Calculus - Revision 2.0

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Abstract

This text demonstrates that how we think about both Mathematics and Physics can be influenced by the mathematical tools that are available to us. The author attempts to predict what Newton might have thought and done if he had known of the works of Euler and Hamilton and had been familiar with the matrix methods of Linear Algebra. The author shows that Newton would have come very close to Special Relativity.

Preface

Knowledge of quaternions and Linear Algebra is required. This essay was written for the 2015 FQXI Essay Contest. This work contains excerpts from other works by the author. All of the work is original. None of the work is published. However, the works are posted to the website viXra.org. The essay contest concerns the mysterious connection between Mathematics and Physics. The author will state simply that Geometric Algebra describes three dimensional space and that Physics occurs within three dimensional space. Therefore, Geometric Algebra is able to describe Physics. The author's objective with this essay is to illustrate the usefulness of Hamilton's work on Geometric Algebra.

"Mathematics is the gate and the key of the sciences." - Roger Bacon

"Measure what is measureable, and make measureable what is not so." - Galileo Galilee

Discussion

Isaac Newton lived from 1642 - 1726. Leonhard Euler lived from 1707 - 1783. William Rowan Hamilton lived from 1805 - 1865. Therefore, Newton preceded both of the other men and hence did not have the benefit of their respective works. The author has recently wondered what might have been if Newton had known of their respective works. The author speculates regarding how Newton's work on Calculus might have been affected by such knowledge.

Newton's muse regarding Calculus was motion. He wanted to be able to describe the motion of objects with respect to time. He was especially interested in acceleration as it pertained to gravity. If Newton had known of Hamilton, the author thinks that he would have thought of position as a vector **x** rather than simply as a distance along one of the axes of the Cartesian coordinate system. This distinction is the subject of the present work.

The author will begin by stating the two main concepts that Newton would have known.

Euler's Equation[1] is written as:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Hamilton[2] defined a quaternion as being the ratio between two generic space vectors. He established the following:

$$\mathbf{Q} = q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k};\ where\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 = \mathbf{ijk}$$

Equation 0:

$$\mathbf{Q} = \frac{\mathbf{y}}{\mathbf{x}} = \frac{y_i\mathbf{i} + y_j\mathbf{j} + y_k\mathbf{k}}{x_i\mathbf{i} + x_j\mathbf{j} + x_k\mathbf{k}}$$

The convention used here is that a quaternion is represented by a **bold**-faced CAPITAL letter. A vector is represented by a **bold**-faced lower case letter. A scalar is represented by a lower case letter in regular font. By combining a scalar and a vector, Hamilton developed a Geometric Algebra. The vectors **i**, **j**, and **k** are unit vectors in the direction of the x, y, and z axes respectively.

Rearranging the definition of a quaternion from Equation 0 above by Hamilton allows the following:

$$\mathbf{Qx} = \left(q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}\right)\left(x_i\mathbf{i} + x_j\mathbf{j} + x_k\mathbf{k}\right) = \mathbf{y} = y_i\mathbf{i} + y_j\mathbf{j} + y_k\mathbf{k}$$

This multiplication results in a system of four simultaneous equations. These can be represented by the following matrix multiplication:

$$\begin{bmatrix} 0 & -x_i & -x_j & -x_k \\ +x_i & 0 & +x_k & -x_j \\ +x_j & -x_k & 0 & +x_i \\ +x_k & +x_j & -x_i & 0 \end{bmatrix}\begin{bmatrix} q_0 \\ q_i \\ q_j \\ q_k \end{bmatrix} = \begin{bmatrix} 0 \\ y_i \\ y_j \\ y_k \end{bmatrix}$$

This system is solved for the elements of **Q** by multiplying by the inverse of the coefficient matrix as follows:

$$\begin{bmatrix} q_0 \\ q_i \\ q_j \\ q_k \end{bmatrix} = \frac{1}{x_i^2 + x_j^2 + x_k^2}\begin{bmatrix} 0 & +x_i & +x_j & +x_k \\ -x_i & 0 & -x_k & +x_j \\ -x_j & +x_k & 0 & -x_i \\ -x_k & -x_j & +x_i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ y_i \\ y_j \\ y_k \end{bmatrix}$$

This allows **Q** to be expressed as follows:

Equation 1:

$$\mathbf{Q} = \frac{\mathbf{y}}{\mathbf{x}} = \frac{1}{\|\mathbf{x}\|^2}(\mathbf{x}\cdot\mathbf{y} + \mathbf{x}\times\mathbf{y});\ q_0 = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|^2}\ and\ \mathbf{q} = \frac{\mathbf{x}\times\mathbf{y}}{\|\mathbf{x}\|^2};\mathbf{Q} = q_0 + \mathbf{q}$$

Please note that this form of **Q** is inversely proportional to the square of the length of vector **x**. This is one of the features of Newton's Law of Gravity. Please note also that the complex conjugate **Q*** satisfies the equation **xQ*** = **y**.

Now let us add another relationship between vector **x** and vector **y**. Please refer to Figure 1 below. Let us make the following definition:

$$\mathbf{y} = \mathbf{x} + \Delta\mathbf{x}$$

When this is substituted into Equation 1, rearranged, and simplified the result is:

Equation 2:

$$\frac{(\mathbf{x}\cdot\Delta\mathbf{x}) + (\mathbf{x}\times\Delta\mathbf{x})}{\|\mathbf{x}\|^2} = \frac{\Delta\mathbf{x}}{\mathbf{x}}$$

The dot product term causes the length of vector **x** to change. The cross product term causes vector **x** to sweep out a surface in space.

The essential concept that Newton needed prior to developing Calculus was that of the limit. Specifically, the concept needed for Calculus is the limiting value of a ratio as both the numerator and the denominator of the ratio approach zero. The derivative is then defined as follows:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[\frac{y(x + \Delta x) - y(x)}{\Delta x} \right]; \ \Delta x = x_2 - x_1$$

The Δ**x** terms of Equation 2 are reduced to d**x** by taking the limit as Δ**x** goes to zero.

Equation 3:

$$\frac{(\mathbf{x} \cdot d\mathbf{x}) + (\mathbf{x} \times d\mathbf{x})}{\|\mathbf{x}\|^2} = \frac{d\mathbf{x}}{\mathbf{x}}$$

As an aside, Equation 3 forms the basis for the definition of the natural logarithm of a vector as follows:

Equation 4:

$$\int_{\mathbf{u}}^{\mathbf{x}} \frac{\mathbf{x} \cdot d\mathbf{x}}{\|\mathbf{x}\|^2} + \int_{\mathbf{u}}^{\mathbf{x}} \frac{\mathbf{x} \times d\mathbf{x}}{\|\mathbf{x}\|^2} = \int_{\mathbf{u}}^{\mathbf{x}} \frac{d\mathbf{x}}{\mathbf{x}} = \ln(\mathbf{x})\,; \mathit{where}\ \mathbf{u}\ \mathit{is\ a\ unit\ vector}$$

Often when dealing with new concepts in Mathematics, it is useful to develop identities that can be used. Therefore, Newton would have likely developed the following matrix representation (or something similar) for the multiplication of two quaternions:

Equation 5:

$$\mathbf{Y} = \mathbf{AX} = \begin{bmatrix} +a_0 & -a_i & -a_j & -a_k \\ +a_i & +a_0 & -a_k & +a_j \\ +a_j & +a_k & +a_0 & -a_i \\ +a_k & -a_j & +a_i & +a_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix} = [a][x]$$

This representation is almost begging to be differentiated! The difficulty is that all four of the terms in **X** and **Y** must all go to zero together. Rather than use the Δ as a difference operator and then take the limit as it reduces to zero, the author proposes to multiply the quaternion **X** by a scalar value λ and then take the limit as λ goes to zero. This will assure that all of the coefficients of **X** go to zero together. The definition for the differential of a quaternion with respect to a quaternion is therefore written as follows:

Equation 6:

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \lim_{\lambda \to 0} \left[\frac{\mathbf{Y}(\mathbf{X} + \lambda\mathbf{X}) - \mathbf{Y}(\mathbf{X})}{\lambda\mathbf{X}} \right]; \ \lambda \in R$$

Using this definition, it is fairly easy to show the following:

For **Y** = **A**, the derivative is:

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \mathbf{0}$$

For **Y** = **AX**, the derivative is:

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \mathbf{A}$$

For **Y** = **AX**^{2}, the derivative is:

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = 2\mathbf{AX}$$

For **Y** = **AX**^{3}, the derivative is:

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = 3\mathbf{AX}^2$$

At this point, Newton would conclude, that at least for polynomial type functions, the differential of a quaternion with respect to a quaternion is exactly the same as what he had previously determined for real functions.

So, where does this lead? The two simplest problems in kinematics are constant velocity and constant acceleration. If the quaternion method is applied to these problems, the results are:

Constant Velocity:

$$\mathbf{X} = \mathbf{VT} + \mathbf{X}_0$$

Constant Acceleration:

$$\mathbf{V} = \mathbf{AT} + \mathbf{V}_0$$

$$\mathbf{X} = \frac{1}{2}\mathbf{AT}^2 + \mathbf{V}_0\mathbf{T} + \mathbf{X}_0$$

At this point, the author will limit the discussion to the constant velocity case. This can then be expressed in matrix form as follows:

Equation 7:

$$\begin{bmatrix} 0 \\ x_i \\ x_j \\ x_k \end{bmatrix} = \begin{bmatrix} 0 & -v_i & -v_j & -v_k \\ +v_i & 0 & -v_k & +v_j \\ +v_j & +v_k & 0 & -v_i \\ +v_k & -v_j & +v_i & 0 \end{bmatrix} \begin{bmatrix} t_0 \\ t_i \\ t_j \\ t_k \end{bmatrix} + \begin{bmatrix} 0 \\ x_i \\ x_j \\ x_k \end{bmatrix}_0$$

Since position and velocity are vectors, they have no scalar terms and hence zero was substituted for these values. If Euler's Equation is considered to be a quaternion and then extended to three dimensions, the quaternion **T** can be represented as:

$$\mathbf{T} = \|\mathbf{T}\|\left[\cos(\theta_0) + \sin(\theta_i)\mathbf{i} + \sin(\theta_j)\mathbf{j} + \sin(\theta_k)\mathbf{k}\right]$$

The kinematic relationship then becomes:

Equation 8:

$$\begin{bmatrix} 0 \\ x_i \\ x_j \\ x_k \end{bmatrix} = \begin{bmatrix} 0 & -v_i & -v_j & -v_k \\ +v_i & 0 & -v_k & +v_j \\ +v_j & +v_k & 0 & -v_i \\ +v_k & -v_j & +v_i & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta_0) \\ \sin(\theta_i) \\ \sin(\theta_j) \\ \sin(\theta_k) \end{bmatrix} \|\mathbf{T}\| + \begin{bmatrix} 0 \\ x_i \\ x_j \\ x_k \end{bmatrix}_0$$

This can then be solved for the time column vector as follows:

Equation 9:

$$\begin{bmatrix} \cos(\theta_0) \\ \sin(\theta_i) \\ \sin(\theta_j) \\ \sin(\theta_k) \end{bmatrix} = \frac{1}{\left(v_i^2 + v_j^2 + v_k^2\right)\|\mathbf{T}\|} \begin{bmatrix} 0 & +v_i & +v_j & +v_k \\ -v_i & 0 & +v_k & -v_j \\ -v_j & -v_k & 0 & +v_i \\ -v_k & +v_j & -v_i & 0 \end{bmatrix} \left\{ \left(\begin{bmatrix} 0 \\ x_i \\ x_j \\ x_k \end{bmatrix} - \begin{bmatrix} 0 \\ x_i \\ x_j \\ x_k \end{bmatrix}_0 \right) \right\}$$

The 4x4 matrix with the velocity terms is the conjugate of the velocity vector. This expression simplifies to:

Equation 10:

$$\begin{bmatrix} \cos(\theta_0) \\ \sin(\theta_i) \\ \sin(\theta_j) \\ \sin(\theta_k) \end{bmatrix} = \frac{1}{\|\mathbf{T}\|} \frac{\Delta\mathbf{x}}{\mathbf{v}^*}$$

Newton would have laid the groundwork for Special Relativity almost 200 years before Einstein! Einstein could then have incorporated Equation 10 or something similar into his 1905 paper as part of the definitions for simultaneity and the velocity of light.

Now let us return to Equation 8. The scalar equation from the matrix multiplication is:

Equation 11.1:

$$-v_i \sin(\theta_i) - v_j \sin(\theta_j) - v_k \sin(\theta_k) = 0$$

This can be satisfied by setting all of the velocity terms to zero. Newton would likely have thought of that solution as being the Aether at absolute rest.

The **i** equation from the matrix multiplication is:

Equation 11.2:

$$x_i \ = \ v_i \cos(\theta_0) \, \|\mathbf{T}\| \ + \ \left[-v_k \sin\left(\theta_j\right) + \ v_j \sin(\theta_k)\right] \|\mathbf{T}\| \ + \ x_i^0$$

The **j** equation from the matrix multiplication is:

Equation 11.3:

$$x_j \ = \ v_j \cos(\theta_0) \, \|\mathbf{T}\| \ + \ [v_k \sin(\theta_i) \ - \ v_i \sin(\theta_k)] \|\mathbf{T}\| \ + \ x_j^0$$

The **k** equation from the matrix multiplication is:

Equation 11.4:

$$x_k \ = \ v_k \cos(\theta_0) \, \|\mathbf{T}\| \ + \ \left[-v_j \sin(\theta_i) + \ v_i \sin\left(\theta_j\right)\right] \|\mathbf{T}\| \ + \ x_k^0$$

Setting the various sine terms equal to the respective velocities divided by c causes the terms in the square brackets [] in Equations 11.2 - 11.4 to sum to zero. Therefore, Hamilton's Geometric Algebra appears to be completely consistent with Special Relativity provided that time is allowed to be the following quaternion:

Equation 12:

$$\mathbf{T} = \ \|\mathbf{T}\| \left[\cos(\theta_0) \ + \ \sin(\theta_i)\mathbf{i} \ + \ \sin\left(\theta_j\right)\mathbf{j} \ + \ \sin(\theta_k)\mathbf{k}\right] \ = \ \|\mathbf{T}\| \left[\sqrt{1 - \frac{v_i^2 + v_j^2 + v_k^2}{c^2}} \ + \ \frac{v_i}{c}\mathbf{i} \ + \ \frac{v_j}{c}\mathbf{j} \ + \ \frac{v_k}{c}\mathbf{k}\right]$$

Conclusions

We have been blessed by the works of many great men and women. Each of their efforts have moved us forward through the strength of their ideas. But, it is only afterwards when it is possible to see several of these ideas together that we can determine the best order for these gifts to be learned and applied. It seems to the author that it is only partially correct to replace the Geometry of Euclid by the space-time from Einstein's Special Relativity. We must also include the mathematical machinery of Hamilton's Geometric Algebra. It is Hamilton's machinery that allows us to describe space. Physics then allows us to make predictions. It is only through the comparison between prediction and observation that we can hope to understand.

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References

1. Thomas, G.B. 1972. Calculus and Analytic Geometry - Alternate Edition, Addison-Wesley, Reading Mass, p. 891.
2. Hamilton, W.R. 1866. Elements of Quaternions Book II, Longmans, Green, & Co., London, p. 160.

Figure 1