ON THE CUBIC COMBINATION AND THE THIRD DEGREE RAMANUJAN IDENTITIES

Nicolae Ion Bratu *
Adina N Cretan **

ABSTRACT. The present paper is a fragment revised from the work [3], published only in Romanian. Using a new function, "cubic combination", we can solve different problems. The novelty of this work consists in the deduction of an infinite number of third degree Ramanujan identities.
Keywords: Cubic diophantine equations. Identities Ramanujan's.

1. Introduction. In actual theory, for the diophantine soluble equation:

$$X^3 + Y^3 + Z^3 = W^3 \qquad (1),$$

the existence and the infinity of solutions is proved by finding some particular solutions and some parametric identities, among which the most well known is a cubic Ramanujan identity.

In 1983, N.I Bratu sent a memoir "Considerations of diophantine analysis and the last theorem Fermat's" to the Romanian Academy. That article contained some new results in the numbers theory. Later, the article was divided in fragments, some of them being published in world wide mathematics magazines. The present paper appeared for the first time in English.

The numerical function called 'cubic combination' was introduced in [3]. The goal of this paper is to present the function and its applications, among which the deduction of some new Ramanujan identities.

2. Present theory. For the equation (1), we can easily find solutions with rational integers, f.e.(-1,1,1,1) and with natural numbers, f.e. (3,4,5,6). The infinity of solutions can be proved using some identities; the identity Ramanujan's is well known. .

(P1) The brilliant mathematician S. Ramanujan found, by intuition, an identity for equation (1):

The solution can have the parametric form:

$X1 = 3u^2 + 5uv - 5v^2$; $\qquad Y1 = 4u^2 - 4uv + 6v^2$; $\qquad$ (2)

$Z1 = 5u^2 - 5uv - 3v^2$; $\qquad W1 = 6u^2 - 4uv + 4v^2$.

$X1^3 + Y1^3 + Z1^3 = W1^3$

(P2*)* The following affirmation is known and was also found by intuition:

If the equation Fermat : $\qquad X^3 + Y^3 = W^3 \qquad$ (3)

has a unordinary solution (x, y. w), then it will be a solutions of the following form::

$x1= w (y^3 - x^3) \qquad y1= x(w^3 +y^3) \qquad z1= 0 \qquad w1= y (w^3 + x^3)$ $\qquad$ (4)

3. Cubic combination.

The equations, in which we are interested, have the form (1).

Definitions **1.** If S1 = (x,y,z,w) and S2 = (a,b,c,d) are two integer solutions of the equation (1), then we define the operation (+) 'the cubic combination':

* Department of Mathematics Craiova University, Romania
** Academy of Economics Studies Bucharest, Romania

S3 = S1 (+) S2, where S3 = (x + at, y + bt, z + ct, w + dt) and

$$t = -\frac{x^2a + y^2b + z^2c - w^2d}{a^2x + b^2y + c^2z - d^2w} \qquad (5)$$

Properties of the operation (+):

(i) a neutral element: E = (0,0,0,0) and S (+) E = S

(ii) a symmetrical element: S (+) S = E

(iii) the simple commutative: S1 (+) S2 = S2 (+) S1

(iV) 'the triangular commutative': S1 (+) S2 = S3, S2 (+) S3 = S1, S3 (+) S1 = S2.

The last property is a new one in the theory of functions

2. We define Si a 'solution with defect', if it has the form (-m, m, e, e).

3. We define the 'selfcombination' a cubic combination, where S1 and S2 have the variables permutated.

3. Results

We will show that the same results, when extended, can be deducted from some provable affirmations.

(R1) We present a way of deduction of some solutions in natural numbers for the equation (1). We search solutions of the form (a+t, b+t, c+t, d+t), where a+b+c= d .
We obtain the equation :

$$2\,t^3 = 3\,(ab + bc + ca)\,(2t+d) - 3abc \qquad (t),$$

In the particular case c=0, the equation (t) has the solution t=3 , where a=1 and b=2.
Thus, the solution deducted for the equation (1) is S1= (3,4,5,6).
Another particular case is a+b =c, where a = -2 and b = 5; the solution for equation (t) will be t=3 and the new solution deducted for the equation (1) is:
S2 = (1,6,8,9). etc

More remarkable results can be obtained using 'cubic combination'.

Proposition 1. Through cubic combination we can deduct an infinity of positive integer solutions for the equation (1).

We have for example: S3= (3,4,5,6) (+) (-1.1,1,1) = (7,14,17,20),
or S4 = (3,4,5,6) (+) (2,1,-2,1) = (97,86, 23,116),
or S5= (3,4,5,6) (+) ((-5,-3,6,4) = (50,163,499,506); etc.

(R2) It can be showed that the proposition **(P2)** can be demonstrated by the cubic combination*:*
If we use the cubic selfcombination: (x,y,0,w) (+) (y,-w, 0,-x) = (x1,y1,0,w1),
the result is

$$t = \frac{y^2w + w^2x - x^2y}{w^2y + y^2x - x^2w},$$ for t = 1, we have a new unordinary solution

$$x1 = w\,(y^3 - x^3) \qquad y1 = x(w^3 + y^3) \qquad z1 = 0 \qquad w1 = y\,(w^3 + x^3) \qquad (4)$$

Remark 1. It can also be showed a specific demonstration for Last Theorem Fermat third degree. Because [x,y,w] = 1, w> y>x, is a reduced solution, then S1 = (x1,y1,w1) is a reduced solution too with: w1> w. Analogous, we'll have another solution S2, with bigger values x2, y2, w2, so that we'll have a row of increasing values: $w < w1 < w2 < \ldots\ldots < wi < \ldots$.

According to the Faltings's result about the finitement of the numerical solutions set, we'll have for any n: Wn = Wn-1, Yn = Yn-1, Xn = X n-1, that means:

X=0, Y=1, W=1. Thus we get the ordinary solution as the only solution.

Proposition 2. It can be demonstrated that there are other identities as **(2)** and even an infinite row of identities in which the solutions of the equation (1) are functions of two parameters, u, v.

We use the cubic combination of Ramanujan's solutions with other integer solutions of the equation (1) ..

1- 'Combined cubical' the solutions **(2)** with the 'solution with defect' (-1, 1, 1, 1), it will result:

$$t = 6(u^2 - v^2) - 16uv \qquad (6) \quad ;$$

so that we can have another identity, in which:

$$X2 = 9u^2 - 11uv - v^2 ; \qquad Y2 = 10u^2 - 20uv + 12v^2 ; \qquad (7)$$
$$Z2 = -u^2 + 11uv - 9v^2 ; \qquad W2 = 12u^2 - 20uv - 10v^2 .$$

The new identity Ramanujan is: $X2^3 + Y2^3 + Z2^3 = W2^3$ (1')

2- If we 'combine cubical' the solutions **(R1)** with solution (1, –1, 1, 1), we obtain:

$$t = -\frac{2(u^2 + v^2) + 32uv}{3} \qquad (8)$$

and we deduct a new Ramanujan identity:

$$X3 = 7u^2 - 17uv - 17v^2 ; \qquad Y3 = 14u^2 - 20uv + 20v^2 ; \qquad (9)$$
$$Z3 = 17u^2 - 17uv - 7v^2 ; \qquad W3 = 20u^2 + 20uv - 14v^2 .$$

with: $X3^3 + Y3^3 + Z3^3 = W3^3$ (1'')

The process of cubic combination can continue.

(R3) In the end, we will study the equation:

$$X^3 + Y^3 + Z^3 = 3 \qquad (10)$$

In 1955, in London Math. Soc., Miller and Woolett, published tables to $x<y<z<3164$, showing that they found only the integer solutions S1 =(1, 1, 1) and S2 = (4, 4, -5)

A question arises. Does the equation (10) have infinity of integer solution?

Proposition 3. Miller-Woolett solutions are the only integer solutions for the equation (10) and we can deduct those solutions. {The full proof is in [3]}.

The condition of solving in rational integers is: $X = Y = Z = 1 \bmod 3$ (10-1)

We search solutions of the following form: (x+3at, y+3bt, z-3ct), where x,y,z t are rational integers and parameters a,b,c are positive integers.
Starting from the solution S1 = (1,1,1), for integer solutions and with simple algebraic considerations, we will have the condition a+b= c.
We will obtain an equation of second degree and, for the condition (10-2), a linear equation:

$$ab(a+b)\, t^2 + [a^2 + b^2 - (a+b)^2]\, t + [a + b - (a+b)] = 0$$

$$(a+b)\, t = 2 \qquad (10\text{-}2)$$

This is solvable only if a=b=1; it results: t=1.
The new solution for the equation (10) is S2 = (4, 4, -5).
Now, starting from the solution S2 = (4, 4, -5)

$$3\, t^2 (a^3 + b^3 - c^3) + 3\, t\,(4a^2 + 4b^2 - 5c^2) + (16a + 16b - 25c) = 0$$

By congruence considerations, with $abc \neq 0$, this equation is solvable and it has integer solutions, if we suppose: a +b = c
With t' = 3t, we obtain an equation in t':

$$t'^2 ab(a+b) + t' [(a + b)^2 + 8ab] + 9\,(a+b) = 0; \quad \text{and } a=b=1:$$

$$t'^2 + 6t' + 9 = 0 \qquad (10\text{-}3)$$

It results t'= -3; and t = -1. We go back to S1 = (1,1,1).
Equation (10) has only two integer solutions

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