A new mathematical duality relating Jacobi Identity and Lie algebra

Syed Afsar Abbas
Jafar Sadiq Research Institute
AzimGreenHome, NewSirSyed Nagar, Aligarh - 202002, India
(e-mail : drafsarabbas@gmail.com)

Abstract

We obtain a new mathematical duality relating the Jacobi Identity and the Lie Algebra. The duality is between two independent but simultaneously existing mathematical structures related to the fundamental relationship between the Lie algebra of a Lie group and the corresponding Jacobi Identity. We show that this new mathematical duality has a physical counterpart within the Eightfold-way model and the SU(3) Lie group.

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A Lie algebra is defined as a set of symbols which may be elements, or opertors or generators X_μ with a combination rule as

$$[X_\lambda, X_\mu] = i f_{\lambda\mu\nu} X_\nu \tag{1}$$

where X's are members of the set and the product [,] is called a commutator [1]. The f's on the right-hand side are called the structure constants of the algebra and these determine the structure of the algebra. Important to note is that these structure constants are independent of the X's. The commutator $[X_\lambda, X_\mu] = X_\lambda X_\mu - X_\mu, X_\lambda$ is antisymmetric $[X_\lambda, X_\mu] = -[X_\mu, X_\lambda]$. In addition these should also satisfy another independent condition, the so called Jacobi Identity

$$[X_\lambda, [X_\mu, X_\nu]] + [X_\mu, [X_\nu, X_\lambda]] + [X_\nu, [X_\lambda, X_\mu]] = 0 \tag{2}$$

Note that this Jacobi Identity is defined in terms of the Lie brackets of the Lie generators, that is entirely in terms of mathematical quantities which occur on the left-hand side of the definition of Lie algebra above.

In terms of the structure constants (that is terms of what occurs entirely on the right-hand side in the definition of the Lie algebra) the above Jacobi Identity becomes

$$f_{\lambda\mu\sigma} f_{\sigma\nu\tau} + f_{\mu\nu\sigma} f_{\sigma\lambda\tau} + f_{\nu\lambda\sigma} f_{\sigma\mu\tau} = 0 \tag{3}$$

The Lie algebra of su(n) is defined in terms of $(n^2 - 1)$ infinitesimal generators X_μ derived from the corresponding SU(n) Lie group and which satisfy the algebra (1) above. Due to the anticommuting property of the commutator, here the Jacobi Identity is satisfied automatically and hence trivially. What this means is that the Jacobi identity is satisfied here entirely by the left-hand side structure in the definition of the Lie algebra. The triviality of satsifying of the Jacobi Identity is based on the fact that mathematically what goes on in the left-hand sie of the definition of the Lie algebra also goes into the Jacobi Identity eqn. (2). Note that this Lie algebra provides us with a defining (or fundamental) representation. This fundamental representation is used to construct other higher dimentional representations (including the adjoint representation).

Next, an indpendent representation of basic significance is the adjoint representation which is provided by the structure constants themselves. Define

$$[F_\lambda]_{\mu\nu} = if_{\lambda\mu\nu} \tag{4}$$

Starting with the Jacobi Identity in terms of the structure functions in eqn. (3) and using the structure function as in eqn. (4), we see immediately that

$$[F_\lambda, F_\mu] = if_{\lambda\mu\nu}F_\nu \tag{5}$$

This algebra is similar and hence isomorphic to the Lie algebra of the infinitesimal generators of the SU(n) given in eqn. (1) above. But note that the above lie algebra given in eqn. (5) is a different algebra and completely independnt of it [1,2,3,4]. We may say that the Jacobi Identity, in terms of the structure constants eqn. (3), **is actually an independent Lie algebra in disguise, giving the adjoint representation**.

What is important is that this different and new algebra is determined (and actually defined as we saw above) by the Jacobi Identity (3) above and which is determined entirely by mathemaical terms arisisng from the right-hand side of the original definition eqn. (1). So Jacobi Identity is the defining and non-trivial condition for the adjoint representation to be a Lie algebra, and thereafter satisfying eqn. (1) trivially.

Note that the equality sign in eqn. (1) express "sameness" of the mathematical expressions on the two sides of the equal sign. While these two sides satisfying an additional condition of the Jacobi Identity, do so differently in terms of eqn. (2) and (3). This points to the equal sign representing an inherent "duality" rather than a simple "sameness".

Let us further ask as to whether the definition of the Lie bracket is more basic than the Jacobi Identity, or the other way round ? Then the answer is that in the first case the Lie bracket is more basic and the Jacobi identity is trivially satisfied and thus provides no further informationa as to to the inherent mathematical structure. This provides us with the defining or the fundamental representation. Next, in the second case, the Jacobi Identity is more basic and the defining expression and thereaftre the Lie bracket is trivially satisfied. This holds only for the adjoint representation

So, there are two independent and coexisting algebras for the adjoint representation. The second one treats the adjoint representation as sacred and independent. While the other one, starting with the defining or the fundamental represenation, reproduces the same adjoint represenation, but in a mathematically different manner. Note that this means that, as to the adjont representaion of a Lie algebra, there are two independent mathematical desciptions.

Is there any mnifestation of this new mathemaytical duality in physics? We seek it in terms of the hadron physics as specified by the su(3) Lie algebra. We thus have re-look at the Eightfold-way model and the su(3) models in hadron phyiscs.

The origin of the Eightfold Way model in 1961 was the realization that there was a systematic parallelism between the $1/2^+$ baryons and the 0^- mesons when one supplements the isospin number with a new quantum number called the hypercharge Y [2]. This is indicated in the following table:

Table

Parallel structure of $1/2^+$ baryons and 0^- mesons

Y	T	$1/2^+$ baryons	0^- mesons
+1	1/2	p,n	K^+ , K^0
0	1	$\Sigma^+, \Sigma^0, \Sigma^-$	π^+, π^0, π^0
0	0	Λ^0	η^0
-1	1/2	Ξ^0, Ξ^-	K^0, K^-

Assigning the baryons and mesons to the octet representation was called the Eightfold-way model [1,2]. Note important and basic points about the Eightfold-way model. First, in order to associate the additive quantum numbers of SU(3) with hypercharge and the U-spin, we must choose T_3 and Y defined such that there is **no baryon number** appearing in the formula for Y [2, p. 277; 3]. Thus there is no baryon number in the Eightfold-way model. Only hypercharge, which is elementary and non-composite, arises here in this model. But clearly in the baryon octet, built from the product 3 x 3 x 3 in SU(3), has a definite baryon number 1. This is due to the fact that in SU(3) the hypercahrge is defined as Y=B+S and which does define a baryon number explicitly.

How come the hypercharge number Y, in the Eightfold-way model and the SU(3) models, are so fundamentally different? Clearly these eigenvalues can not be of the same operator. But how can that be, as in SU(3) there is but one more diagonal generator, besides the isospin T_3? This threw up a puzzle. Now what is the common and popular opinion at present on this issue?

After introducing the Eightfold-way model and noting that there is no baryon number in the corresponding baryon octet, Costa and Fogli in the year 2012 say [1, p.144], "A complete description of the meson and baryon octets **P** and **B** should require the introduction of the baryon number which differentiates the two octets. Then one should go from SU(3) to U(3)". Another popular way: if we have the Lie algebra $u(3) = su(3) \oplus A_0$ where A_0 is a one dimensional Lie algebra representing the baryon number transormation. Then there are only five possibilities [4, p. 27] for the corresponding connected Lie group associated with the above Lie algebra, and which are:

(a) $SU(3) \otimes R$. (b) $\frac{SU(3)}{Z_3} \otimes R$. (c) $SU(3) \otimes T$. (d) $\frac{SU(3)}{Z_3} \otimes T$. (e) U(3).

In Gourdin's notation, the irreducible representation of R are characterized by a real number r which is an integer for the representations of T. If we demand that the electric charge and the baryon number be an integer numbers for all the irreducible representations, then the possibilities (a), (b) and (c) are excluded and then only options of (d) and (e) as being physicially valid are left [7]. Then the Eightfold-Way model octet baryon is explained by the option (d) with an external baryon number arising from the group T. The group U(3) in (e) was used as above [1].

Note that in these standard way of understanding the role of the baryon number in the Eightfold-way model in (d) and (e) above, the baryon number arises fron outside the group SU(3). But we know that in the SU(3) model, the second diagonal generator of SU(3) defines hypercharge Y as Y= B + S ; so it is a composite of the baryon number B and the strangeness S [1,2,3,4]. Thus in SU(3) the baryon number is arising from within SU(3) itself. Therefore clearly the above canonical models are going beyond the group SU(3) to accomodate a baryon number for the baryons arisisng from the Eightfold-way model, and thus are unable to give a consistent description of the octet baryons in the Eightfold-way model and the SU(3) models.

The above also hints at what may be basically wrong. In the conventional way, to be able to define a baryon number, one has to go outside the group SU(3) and this is in conflict wth the SU(3) itself, as there, the baryon number aries internally. But what particular empirical fact forces us to introduce a baryon number to the Eightfold-way model? All that this model in itself is pointing out, is a paralelleism between the octet baryons and the mesons, as per the above Table. And this does need a new hypercharge, but only as defined in the Table, and with no baryon number required. However it so happens that these baryon and meson octets also arise in the SU(3) model, but with the additional provision that there be a baryon number. Thus there is only a duality as to the octet representations. Thus in itself the Eightfold-way does not require a baryon number. We are forcing one upon it. So this points to a similarity between the two octets, but also at the same time, points to a dissimilarity between them. How do we understand this puzzle?

How come the hypercharge number Y, in the Eightfold Way model and the SU(3) models, are so fundamentally different? Clearly these eigenvalues can not be of the same operator. But how can that be, as in SU(3) group itself, there is but one more diagonal generator, besides the isospin T_3 ?

The above mathematical duality between the Lie algbra and the Jacobi identity comes to our rescue. As the octet of the baryons in the Eightfold-way model is the adjoint representation of the su(3) algebra it corresponds to the second structure of the above mathematical duality. And the octet arising in the 3 x 3 x 3 structure corresponds to the first structure. Clearly being independent, the second diagonal generator in SU(3) is allowed to have different eigenvalues, e,g, the baryon-number-less hypecharge in the Eightfold Way model, in contrast to the composite hypercharge with baryon number in it, as in the SU(3)-flavour model. This unambiguously supports the new mathematical duality related to the Lie Algebra and the Jacobi Identity.

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