Higher-order expansion for moment of extreme for generalized Maxwell distribution

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Abstract. In this paper, the higher-order asymptotic expansion of the moment of extreme from generalized Maxwell distribution is gained, by which one establishes the rate of convergence of the moment of the normalized partial maximum to the moment of the associate Gumbel extreme value distribution.

Keywords. Expansion; Extreme; Generalized Maxwell distribution; Moment.

1 Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with common distribution function (df) F_k obeying the generalized Maxwell distribution with scale parameter $k > 0$, denoted by $F_k \sim GMD(k)$, and let $M_n = \max\{X_k,\ 1 \leq k \leq n\}$ represent the partial maximum of $\{X_n,\ n \geq 1\}$. The probability density of GMD(k) is given by

$$f_k(x) = \frac{k}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)} x^{2k} \exp\left(-\frac{x^{2k}}{2\sigma^2}\right),\ x > 0,$$

where σ is positive, the scale parameter k is positive and $\Gamma(\cdot)$ represents the Gamma function. As $k = 1$, GMD(k) are reduced to the ordinary Maxwell distribution.

Vadǎ(2009) introduced the generalized Maxwell distribution when he studied a modified Weibull hazard rate. It has plenty of important applications in a large number of areas which include: statistics, physics, statistical mechanics and so on. Some recent examples of applications contain: constructing fractional rheological constitutive equations (Schiessel et al., 1995); be friction model suitable for quick simulation and control (Farid et al., 2005); forecasting the temporal change of opening angle in multiple time scales and electroscalar wave (Zhang et al., 2008; Arbab and Satti, 2009); project of the time related to behavior of viscoelastic materials (Monsia, 2011). The asymptotic properties of this distribution have been investigated in recent literature. For more details, see Liu and Liu (2013) and Huang et al.(2014).

One very important problem in extreme values analysis, the asymptotic expansions of moments of extremes from given distributions have been considered in plenty of literature in past decades. Mocord(1964) and Pickands (1968) investigated the problems on moments convergence of normalized extremes. Withers and Nadarajh(2011) considered expansions for quantiles and multivariate moments of extremes for distributions of Pareto type. Our objective is to establish asymptotic expansion of the moment of normalized maximum from independent and identical GMD(k) random variables, from which we can obtain the convergence rate of the moment of maximum tending to the moment of the corresponding extreme value distribution. For more study, see Hill and Sptuill(1994), Hüsler et al. (2003), Peng et al. (2010) and Liao et al.(2013).

The rest of this paper are organized as follows. Section 2 gives the main result on asymptotic expansions for the moment of partial maxima of the GMD(k) with $k > 0$. Some auxiliary lemmas needed to prove the main result and related proofs are given in Section 3. The proof of the main result is given in section 4. In the sequel we shall assume that the parameter $k > 0$.

2 Main result

In this section, we give the main result. In the sequel, for $r > 0$ let

$$m_r(n) = E\left(\frac{M_n - b_n}{a_n}\right)^r = \int_{-\infty}^{+\infty} x^r \, \mathrm{d}F_k^n(a_n x + b_n)$$

and

$$m_r = EX^r = \int_{-\infty}^{+\infty} x^r \, \mathrm{d}\Lambda(x)$$

respectively represent the rth moments of $(M_n - b_n)/a_n$ and $X \sim \Lambda(x) = \exp(-\exp(-x))$, and the norming constans a_n and b_n are defined by (2.1). The following result shows asymptotic expansion for the moment of GMD(k) extreme.

Theorem 2.1. *Let $\{X_n,\ n \geq 1\}$ be a sequence of iid random variables with common df F_k following the GMD(k). Then,*

$$\begin{aligned}
&b_n^{2k}\left[b_n^{2k}\Big(m_r(n) - m_r\Big) + 2^{-1}k^{-1}\sigma^2 r\Big((2k-1)m_{r+1} - 2m_r\Big)\right] \\
&\to -rk^{-2}\sigma^4\left\{\left[-\frac{1}{8}(2k-1)^2(r+3) + \frac{1}{3}(k-1)(2k-1)\right]m_{r+2} + \frac{1}{2}\Big[(2k-1)(r+2) - 1\Big]m_{r+1}\right. \\
&\quad \left. + \left[2k - \frac{1}{2}(r+1)\right]m_r\right\},
\end{aligned}$$

as $n \to \infty$, where the normalizing constants a_n and b_n are defined by

$$1 - F_k(b_n) = n^{-1}, \ a_n = k^{-1}\sigma^2 b_n^{1-2k}. \tag{2.1}$$

Remark 2.1. *For the case of $k = 1$, i.e., the ordinary Maxwell distribution case, the corresponding result is stated as following:*

$$\begin{aligned}
&b_n^2\left[b_n^2(m_r(n) - m_r) + 2^{-1}r\sigma^2(m_{r+1} - 2m_r)\right] \\
&\to r\sigma^4\left[2^{-1}(r-3)m_r - 2^{-1}(r+1)m_{r+1} + 8^{-1}(r+3)m_{r+2}\right]
\end{aligned}$$

as $n \to \infty$, where the normalizing constants a_n and b_n are determined by

$$1 - F_1(b_n) = n^{-1}, \ a_n = \sigma^2 b_n^{-1}.$$

Corollary 2.1. *Set $\triangle_r(n) = E((M_n - b_n)/a_n)^r - \int_{-\infty}^{+\infty} x^r \,\mathrm{d}\Lambda(x)$. For the moment of normalized partial maximum of GMD(k), we have*

$$\triangle_r(n) \sim -\frac{r\Big((2k-1)m_{r+1} - 2m_r\Big)}{4k\log n}$$

for large n.

3 Auxiliary results and proofs

In order to prove the main result, we need some auxiliary results. Lemma 3.1 follows from Huang et al. (2014).

Lemma 3.1. *Let $F_k(x)$ and $f_k(x)$ respectively represent the cumulative distribution function (cdf) and probability density function (pdf) of GMD(k). For all $x > 0$, we have*

$$\frac{\sigma^2}{k}x^{1-2k} < \frac{1-F_k(x)}{f_k(x)} < \frac{\sigma^2}{k}x^{1-2k}\left(1+\left(\frac{\sigma^2}{k}x^{2k}-1\right)^{-1}\right),$$

where scale parameter $k > \frac{1}{2}$, σ is positive.

Lemma 3.2 follows from Huang and Liu (2014).

Lemma 3.2. *Let $F_k(x)$ represent the cdf of GMD(k). For norming constants a_n and b_n given by (2.1), we have*

$$b_n^{2k}\left[b_n^{2k}\Big(F_k^n(a_n x + b_n) - \Lambda(x)\Big) - l_k(x)\Lambda(x)\right] \to \left(w_k(x) + \frac{l_k^2(x)}{2}\right)\Lambda(x)$$

as $n \to \infty$, where $l_k(x)$ and $w_k(x)$ are respectively given by

$$l_k(x) = \frac{1}{2}k^{-1}\sigma^2\Big[(2k-1)x^2 - 2x\Big]e^{-x}$$

and

$$w_k(x) = -\frac{1}{24}k^{-2}\sigma^4\Big[3(2k-1)^2x^4 - 4(2k+1)(2k-1)x^3 + 24x^2 - 48kx\Big]e^{-x}.$$

Lemma 3.3. *For any constant $0 < d < 1$ and arbitrary nonnegative real numbers i and j, we have*

$$\lim_{n\to\infty}\int_{db_n^{\frac{2}{3}k}}^{\infty} b_n^i x^j \,\mathrm{d}\Lambda(x) = 0 \ \textit{and}\ \lim_{n\to\infty}\int_{db_n^{\frac{2}{3}k}}^{\infty} b_n^i x^j (1-\Lambda(x))\,\mathrm{d}x = 0.$$

Proof. By the fact that $1 - x < e^{-x} < 1$ as $x > 0$, we have

$$\begin{aligned}\int_{db_n^{\frac{2}{3}k}}^{\infty} b_n^i x^j \,\mathrm{d}\Lambda(x) &\le \int_{db_n^{\frac{2}{3}k}}^{\infty} b_n^i x^j e^{-x}\,\mathrm{d}x \\ &\le b_n^i \exp\Big(-\frac{2}{3}db_n^{\frac{2}{3}k}\Big)\int_{db_n^{\frac{2}{3}k}}^{\infty} x^j \exp\Big(-\frac{x}{3}\Big)\,\mathrm{d}x\end{aligned}$$

$$\to 0$$

as $n \to \infty$. Similarly,

$$\int_{db_n^{\frac{2}{3}k}}^{\infty} b_n^i x^j (1-\Lambda(x))\,\mathrm{d}x \le \int_{db_n^{\frac{2}{3}k}}^{\infty} b_n^i x^j e^{-x}\,\mathrm{d}x \to 0$$

as $n \to \infty$. The proof is complete. □

Lemma 3.4. *For any constant $0 < c < 1$ and arbitrary nonnegative real numbers i and j, we have*

$$\lim_{n\to\infty} \int_{-\infty}^{-c\log b_n} b_n^i |x|^j \Lambda(x)\,\mathrm{d}x = 0, \quad \lim_{n\to\infty} \int_{-\infty}^{-c\log b_n} b_n^i |x|^j \,\mathrm{d}\Lambda(x) = 0$$

and

$$\lim_{n\to\infty} \int_{-\infty}^{-c\log b_n} b_n^i |x|^j F_k^n(a_n x + b_n)\,\mathrm{d}x = 0.$$

Proof. Observe that $b_n \to \infty$ as $n \to \infty$ because of $1 - F_k(b_n) = n^{-1}$. For $0 < c < 1$, since $\int_{-\infty}^{-1} |x|^j e^{-x} \exp\left(-\frac{e^{-x}}{3}\right)\mathrm{d}x$ is finite, we have

$$\begin{aligned}\int_{-\infty}^{-c\log b_n} b_n^i |x|^j \Lambda(x)\,\mathrm{d}x &\le b_n^i \exp\left(-\frac{2}{3}b_n^c\right) \int_{-\infty}^{-1} |x|^j \exp\left(-\frac{e^{-x}}{3}\right)\mathrm{d}x \\ &\to 0\end{aligned}$$

as $n \to \infty$ and

$$\begin{aligned}\int_{-\infty}^{-c\log b_n} b_n^i |x|^j \,\mathrm{d}\Lambda(x) &\le b_n^i \exp\left(-\frac{2}{3}b_n^c\right) \int_{-\infty}^{-1} |x|^j e^{-x} \exp\left(-\frac{e^{-x}}{3}\right)\mathrm{d}x \\ &\to 0\end{aligned}$$

as $n \to \infty$.

Observe that $a_n = k^{-1}\sigma^2 b_n^{1-2k}$, so we have $b_n - ca_n \log b_n = b_n - ck^{-1}\sigma^2 b_n^{1-2k}\log b_n \to \infty$ as $n \to \infty$.

For $k > \frac{1}{2}$, we obtain

$$\begin{aligned}&b_n^k F_k^n(b_n - ca_n \log b_n) \\ <&b_n^k \exp\Big(-n(1-F_k(b_n - ca_n\log b_n))\Big) \\ <&b_n^k \exp\left(-\frac{1-ck^{-1}\sigma^2 b_n^{-2k}\log b_n}{1+b_n^{-2k}(k^{-1}\sigma^2 - b_n^{-2k})^{-1}} \exp\left(-\frac{b^{2k}}{2\sigma^2}(1-ck^{-1}\sigma^2 b_n^{-2k}\log b_n)^{2k} + \frac{b^{2k}}{2\sigma^2}\right)\right) \\ <&b_n^k \exp\left(-\frac{1-ck^{-1}\sigma^2 b_n^{-2k}\log b_n}{1+b_n^{-2k}(k^{-1}\sigma^2 - b_n^{-2k})^{-1}} b_n \exp\left(-c^2(2-k^{-1})\sigma^2 b_n^{-2k}(\log b_n)^2\right)\right) \\ \to&0\end{aligned}$$

as $n \to \infty$ by using the inequality $(1-x)^\alpha < 1 - \alpha x + \alpha(\alpha-1)x^2$ for $0 < x < \frac{1}{4}$, $\alpha > 1$ and Lemma 3.1. Thus, we have

$$b_n^i \int_{-\infty}^{-c\log b_n} |x|^j F_k^n(a_n x + b_n)\,\mathrm{d}x$$

$$
\begin{aligned}
&\leq b_n^i a_n^{-j-1} F_k^{n-1}(b_n - c a_n \log b_n) \int_{-\infty}^{b_n - c a_n \log b_n} |y - b_n|^j F_k(y)\,\mathrm{d}y \\
&\leq b_n^i a_n^{-j-1} F_k^{n-1}(b_n - c a_n \log b_n) \int_{-\infty}^{0} |y - b_n|^j F_k(y)\,\mathrm{d}y \\
&\quad + b_n^{i+j+1} a_n^{-j-1} F_k^{n}(b_n - c a_n \log b_n) \int_{0}^{1 - c a_n b_n^{-1} \log b_n} |y - 1|^j\,\mathrm{d}y \\
&= \sum_{s=0}^{j} C_s^j a_n^{-j-1} b_n^{i+s} F_k^{n-1}(b_n - c a_n \log b_n) \int_0^\infty y^{j-s} F_k(-y)\,\mathrm{d}y \\
&\quad + a_n^{-j-1} b_n^{i+j+1} F_k^{n}(b_n - c a_n \log b_n) \int_0^1 (1-y)^j\,\mathrm{d}y \\
&\to 0
\end{aligned}
$$

as $n \to \infty$ since $\int_0^\infty y^r F_k(-y)\,\mathrm{d}y < \infty$ for all $r > 0$. The proof is finished. □

Lemma 3.5. *For any constant $0 < d < 1$ and arbitrary nonnegative real numbers i and j, we have*

$$
\lim_{n\to\infty} b_n^i \int_{d b_n^{\frac{2}{3}k}}^\infty x^j (1 - F_k^n(a_n x + b_n))\,\mathrm{d}x = 0
$$

and

$$
\lim_{x\to\infty} x^i (1 - F_k^n(a_n x + b_n)) = 0. \tag{3.1}
$$

Proof. By Corollary 3.1 in Huang et al. (2014), we have

$$
1 - F_k(x) = c(x) \exp\left(-\int_1^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right)
$$

for $x > 0$, where $c(x) \to c > 0$, $g(x) \to 1$ as $x \to \infty$, and the auxiliary function $f(x) = k^{-1}\sigma^2 x^{1-2k}$ on $(1, \infty)$ is absolutely continuous with $\lim_{x\to\infty} f'(x) = 0$. Recall that $1 - F_k(b_n) = n^{-1}$ and $a_n = f(b_n)$. By arguments similar to Lemma 2.2(a) in Resnick (1987), we have the following inequality

$$
1 - F_k^n(a_n x + b_n) \leq (1+\varepsilon)^2 (1 + \varepsilon x)^{-\varepsilon^{-1}+1}. \tag{3.2}
$$

for $x > 0$, arbitrary $\varepsilon > 0$ and large n. Thus, for $0 < \varepsilon < 2k/(3i + 2kj + 4k)$, by (3.2) we have

$$
\begin{aligned}
0 &\leq b_n^i \int_{d b_n^{\frac{2}{3}k}}^\infty x^j (1 - F_k^n(a_n x + b_n))\,\mathrm{d}x \\
&\leq (1+\varepsilon)^2 (b_n^{-\frac{2}{3}k} + d\varepsilon)^{-\frac{3i}{2k}} \int_{d b_n^{\frac{2}{3}k}}^\infty x^j (1 + \varepsilon x)^{-j-1}\,\mathrm{d}x \\
&\to 0
\end{aligned}
$$

since $\int_1^\infty x^j (1 + \varepsilon x)^{-j-1}\,\mathrm{d}x < \infty$ for all nonnegative real number j.

Again by using (3.2), for $0 < \varepsilon < 1/(i+2)$, we have

$$
0 \leq \limsup_{x\to\infty} x^i (1 - F_k^n(a_n x + b_n)) \leq \lim_{x\to\infty} x^i (1+\varepsilon)^2 (1 + \varepsilon x)^{-i-1} = 0.
$$

The desired result follows. □

Lemma 3.6. *Let $h(n,x) = n\log F_k(a_nx+b_n)+e^{-x}$, where norming constants a_n and b_n are given by (2.1). For sufficiently large n, we have*

$$|h(n,x)| < 3$$

uniformly for $-c\log b_n < x < db_n^{\frac{2}{3}k}$.

Proof. By using partial integrations, we have

$$\begin{aligned} 1-F_k(x) &= k^{-1}\sigma^2 f_k(x)x^{1-2k} + r(x) \\ &= k^{-1}\sigma^2 f_k(x)x^{1-2k}(1+k^{-1}\sigma^2 x^{-2k}) - s(x), \end{aligned} \tag{3.3}$$

for large $x > 0$ and $k > \frac{1}{2}$, where

$$0 < r(x) < k^{-2}\sigma^4 f_k(x)x^{1-4k} \text{ and } s(x) > 0. \tag{3.4}$$

Let $\phi_n(x) = 1 - F_k(a_nx+b_n)$ and

$$n\log F_k(a_nx+b_n) = -n\phi_n(x) - R_n(x),$$

where

$$0 < R_n(x) < \frac{n\phi_n^2(x)}{2(1-\phi_n(x))}$$

by the inequality $-x - \frac{x^2}{2(1-x)} < \log(1-x) < -x$ for $0 < x < 1$. Therefore,

$$|h(n,x)| = |-n\phi_n(x) + e^{-x} - R_n(x)| \le |-n\phi_n(x)+e^{-x}| + R_n(x). \tag{3.5}$$

For large n and $-c\log b_n < x < db_n^{\frac{2}{3}k}$, it is easy to check that

$$\phi_n(x) < \phi_n(-c\log n) = 1 - F_k(b_n - ca_n\log b_n) < c_0 < 1$$

and by combining Lemma 3.1 with the inequality $1+\alpha x \le (1+x)^\alpha$ as $-1<x<1$ for $\alpha > 1$, we have

$$\begin{aligned} 0 < R_n(x) &< \frac{1}{2(1-c_0)}\frac{(1-F_k(a_nx+b_n))^2}{1-F_k(b_n)} \\ &< \frac{b_n(1+k^{-1}\sigma^2 b_n^{-2k}x)^2}{2^{1+\frac{k}{2}}(1-c_0)\sigma^{\frac{1}{k}}\Gamma(1+\frac{k}{2})}\exp\left(-\frac{b_n^{2k}}{2\sigma^2} - 2x\right) \\ &< \frac{b_n^{1+2c}(1+dk^{-1}\sigma^2 b_n^{-\frac{4}{3}k})^2}{2^{1+\frac{k}{2}}(1-c_0)\sigma^{\frac{1}{k}}\Gamma(1+\frac{k}{2})}\exp\left(-\frac{b_n^{2k}}{2\sigma^2}\right) \\ &< 1. \end{aligned} \tag{3.6}$$

For $x \ge 0$, we have

$$|-n\phi_n(x)+e^{-x}| \le n\phi_n(x) + e^{-x} \le n(1-F_k(b_n)) + 1 = 2. \tag{3.7}$$

Hence, $|h(n,x)| < 3$ as $0 \le x < db_n^{\frac{2}{3}k}$ by combining (3.6) and (3.7).

Next, we consider the case of $-c\log b_n < x < 0$. By (3.3) and (3.4), we have

$$\begin{aligned}
-n\phi_n(x)+e^{-x} &= -\frac{1-F_k(a_nx+b_n)}{1-F_k(b_n)}+e^{-x}\\
&= e^{-x}(1+k^{-1}\sigma^2b_n^{-2k}x)\Big((1+k^{-1}\sigma^2b_n^{-2k}x)^{-1}-\frac{1-\delta(a_nx+b_n)}{1-\delta(b_n)}\\
&\quad\times\exp\Big(-\frac{b_n^{2k}}{2\sigma^2}\sum_{i=2}^{\infty}C_{2k}^i(k^{-1}\sigma^2b_n^{-2k}x)^i\Big)\Big)\\
&= e^{-x}(1+k^{-1}\sigma^2b_n^{-2k}x)d_n(x),
\end{aligned}$$

where $\delta(x)=2^{\frac{k}{2}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{k}{2})x^{-1}r(x)\exp(\frac{x^{2k}}{2\sigma^2})$ satisfying $0<\delta(x)<k^{-1}\sigma^2x^{-2k}$ for large $x>0$ and

$$d_n(x)=(1+k^{-1}\sigma^2b_n^{-2k}x)^{-1}-\frac{1-\delta(a_nx+b_n)}{1-\delta(b_n)}\exp\Big(-\frac{b_n^{2k}}{2\sigma^2}\sum_{i=2}^{\infty}C_{2k}^i(k^{-1}\sigma^2b_n^{-2k}x)^i\Big).$$

For $-c\log b_n < x < 0$, set

$$u_n(x)=\sum_{i=2}^{\infty}C_{2k}^i(k^{-1}\sigma^2b_n^{-2k}x)^i,$$

noting that $1+\alpha x\le(1+x)^\alpha<1$ as $-1<x<0$ and $\alpha>1$, we have $u_n(x)>0$. Since $1-x<e^{-x}<1$ for $x>0$, we have

$$\begin{aligned}
d_n(x) &< (1+k^{-1}\sigma^2b_n^{-2k}x)^{-1}-(1-\delta(a_nx+b_n))\Big(1-\frac{b_n^{2k}}{2\sigma^2}u_n(x)\Big)\\
&< -k^{-1}\sigma^2b_n^{-2k}x(1+k^{-1}\sigma^2b_n^{-2k}x)^{-1}+\frac{b_n^{2k}}{2\sigma^2}u_n(x)+k^{-1}\sigma^2(a_nx+b_n)^{-2k}
\end{aligned}$$

and

$$\begin{aligned}
d_n(x) &> (1+k^{-1}\sigma^2b_n^{-2k}x)^{-1}-\frac{1-\delta(a_nx+b_n)}{1-\delta(b_n)}\\
&> (1+k^{-1}\sigma^2b_n^{-2k}x)^{-1}-\frac{1}{1-\delta(b_n)}\\
&> -k^{-1}\sigma^2b_n^{-2k}x(1+k^{-1}\sigma^2b_n^{-2k}x)^{-1}-2k^{-1}\sigma^2b_n^{-2k}.
\end{aligned}$$

Thus, as $-c\log b_n<x<0$ we have

$$|d_n(x)|<2k^{-1}\sigma^2b_n^{-2k}|x|(1+k^{-1}\sigma^2b_n^{-2k}x)^{-1}+\Big|\frac{b_n^{2k}}{2\sigma^2}u_n(x)\Big|+3k^{-1}\sigma^2(a_nx+b_n)^{-2k}$$

for large n. It is easy to check that for large n

$$(a_nx+b_n)^{-2k}\le b_n^{-2k}(1-ck^{-1}\sigma^2b_n^{-2k}\log b_n)^{-2k}$$

and

$$\Big|\frac{b_n^{2k}}{2\sigma^2}u_n(x)\Big|\le c^2(1-2^{-1}k^{-1})\sigma^2b_n^{-2k}(\log b_n)^2$$

hold uniformly for all $-c\log b_n < x < 0$, so

$$|d_n(x)| < c_1 b_n^{-2k}(\log b_n)^2$$

with c_1 being a positive constant.

Thus, for large enough n,

$$\begin{aligned}|-n\phi_n(x)+e^{-x}| &= e^{-x}(1+k^{-1}\sigma^2 b_n^{-2k}x)|d_n(x)| \\ &< c_1(1+ck^{-1}\sigma^2 b_n^{-2k}\log b_n)b_n^{c-2k}(\log b_n)^2 \\ &< 2 \end{aligned} \tag{3.8}$$

uniformly for $-c\log b_n < x < 0$. By (3.6) and (3.8), we have $|h(n,x)| < 3$ uniformly for $-c\log b_n < x < 0$. The proof is complete. □

Lemma 3.7. *For large n and all $-c\log b_n < x < db_n^{\frac{2}{3}k}$,*

$$x^r b_n^{2k}[b_n^{2k}(F_k^n(a_nx+b_n)-\Lambda(x)) - l_k(x)\Lambda(x)]$$

is bounded by integrable functions independent of n, with $r > 0$, $0 < c < 1$ and $0 < d < 1$.

Proof. Utilizing Lemma 3.6, for large n we have

$$\begin{aligned}&b_n^{2k}[b_n^{2k}(F_k^n(a_nx+b_n)-\Lambda(x)) - l_k(x)\Lambda(x)] \\ <&b_n^{2k}[b_n^{2k}h(n,x)-l_k(x)]\Lambda(x) + b_n^{4k}h^2(n,x)[2^{-1}+\exp(|h(n,x)|)]\Lambda(x) \\ <&b_n^{2k}[b_n^{2k}h(n,x)-l_k(x)]\Lambda(x) + b_n^{4k}h^2(n,x)[2^{-1}+e^3]\Lambda(x),\end{aligned}$$

where $h(n,x) = n\log F_k(a_nx+b_n)+e^{-x}$.

The work below we will give that $|b_n^{2k}(b_n^{2k}h(n,x) - l_k(x))|$ and $|b_n^{2k}h(n,x)|$ are bounded by $p(x)e^{-x}$, where $p(x)$ is a polynomial on x. In this we only prove the former because the arguments of the two cases are similar. Rewrite

$$b_n^{2k}(b_n^{2k}h(n,x)-l_k(x)) = b_n^{4k}(-n\phi_n(x)+e^{-x}-b_n^{-2k}l_k(x)) - b_n^{4k}R_n(x). \tag{3.9}$$

By (3.6), for large n and $-c\log b_n < x < db_n^{\frac{2}{3}k}$ we have

$$\begin{aligned}b_n^{4k}R_n(x) &< \frac{b_n^{1+4k}(1+k^{-1}\sigma^2 b_n^{-2k}x)^2}{2^{1+\frac{k}{2}}(1-c_0)\sigma^{\frac{1}{k}}\Gamma(1+\frac{k}{2})}e^{-x}\exp\left(-\frac{b_n^2}{2\sigma^2}+c\log b_n\right) \\ &< \frac{b_n^{1+4k+c}(1+dk^{-1}\sigma^2 b_n^{-\frac{4}{3}k})^2}{2^{1+\frac{k}{2}}(1-c_0)\sigma^{\frac{1}{k}}\Gamma(1+\frac{k}{2})}e^{-x}\exp\left(-\frac{b_n^2}{2\sigma^2}\right) \\ &< e^{-x}.\end{aligned} \tag{3.10}$$

For large n, easily check that $a_nx+b_n > 0$ for $-c\log b_n < x < db_n^{\frac{2}{3}k}$. By Lemma 3.1 and applying the inequality $1+\alpha x \le (1+x)^\alpha$ as $-1 < x < 1$ and $\alpha > 1$, for $-c\log b_n < x < db_n^{\frac{2}{3}k}$ we have

$$\frac{1-F_k(a_nx+b_n)}{1-F_k(b_n)} < \frac{\frac{\sigma^2}{k}(a_nx+b_n)^{1-2k}(1+(\frac{\sigma^2}{k}(a_nx+b_n)^{2k}-1)^{-1})f_k(a_nx+b_n)}{\frac{\sigma^2}{k}b_n^{1-2k}f_k(b_n)}$$

$$
\begin{aligned}
=&(1+k^{-1}\sigma^2 b_n^{-2k}x)(1+(k^{-1}\sigma^2 b_n^{2k}(1+k^{-1}\sigma^2 b_n^{-2k}x)^{2k}-1)^{-1})\\
&\times\exp\bigg(-\frac{b_n^2}{2\sigma^2}((1+k^{-1}\sigma^2 b_n^{-2k}x)^{2k}-1)\bigg)\\
<&2e^{-x} \qquad (3.11)
\end{aligned}
$$

for large n. By Lemma 3.2 in Huang and Liu (2014), we have

$$
\frac{1-F_k(b_n)}{1-F_k(a_nx+b_n)}e^{-x}=B_k(n,x)\exp\bigg(\int_0^x\bigg(\frac{ka_n(a_nt+b_n)^{2k-1}}{\sigma^2}-\frac{a_n}{a_nt+b_n}-1\bigg)\,\mathrm{d}t\bigg),
$$

where

$$
B_k(n,x)=\frac{1+k^{-1}\sigma^2 b_n^{-2k}+k^{-2}(1-2k)\sigma^4 b_n^{-4k}+O(b_n^{-6k})}{1+k^{-1}\sigma^2(a_nx+b_n)^{-2k}+k^{-2}(1-2k)\sigma^4(a_nx+b_n)^{-4k}+O(b_n^{-6k})}
$$

with $\lim_{n\to\infty}B_k(n,x)=1$ uniformly for all $-c\log b_n<x<db_n^{\frac{2}{3}k}$. Rewrite

$$
\begin{aligned}
&b_n^{4k}[-n\phi_n(x)+e^{-x}-b_n^{-2k}l_k(x)]\\
=&\frac{1-F_k(a_nx+b_n)}{1-F_k(b_n)}b_n^{4k}\bigg[-1+\frac{1-F_k(b_n)}{1-F_k(a_nx+b_n)}e^{-x}\bigg(1-\frac{1}{2}k^{-1}\sigma^2((2k-1)x^2-2x)b_n^{-2k}\bigg)\bigg]\\
=&\frac{1-F_k(a_nx+b_n)}{1-F_k(b_n)}[A_k(n,x)+D_k(n,x)-E_k(n,x)+G_k(n,x)], \qquad (3.12)
\end{aligned}
$$

where

$$
\begin{aligned}
A_k(n,x)&=b_n^{4k}(B_k(n,x)-1),\\
D_k(n,x)&=b_n^{4k}B_k(n,x)\bigg(\int_0^x\bigg(\frac{ka_n(a_nt+b_n)^{2k-1}}{\sigma^2}-\frac{a_n}{a_nt+b_n}-1\bigg)\,\mathrm{d}t-\frac{1}{2}k^{-1}\sigma^2((2k-1)x^2-2x)b_n^{-2k}\bigg),\\
E_k(n,x)&=\frac{1}{2}b_n^{2k}B_k(n,x)k^{-1}\sigma^2((2k-1)x^2-2x)\int_0^x\bigg(\frac{ka_n(a_nt+b_n)^{2k-1}}{\sigma^2}-\frac{a_n}{a_nt+b_n}-1\bigg)\,\mathrm{d}t\\
G_k(n,x)&=b_n^{4k}B_k(n,x)\bigg(1-2^{-1}k^{-1}\sigma^2((2k-1)x^2-2x)b_n^{-2k}\bigg)\\
&\qquad\times\sum_{i=2}^{\infty}\frac{\bigg(\int_0^x\bigg(\frac{ka_n(a_nt+b_n)^{2k-1}}{\sigma^2}-\frac{a_n}{a_nt+b_n}-1\bigg)\,\mathrm{d}t\bigg)^i}{i!}.
\end{aligned}
$$

First of all, we consider the bound of $A_k(n,x)$. Noting that $1-\alpha x<(1+x)^{-\alpha}<1$ as $x>0$ and $\alpha>0$, for the case of $0\le x<db_n^{\frac{2}{3}k}$ we have

$$
\begin{aligned}
&|A_k(n,x)|\\
<&(1-k^2(2k-1)\sigma^4(a_nx+b_n)^{-4k})^{-1}b_n^{4k}\\
&\times|k^{-1}\sigma^2 b_n^{-2k}(1-(1+k^{-1}\sigma^2 b_n^{-2k}x)^{-2k})+k^{-2}(1-2k)\sigma^4 b_n^{-4k}(1-(1+k^{-1}\sigma^2 b_n^{-2k}x)^{-4k})+O(b_n^{-6k})|\\
<&(1-k^2(2k-1)\sigma^4 b_n^{-4k})^{-1}[2k^{-1}\sigma^4 x-4k^{-2}(1-2k)\sigma^6 b_n^{-2k}x]\\
<&4k^{-2}\sigma^4(k+2(2k-1)\sigma^2)x. \qquad (3.13)
\end{aligned}
$$

Next consider the case of $-c\log b_n<x<0$. For large n, we have

$$
|A_k(n,x)|
$$

$$
\begin{aligned}
&<(1-k^2(2k-1)\sigma^4(b_n-ca_n\log b_n)^{-4k})^{-1}b_n^{4k}\\
&\quad\times|k^{-1}\sigma^2 b_n^{-2k}(1-ck^{-1}\sigma^2 b_n^{-2k}\log b_n)^{-2k}((1+k^{-1}\sigma^2 b_n^{-2k}x)^{2k}-1)\\
&+k^{-2}(1-2k)\sigma^4 b_n^{-4k}(1-ck^{-1}\sigma^2 b_n^{-2k}\log b_n)^{-4k}((1+k^{-2}\sigma^2 b_n^{-2k}x)^{4k}-1)+O(b_n^{-6k})|\\
&<4k^{-2}\sigma^4(k+2(2k-1)\sigma^2)|x| \qquad (3.14)
\end{aligned}
$$

Since $1+\alpha x<(1+x)^\alpha<1$ as $-1<x<0$ and $\alpha>1$. Similarly, for the bounds of $D_k(n,x)$, $E_k(n,x)$ and $G_k(n,x)$, we have

$$
|D_k(n,x)|<2\left(\frac{1}{3}|1-k^{-1}|(2-k^{-1})\sigma^4|x|^3+\frac{1}{2}k^{-1}\sigma^2|k\sigma^{-2}-c|^{-1}x^2\right), \qquad (3.15)
$$

$$
\begin{aligned}
|E_k(n,x)|<k^{-1}\sigma^2\left((2k-1)x^2+2|x|\right)\Bigg(&|k^{-1}\sigma^2-c|^{-1}|x|+\frac{1}{2}(2-k^{-1})\sigma^2x^2\\
&+\frac{1}{3}|1-k^{-1}|(2-k^{-1})\sigma^4|x|^3\Bigg), \qquad (3.16)
\end{aligned}
$$

and

$$
\begin{aligned}
|G_k(n,x)|<&\left(1+\frac{1}{2}k^{-1}\sigma^2\left((2k-1)x^2+2|x|\right)\right)\Bigg(|k^{-1}\sigma^2-c|^{-1}|x|+\frac{1}{2}(2-k^{-1})\sigma^2x^2\\
&+\frac{1}{3}|1-k^{-1}|(2-k^{-1})\sigma^4|x|^3\Bigg)^2\exp\left(c|k^{-1}\sigma^2-c|^{-1}+\frac{1}{2}c^2(2-k^{-1})\sigma^2\right) \qquad (3.17)
\end{aligned}
$$

for $-c\log b_n<x<db_n^{\frac{2}{3}k}$ and large n. Thus, we complete the proof of the lemma by combining (3.9)-(3.17) together. □

4 Proof of main result

Note the fact that $\int_{-\infty}^0|x|^rf_k(x)\,\mathrm{d}x$ is finite for all integers $r>0$ and by Proposition 2.1(iii) in Resnick (1987), we have

$$
\lim_{n\to\infty}m_r(n)=\lim_{n\to\infty}E\left(\frac{M_n-b_n}{a_n}\right)^r=m_r=\int_{-\infty}^{+\infty}x^r\,\mathrm{d}\Lambda(x)=(-1)^r\Gamma^{(r)}(1),
$$

where $\Gamma^{(r)}(1)$ denotes the rth derivative of the Gamma function at $x=1$. Thus, $m_r(n)<\infty$ for large n and

$$
\begin{aligned}
m_r(n)-m_r&=\int_{-\infty}^{+\infty}x^r\left(F_k^n(a_nx+b_n)-\Lambda(x)\right)'\,\mathrm{d}x\\
&=\int_{-\infty}^{+\infty}x^r\,\mathrm{d}(F_k^n(a_nx+b_n)-\Lambda(x)).
\end{aligned}
$$

Observing that $\int_{-\infty}^0|x|^rf_k(x)\,\mathrm{d}x<\infty$, we have $\lim_{x\to-\infty}|x|^rF_k(x)=0$, and utilizing the C_r-inequality, implies

$$
0\le\limsup_{x\to-\infty}|x|^rF_k^n(a_nx+b_n)\le\lim_{y\to-\infty}\frac{2^{r-1}(|y|^r+|b_n|^r)}{a_n^r}F_k^n(y)=0,
$$

which induces

$$\lim_{x\to-\infty} x^r F_k^n(a_nx+b_n)=0. \tag{4.1}$$

Therefore, by (3.1) and (4.1), we have

$$\lim_{x\to+\infty} x^r(F_k^n(a_nx+b_n)-\Lambda(x))=\lim_{x\to+\infty} x^r(1-\Lambda(x))-\lim_{x\to+\infty} x^r(1-F_k^n(a_nx+b_n))=0$$

and

$$\lim_{x\to-\infty} x^r(F_k^n(a_nx+b_n)-\Lambda(x))=\lim_{x\to-\infty} x^r F_k^n(a_nx+b_n)-\lim_{x\to-\infty} x^r\Lambda(x)=0.$$

Hence, by using partial integrations, we have

$$m_r(n)-m_r=-r\int_{-\infty}^{+\infty} x^{r-1}(F_k^n(a_nx+b_n)-\Lambda(x))\,\mathrm{d}x \tag{4.2}$$

and

$$\int_{-\infty}^{+\infty} x^{r+1}e^{-2x}\Lambda(x)\,\mathrm{d}x=-(r+1)m_r+m_{r+1}. \tag{4.3}$$

By combining (4.2) and (4.3), with Lemma 3.2 – 3.7 and the dominated convergence theorem, we have

$$\begin{aligned}
& b_n^{2k}\left[b_n^{2k}(m_r(n)-m_r)+2^{-1}k^{-1}\sigma^2 r\Big((2k-1)m_{r+1}-2m_r\Big)\right]\\
=& -r\int_{-\infty}^{+\infty} b_n^{2k}\left[b_n^{2k}x^{r-1}(F_k^n(a_nx+b_n)-\Lambda(x))-x^{r-1}l_k(x)\Lambda(x)\right]\mathrm{d}x\\
=& -r\int_{db_n^{\frac{2}{3}k}}^{+\infty} b_n^{2k}\left[b_n^{2k}x^{r-1}(F_k^n(a_nx+b_n)-\Lambda(x))-x^{r-1}l_k(x)\Lambda(x)\right]\mathrm{d}x\\
& -r\int_{-c\log b_n}^{db_n^{\frac{2}{3}k}} b_n^{2k}\left[b_n^{2k}x^{r-1}(F_k^n(a_nx+b_n)-\Lambda(x))-x^{r-1}l_k(x)\Lambda(x)\right]\mathrm{d}x\\
& -r\int_{-\infty}^{-c\log b_n} b_n^{2k}\left[b_n^{2k}x^{r-1}(F_k^n(a_nx+b_n)-\Lambda(x))-x^{r-1}l_k(x)\Lambda(x)\right]\mathrm{d}x\\
\to & -r\int_{-\infty}^{+\infty}\left(w_k(x)+\frac{1}{2}l_k^2(x)\right)x^{r-1}\Lambda(x)\,\mathrm{d}x\\
=& -rk^{-2}\sigma^4\Bigg\{\left[-\frac{1}{8}(2k-1)^2(r+3)+\frac{1}{3}(k-1)(2k-1)\right]m_{r+2}+\frac{1}{2}\Big[(2k-1)(r+2)-1\Big]m_{r+1}\\
& +\left[2k-\frac{1}{2}(r+1)\right]m_r\Bigg\}
\end{aligned}$$

as $n\to\infty$.

We obtain the desired result.

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