Minimal Fractal Manifold as Asymptotic Regime of Non-Commutative Field Theory

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Abstract

The *minimal fractal manifold* (MFM) defines a space-time continuum endowed with arbitrarily small deviations from four-dimensions $(\varepsilon = 4 - D,\ \varepsilon \ll 1)$. It was recently shown that MFM is a natural consequence of the Renormalization Group and that it brings up a series of unforeseen solutions to the challenges raised by the Standard Model. In this brief report we argue that MFM may be treated as asymptotic manifestation of Non-Commutative (NC) Field Theory near the electroweak scale. Our provisional findings may be further expanded to bridge the gap between MFM and NC Field Theory.

Key words: Minimal Fractal Manifold, Non-Commutative Field Theory, Standard Model, Renormalization Group, Dimensional Flow, Dimensional Reduction, Fractal Operators, Fractional Calculus.

1. Introduction

Non-Commutative field theory represents a generalization of standard Quantum Field Theory (QFT) to space-times having non-commuting coordinates. It is based on the premise that coordinates may be promoted to hermitean operators x^{μ} ($\mu = 0,1,2,3$) obeying the commutation rules [1, 3-5]

$$\left[x^{\mu}, x^{\nu}\right] = i\theta^{\mu\nu} \tag{1}$$

where $\theta^{\mu\nu}$ is a real-valued and anti-symmetric matrix of dimension (length)^2. If $\theta^{\mu\nu}$ is constant, the commutators define a *Heisenberg algebra* and imply the space-time uncertainty

$$\Delta x^{\mu}\Delta x^{\nu} > \frac{1}{2}\left|\theta^{\mu\nu}\right| \tag{2}$$

It is known that space-time quantization (1) involves a number of difficulties when gauged against the geometry of four-dimensional continuum. For example, the condition $\theta^{0i} \neq 0$, $i = 1, 2, 3$ implies a theory that violates *causality* and *unitarity*. Likewise, (1) stands in conflict with *Lorentz invariance*: the choice $\theta^{12} \neq 0$ leads to breaking of Lorentz invariance to the residual $SO(1,1) \times SO(2)$ symmetry generated by boosts along the third space direction (3) and rotations in the $(1, 2)$ directions [6].

As with any compelling efforts aimed at developing QFT beyond its present boundaries, NC field theory must be able to recover the physics of the Standard Model in the appropriate limit. In particular it has to fulfill all consistency requirements mandated by the Standard Model near the electroweak scale. It is our opinion that NC field theory, despite advancing many attractive claims, is not yet at this stage. As explained in the text, there are reasons to believe that the only way NC field theory can make sensible contact with the physics of the Standard Model is to conjecture that (1) can be mapped to a *continuous deformation* of conventional commutation rules. Moreover, this deformation must be dependent on a parameter that vanishes identically on the four-dimensional space-time. The goal of our short and informal report is to point out that the concept of *Minimal Fractal Manifold* (MFM) provides a natural choice for this conjecture.

To make the report self-contained, we outline below the main motivation for the introduction of the MFM [8-13].

A counterintuitive outcome of field theory is that the exact continuum limit of a local QFT formulated on flat spacetime has, strictly speaking, *no correlate to physical reality* [7]. The

Minkowski metric of Special Relativity underlies the most basic aspect of QFT, namely the space-like commutativity of local observables, yet is considered only an "emergent" phenomenon and an approximate description of an underlying fundamental theory.

Considerations based on the Renormalization Group suggest that the smooth four-dimensional space-time turns into a manifold with arbitrarily small deviations from four dimensions near the electroweak scale [8-13]. Topological structures of this kind are called MFM's and are defined as continuous space-times of dimension $D = 4 \pm \varepsilon$, $\varepsilon << 1$. The cross-over regime between $\varepsilon \neq 0$ and $\varepsilon = 0$ is the *only sensible setting* where the dynamics of interacting fields asymptotically meets all consistency requirements mandated by the Standard Model. Hence, a key feature of the MFM is that the assumption ε << 1, postulated near or above the electroweak scale, is the only realistic way of asymptotically matching the Standard Model in the limit of vanishing fractality ε = 0. Large deviations from four dimensions ($\varepsilon \sim O(1)$) are likely to stand in direct conflict with both QFT and Standard Model. Particular attention needs to be paid, for example, to the potential violation of Lorentz invariance in Quantum Gravity theories advocating the emergence of space-time of lower dimensionality at high energy scales. Similarly, large departures from four-dimensionality imply non-differentiability of space-time trajectories in the conventional sense. This in turn, spoils the very concept of "speed of light" which becomes manifestly incompatible with the Lorentz symmetry.

2. Non-commutativity of fractal operators

In a nut-shell, fractal (or fractional) operators are differential derivatives and integrals of arbitrary non-integer order. They offer novel tools for the analysis of interacting systems that are embedded on fractal supports or in dynamic environments falling outside equilibrium conditions.

We survey next the commutativity of fractal operators with emphasis on the setting describing minimal fractality (ε << 1). Let

$$n-1 < \alpha < n, \; m-1 < \beta < m \text{ where } n,m \in N^{+} \tag{3}$$

denote the fractional order for two Caputo operators O^{α}, O^{β} working on a generic function $f(x)$ [2]. Their commutator is given by

$$\left[O^{\alpha}, O^{\beta}\right] = O^{\alpha}O^{\beta} - O^{\beta}O^{\alpha} \tag{4a}$$

We introduce the convention

$$O^{\alpha} = \partial^{\alpha}, \text{ if } \alpha > 0 \tag{4b}$$

$$O^{\alpha} = I^{\alpha}, \text{ if } \alpha < 0 \tag{4c}$$

To model the behavior of (4b-c) on the MFM and establish connection to the NC field theory, we take

$$\alpha = \varepsilon << 1, \;\; \varepsilon = 4 - D > 0 \tag{5a}$$

$$\beta = \varepsilon' << 1, \;\; \varepsilon' = 4 - D < 0 \tag{5b}$$

$$f(x^{\mu}) = x^{\mu} \tag{5c}$$

(5c) asymptotically converges to the conventional space-time coordinates in the limit $\varepsilon, \varepsilon' \to 0$, that is,

$$\lim_{\varepsilon \to 0} (\partial^{\varepsilon})(x^{\mu}) = x^{\mu} \tag{6a}$$

$$\lim_{\varepsilon' \to 0} (I^{-\varepsilon'})(x^{\nu}) = x^{\nu} \tag{6b}$$

Using calculations detailed in [2] yields

$$\left[\partial^{\varepsilon}, I^{-\varepsilon'}\right] f(x) = 2\sum_{j=0}^{n-1} \frac{(x-x_0)^{j-\varepsilon'-\varepsilon}}{\Gamma(j+1-\varepsilon'-\varepsilon)} (\partial^j f)(x_0) \tag{7}$$

where the space-time index μ, ν is omitted for the sake of clarity. The commutator vanishes if

$$(\partial^j f)(x_0) = 0, \quad j = 0,1,2....,n-1. \tag{8}$$

which fails to be true unless $x_0 = 0$. Same conclusion applies to the case where the two operators are of the Riemann-Liouville type [2]. It is readily seen from (6a-b) and (7), (8) that fractal operators working on the MFM enable a *continuous deformation* of space-time commutativity into the quantization condition (1). The deformation goes away as $\varepsilon, \varepsilon' \to 0$, a setting that recovers the familiar geometry of the four-dimensional continuum.

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