring its stability. The same regularization scheme applied to the expression  $\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^3} d^3r' \partial J(t - \varepsilon, r') / \partial t$  does not change its value.

Thus, making use of the expressions (3.18), (3.19) the Lagrangian function (3.17) brings about

$$(3.21) \quad \mathcal{L}_{f-p}^{(t)} = \frac{\mathcal{E}^{es}}{6c^2} |dr/dt|^2 - \langle k(t), dr/dt \rangle - \mathcal{E}_{es} (1 - |u|^2/c^2) - m_\xi |u|^2/2,$$

where we have denoted by

$$(3.22) \quad \mathcal{E}_{es} := \frac{1}{2} \int_{\mathbb{R}^3} d^3r \int_{\mathbb{R}^3} d^3r' \frac{\rho(t, r') \rho(t, r)}{|r - r'|}$$

the own charged particle  $\xi$  electrostatic energy.

To obtain the corresponding evolution equation for our charged particle  $\xi$  we need, within the Feynman proper time paradigm, to transform the Lagrangian function (3.21) to that with respect to charged particle proper time reference frame  $\mathcal{K}(\tau; r)$ :

$$\begin{aligned}
(3.23) \quad \mathcal{L}_{f-p}^{(\tau)} &= (m_{es}/6) |\dot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} - m_{es} c^2 (1 + |\dot{r}|^2/c^2)^{-1/2} - \\
&- \langle k(t), \dot{r} \rangle - m_\xi |\dot{r}|^2 / 2 (1 + |\dot{r}|^2/c^2)^{-1/2},
\end{aligned}$$

where, for brevity, we have denoted by  $\dot{r} := dr/d\tau$  the charged particle  $\xi$  velocity with respect to the proper reference frame  $\mathcal{K}(\tau; r)$  and by  $m_{es} := \mathcal{E}_{es}/c^2$  the its so called electrostatic mass.

Thus, the generalized charged particle  $\xi$  momentum up to  $O(1/c^4)$  equals

$$(3.24) \quad \begin{aligned} \pi_p &:= \partial \mathcal{L}_{f-p}^{(\tau)} / \partial r = \frac{1}{3} \frac{m_{es} \dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}} - \frac{m_{es} |\dot{r}|^2 \dot{r}}{6c^2(1+|\dot{r}|^2/c^2)^{3/2}} + \frac{m_{es} \dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}} \\ &- k(t) - \frac{m_{\xi} \dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}} + \frac{m_{\xi} |\dot{r}|^2 \dot{r}}{2(1+|\dot{r}|^2/c^2)^{1/2}} \simeq \frac{1}{3} m_{es} u (1 - |u|^2/c^2)^{1/2} + \\ &+ m_{es} u (1 - |u|^2/c^2)^{1/2} - k(t) - m_{\xi} u \simeq (-m_{\xi} + \frac{4}{3} m_{es}) u - k(t), \end{aligned}$$

where we denoted, as before, by  $u := dr/dt$  the charged particle  $\xi$  velocity with respect to the laboratory reference frame  $\mathcal{K}(t; r)$ . The generalized momentum (3.24) satisfies with respect to the proper reference frame  $\mathcal{K}(\tau; r)$  the evolution equation

$$(3.25) \quad d\pi_p/d\tau := \partial \mathcal{L}_{f-p}^{(\tau)} / \partial r = 0,$$

being equivalent with respect to the laboratory reference frame  $\mathcal{K}(t; r)$  to the Lorentz type equation

$$(3.26) \quad \frac{d}{dt} (-m_{\xi} u + \frac{4}{3} m_{es} u) = -dk(t)/dt.$$

The evolution equation (3.25) allows the corresponding canonical Hamiltonian formulation on the phase space  $T^*(\mathbb{R}^3)$  with the Hamiltonian function

$$(3.27) \quad \begin{aligned} H_{f-p} &:= \langle \pi_p, r \rangle - \mathcal{L}_{f-p}^{(\tau)} \simeq \frac{1}{3} \frac{m_{es} \dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}} + \frac{m_{es} \dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}} - \\ &- k(t) - \frac{m_{\xi} \dot{r}}{(1+|\dot{r}|^2/c^2)^{1/2}}, \dot{r} \rangle - (m_{es}/6) |\dot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} + \\ &+ m_{es} c^2 (1 + |\dot{r}|^2/c^2)^{-1/2} + \langle k(t), \dot{r} \rangle + (m_{\xi}/2) |\dot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} = \\ &= \frac{1}{3} m_{es} |\dot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} + m_{es} |\dot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} - \langle k(t), \dot{r} \rangle - \\ &- m_{\xi} |\dot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} - (m_{es}/6) |\dot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} + \\ &+ m_{es} c^2 (1 + |\dot{r}|^2/c^2)^{-1/2} + \langle k(t), \dot{r} \rangle + (m_{\xi}/2) |\dot{r}|^2 (1 + |\dot{r}|^2/c^2)^{-1/2} \\ &\simeq \left[ \frac{(-m_{\xi} + m_{es}/3) |\pi_p + k(t)|^2}{2(-m_{\xi} + 4m_{es}/3)^2} + m_{es} c^2 \right] \left( 1 - \frac{|\pi_p + k(t)|^2}{(-m_{\xi} + 4m_{es}/3)^2 c^2} \right)^{-1/2}, \end{aligned}$$

satisfying for all  $\tau$  and  $t \in \mathbb{R}$  the conservation conditions

$$(3.28) \quad \frac{d}{d\tau} H_{f-p} = 0 = \frac{d}{dt} H_{f-p}.$$

To determine the damping radiation force  $k(t) \in \mathbb{E}^3$ , we can make use of the Lorentz type force expression (2.28) in the proper case  $u = u_f \in T(\mathbb{R}^3)$  and obtain, similarly to [29], up to  $O(1/c^4)$  accuracy, the resulting Abraham-Lorentz force as

$$(3.29) \quad \frac{d}{dt} (-m_{\xi} u + \frac{4}{3} m_{es} u) = \frac{2\xi^2}{3c^3} d^2 u / dt^2.$$

Comparing the force expressions (3.26) and (3.29) one ensues up to  $O(1/c^4)$  accuracy that

$$(3.30) \quad k(t) = \frac{2\xi^2}{3c^3} du/dt,$$

which should be understood as a smooth function of the temporal parameter  $t \in \mathbb{R}$ . Moreover, looking at the equation (3.29) one can define the physical observable charged particle  $\xi$  mass parameter as

$$(3.31) \quad m_{phys} := -m_{\xi} + \frac{4}{3} m_{es}.$$

For the mass parameter  $m_{\xi} \in \mathbb{R}$  in the expression (3.31) to be determined, we need to analyze in details the charged particle  $\xi$  stability condition and try to understand its relationship to the additional momentum production. Before proceeding to this analysis, we will review some important results devoted to the stability problem of a charged particle like electron and try to conceive a related additional momentum generation mechanism.

*Remark 3.1.* Some years ago there was suggested in the work [38] a "solution" to the mentioned before "4/3-electron mass" problem, expressed by the physical mass mass relationship (3.31) and formulated more than one hundred years ago by H. Lorentz and M. Abraham. To the regret, the above mentioned "solution" appeared to be fake that one can easily observe from the main not correct assumptions on which the work [38] has been based: the first one is about the particle-field momentum conservation, taken in the form

$$(3.32) \quad \frac{d}{dt}(p + \xi A) = 0,$$

and the second one is a speculation about the 1/2-coefficient imbedded into the calculation of the Lorentz type self-interaction force

$$(3.33) \quad F := -\frac{1}{2c} \int_{\mathbb{R}^3} d^3r \rho(t; r) \partial A(t; r) / \partial t,$$

being not correctly argued by the reasoning that the expression (3.33) represents "... the interaction of a given element of charge with all other parts, otherwise we count twice that reciprocal action" (cited from [38], page 2710). This claim is fake as there was not taken into account the important fact that the interaction in the integral (3.33) is, in reality, retarded and its impact into it should be considered as that calculated for two virtually different charged particles, as it has been done in the classical works of H. Lorentz and M. Abraham. Subject to the first assumption (3.32) it is enough to recall that a vector of the field momentum  $\xi A \in \mathbb{E}^3$  is not independent and is, within the charged particle model considered, strongly related with the local flow of the electromagnetic energy in the Lorentz constraint form:

$$(3.34) \quad \partial(\xi\varphi)/\partial t + \langle \nabla, c\xi A \rangle = 0,$$

under which there hold the exploited in the work [38] Lienard-Wiechert potentials (3.17) for calculation of the integral (3.33). Thus, the equation (3.32), following the classical Newton second law, should be replaced by

$$(3.35) \quad \frac{d}{dt'}(p + \xi A) = -\nabla(\xi\varphi'),$$

written with respect to the reference frame  $\mathcal{K}(t'; r)$  subject to which the charged particle  $\xi$  is at rest. Taking into account that with respect to the laboratory reference frame  $\mathcal{K}(t; r)$  there hold the relativistic relationships  $dt = dt'(1 - |u|^2/c^2)^{1/2}$  and  $\varphi' = \varphi(1 - |u|^2/c^2)^{1/2}$ , from (3.35) one easily obtains that

$$(3.36) \quad \begin{aligned} \frac{d}{dt}(p + \xi A) &= -\xi \nabla \varphi (1 - |u|^2/c^2) = \\ &= -\xi \nabla \varphi + \frac{\xi}{c} \nabla \langle u, u\varphi/c \rangle = -\xi \nabla \varphi + \frac{\xi}{c} \nabla \langle u, A \rangle. \end{aligned}$$

Here we made use of the well-known relationship  $A = u\varphi/c$  for the vector potential generated by this charged particle  $\xi$  moving in space with the velocity  $u \in T(\mathbb{R}^3)$  with respect to the laboratory reference frame  $\mathcal{K}(t; r)$ . Based now on the equation (3.36) one can derive the final expression for the evolution of the charged particle  $\xi$  momentum:

$$(3.37) \quad \begin{aligned} dp/dt &= -\xi \nabla \varphi - \frac{\xi}{c} dA/dt + \frac{\xi}{c} \nabla \langle u, A \rangle = \\ &= -\xi \nabla \varphi - \frac{\xi}{c} \partial A / \partial t - \frac{\xi}{c} \langle u, \nabla \rangle A + \frac{\xi}{c} \nabla \langle u, A \rangle = \\ &= \xi E + \frac{\xi}{c} u \times (\nabla \times A) = \xi E + \frac{\xi}{c} u \times B, \end{aligned}$$

that is exactly the well known Lorentz force expression, used in the works of H. Lorentz and M. Abraham.

Recently enough there appeared other interesting works devoted to this "4/3-electron mass" problem, amongst which we would like to mention [54, 41], whose argumentations are close to each other and based on the charged shell electron model, within which there is assumed a virtual interaction of the electron with the ambient "dark" radiation energy. The latter was clearly demonstrated in [54], where a suitable compensation of the related singular electrostatic Coulomb electron energy and the wide band vacuum electromagnetic radiation energy fluctuations deficit

inside the electron shell is harmonically realized as the electron shell radius  $a \rightarrow 0$ . Moreover, this compensation happens exactly when the induced outward directed electrostatic Coulomb pressure on the whole electron coincides, up to the sign, with that induced by the mentioned above vacuum electromagnetic energy fluctuations outside the electron shell, since there was manifested their absence inside the electron shell.

Really, the outward directed electrostatic spatial Coulomb pressure on the electron equals

$$(3.38) \quad \eta_{coul} := \lim_{a \rightarrow 0} \frac{\varepsilon_0 |E|^2}{2} \Big|_{r=a} = \lim_{a \rightarrow 0} \frac{\xi^2}{32\varepsilon_0\pi^2 a^4},$$

where  $E = \frac{\xi r}{4\pi\varepsilon_0|r|^3} \in \mathbb{E}^3$  is the electrostatic field at point  $r \in \mathbb{R}$  subject to the electron center at the point  $r = 0 \in \mathbb{R}$ . The related inward directed vacuum electromagnetic fluctuations spatial pressure equals

$$(3.39) \quad \eta_{vac} := \lim_{\Omega \rightarrow \infty} \frac{1}{3} \int_0^\Omega d\mathcal{E}(\omega),$$

where  $d\mathcal{E}(\omega)$  is the electromagnetic energy fluctuations density for a frequency  $\omega \in \mathbb{R}$ , and  $\Omega \in \mathbb{R}$  is the corresponding electromagnetic frequency cutoff. The integral (3.39) can be calculated if to take into account the quantum statistical recipe [17, 28, 9] that

$$(3.40) \quad d\mathcal{E}(\omega) := \hbar\omega \frac{d^3p(\omega)}{h^3},$$

where the Plank constant  $h := 2\pi\hbar$  and the electromagnetic wave momentum  $p(\omega) \in \mathbb{E}^3$  satisfies the relativistic relationship

$$(3.41) \quad |p(\omega)| = \hbar\omega/c.$$

Whence by substituting (3.41) into (3.40) one obtains

$$(3.42) \quad d\mathcal{E}(\omega) = \frac{\hbar\omega^3}{2\pi^2 c^3} d\omega,$$

which entails, owing to (3.39), the following vacuum electromagnetic energy fluctuations spatial pressure

$$(3.43) \quad \eta_{vac} = \lim_{\Omega \rightarrow \infty} \frac{\hbar\Omega^4}{24\pi^2 c^3}.$$

For the charged electron shell model to be stable at rest it is necessary to equate the inward (3.43) and outward (3.38) spatial pressures:

$$(3.44) \quad \lim_{\Omega \rightarrow \infty} \frac{\hbar\Omega^4}{24\pi^2 c^3} = \lim_{a \rightarrow 0} \frac{\xi^2}{32\varepsilon_0\pi^2 a^4},$$

giving rise to the balance electron shell radius  $a_b \rightarrow 0$  limiting condition:

$$(3.45) \quad a_b = \lim_{\Omega \rightarrow \infty} \left[ \Omega^{-1} \left( \frac{3\xi^2 c^2}{2\hbar} \right)^{1/4} \right].$$

Simultaneously we can calculate the corresponding Coulomb and electromagnetic fluctuations energy deficit inside the electron shell:

$$(3.46) \quad \Delta W_b := \frac{1}{2} \int_{a_b}^\infty \varepsilon_0 |E|^2 d^3r - \int_0^{a_b} d^3r \int_0^\Omega d\mathcal{E}(\omega) = \frac{\xi^2}{8\pi\varepsilon_0 a_b} - \frac{\hbar\Omega^4 a_b^3}{6\pi c^3} = 0,$$

additionally ensuring the electron shell model stability.

Another important consequence from this pressure-energy compensation mechanism can be rederived concerning the electron mass component  $m_\xi \in \mathbb{R}$ , entering the momentum expression (3.24) in the case of the electron movement. Namely, following the reasonings from [41], one can observe that during the electron movement there arises an additional hidden velocity  $u \in T(\mathbb{R}^3)$  directed not compensated electrostatic Coulomb surface self-pressure acting only on the *front half part* of the electron shell and equal to

$$(3.47) \quad \eta_{surf} := \frac{|E\xi|}{4\pi a_b^2} \frac{1}{2} = \frac{\xi^2}{32\pi\varepsilon_0 a_b^4},$$

coinciding, evidently, with the already compensated outward directed electrostatic Coulomb spatial pressure (3.38). As, evidently, during the electron motion in space its surface electric current energy flow is not vanishing [41], one ensues that the electron momentum gains an additional mechanical impact, which can be expressed as

$$(3.48) \quad \pi_\xi := -\eta_{surf} \frac{4\pi a_b^3}{3c^2} u = -\frac{1}{3} \frac{\xi^2}{8\pi\epsilon_0 a_b c^2} u = -\frac{1}{3} m_{es} u,$$

where we took into account that within this electron shell model the corresponding electrostatic electron mass equals its electrostatic energy

$$(3.49) \quad m_{es} = \frac{\xi^2}{8\pi\epsilon_0 a_b c^2}.$$

Thus, one can claim that, owing to the structural stability of the electron shell model, its generalized self-interaction momentum  $\pi_p \in T^*(\mathbb{R}^3)$  gains during the movement with velocity  $u = dr/dt \in T(\mathbb{R}^3)$  the additional backward directed hidden impact (3.48), which can be identified with the momentum component

$$(3.50) \quad \pi_\xi = -m_\xi u,$$

entering the momentum expression (3.24). The latter, owing to (3.30), becomes then as

$$(3.51) \quad \begin{aligned} \pi_p &= (-m_\xi + \frac{4}{3}m_{es})u - \frac{2\xi^2}{3c^3}d^2u/dt^2 = \\ &= (-\frac{1}{3}m_{es} + \frac{4}{3}m_{es})u - \frac{2\xi^2}{3c^3}d^2u/dt^2 = \\ &= m_{es}u - \frac{2\xi^2}{3c^3}d^2u/dt^2, \end{aligned}$$

strongly supporting the electromagnetic origin of the electron mass for the first time conceived by H. Lorentz and M. Abraham.

The result above makes it possible to reanalyze the calculation of the Lagrangian function (3.21), based on the averaged limiting integral expression (3.20), taking into account the electron shell model and its dynamical stability. Namely, the averaged limiting integral expression (3.20) can be calculated within the accepted above dynamically stable electron shell model as follows:

$$(3.52) \quad \begin{aligned} &\lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^3} d^3r \rho(t; r) \int_{\mathbb{R}^3} d^3r' < \frac{\epsilon u}{|r'-r|}, \frac{r'-r}{|r'-r|^2} > \rho(t; r') \simeq \\ &\simeq \frac{1}{2} \lim_{\epsilon \downarrow 0} \frac{1}{3} \int_{\mathbb{R}^3} d^3r \rho(t; r) \int_{\mathbb{R}^3} d^3r' < \frac{\epsilon u}{|r'-r|}, \frac{\epsilon u}{\epsilon^2 c^2} > \rho(t; r') = \\ &= \frac{2\xi_{es}}{6c^2} |u|^2 = \frac{1}{3} m_{es} |u|^2 := m_\xi |u|^2, \end{aligned}$$

where we took into account that, owing to the retarded electron self-interaction, only one half the charged electron shell, separated by the distance  $|r' - r| = \epsilon c$ , generates an additional impact into the Lagrangian function (3.21), as the second half is shadowed by the electron shell interior with the absent electric field. Thus, having substituted the found above value  $m_\xi = \frac{1}{3}m_{es}$  into the final electron physical mass expression (3.31), one ensues that

$$(3.53) \quad m_{phys} := -\frac{1}{3}m_{es} + \frac{4}{3}m_{es} = m_{es},$$

additionally supporting the Abraham-Lorentz suggestion about the electromagnetic electron mass origin.

#### 4. CONCLUSION.

In our work the electromagnetic mass origin problem there was reanalyzed in details within the Feynman proper time paradigm and related vacuum field theory approach by means of the fundamental least action principle and the Lagrangian and Hamiltonian formalisms. The resulting electron inertia appeared to coincide in part, in the quasi-relativistic limit, with the momentum expression obtained more than one hundred years ago by M. Abraham and H. Lorentz [1, 35, 36, 37], yet it proved to contain an additional hidden impact owing to the imposed electron stability constraint, which was taken into account in the original action functional as some preliminarily

undetermined constant component. As it was demonstrated in [54, 41], this stability constraint can be successfully realized within the charged shell model of electron at rest, if to take into account the existing ambient electromagnetic "dark" energy fluctuations, whose inward directed spatial pressure on the electron shell is compensated by the related outward directed electrostatic Coulomb spatial pressure as the electron shell radius satisfies some limiting compatibility condition. The latter also allows to compensate simultaneously the corresponding electromagnetic energy fluctuations deficit inside the electron shell, thereby forbidding the external energy to flow into the electron. In contrary to the lack of energy flow inside the electron shell, during the electron movement the corresponding internal momentum flow is not vanishing owing to the nonvanishing hidden electron momentum flow caused by the surface pressure flow and compensated by the suitably generated surface electric current flow. As it was shown, this backward directed hidden momentum flow makes it possible to justify the corresponding self-interaction electron mass expression and to state, within the electron shell model, the fully electromagnetic electron mass origin, as it has been conceived by H. Lorentz and M. Abraham and strongly supported in his Lectures by R. Feynman. This consequence is also independently supported by means of the least action approach, based on the Feynman proper time paradigm and the suitably calculated regularized retarded electric potential impact into the charged particle Lagrangian function.

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