

A note on the harmonic series and the logarithm

Martin Schlueter

3 Jan 2015

Abstract

An (assumed) new relationship between the harmonic series H_n and the natural logarithm $\log(n)$ is presented.

The harmonic series is:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots$$

The logarithm can be understood as part(s) of the harmonic series:

$$\underbrace{\frac{1}{1} + \dots + \frac{1}{N}}_{H_N} + \underbrace{\dots \frac{1}{N^2}}_{\approx \log(N)} + \underbrace{\dots \frac{1}{N^3}}_{\approx \log(N)} + \underbrace{\dots \frac{1}{N^4}}_{\approx \log(N)} + \dots$$

which implies the (arguable) most elementary formulation of the Euler-Mascheroni constant:

$$\gamma = \lim_{n \rightarrow \infty} \left(\frac{1}{1} + \dots + \frac{1}{n} - \dots - \frac{1}{n^2} \right)$$

Numerical results:

Approximation of $\log(n)$	n = 1,000	n = 10,000	n = 100,000
$\log(n) - (H_{n^2+n} - H_n)$	-5.0008e-04	-5.0001e-05	-5.0000e-06
$\log(n) - (H_{n^3+n} - H_{n^2+n})$	9.9900e-04	9.9990e-05	9.9999e-06
$\log(n) - (H_{n^4+n} - H_{n^3+n})$	9.9950e-07	9.9995e-09	1.0000e-10

Approximation of γ	n = 1,000	n = 10,000	n = 100,000
$\gamma - \left(\frac{1}{1} + \dots + \frac{1}{n} - \dots - \frac{1}{n^2} \right)$	-9.9933e-04	-9.9993e-05	9.9999e-06
$\gamma - (H_n - (H_{n^2+n} - H_n))$	1.6650e-07	1.6666e-09	1.6665e-11
$\gamma - (H_n - (H_{n^3+n} - H_{n^2+n}))$	-0.0014989	-1.4999e-04	-1.5000e-05
$\gamma - (H_n - (H_{n^4+n} - H_{n^3+n}))$	-5.0092e-04	-5.0009e-05	-5.0001e-06
$\gamma - (H_n - \log(n))$	-4.9992e-04	-4.9999e-05	-5.0000e-06

Proof:

$$\begin{aligned} H_{n^{a+1}} - H_{n^a} &\approx (\log(n^{a+1}) + \gamma) - (\log(n^a) + \gamma) \\ &\approx (\log(n^a) + \log(n) + \gamma) - (\log(n^a) + \gamma) \approx \log(n) \end{aligned}$$

And in general for any $x \in \mathbb{R}^+$:

$$\log(x) = \lim_{N \rightarrow \infty} H_{xN} - H_N$$

Proof:

$$\begin{aligned} \lim_{N \rightarrow \infty} H_{xN} - H_N &= (\log(xN)) + \gamma - (\log(N) + \gamma) \\ &= (\log(x) + \log(N) + \gamma) - (\log(N) + \gamma) = \log(x) \end{aligned}$$

Example:

$$\log(123.456) \approx 4.815885$$

$$H_{123.456 \cdot 1000000} - H_{1000000} \approx 4.815884$$