A nonstandard cubic equation

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A nonstandard cubic equation is shown to have an unusually economical solution, where this solution incorporates an angle that serves as the equation's discriminant.

This equation is treated as four special cases:

- In Sec. I the equation has just two independent constants m and Z.
- In Sec. II the equation has three independent constants m, Z, and k, and represents the general case.
- In Sec. III the equation has just a single independent constant m, but is especially interesting as it possesses the simple approximate solution $x \approx 1 - \frac{1}{3(m+1)^4}$.
- In Sec. IV the equation again has one independent constant m, but $3m$ must be a perfect cube.

The solution to the standard cubic equation is given in Appendix A.

I. THE CUBIC EQUATION WITH TWO CONSTANTS

We begin with a theorem providing the solution to the nonstandard cubic equation having just two constants.

Theorem 1. *Define the cubic equation*

$$\frac{(m+x)^3}{3m} + (m+x)^2 = Z \quad , \tag{1.1}$$

having positive constants m and Z, and the variable x. Zero out x from the above equation to define

$$W = \frac{m^3}{3m} + m^2 \tag{1.2}$$

and let

$$\sin\theta = \sqrt{1 - \frac{W}{Z}} \tag{1.3}$$

and

$$v = \frac{1+\sin\theta}{1-\sin\theta} \quad . \tag{1.4}$$

Then

$$x = m\left(\sqrt[3]{v} + \sqrt[3]{\frac{1}{v}}\right) - 2m \tag{1.5}$$

solves Eq. (1.1).

Proof. We will expand Eq. (1.1) into the standard cubic equation, identify its coefficients, and then solve it by using its classical solution. This solution will then be simplified by a series of substitutions until Eqs. (1.4) and (1.5) are recovered.

The standard cubic equation

$$ax^3 + bx^2 + cx + d = 0 \qquad (a = 1) \tag{1.6}$$

has this solution

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - r \quad , \tag{1.7}$$

where

$$\left.\begin{aligned} p &= c - \frac{b^2}{3} \\ q &= \frac{-2b^3}{27} + \frac{bc}{3} - d \\ r &= \frac{b}{3} \end{aligned}\right\} \tag{1.8}$$

(see Appendix A for proof). When Eq. (1.1) is expanded we get

$$\frac{x^3 + 3mx^2 + 3m^2x + m^3}{3m} + x^2 + 2mx + m^2 = Z$$

or

$$\frac{x^3 + 6mx^2 + 9m^2x + 4m^3}{3m} = Z \quad ,$$

so that

$$x^3 + 6mx^2 + 9m^2x + 4m^3 - 3mZ = 0 \quad .$$

This produces coefficients of

$$\left.\begin{aligned} a &= 1 \\ b &= 6m \\ c &= 9m^2 \\ d &= 4m^3 - 3mZ \\ &= 3m(W - Z) \end{aligned}\right\} \tag{1.9}$$

for Eq. (1.6).

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Substituting these values into Eq. (1.8) gives

$$\left.\begin{aligned} p &= 9m^2 - \frac{(6m)^2}{3} \\ q &= \frac{-2\times(6m)^3}{27} + \frac{6m\times 9m^2}{3} - 4m^3 + 3mZ \\ r &= \frac{6m}{3} \end{aligned}\right\} \tag{1.10}$$

which simplifies to

$$\left.\begin{aligned} p &= -3m^2 \\ q &= 3mZ - 2m^3 \\ r &= 2m \end{aligned}\right\} \quad . \tag{1.11}$$

Substituting these coefficients into Eq. (1.7) gives

$$\begin{aligned} x = &\sqrt[3]{\frac{3mZ-2m^3}{2} + \sqrt{\frac{(3mZ-2m^3)^2}{4} + \frac{(-3m^2)^3}{27}}} \\ &+ \sqrt[3]{\frac{3mZ-2m^3}{2} - \sqrt{\frac{(3mZ-2m^3)^2}{4} + \frac{(-3m^2)^3}{27}}} \\ &- 2m \end{aligned}$$

or

$$\begin{aligned} x = &\sqrt[3]{\left(\frac{3mZ}{2} - m^3\right) + \sqrt{\left(\frac{3mZ}{2} - m^3\right)^2 - m^6}} \\ &+ \sqrt[3]{\left(\frac{3mZ}{2} - m^3\right) - \sqrt{\left(\frac{3mZ}{2} - m^3\right)^2 - m^6}} \\ &- 2m \quad . \end{aligned}$$

Factoring out m gives

$$\begin{aligned} x = m&\sqrt[3]{\left(\frac{3Z}{2m^2} - 1\right) + \sqrt{\left(\frac{3Z}{2m^2} - 1\right)^2 - 1}} \\ &+ m\sqrt[3]{\left(\frac{3Z}{2m^2} - 1\right) - \sqrt{\left(\frac{3Z}{2m^2} - 1\right)^2 - 1}} \\ &- 2m \quad . \end{aligned} \tag{1.12}$$

Because the values in the above two outer radicals are reciprocals of each other, it follows that letting

$$u = \left(\frac{3Z}{2m^2} - 1\right) + \sqrt{\left(\frac{3Z}{2m^2} - 1\right)^2 - 1} \tag{1.13}$$

allows Eq. (1.12) to be rewritten

$$x = m\left(\sqrt[3]{u} + \sqrt[3]{\frac{1}{u}}\right) - 2m \quad . \tag{1.14}$$

But this equation is identical to Eq. (1.5) except that u has replaced v. It follows that Eq. (1.5) (our goal) holds provided that

$$u = v \quad , \tag{1.15}$$

which is to say if

$$u = \frac{1+\sin\theta}{1-\sin\theta} \quad . \tag{1.16}$$

But this is easily shown: Observe that Eq. (1.2) gives

$$m^2 = \frac{3}{4}W \quad .$$

This allows removing m^2 from Eq. (1.13) by substituting $\frac{3}{4}W$ to get

$$\begin{aligned} u &= \left(\frac{3Z}{2\times\frac{3}{4}W} - 1\right) + \sqrt{\left(\frac{3Z}{2\times\frac{3}{4}W} - 1\right)^2 - 1} \\ &= 2\frac{Z}{W} - 1 + \sqrt{\left(2\frac{Z}{W} - 1\right)^2 - 1} \\ &= 2\frac{Z}{W} - 1 + \sqrt{\left(2\frac{Z}{W}\right)^2 - 4\frac{Z}{W} + 1 - 1} \\ &= 2\frac{Z}{W} - 1 + 2\sqrt{\left(\frac{Z}{W}\right)^2 - \frac{Z}{W}} \\ &= 2\frac{Z}{W} - 1 + 2\frac{Z}{W}\sqrt{1 - \frac{W}{Z}} \quad . \end{aligned}$$

We now need to eliminate Z and W by substituting $\sin\theta$. A glance at Eq. (1.3) shows that this requires rewriting the above equation using powers of $\sqrt{1-\frac{W}{Z}}$. So, we divide the above numerator and denominator by $\frac{W}{Z}$ to get

$$u = \frac{2 - \frac{W}{Z} + 2\sqrt{1-\frac{W}{Z}}}{\frac{W}{Z}}$$

and rearrange terms so that

$$u = \frac{1 + 2\sqrt{1-\frac{W}{Z}} + \left(1-\frac{W}{Z}\right)}{1 - \left(1-\frac{W}{Z}\right)} \quad . \tag{1.17}$$

Now we can eliminate powers of $\sqrt{1-\frac{W}{Z}}$ by substituting powers of $\sin\theta$ as defined by Eq. (1.3). This gives

$$u = \frac{1 + 2\sin\theta + \sin^2\theta}{1-\sin^2\theta} \quad ,$$

which factors into

$$u = \frac{1+\sin\theta}{1-\sin\theta} \times \frac{1+\sin\theta}{1+\sin\theta} \quad ,$$

so that

$$u = \frac{1+\sin\theta}{1-\sin\theta} \quad .$$

Finally, we substitute into Eq. (1.14) to recover Eq. (1.5). □

Remark 1. If $\theta = 0$ then Eq. (1.1) has two distinct real roots.
Remark 2. If $0 < \theta < \pi/2$ then Eq. (1.1) has one real and two complex roots.
Remark 3. If θ is purely imaginary then Eq. (1.1) has three distinct real roots. Note: $\sin i\theta = i \sinh\theta$.
Remark 4. As a side issue, note the use of $W - Z$ in the simple alternate expression for d in Eq. (1.9).

II. THE CUBIC EQUATION WITH THREE CONSTANTS

It is possible to modify Eq. (1.1) slightly by joining x with a new real constant k, so as to create a general version of Eq. (1.1). In

$$\frac{(m+k+x)^3}{3m} + (m+k+x)^2 = Z \tag{2.1}$$

m and Z are (again) positive constants, but the expression $k+x$ now serves in the role earlier served by x alone. Hence, Eq. (1.5) becomes

$$k + x = m\left(\sqrt[3]{v} + \sqrt[3]{\frac{1}{v}}\right) - 2m \quad , \tag{2.2}$$

so that the solution to Eq. (2.1) is

$$x = m\left(\sqrt[3]{v} + \sqrt[3]{\frac{1}{v}}\right) - 2m - k \quad , \tag{2.3}$$

where W, θ, and v are defined as in Eqs. (1.2)–(1.4). (Note that this use of k does not affect the usefulness of θ as the discriminant.)

Equation (2.1) produces coefficients of

$$\left.\begin{aligned} a &= 1 \\ b &= 6m + 3k \\ c &= 9m^2 + 12mk + 3k^2 \\ d &= 4m^3 - 3mZ + 9m^2k + 6mk^2 + k^3 \\ &= 3m(W - Z) + k[c - k(b - ka)] \end{aligned}\right\} \tag{2.4}$$

for Eq. (1.6), where $k = 0$ recovers Eq. (1.9).

III. THE CUBIC EQUATION WITH ONE CONSTANT

Now suppose that Z ceases to be an *independent* constant, but instead derives from m and M as follows

$$Z = \frac{M^3 - M^{-3}}{3m} + M^2 - M^{-3} \quad , \tag{3.1}$$

where

$$M = m + 1 \quad ,$$

but where now

$$m \geq 9 \quad .$$

Then, a surprisingly simple, but accurate, approximate solution to Eq. (1.1) becomes possible: namely,

$$x \approx 1 - \frac{1}{3 \times M^4} \quad . \tag{3.2}$$

In the theorem that follows the extremely small size computed for ϵ is not proof of the accuracy of the above approximate solution — but the proof does help explain why the approximation is so accurate.

Theorem 2. *Let*

$$\begin{aligned} \epsilon \;=\; & \left[\frac{(M-y)^3}{3m} + (M-y)^2\right] \\ & - \left[\frac{M^3 - M^{-3}}{3m} + M^2 - M^{-3}\right] \quad , \end{aligned} \tag{3.3}$$

where

$$y = \frac{1}{3 \times M^4} \quad , \tag{3.4}$$

and m and M are positive constants such that

$$M = m + 1 \quad , \tag{3.5}$$

where

$$m \geq 9 \quad . \tag{3.6}$$

Then

$$\epsilon = \frac{1}{9M^7 m} + \frac{1}{9M^8} - \frac{1}{81M^{12}m} \quad . \tag{3.7}$$

Remark 5. Informally speaking, the absolute value for ϵ equals the difference between the value for Z produced by Eq. (1.1) when $x = 1 - \frac{1}{3M^4}$, versus that produced by Eq. (3.1). Moreover, as Eq. (3.7) makes clear, for ever larger M the (necessarily small) value for ϵ shrinks rapidly.

Proof. Substituting y, as defined by Eq. (3.4), into Eq. (3.3) gives

$$\begin{aligned} \epsilon \;=\; & \frac{\left(M - \frac{1}{3M^4}\right)^3}{3m} + \left(M - \frac{1}{3M^4}\right)^2 \\ & - \left(\frac{M^3 - M^{-3}}{3m} + M^2 - M^{-3}\right) \quad . \end{aligned} \tag{3.8}$$

This expands and simplifies to

$$\begin{aligned} \epsilon &= \frac{-27M^{10} + 9M^5 - 1}{81M^{12}m} + \frac{-6M^5 + 1}{9M^8} \\ &\quad - \left(-\frac{M^{-3}}{3m} - M^{-3}\right) \\ &= \frac{-27M^{10} + 9M^5 - 1 - 54M^9 m + 9M^4 m}{81M^{12}m} \\ &\quad + \frac{27M^9 + 81M^9 m}{81M^{12}m} \quad . \end{aligned}$$

TABLE I: Values produced by Eq. (1.1) when Z is determined by Eq. (3.1). Values are computed for the two smallest m for which $3m$ is a perfect cube. The values in the first row derive from Eq. (4.2).

m	Z	Cubed expression	Squared expression
9^a	137.036	$\frac{10}{3} - \frac{1}{3\times 29\,999.932\ldots}$	$\frac{10}{1} - \frac{1}{29\,999.932\ldots}$
72	$7130.004\ldots$	$\frac{73}{6} - \frac{1}{6\times 85\,194\,722.990\ldots}$	$\frac{73}{1} - \frac{1}{85\,194\,722.990\ldots}$

[a]Minimal case.

Combining large and small terms separately gives

$$\epsilon = \frac{(27M^9 m - 27M^{10} + 27M^9) + (9M^5 + 9M^4 m - 1)}{81M^{12}m} \quad . \tag{3.9}$$

But the large terms of the above numerator sum to 0; that is to say, given Eq. (3.5), it follows that

$$\begin{aligned} &27M^9 m - 27M^{10} + 27M^9 \\ &= M^9 m - M^9 (M-1) \\ &= M^9 m - M^9 m \\ &= 0 \quad . \end{aligned}$$

So, the effects of $\frac{1}{3M^4}$ and M^{-3} in Eq. (3.8) *almost* completely cancel. What does not cancel is this relatively small amount

$$\epsilon = \frac{9M^5 + 9M^4 m - 1}{81M^{12}m} \quad . \tag{3.10}$$

This fraction, which has only comparatively small powers of M in its numerator, gives

$$\epsilon = \frac{1}{9M^7 m} + \frac{1}{9M^8} - \frac{1}{81M^{12}m} \quad . \tag{3.11}$$

□

Remark 6. In the numerator of Eq. (3.9) all large (ninth and tenth) powers of M, which might otherwise contribute greatly to approximation error, completely cancel; this leaves only the much smaller (fourth and fifth) powers of M as the major sources of error. It follows from Eqs. (3.5), (3.6), and (3.11) that

$$\epsilon \le \frac{1\,709\,999}{729\,000\,000\,000\,000} \quad .$$

IV. THE CUBIC EQUATION WITH ONE CONSTANT AND $3m$ A PERFECT CUBE

If $m = 9$, then Eq. (3.5) gives $M = 10$, so that Eq. (3.1) gives

$$\begin{aligned} Z &= \frac{10^3 - 10^{-3}}{3\times 9} + 10^2 - 10^{-3} \\ &= \frac{999.999}{27} + 99.999 \\ &= 137.036 \quad . \end{aligned}$$

TABLE II: Values produced by Eq. (1.1) when Z is determined by Eq. (3.1). Values are computed for the two smallest m for which $3m$ is a perfect cube.

m	$\sqrt[3]{3m}$	M	W	Z	$\sim 1/(1-x)$	$\sim\sin^2\theta$
9^a	3	10	108	137.036	$29\,999.932^b$	0.2119^c
72	6	73	6912	$7130.004\ldots$	$85\,194\,722.991^d$	0.0306

[a]Minimal case.
[b]Approximately $3\times 10^4 = 30\,000$. See Eqs. (4.4) and (4.5).
[c]So, $\cos^2\theta = \frac{108}{137.036}$ where $\theta \approx 27.407\,157°$.
[d]Approximately $3\times 73^4 = 85\,194\,723$.

Because $3m = 3\times 9$ is a perfect cube this may be rewritten

$$\begin{aligned} Z &= \left(\frac{10}{3}\right)^3 - \left(\frac{1}{10\times 3}\right)^3 + 10^2 - 10^{-3} \\ &= 137.036 \quad . \end{aligned} \tag{4.1}$$

With $3m$ a perfect cube, Eq. (1.1) can likewise be rewritten. So, substituting the above values for m and Z into Eq. (1.1) gives

$$\begin{aligned} Z &= \left(\frac{10}{3} - \frac{1}{3\times 29\,999.932\ldots}\right)^3 \\ &\quad + \left(\frac{10}{1} - \frac{1}{29\,999.932\ldots}\right)^2 \\ &= 137.036 \quad . \end{aligned} \tag{4.2}$$

It is these values which appear in the first rows of Tables I and II. Because $m = 9$ is the smallest positive number for which $3m$ is a perfect cube it follows that $m = 9$ and $Z = 137.036$ represent a *minimal case.*

All of this shows that at the outset we might have chosen as a different starting point this logical alternative to Eq. (1.1)

$$\left(\frac{m+x}{n}\right)^3 + (m+x)^2 = Z \quad , \tag{4.3}$$

where $n^3 = 3m$.

And, finally, note that for the above m and Z, Eq. (1.1) produces

$$x \approx 1 - \frac{1}{29\,999.932\,142\,743\,338} \quad , \tag{4.4}$$

a value very close to the approximate value for x given by Eq. (3.2), namely

$$\begin{aligned} x &\approx 1 - \frac{1}{3\times M^4} \\ &\approx 1 - \frac{1}{30\,000} \quad . \end{aligned} \tag{4.5}$$

APPENDIX A: THE SOLUTION TO THE STANDARD CUBIC EQUATION

Theorem 3. *The standard cubic equation*

$$ax^3 + bx^2 + cx + d = 0 \qquad (a = 1) \tag{A1}$$

has the solution

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - r \tag{A2}$$

provided that

$$\left.\begin{aligned} p &= c - \frac{b^2}{3} \\ q &= \frac{-2b^3}{27} + \frac{bc}{3} - d \\ r &= \frac{b}{3} \end{aligned}\right\} \quad . \tag{A3}$$

Proof. We introduce y as follows

$$x = y - r \tag{A4}$$

and substitute r as defined by Eq. (A3) to get

$$x = y - \frac{b}{3} \quad .$$

Substituting into Eq. (A1) gives

$$\left(y - \frac{b}{3}\right)^3 + b\left(y - \frac{b}{3}\right)^2 + c\left(y - \frac{b}{3}\right) + d = 0 \quad .$$

This expands and simplifies to

$$y^3 - q + py = 0 \tag{A5}$$

with p and q from Eq. (A3) neatly replacing all instances of b, c, and d. (Note the absence of a y^2 term: the point of this substitution.)

We introduce z as follows

$$y = z - \frac{p}{3z} \tag{A6}$$

and make *Vieta's* substitution into Eq. (A5) to get

$$\left(z - \frac{p}{3z}\right)^3 - q + p\left(z - \frac{p}{3z}\right) = 0 \quad .$$

This expands and neatly simplifies to

$$z^3 - q - \frac{p^3}{27}z^{-3} = 0 \quad . \tag{A7}$$

We turn this into a quadratic equation in z^3 by multiplying through by z^3 to get

$$\left(z^3\right)^2 - q\left(z^3\right) - \frac{p^3}{27} = 0 \tag{A8}$$

(the point of *Vieta's* substitution). The standard quadratic formula then gives

$$\begin{aligned} z^3 &= \frac{-(-q) \pm \sqrt{(-q)^2 - (4)(1)\left(-\frac{p^3}{27}\right)}}{(2)(1)} \\ &= \frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \quad . \end{aligned} \tag{A9}$$

We are now close to recovering Eq. (A2), which we have to reassemble from the trail of parts we left behind. Essentially, we need to roll back the $y - r$ and $z - \frac{p}{3z}$ substitutions made earlier. We proceed in reverse order by eliminating $z - \frac{p}{3z}$ first.

From Eq. (A9) we know that

$$z = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad . \tag{A10}$$

(The inner radical we arbitrarily give a plus sign, but a minus sign would lead to identical results.) We now introduce this identity

$$-\frac{p}{3} = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \times \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

into which we substitute z from Eq. (A10) to get

$$-\frac{p}{3} = z \times \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad .$$

By moving z to the left, we then also know that

$$-\frac{p}{3z} = \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad . \tag{A11}$$

Substituting the above values for z and $-\frac{p}{3z}$ into Eq. (A6) gives

$$y = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad , \tag{A12}$$

undoing *Vieta's* substitution.

Finally, we undo the first substitution by plugging this y into Eq. (A4) to recover Eq. (A2). □

Remark 7. The discriminant of Eq. (A1) can be shown to be

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 \quad (a = 1) \quad .$$

Compare this against the economy of the discriminant θ, discussed in Remarks 1, 2, and 3. By playing a central role in the solutions to Eqs. (1.1) and (2.1), the simple discriminant θ shows these two equations to be — at least in this limited respect — more fundamental than Eq. (A1).