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A Theoretical Reformulation of the Classical Double Slit Interference Experiment

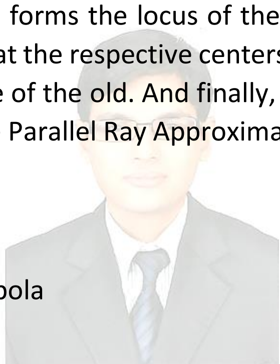
Abstract

In 1801, Thomas Young devised what is now known as the Classical Double Slit Experiment. In this experiment, light waves emanating from two separate sources, interfere to form a pattern of alternating bright and dark fringes on a distant screen. By measuring the position of individual fringe centers, the fringe widths and the variation of average light intensity on the screen, it is possible to compute the wavelength of light itself. The original theoretical analysis of the experiment employs a set of geometric assumptions which are collectively referred to here as the Parallel Ray Approximation. Accordingly, any two rays of light arising from either source and convergent on an arbitrary point on the screen, are considered very nearly parallel to each other in the vicinity of the sources. This approximation holds true only when the screen to source distance is very large and the inter-source distance is much larger than the wavelength of light. The predictions that naturally follow are valid only for fringes located near the center of the screen (e.g. the equal spacing of fringes). But for those fringes located further away from the screen center, the precision of these predictions rapidly wanes. Also when the screen to source distance is comparable to the inter-source separation or when the inter-source distance is comparable to the wavelength of light, the original analysis is no longer applicable.

In this paper, the theoretical foundations of Young's experiment are re-formulated using a newly derived analytical equation of a hyperbola, which forms the locus of the points of intersections of two expanding circular wavefronts (with sources located at the respective centers of expansion). The ensuing predictions of the new analysis are compared with those of the old. And finally, it is shown that the latter approach is just a special instance of the former, when the Parallel Ray Approximation can be said to hold true.

Keywords

Superposition, interference, fringe, hyperbola



Acknowledgements

Gloria in excelsis Deo

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1. Introduction [1]

1.1. Interference

Interference is a Wave phenomenon. Its basis is the Principle of Superposition. When two separate wave-fronts originating from their respective sources, superimpose (i.e. add together) at a particular point in space and in time, the resultant wave has an amplitude that is either greater or lesser than either of the individual waves. When the resultant wave has an amplitude greater than either of the individual waves, interference is said to be constructive. When the resultant wave has an amplitude lesser than either of the individual waves, interference is said to be destructive.

1.2. Principle of Superposition

When two or more waves of amplitudes $(y_1, y_2, \dots, y_n)$ traverse the same space, the net amplitude of the resultant wave $(y_{resultant})$, is the algebraic sum of the amplitudes of the individual waves. Mathematically,

$$y_{resultant} = y_1 + y_2 + \cdots + y_n \qquad \text{...(1)}$$

1.3. Conditions to be fulfilled in order to observe Interference between Light Waves

(i) The Light Sources must be Coherent, i.e., they must maintain a constant phase relationship with respect to each other. This is necessary in order to produce a stable interference pattern.

(ii) The Light Sources must be Monochromatic, i.e., they must be of a single wavelength. This is achieved by using a single Monochromatic Source to illuminate a barrier with two small openings which are in the shape of slits. The light emerging from both slits will be coherent, because any changes that occur in the light of the primary source gets transferred simultaneously to both the secondary sources. In this manner, two coherent sources can be set up from a single light source.

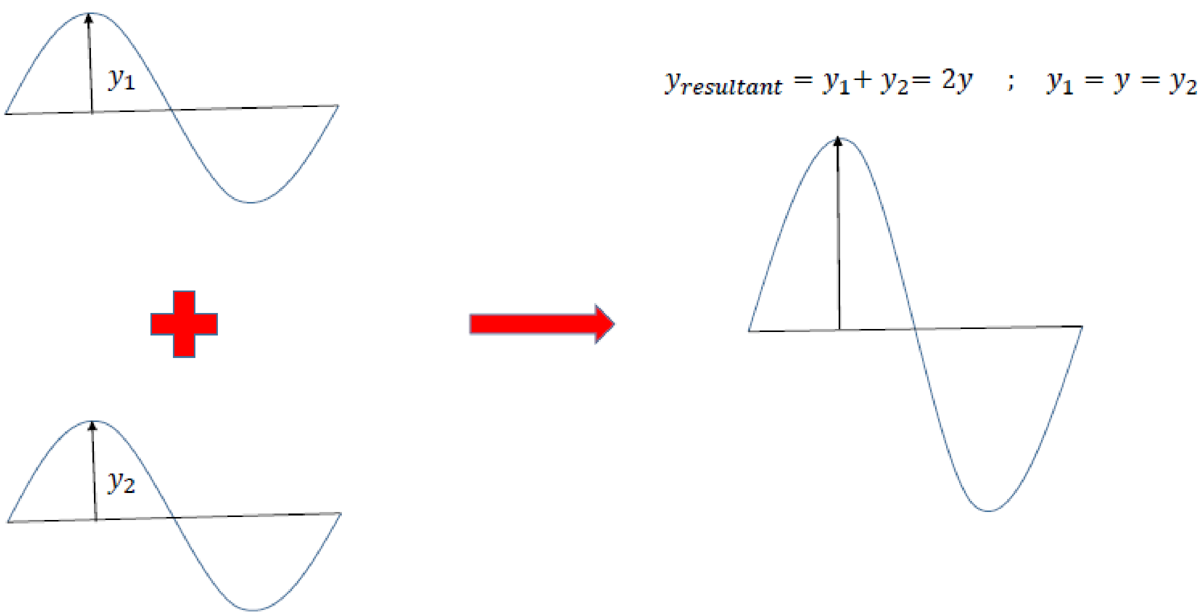


Figure 1.1: *Illustration of Constructive Interference*

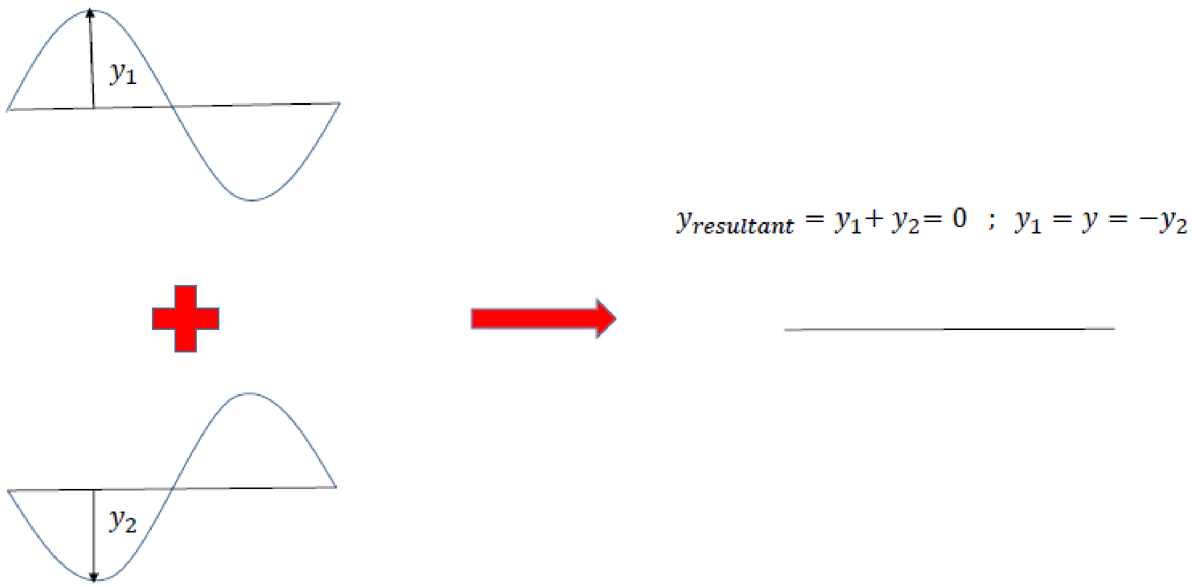


Figure 1.2: *Illustration of Destructive Interference*

2. Young's Double-Slit Experiment [1]

2.1. Qualitative Aspects

The first experiment done to demonstrate the phenomenon of light interference was devised by the Physician cum Physicist, Thomas Young in 1801. The apparatus he used, consists of two barriers. The first barrier has a single slit S and the second barrier placed just infront of the first, has two slits S_1 and S_2. Light in the form of a plane wave-front when incident on the first barrier, emerges out of S in the form of circular wave-fronts. Upon arrival at the second barrier, the single circular wave-front is split into two circular wave-fronts by slits S_1 and S_2. S_1 and S_2 behave as a pair of coherent light sources because the light waves emerging from them are derived from the same wave-front and therefore bear a constant phase relationship with time.

A viewing screen is situated some distance infront of the second barrier. Light from both slits S_1 and S_2 combine either constructively or destructively at various points on this screen, giving a visible pattern of alternating dark and bright parallel bands, called fringes. Constructive interference gives rise to a bright fringe and destructive interference to a dark fringe.

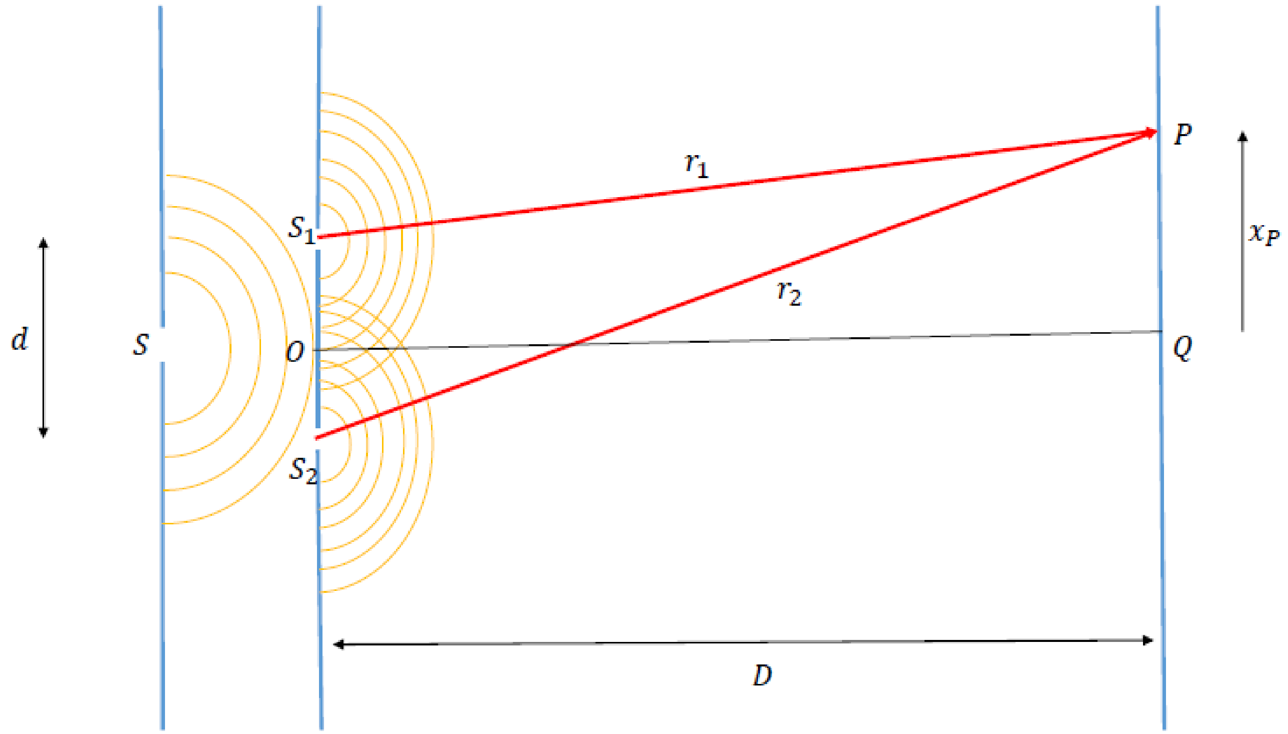


Figure 2.1: *Young's Double Slit Apparatus*

2.2. Quantitative Analysis

Let the viewing screen be situated at a distance D from the double slit barrier and the distance between the two slits S_1 and S_2 be d. Also let the wavelength of monochromatic light used be λ. In order to reach any arbitrary point P on the viewing screen, a wave from one of the slits must travel a distance equal to d.Sinθ farther than the wave from the other slit. This disparity in the distance traversed by the waves from either slit, when convergent on a single point on the screen, is called the path difference δ. If the distances of S_1 and S_2 from the point P on the screen, (denoted by r_1 and r_2 respectively), are taken to be approximately parallel (justification: D>> d), then δ can be given by:

$$\delta = r_2 - r_1 = d.sin\theta \qquad ...(2.1)$$

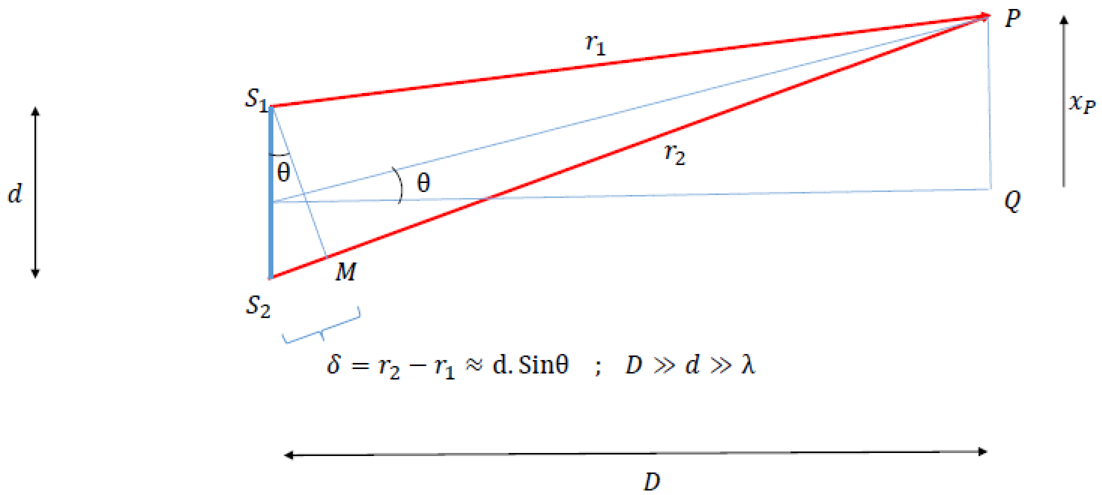


Figure 2.2: *Geometric Construction of Young's Experiment*

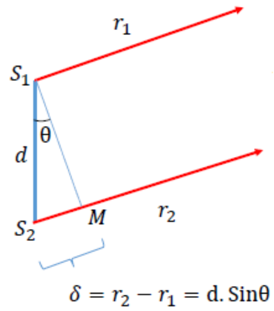


Figure 2.3: *Parallel Ray Approximation: Rays headed towards the same arbitrary point P on the screen P are approximately parallel to each other in the vicinity of S_1 and S_2*

The value of δ determines whether two waves from either slit arrive at point P on the screen, in phase or out of phase. If δ is an integer multiple of λ, then the two waves from S_1 and S_2 are in phase and constructive interference results. However, if δ is an odd integer multiple of λ/2, the two waves from S_1 and S_2 are 180° out of phase and destructive interference results.

<u>*Condition for Constructive Interference (Bright Fringe Formation)*</u>

$$\delta = d.sin\theta = n.\lambda \quad ; \quad n = 0, 1, 2, 3, \ldots \qquad \ldots(2.2)$$

<u>*Condition for Destructive Interference (Dark Fringe Formation)*</u>

$$\delta = d.sin\theta = (2n+1).\frac{\lambda}{2} \quad ; \quad n = 0, 1, 2, 3, \ldots \qquad \ldots(2.3)$$

Where n is referred to as the *Order of the Fringe*. The angle θ is very small when the following assumptions are made:

(i) $D >> d$
(ii) $d >> \lambda$

The relative magnitudes of D, d and λ are of the orders of fraction of a meter, millimeter and micrometer, respectively. Under these conditions, we can take $\approx tan\theta$. From figure 2.2, it is clear that $tan\theta = \pm\frac{x_P}{D}$. The introduction of the ± sign on the RHS of the $tan\theta$ expression is justified on the grounds that the angle θ is positive when measured in the anti-clockwise direction and negative when measured in the clockwise direction. The advantage of this departure from textbook convention, is that the Order of the Fringe n can be taken as a set of non-negative integers. Consequently, the order of any fringe, be it bright or dark, can be denoted by the same symbol n. This helps avoid any confusion while making graphical simulations of fringe position, fringe width and variation of light intensity. Because for both types of fringes, the reference point is fixed to the Center Q of the screen.

The position of the bright fringe (x_{bright}) measured from the center of the screen Q, can be found by substituting the $tan\theta$ expression into the condition for constructive interference.

That is, $$d.\left(\pm\frac{x_P}{D}\right) = n.\lambda \Rightarrow x_{bright} = \pm n.\frac{D\lambda}{d}$$

Similarly, the position of the dark fringes (x_{dark}) can be found,

$$d.\left(\pm\frac{x_P}{D}\right) = (2n+1).\frac{\lambda}{2} \Rightarrow x_{dark} = \pm\frac{(2n+1)}{2}.\frac{D\lambda}{d}$$

2.2.1. Position of Fringe Centers in the Interference Pattern

(i) Position of the Centers of Bright Fringes

$$x_{bright} = \pm n.\frac{D\lambda}{d} \quad ; \quad n = 0,\ 1,\ 2,\ 3, \ldots \qquad \ldots(2.4)$$

(a) Position of Central Maxima ($n = 0$) : $x_0 = 0$

(b) Position of First Order Bright Fringe ($n = 1$) : $x_1 = \pm\frac{D\lambda}{d}$

(c) Position of Second Order Bright Fringe ($n = 2$) : $x_2 = \pm\frac{2D\lambda}{d}$

… … … … … …

(d) Position of $(n-1)th$ Order Bright Fringe ($n = (n-1)$) : $x_{(n-1)} = \pm\frac{(n-1)D\lambda}{d}$

(e) Position of nth Order Bright Fringe ($n = n$) : $x_n = \pm\frac{nD\lambda}{d}$

(ii) Position of the Centers of Dark Fringes

$$x_{dark} = \pm\frac{(2n+1)}{2}.\frac{D\lambda}{d} \quad ; \quad n = 0,\ 1,\ 2,\ 3, \ldots \qquad \ldots(2.5)$$

(a) Position of Zeroth Order Dark Fringe ($n = 0$) : $x_0 = \pm\frac{D\lambda}{2d}$

(b) Position of First Order Dark Fringe ($n = 1$) : $x_1 = \pm\frac{3D\lambda}{2d}$

(c) Position of Second Order Dark Fringe ($n = 2$) : $x_2 = \pm\frac{5D\lambda}{2d}$

… … … … … …

(d) Position of $(n-1)th$ Order Bright Fringe ($n = (n-1)$) : $x_{(n-1)} = \pm\frac{(2n-1)}{2}.\frac{D\lambda}{d}$

(e) Position of nth Order Bright Fringe ($n = n$) : $x_n = \pm\frac{(2n+1)}{2}.\frac{D\lambda}{d}$

2.2.2. Width of Fringes in the Interference Pattern

(i) Width of the Bright Fringes

(a) The positon of the center of the Central Maxima coincides with the center Q of the screen ($x_0 = 0$). It lies sandwiched between the two Zeroth Order Dark Fringes $x_{0+} = \frac{D\lambda}{2d}$ and $x_{0-} = -\frac{D\lambda}{2d}$. So its width is equal to $x_{0+} - x_{0-} = \frac{D\lambda}{d}$.

(b) There are two First Order Bright Fringes on either sides of Center Q of the screen. One lies sandwiched between the First Order Dark Fringe $x_{1+} = \frac{3D\lambda}{2d}$ and the Zeroth Order Dark Fringe $x_{0+} = \frac{D\lambda}{2d}$. So its width is equal to $x_{1+} - x_{0+} = \frac{D\lambda}{d}$. The other lies sandwiched between the Zeroth Order Dark Fringe $x_{0-} = -\frac{D\lambda}{2d}$ and the First Order Dark Fringe $x_{1-} = -\frac{3D\lambda}{2d}$. So its width is equal to $x_{0-} - x_{1-} = \frac{D\lambda}{d}$.

(c) It can be similarly shown that the width of any nth order bright fringe, in general on either side of the Center Q of the screen, is equal to the quantity $\frac{D\lambda}{d}$.

(ii) Width of the Dark Fringes

A similar analysis can be carried out in the case of the Dark Fringes, by defining the width of a Dark Fringe as equal to the spacing between the centers of two successive Bright Fringes. This can be shown to be equal to the quantity $\frac{D\lambda}{d}$. Therefore, successive Dark Fringes are equally spaced, just as in the case of the Bright Fringes.

2.2.3. Distribution of Light Intensity in the Interference Pattern

Let the waves (sinusoidal) emanated from the two coherent sources S_1 and S_2 have the same amplitude E_0 (Electric Field Component), frequency ω and constant phase difference ϕ. Their instantaneous electric field displacements E_1 and E_2 at an arbitrary point P on the screen can be written as:

$$E_1 = E_0.sin\omega t \quad \text{...(2.6)}$$

$$E_2 = E_0.sin(\omega t + \phi) \quad \text{...(2.7)}$$

Note that at the slits S_1 and S_2 the waves are in phase, however at the point P on the screen, the phase difference ϕ depends on the path difference δ. The relationship between ϕ and δ is given by:

$$\phi = \left(\frac{2\pi}{\lambda}\right)\delta \quad \text{...(2.8)}$$

The magnitude of the resultant field at point P is found using the Principle of Superposition:

$$E_P = E_1 + E_2 = E_0(sin\omega t + \sin(\omega t + \phi)) = 2E_0 \cos\left(\frac{\phi}{2}\right)\sin\left(\omega t + \frac{\phi}{2}\right) \quad \text{...(2.9)}$$

The intensity of light at point P is directly proportional to the square of the resultant electric field amplitude at that point:

$$I \propto E_P^2 = 4E_0^2.cos^2\left(\frac{\phi}{2}\right).sin^2\left(\omega t + \frac{\phi}{2}\right) \quad \text{...(2.10)}$$

Over one cycle, the time averaged value of $sin^2\left(\omega t + \frac{\phi}{2}\right)$ is ½. And since most light detecting instruments, measure only the time averaged light intensity, we can write the average light intensity at point P as:

$$I = I_{max}cos^2\left(\frac{\phi}{2}\right) \quad \text{...(2.11)}$$

Where I_{max} is the maximum intensity on the screen.

Substituting (2.8) and (2.1) in (2.11),

$$I = I_{max}cos^2\left(\frac{\pi d sin\theta}{\lambda}\right) \quad \text{...(2.12)}$$

Since θ is small, we can write $sin\theta \approx tan\theta = \frac{x}{D}$. Therefore,

$$I = I_{max}cos^2\left(\frac{\pi dx}{D\lambda}\right) \quad \text{...(2.13)}$$

From the above, it is clear that $I = I_{max}$ when the argument $\frac{\pi dx}{D\lambda}$`is an integral multiple of $\pm\pi$. That is, when $x = \pm\frac{nD\lambda}{d}$ where $n = 0, 1, 2, 3, \ldots$ This is consistent with the condition for constructive interference. Also, $I = 0$ when the argument $\frac{\pi dx}{D\lambda}$ is an odd integral multiple of $\pm\pi/2$. That is, when $x = \pm\frac{(2n+1)D\lambda}{2d}$ where $n = 0, 1, 2, 3, \ldots$ This is consistent with the condition for destructive interference.

The graphical plots of Light Intensity (I) versus Path Difference $(dsin\theta)$ and Light Intensity (I) versus Fringe Position (x) on the screen, shows that *the interference pattern consists of equally spaced fringes of equal intensity*. However, this result suffers from the limitation that it is valid only under certain prescribed conditions listed below. From here on they shall be collectively referred to as the *Parallel Ray Approximation*:

(i) $D \gg d$
(ii) $Very\ small\ \theta$

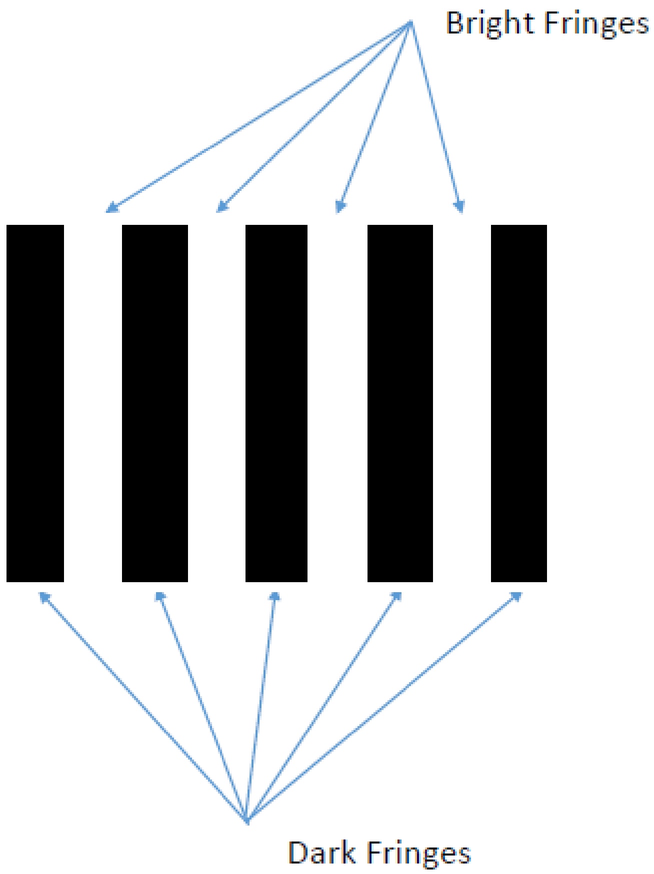


Figure 2.4: *Classical Double Slit Interference Fringe Pattern*

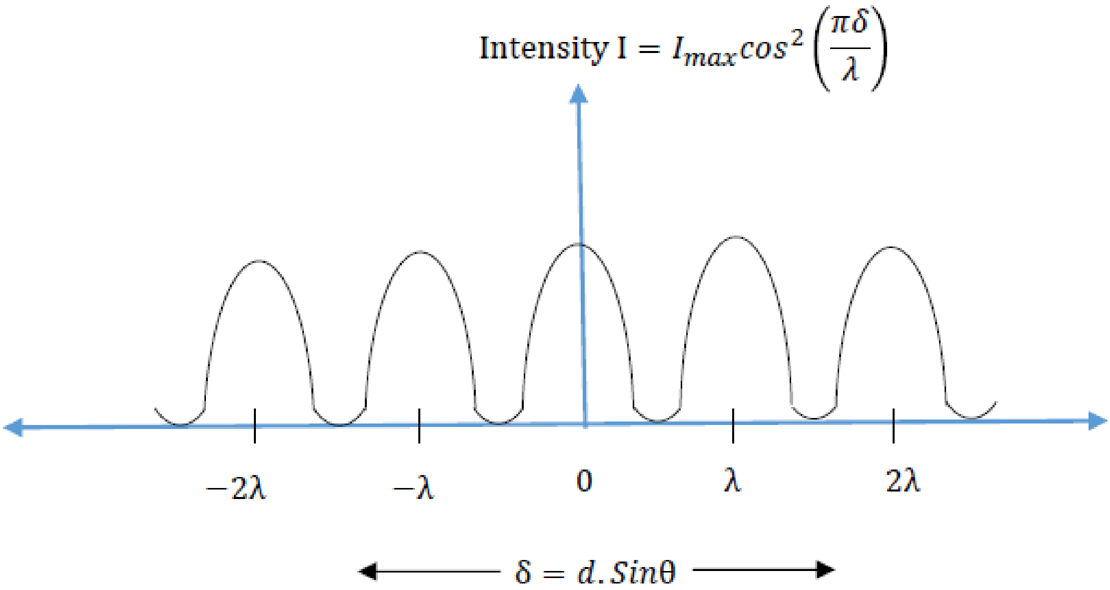


Figure 2.5: *Fringe Intensity (I) versus Path Difference (δ)*

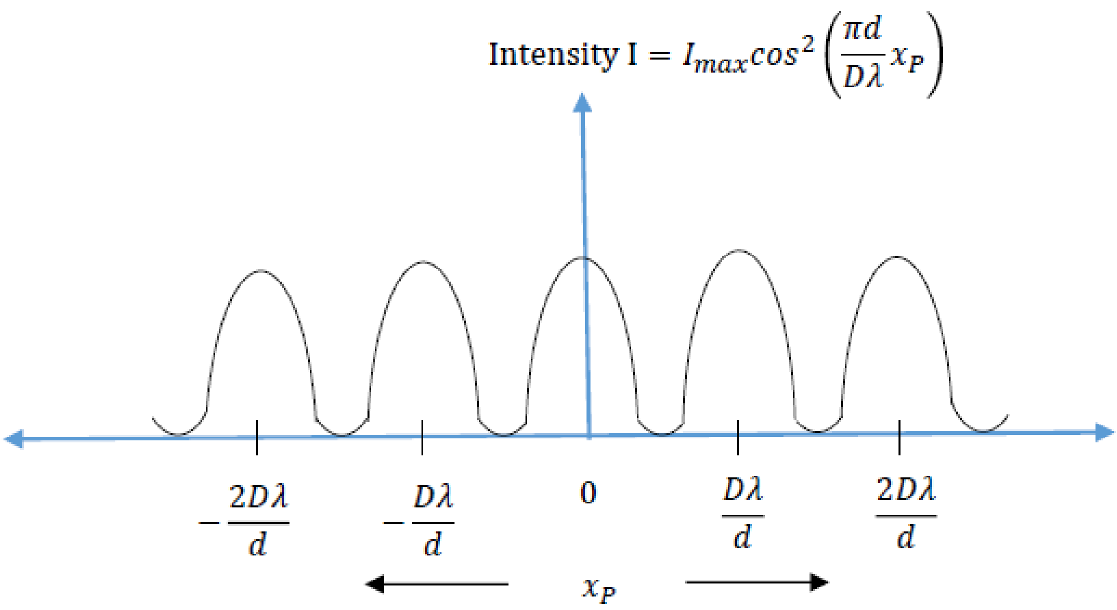


Figure 2.6: *Fringe Intensity (I) versus Fringe Position on the screen (x_P)*

3. Derivation of the Analytical Equation of the Hyperbola formed as the locus of the Intersection Points of two expanding Circular Wavefronts [2,3]

Consider two point Sources A and B located at positions $(-a, 0)$ and $(a, 0)$, respectively in a two dimensional XY-plane, with the Origin $O(0,0)$ lying mid-way between them. Say that the Source A emits a circular wavefront at an instant of time t_A and Source B emits a similar circular wavefront, at a later instant t_B. Also assume that the speed of propagation u of both wavefronts is equal and uniform in all directions. Then the equation of the circular wavefront emanating from source $A(-a, 0)$, at a given time $t > t_A$, can be written as:

$$(x + a)^2 + y^2 = R^2 \qquad ...(3.1)$$

Similarly, the equation of the circular wavefront emanating from source $B(a, 0)$, at the instant $t > t_B$, can be written as:

$$(x - a)^2 + y^2 = r^2 \qquad ...(3.2)$$

Where R and r are the instantaneous radii of the wavefronts emanating from sources A and B, respectively. Note that, $R > r$ for $t_A < t_B$. Recall that the speed of propagation of both wavefronts u is equal and uniform in all directions, given by:

$$u = \frac{dR}{dt} = \frac{dr}{dt} \qquad ...(3.3)$$

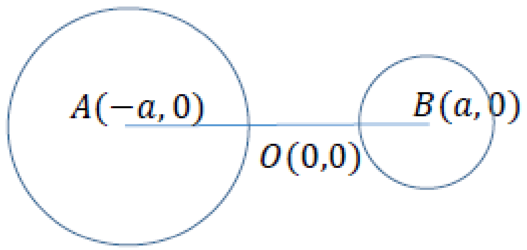


Figure 3.1: *Sources A and B emitting circular wavefronts in temporal succession*

Subtracting (3.2) from (3.1),

$$(x + a)^2 - (x - a)^2 = R^2 - r^2$$

On simplifying,

$$x = \frac{(R^2 - r^2)}{4a} \qquad ...(3.4)$$

Squaring (3.4),

$$x^2 = \frac{(R^2 - r^2)^2}{16a^2} \qquad ....(3.5)$$

Differentiating (3.5) with respect to time,

$$2x\frac{dx}{dt} = \frac{2(R^2 - r^2)(2R.\frac{dR}{dt} - 2r.\frac{dr}{dt})}{16a^2}$$

$$2x\frac{dx}{dt} = \frac{4u(R^2-r^2)(R-r)}{16a^2} \qquad \text{(By (3.3))}$$

$$2x\frac{dx}{dt} = \frac{4u(R+r)(R-r)^2}{16a^2} \qquad \text{....(3.6)}$$

Substituting (3.4) in (3.1),

$$y^2 = R^2 - (x + a)^2$$

$$= R^2 - \left(\frac{(R^2 - r^2)}{4a} + a\right)^2$$

$$= \left(R + \left(\frac{(R^2 - r^2)}{4a} + a\right)\right)\left(R - \left(\frac{(R^2 - r^2)}{4a} + a\right)\right)$$

$$= \frac{(R^2 - r^2 + 4a^2 + 4aR).(-R^2 + r^2 - 4a^2 + 4aR)}{16a^2}$$

$$= -\frac{(R^4 + r^4 + 16a^4 - 2R^2r^2 - 8a^2R^2 - 8a^2r^2)}{16a^2}$$

$$= -\frac{\left[(R^2 + r^2 - 4a^2)^2 - 4R^2r^2\right]}{16a^2}$$

$$= -\frac{\left[((R-r)^2 + 2Rr - 4a^2)^2 - 4R^2r^2\right]}{16a^2}$$

$$= -\frac{[((R-r)^2 + 2Rr - 4a^2) + 2Rr][((R-r)^2 + 2Rr - 4a^2) - 2Rr]}{16a^2}$$

$$= -\frac{((R-r)^2 + 4Rr - 4a^2)((R-r)^2 - 4a^2)}{16a^2}$$

$$y^2 = -\frac{((R+r)^2 - 4a^2)((R-r)^2 - 4a^2)}{16a^2} \qquad \text{...(3.7)}$$

From (3.7), it is clear that in order for $y \in \mathbb{R}$, either one of the following two conditions must hold true:

(i) $R + r > 2a$ and $R - r < 2a$, or

(ii) $R + r < 2a$ and $R - r > 2a$

In order that the two circular wavefronts intersect each other to trace out the locus of some curve, (it will be later shown that the curve is a branch of a hyperbola with vertex V lying somewhere on the line AB joining the point sources A and B), it is necessary that condition (i) holds true. Condition (ii) would geometrically imply that the circles intersect nowhere in the XY-plane and is therefore rejected. So provided condition (i) holds true, we can write:

$$y = \pm\sqrt{-\frac{((R+r)^2 - 4a^2)((R-r)^2 - 4a^2)}{16a^2}} \in \mathbb{R} \qquad \text{...(3.8)}$$

Differentiating (3.7) with respect to time,

$$2y.\frac{dy}{dt} = -\frac{\left[((R+r)^2 - 4a^2).2(R-r)\left(\frac{dR}{dt} - \frac{dr}{dt}\right) + ((R-r)^2 - 4a^2).2(R+r)\left(\frac{dR}{dt} + \frac{dr}{dt}\right)\right]}{16a^2}$$

$$\Rightarrow \qquad 2y.\frac{dy}{dt} = -\frac{4u(R+r)((R-r)^2 - 4a^2)}{16a^2} \qquad \text{...(3.9)} \qquad \text{(By (3.3))}$$

To re-iterate, t_A and t_B are the instants at which the sources A and B emit circular wavefronts, respectively ($t_A < t_B$). Additionally, let us assume τ to be the instant at which both these expanding wavefronts come to meet at a common point V lying on the line AB. We can therefore reason that the wavefront arising from source A, would have grown from an initial radius $R = 0$ to $R = R(\tau)$ in the time interval spanning t_A to τ. Similarly, the wavefront arising from source B, would have grown from an initial radius $r = 0$ to $r = r(\tau)$ in the time interval spanning t_B to τ. So it should be possible to integrate equation (3.3), keeping in mind that the speed of propagation of both wavefronts is equal and uniform in all directions and that $t_A < t_B < \tau$:

$$\int_0^{R(\tau)} dR = \int_{t_A}^{\tau} u.dt \Rightarrow R(\tau) = u(\tau - t_A) \qquad ...(3.10)$$

$$\int_0^{r(\tau)} dr = \int_{t_B}^{\tau} u.dt \Rightarrow r(\tau) = u(\tau - t_B) \qquad ...(3.11)$$

At the instant, $t = \tau$, both wavefronts meet at the point V on the line $AB = 2a$. So we can write,

$$R(\tau) + r(\tau) = 2a \qquad ...(3.12)$$

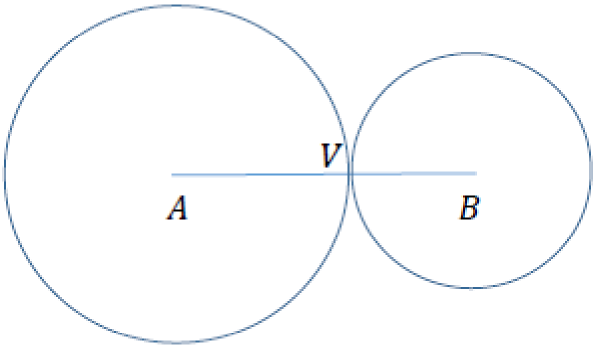


Figure 3.2: *Circular wavefronts expand to meet at a single point V lying on the line joining A and B*

Subtracting (3.11) from (3.10),

$$R(\tau) - r(\tau) = u(t_B - t_A) = u.\Delta t_{AB} \qquad ...(3.13)$$

The two expanding circular wavefronts will intersect each other at two points, call them P and P', after time $t > \tau$. The (x, y) co-ordinates of these point-pair intersections are given by equations (3.4) and (3.8):

$$\left(\frac{(R(t)^2 - r(t)^2)}{4a}, \pm\sqrt{-\frac{((R(t)+r(t))^2 - 4a^2)((R(t)-r(t))^2 - 4a^2)}{16a^2}}\right) \qquad ...(3.14)$$

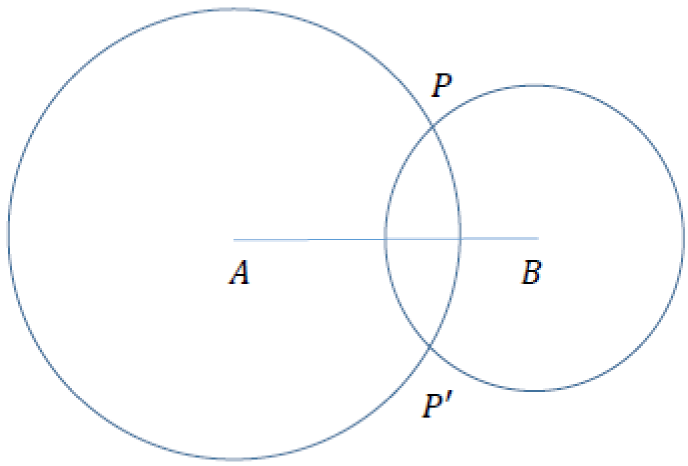


Figure 3.3: *Circular wavefronts expand to intersect each other at two points P and P'*

The co-ordinate of the point V lying on AB can be found by substituting (3.12) & (3.13) in (3.14):

$$\left(\frac{u\Delta t_{AB}}{2}, 0\right) \qquad \text{...(3.15)}$$

Since the two circular wavefronts propagate outwards at the same expansion rate u, we can expect that the instantaneous difference in their radii, $R(t) - r(t)$ to be constant with time. A formal justification of this statement can be made as follows:

$$\frac{d(R(t)-r(t))}{dt} = \frac{dR}{dt} - \frac{dr}{dt} = u - u = 0 \qquad \text{(By (3.3))}$$

$$\Rightarrow R(t) - r(t) = constant$$

This would imply that Equation (3.13) should hold true for all times, $t \geq \tau$. That is,

$$R(t) - r(t) = u(t_B - t_A) = u.\Delta t_{AB} \qquad \text{...(3.16)}$$

This satisfies the defining property of a hyperbola, as the locus of the point whose difference in the distances from two fixed points (foci), is a constant. That implies, the locus of the point of intersections of two circular wavefronts emanating from sources A and B, takes the shape of a hyperbola, since the differences in their instantaneous radii have been shown to be constant. Therefore, $V\left(\frac{u\Delta t_{AB}}{2}, 0\right)$ will be the co-ordinate of the Vertex of one branch of a hyperbola, generated when source A emits a circular wavefront before source B. The Vertex of the complementary branch of the hyperbola is generated when source B emits a circular wavefront before source A and has its vertex at the co-ordinate $V'\left(-\frac{u\Delta t_{BA}}{2}, 0\right)$, since $\Delta t_{AB} = t_B - t_A = -(t_A - t_B) = -\Delta t_{BA}$.

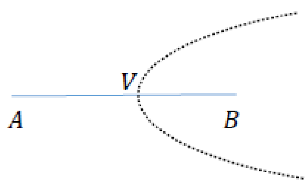


Figure 3.4: *Locus of the Intersection Points when Source A emits before Source B*

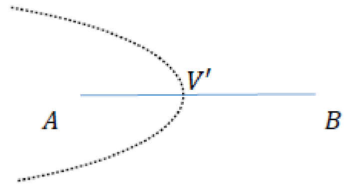


Figure 3.5: *Locus of the Intersection Points when Source B emits before Source A*

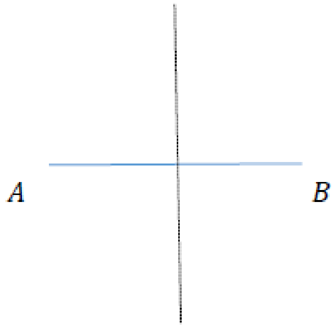


Figure 3.6: *Locus of the Intersection Points when Sources A and B emit simultaneously*

The general equation of a hyperbola with center at origin and transverse axis along the X-axis is:

$$\frac{x^2}{C^2} - \frac{y^2}{D^2} = 1 \qquad \text{...(3.17)}$$

Where C and D are the semi-lengths of the transverse and conjugate axes respectively. The value of the constant C is already known to us from (3.15) since it represents the distance of the vertex of the hyperbola from the origin. That is,

$$C = \frac{u\Delta t_{AB}}{2} \qquad \text{...(3.18)}$$

However, the value of the constant D is yet to be determined. Once D is found and put into (3.17), we would have arrived at the required equation of the hyperbola. (Note that the sources $A(-a, 0)$ and $B(a, 0)$ lie at the foci of the hyperbola).

Differentiating Equation (3.17) with respect to time,

$$\frac{1}{C^2} 2x \frac{dx}{dt} - \frac{1}{D^2} 2y \frac{dy}{dt} = 0$$

The above equation should hold true for all times $t \geq \tau > t_B > t_A$. This would mean that for $t = \tau$,

$$\frac{1}{C^2} . 2x \frac{dx}{dt}_{t=\tau} - \frac{1}{D^2} . 2y \frac{dy}{dt}_{t=\tau} = 0 \qquad \text{...(3.19)}$$

From Equations (3.6), (3.12) and (3.13),

$$2x\left.\frac{dx}{dt}\right|_{t=\tau} = \frac{4u(R(\tau)+r(\tau))(R(\tau)-r(\tau))^2}{16a^2} = 4u.\,2a.\frac{(u\Delta t_{AB})^2}{16a^2} = \frac{u^3(\Delta t_{AB})^2}{2a} \qquad \text{...(3.20)}$$

From Equations (3.9), (3.12) and (3.13),

$$2y.\left.\frac{dy}{dt}\right|_{t=\tau} = -\frac{4u(R(\tau)+r(\tau))\left((R(\tau)-r(\tau))^2-4a^2\right)}{16a^2} = -4u.\frac{2a\left((u.\Delta t_{AB})^2-4a^2\right)}{16a^2} = -\frac{u\left((u.\Delta t_{AB})^2-4a^2\right)}{2a} \qquad \text{...(3.21)}$$

Substituting (3.20), (3.21) and (3.18) in Equation (3.19),

$$\frac{1}{\left(\frac{u\Delta t_{AB}}{2}\right)^2}\frac{u^3(\Delta t_{AB})^2}{2a} - \frac{1}{D^2}\left(-\frac{u\left((u.\Delta t_{AB})^2-4a^2\right)}{2a}\right) = 0$$

On algebraic simplification of the above, we get:

$$D^2 = a^2 - \frac{u^2(\Delta t_{AB})^2}{4} = a^2 - \left(\frac{u\Delta t_{AB}}{2}\right)^2 = a^2 - C^2 \qquad \text{...(3.22)} \quad \text{(By (3.18))}$$

Substituting (3.22) and (3.18) in (3.17), we finally arrive at,

$$\boxed{\frac{x^2}{\left(\frac{u\Delta t_{AB}}{2}\right)^2} - \frac{y^2}{a^2-\left(\frac{u\Delta t_{AB}}{2}\right)^2} = 1}$$

This is the analytical equation of the hyperbola representing the locus of all the points of intersection between two circular wavefronts emanating from sources A and B, emitted at times t_A and t_B, respectively ($t_A < t_B$) . It is expressed in terms of the Inter-Source Interval Δt_{AB}, the speed of propagation of the circular wavefront u and the position of the sources $(\pm a, 0)$ with respect to the origin O, which lies midway between them.

3.1. Dual Interpretations of the Quantity Δt_{AB}

(i) Δt_{AB} represents the time interval between the emanation of circular wavefronts from sources A and B. It is mathematically expressed as $\Delta t_{AB} = t_B - t_A$, when A generates a wavefront before B and as $\Delta t_{BA} = t_A - t_B$, when B generates a wavefront before A. For this reason, Δt may be called the Inter-Source Interval (ISI) or Inter-Pulse Interval (IPI).

(ii) Re-iterating equation (3p),

$$R(t) - r(t) = u(t_B - t_A) = u.\Delta t_{AB}$$

$$\Rightarrow \frac{R(t)-r(t)}{u} = t_B - t_A$$

$$\Rightarrow \frac{R(t)}{u} - \frac{r(t)}{u} = t_B - t_A$$

$$\Rightarrow t_{AP} - t_{BP} = t_B - t_A$$

Where,

$$t_{AP} = time\ taken\ for\ circular\ wavefront\ to\ make\ the\ transit\ from\ A \to P$$

$$t_{BP} = time\ taken\ for\ circular\ wavefront\ to\ make\ the\ transit\ from\ B \to P$$

$$t_A = Instant\ at\ which\ A\ emits\ a\ circular\ wavefront$$

$$t_B = Instant\ at\ which\ B\ emits\ a\ circular\ wavefront$$

Therefore, we may conclude that the difference in the times of arrival (TDOA) of the circular wavefronts from the sources A and B at an arbitrary point P, is equal to the Inter-Source Interval.

$$i.e. \qquad TDOA = ISI = \Delta t$$

4. Re-analyzing Young's Double Slit Interference Experiment using the Equation of the Hyperbola that forms the locus of the Intersection Points of two expanding Circular Wavefronts

Re-iterating the newly derived equation of the hyperbola below,

$$\frac{x^2}{\left(\frac{u.\Delta t_{AB}}{2}\right)^2} - \frac{y^2}{a^2 - \left(\frac{u.\Delta t_{AB}}{2}\right)^2} = 1 \qquad \text{...(4.1)}$$

Where a is the source position, u is the speed of propagation of the circular wavefront and Δt_{AB} may be interpreted either as the Inter-Source Interval (ISI) or the Time Difference of Arrival (TDOA) of wavefronts from A and B at an arbitrary point P in the XY plane.

It is possible to adapt the above parameters $(x, y, u, \Delta t_{AB}, a)$, to the arrangement of the apparatus used in Young's Double Slit Experiment by taking the X-axis to lie along the second barrier, with origin O lying midway between the slits S_1 and S_2. Here, the slits play the role of sources A and B. Also, recall that the screen is placed at a distance D from the second barrier and the distance between the slits is d. Finally, the speed of propagation of the circular wavefronts is equal to the speed of light c.

We are therefore justified in replacing the parameters $(x, y, u, \Delta t_{AB}, a)$ in (4.1) as follows, in order to specify the location of an arbitrary point P on the screen,

$$x = x_P; \qquad y = D\,; \qquad u = c\,; \qquad \Delta t_{AB} = \Delta t_{S_1 S_2} = \tau\,; \qquad a = \frac{d}{2}$$

$$\frac{{x_P}^2}{\left(\frac{c.\tau}{2}\right)^2} - \frac{D^2}{\frac{d^2}{4} - \left(\frac{c.\tau}{2}\right)^2} = 1 \qquad \text{...(4.2)}$$

$$\Rightarrow {x_P}^2 = \left(\frac{c.\tau}{2}\right)^2 \left(1 + \frac{D^2}{\frac{d^2}{4} - \left(\frac{c.\tau}{2}\right)^2}\right) \qquad \text{...(4.3)}$$

Let S_1P and S_2P be two rays emerging from the slits S_1 & S_2 that are convergent on an arbitrary point P on the screen. Then their Path Difference δ is given by,

$$\delta = S_1P - S_2P \qquad \text{...(4.4)}$$

By the previously stated definition,

$$\tau = TDOA\ at\ P = t_{S_1P} - t_{S_2P} = \frac{S_1P}{c} - \frac{S_2P}{c} = \frac{S_1P - S_2P}{c} = \frac{\delta}{c}$$

Therefore, the Path Difference can be expressed as,

$$\delta = c.\tau \qquad \text{...(4.5)}$$

Substituting (4.5) in (4.3),

$$x_P{}^2 = \left(\frac{\delta}{2}\right)^2 \left(1 + \frac{D^2}{\frac{d^2}{4} - \left(\frac{\delta}{2}\right)^2}\right) \qquad \text{...(4.6)}$$

$$\Rightarrow x_P{}^2 = \frac{\delta^2}{4} + \frac{D^2.\delta^2}{d^2 - \delta^2}$$

$$\Rightarrow \quad x_P = \pm\sqrt{\frac{\delta^2}{4} + \frac{D^2.\delta^2}{d^2 - \delta^2}} \qquad \text{...(4.7)}$$

Equation (4.7) expresses the *exact* position (along the abscissa) of an arbitrary point $P(x_P, D)$ on the screen, in terms of the path difference δ, screen distance D and inter-slit separation d. Unlike in the original analysis of §2, the *Parallel Ray Approximation* was not invoked. And for this reason, the ensuing results can claim precision in their predictions.

4.1. Position of Fringe Centers in the Interference Pattern

4.1.1. Condition for Constructive Interference (Bright Fringe Formation)

$$\delta = n\lambda \quad ; \quad n = 0, 1, 2, 3, \ldots$$

Substituting δ in (4.7),

$$x_{bright} = \pm\sqrt{\frac{n^2\lambda^2}{4} + \frac{D^2.n^2\lambda^2}{d^2 - n^2\lambda^2}} \qquad \text{...(4.8)}$$

(a) Position of Central Maximum ($n = 0$),

$$x_0 = 0$$

(b) Position of 1st Order Maximum ($n = 1$),

$$x_1 = \pm\sqrt{\frac{\lambda^2}{4} + \frac{D^2.\lambda^2}{d^2 - \lambda^2}}$$

(c) Position of 2nd Order Maximum ($n = 2$),

$$x_2 = \pm\sqrt{\lambda^2 + \frac{4D^2.\lambda^2}{d^2 - 4\lambda^2}}$$

… … … … … …

(d) Position of $(n-1)^{th}$ Order Maximum $(n = n-1)$,

$$x_{(n-1)} = \pm\sqrt{\frac{(n-1)^2\lambda^2}{4} + \frac{D^2.(n-1)^2\lambda^2}{d^2-(n-1)^2\lambda^2}}$$

(e) Position of n^{th} Order Maximum $(n = n)$,

$$x_n = \pm\sqrt{\frac{n^2\lambda^2}{4} + \frac{D^2.n^2\lambda^2}{d^2-n^2\lambda^2}}$$

4.1.2. Condition for Destructive Interference (Dark Fringe Formation)

$$\delta = \left(n + \frac{1}{2}\right)\lambda \qquad ; \qquad n = 0, 1, 2, 3, \ldots$$

Substituting δ in (4.7),

$$x_{dark} = \pm\sqrt{\frac{(2n+1)^2\lambda^2}{16} + \frac{D^2.(2n+1)^2\lambda^2}{4\left(d^2-\frac{(2n+1)^2}{4}\lambda^2\right)}} \qquad \ldots(4.9)$$

(a) Position of 0^{th} Order Minimum $(n = 0)$,

$$x_0 = \pm\sqrt{\frac{\lambda^2}{16} + \frac{D^2.\lambda^2}{4\left(d^2-\frac{\lambda^2}{4}\right)}}$$

(b) Position of 1^{st} Order Minimum $(n = 1)$,

$$x_1 = \pm\sqrt{\frac{9\lambda^2}{16} + \frac{9D^2.\lambda^2}{4\left(d^2-\frac{9}{4}\lambda^2\right)}}$$

(c) Position of 2^{nd} Order Minimum $(n = 2)$,

$$x_2 = \pm\sqrt{\frac{25}{16}\lambda^2 + \frac{25D^2.\lambda^2}{4\left(d^2-\frac{25}{4}\lambda^2\right)}}$$

… … … … … …

(d) Position of $(n-1)^{th}$ Order Minimum $(n = n-1)$,

$$x_{(n-1)} = \pm\sqrt{\frac{(2n-1)^2\lambda^2}{16} + \frac{D^2.(2n-1)^2\lambda^2}{4\left(d^2-\frac{(2n-1)^2}{4}\lambda^2\right)}}$$

(e) Position of n^{th} Order Minimum $(n = n)$,

$$x_n = \pm\sqrt{\frac{(2n+1)^2\lambda^2}{16} + \frac{D^2.(2n+1)^2\lambda^2}{4\left(d^2-\frac{(2n+1)^2}{4}\lambda^2\right)}}$$

4.2. Fringe Widths

The distance between the Centers of any two successive Dark Fringes is equal to the width of a Bright Fringe. Similarly, the distance between the Centers of any two successive Bright Fringes is equal to the width of a Dark Fringe. The Central Maximum lies sandwiched between the 0th Order Minima on either sides of the Center Q of the screen.

4.2.1. Width of Bright Fringes

(a) $$Width\ of\ the\ Central\ Maximum = |x_{0+} - x_{0-}| = 2.\sqrt{\frac{\lambda^2}{16} + \frac{D^2.\lambda^2}{4\left(d^2 - \frac{\lambda^2}{4}\right)}}$$

(b) $$Width\ of\ the\ 1st\ Bright\ Fringe\ (on\ one\ side\ of\ Q) = |x_{1+} - x_{0+}|$$

$$= \left|\sqrt{\frac{9\lambda^2}{16} + \frac{9D^2.\lambda^2}{4\left(d^2 - \frac{9}{4}\lambda^2\right)}} - \sqrt{\frac{\lambda^2}{16} + \frac{D^2.\lambda^2}{4\left(d^2 - \frac{\lambda^2}{4}\right)}}\right|$$

(c) $$Width\ of\ the\ 1st\ Bright\ Fringe\ (on\ other\ side\ of\ Q) = |x_{0-} - x_{1-}|$$

$$= \left|-\sqrt{\frac{\lambda^2}{16} + \frac{D^2.\lambda^2}{4\left(d^2 - \frac{\lambda^2}{4}\right)}} + \sqrt{\frac{9\lambda^2}{16} + \frac{9D^2.\lambda^2}{4\left(d^2 - \frac{9}{4}\lambda^2\right)}}\right|$$

(d) $$Width\ of\ the\ 2nd\ Bright\ Fringe\ (on\ one\ side\ of\ Q) = |x_{2+} - x_{1+}|$$

$$= \left|\sqrt{\frac{25}{16}\lambda^2 + \frac{25D^2.\lambda^2}{4\left(d^2 - \frac{25}{4}\lambda^2\right)}} - \sqrt{\frac{9\lambda^2}{16} + \frac{9D^2.\lambda^2}{4\left(d^2 - \frac{9}{4}\lambda^2\right)}}\right|$$

(e) $$Width\ of\ the\ 2nd\ Bright\ Fringe\ (on\ other\ side\ of\ Q) = |x_{1-} - x_{2-}|$$

$$= \left|-\sqrt{\frac{9\lambda^2}{16} + \frac{9D^2.\lambda^2}{4\left(d^2 - \frac{9}{4}\lambda^2\right)}} + \sqrt{\frac{25}{16}\lambda^2 + \frac{25D^2.\lambda^2}{4\left(d^2 - \frac{25}{4}\lambda^2\right)}}\right|$$

… … … … … …

(f) $$Width\ of\ the\ nth\ Bright\ Fringe\ (on\ one\ side\ of\ Q) = \left|x_{n+} - x_{(n-1)+}\right|$$

$$= \left|\sqrt{\frac{(2n+1)^2\lambda^2}{16} + \frac{D^2.(2n+1)^2\lambda^2}{4\left(d^2 - \frac{(2n+1)^2}{4}\lambda^2\right)}} - \sqrt{\frac{(2n-1)^2\lambda^2}{16} + \frac{D^2.(2n-1)^2\lambda^2}{4\left(d^2 - \frac{(2n-1)^2}{4}\lambda^2\right)}}\right|$$

(g) $$Width\ of\ the\ nth\ Bright\ Fringe\ (on\ other\ side\ of\ Q) = \left|x_{(n-1)-} - x_{n-}\right|$$

$$= \left|-\sqrt{\frac{(2n-1)^2\lambda^2}{16} + \frac{D^2.(2n-1)^2\lambda^2}{4\left(d^2 - \frac{(2n-1)^2}{4}\lambda^2\right)}} + \sqrt{\frac{(2n+1)^2\lambda^2}{16} + \frac{D^2.(2n+1)^2\lambda^2}{4\left(d^2 - \frac{(2n+1)^2}{4}\lambda^2\right)}}\right|$$

4.2.2. Width of Dark Fringes

(a) $Width\ of\ the\ 1st\ Dark\ Fringe\ (on\ one\ side\ of\ Q)$

$$= |x_{1+} - x_{0+}|$$

$$= \left|\sqrt{\frac{\lambda^2}{4} + \frac{D^2.\lambda^2}{d^2 - \lambda^2}} - 0\right|$$

$$= \sqrt{\frac{\lambda^2}{4} + \frac{D^2.\lambda^2}{d^2 - \lambda^2}}$$

(b) $Width\ of\ the\ 1st\ Dark\ Fringe\ (on\ other\ side\ of\ Q)$

$$= |x_{0-} - x_{1-}|$$

$$= \left|0 + \sqrt{\frac{\lambda^2}{4} + \frac{D^2.\lambda^2}{d^2 - \lambda^2}}\right|$$

$$= \sqrt{\frac{\lambda^2}{4} + \frac{D^2.\lambda^2}{d^2 - \lambda^2}}$$

(c) $Width\ of\ the\ 2nd\ Dark\ Fringe\ (on\ one\ side\ of\ Q)$

$$= |x_{2+} - x_{1+}|$$

$$= \left|\sqrt{\lambda^2 + \frac{4D^2.\lambda^2}{d^2 - 4\lambda^2}} - \sqrt{\frac{\lambda^2}{4} + \frac{D^2.\lambda^2}{d^2 - \lambda^2}}\right|$$

(d) $Width\ of\ the\ 2nd\ Dark\ Fringe\ (on\ other\ side\ of\ Q)$

$$= |x_{1-} - x_{2-}|$$

$$= \left|-\sqrt{\frac{\lambda^2}{4} + \frac{D^2.\lambda^2}{d^2 - \lambda^2}} + \sqrt{\lambda^2 + \frac{4D^2.\lambda^2}{d^2 - 4\lambda^2}}\right|$$

… … … … … …

(e) $Width\ of\ the\ nth\ Dark\ Fringe\ (on\ one\ side\ of\ Q)$

$$= \left|x_{n+} - x_{(n-1)+}\right|$$

$$= \left|\sqrt{\frac{n^2\lambda^2}{4} + \frac{D^2.n^2\lambda^2}{d^2 - n^2\lambda^2}} - \sqrt{\frac{(n-1)^2\lambda^2}{4} + \frac{D^2.(n-1)^2\lambda^2}{d^2 - (n-1)^2\lambda^2}}\right|$$

(f) $Width\ of\ the\ nth\ Dark\ Fringe\ (on\ other\ side\ of\ Q)$

$$= \left|x_{(n-1)-} - x_{n-}\right|$$

$$= \left|-\sqrt{\frac{(n-1)^2\lambda^2}{4} + \frac{D^2.(n-1)^2\lambda^2}{d^2 - (n-1)^2\lambda^2}} + \sqrt{\frac{n^2\lambda^2}{4} + \frac{D^2.n^2\lambda^2}{d^2 - n^2\lambda^2}}\right|$$

4.3 Derivation of the Exact Formula for Variation of Fringe Intensity on the Screen

The waves (sinusoidal) emanated from the two coherent sources S_1 and S_2, having the same amplitude E_0 (Electric Field Component), frequency ω and constant phase difference ϕ, will produce fringes of Intensity I on the screen. The Intensity Variation Formula is given by equation (2.11),

$$I = I_{max} cos^2 \left(\frac{\phi}{2}\right) \quad ...(4.10)$$

Where I_{max} is the maximum intensity on the screen.

Recall that the relationship between phase difference ϕ and path difference δ is given by equation (2.8),

$$\phi = \left(\frac{2\pi}{\lambda}\right) \delta \quad ...(4.11)$$

Re-iterating equation (3.16) below,

$$R(t) - r(t) = u(t_B - t_A) = u.\Delta t_{AB}$$

Replacing the speed of wavefront propagation u with the speed of light c and the Inter-Source Interval (or equivalently, Time Difference of Arrival TDOA) Δt_{AB} with the symbol τ,

$$R(t) - r(t) = c.\tau \quad ...(4.12)$$

The quantity $R(t) - r(t)$ on the LHS of equation (4.12), denotes the path difference δ, of the waves that emanate from slits S_1 and S_2 to reach an arbitrary point P on the screen, at some instant of time *t*. We can therefore re-write this equation as,

$$\delta = c.\tau \quad ...(4.13)$$

The relationship between speed of wave propagation c, wave frequency υ and wavelength λ is given by,

$$c = \upsilon.\lambda \quad ...(4.14)$$

From (4.11), (4.13) and (4.14),

$$\phi = \frac{2\pi}{\lambda} c.\tau = 2\pi\upsilon\tau = \omega\tau \quad ...(4.15)$$

Substituting (4.15) in (4.10),

$$I = I_{max} Cos^2 \left(\frac{\omega\tau}{2}\right) \quad ...(4.16)$$

The Maximum Intensity $I = I_{max}$ is obtained when the argument $\frac{\omega\tau}{2}$ of the cosine squared function is an integral multiple of π. That is,

$$\frac{\omega\tau}{2} = \pm n\pi \Rightarrow \tau = \pm\frac{2n\pi}{\omega} = \pm\frac{n}{\upsilon} \text{ where } n = 0,1,2,3,\ldots \quad ...(4.17)$$

Therefore, the TDOA of light rays (τ) at a point P on the screen must be equal to the integral multiple of the reciprocal of the frequency of light υ, for constructive interference (bright fringe formation) to occur.

The Minimum Intensity $I = 0$ is obtained when the argument $\frac{\omega\tau}{2}$ of the cosine squared function is an odd integral multiple of π/2. That is,

$$\frac{\omega\tau}{2} = \pm\frac{(2n+1)}{2}\pi \Rightarrow \tau = \pm\frac{(2n+1)\pi}{\omega} = \pm\frac{(2n+1)}{2\upsilon} \text{ where } n = 0,1,2,3,\ldots \quad ...(4.18)$$

Therefore, the TDOA of light rays (τ) at a point P on the screen must be equal to the half-odd integral multiple of the reciprocal of the frequency of light υ, for destructive interference (dark fringe formation) to occur.

The relationship between Time Difference of Arrival (τ) of light rays from the two slits at an arbitrary point P on the screen (x_p) is given by equation (4.7), which is re-iterated below,

$$x_P = \pm\sqrt{\frac{\delta^2}{4} + \frac{D^2.\delta^2}{d^2-\delta^2}} \quad ...(4.19)$$

On rearranging the terms of (4.19), we get the following biquadratic equation in δ,

$$\delta^4 - \left(d^2 + 4{x_P}^2 + 4D^2\right)\delta^2 + 4{x_P}^2 d^2 = 0 \quad ...(4.20)$$

On solving (4.20), we get,

$$\delta = \pm\sqrt{\frac{\left(d^2+4{x_P}^2+4D^2\right) \pm \sqrt{\left(d^2+4{x_P}^2+4D^2\right)^2 - 16{x_P}^2 d^2}}{2}} \quad ...(4.21)$$

From (4.13) and (4.21),

$$\tau = \pm\frac{1}{c}\sqrt{\frac{\left(d^2+4{x_P}^2+4D^2\right) \pm \sqrt{\left(d^2+4{x_P}^2+4D^2\right)^2 - 16{x_P}^2 d^2}}{2}} \quad ...(4.22)$$

Multiplying both sides of (4.22) by ω/2,

$$\frac{\omega}{2}.\tau = \pm\frac{\omega}{2}.\frac{1}{c\sqrt{2}}\sqrt{\left(d^2 + 4{x_P}^2 + 4D^2\right) \pm \sqrt{\left(d^2 + 4{x_P}^2 + 4D^2\right)^2 - 16{x_P}^2 d^2}} \quad ...(4.23)$$

From (4.14) and (4.23),

$$\frac{\omega}{2}.\tau = \pm\frac{\pi}{\sqrt{2}\lambda}\sqrt{\left(d^2 + 4{x_P}^2 + 4D^2\right) \pm \sqrt{\left(d^2 + 4{x_P}^2 + 4D^2\right)^2 - 16{x_P}^2 d^2}} \quad ...(4.24)$$

From (4.16) and (4.24),

$$I = I_{max}Cos^2\left(\frac{\pi}{\sqrt{2}\lambda}\sqrt{\left(d^2 + 4{x_P}^2 + 4D^2\right) \pm \sqrt{\left(d^2 + 4{x_P}^2 + 4D^2\right)^2 - 16{x_P}^2 d^2}}\right) \quad ...(4.25)$$

Equation (4.25), gives the exact variation of fringe intensity (I) with fringe position on the screen (x_P). For reasons that will become more clearer in *§6 Results of a Simulation Study*, the positive signed inner square root of the cosine squared function's argument is rejected in favor of the negative one.

5. A Comparative Summary of the Old and New Analysis

S.No.	Parameter	Old Analysis	New Analysis
1.	Path Difference	The path difference δ is obtained after making the *Parallel Ray Approximation*. By this assumption, the rays that emanate from either slit and are convergent onto a single point P on the screen, are taken to be approximately parallel to each other when near the second barrier. Further, δ is expressed in terms of the inter-slit distance d and the angle θ. $\delta = r_2 - r_1 \approx d.sin\theta$ (Ref. eq. (2.1))	The path difference δ is obtained following an exact geometrical analysis of two expanding circular wavefronts of light, emanating from either slit source. No approximation is used in its calculation. Further, δ is expressed in terms of the speed of light c and the difference in the time of arrival of the rays (τ) at a common arbitrary point P on the screen. $\delta = R(t) - r(t) = c.\tau$ (Ref. eq. (4.12) and (4.13))
2.	Phase Difference	$\phi = \dfrac{2\pi d sin\theta}{\lambda} = \dfrac{2\pi dx}{D\lambda}$ (Ref. eq. (2.11), (2.12), (2.13))	$\phi = \omega\tau$ (Ref. eq. (4.15))

3.	Conditions for Fringe Formation	*<u>Condition for Constructive Interference (Bright Fringe Formation)</u>* When the path difference δ between the two waves emanating from S_1 and S_2 is an integer multiple of λ, then they are in phase at the point P on the screen and constructive interference results. $\delta = d.sin\theta = n.\lambda \;;\quad n = 0, 1, 2, 3, \ldots$ (Ref. eq. (2.2)) *<u>Condition for Destructive Interference (Dark Fringe Formation)</u>* When the path difference δ between the two waves emanating from S_1 and S_2 is an odd integer multiple of λ/2, then they are 180° out of phase at the point P on the screen and destructive interference results. $\delta = d.sin\theta = (2n+1).\frac{\lambda}{2} \;;\; n = 0, 1, 2, 3, \ldots$ (Ref. eq. (2.3))	*<u>Condition for Constructive Interference (Bright Fringe Formation)</u>* The TDOA of light rays (τ) at a point P on the screen must be equal to the integral multiple of the reciprocal of the frequency of light υ, for constructive interference to occur. $\tau = \pm\frac{n}{\upsilon} \quad ; \quad n = 0, 1, 2, 3, \ldots$ (Ref. eq. (4.17)) *<u>Condition for Destructive Interference (Dark Fringe Formation)</u>* The TDOA of light rays (τ) at a point P on the screen must be equal to the half-odd integral multiple of the reciprocal of the frequency of light υ, for destructive interference to occur. $\tau = \pm\frac{(2n+1)}{2\upsilon} \quad ; \quad n = 0,1,2,3, \ldots$ (Ref. eq. (4.18))
4.	Fringe Position	Position of the nth Order Bright Fringe $x_n = \pm\frac{nD\lambda}{d} \quad ; \quad n = 0, 1, 2, 3, \ldots$ (Ref. eq. (2.4)) Position of the nth Order Dark Fringe $x_n = \pm\frac{(2n+1)}{2}.\frac{D\lambda}{d} \quad ; \quad n = 0, 1, 2, 3, \ldots$ (Ref. eq. (2.5))	Position of the nth Order Bright Fringe $x_n = \pm\sqrt{\frac{n^2\lambda^2}{4} + \frac{D^2.n^2\lambda^2}{d^2-n^2\lambda^2}} \quad ; \quad n = 0, 1, 2, 3, \ldots$ (Ref. eq. (4.8)) Position of the nth Order Dark Fringe $x_n = \pm\sqrt{\frac{(2n+1)^2\lambda^2}{16} + \frac{D^2.(2n+1)^2\lambda^2}{4(d^2-\frac{(2n+1)^2}{4}\lambda^2)}} \;;\; n = 0, 1, 2, 3, \ldots$ (Ref. eq. (4.9))

5.	Fringe Width	Width of Central Maxima (Refer § 2.2.2.(ia)) $W_n = \frac{D\lambda}{d} \quad ; \quad n = 0$ Width of nth Bright Fringe (Refer § 2.2.2.(ic)) $W_n = \frac{D\lambda}{d}$ $n = 1, 2, 3, \ldots$ Width of nth Dark Fringe (Refer § 2.2.2.(ii)) $W_n = \frac{D\lambda}{d}$ $n = 1, 2, 3, \ldots$	Width of Central Maxima (Refer § 4.2.1.(a)) $W_n = 2.\sqrt{\frac{\lambda^2}{16} + \frac{D^2.\lambda^2}{4\left(d^2 - \frac{\lambda^2}{4}\right)}} \quad ; \quad n = 0$ Width of nth Bright Fringe (Refer § 4.2.1) $W_n = \left\vert \sqrt{\frac{(2n+1)^2\lambda^2}{16} + \frac{D^2.(2n+1)^2\lambda^2}{4\left(d^2 - \frac{(2n+1)^2}{4}\lambda^2\right)}} - \sqrt{\frac{(2n-1)^2\lambda^2}{16} + \frac{D^2.(2n-1)^2\lambda^2}{4\left(d^2 - \frac{(2n-1)^2}{4}\lambda^2\right)}} \right\vert$ $n = 1, 2, 3, \ldots$ With of nth Dark Fringe (Refer § 4.2.2) $W_n = \left\vert \sqrt{\frac{n^2\lambda^2}{4} + \frac{D^2.n^2\lambda^2}{d^2 - n^2\lambda^2}} - \sqrt{\frac{(n-1)^2\lambda^2}{4} + \frac{D^2.(n-1)^2\lambda^2}{d^2 - (n-1)^2\lambda^2}} \right\vert$ $n = 1, 2, 3, \ldots$
6.	Fringe Intensity Variation	$I = I_{max} cos^2\left(\frac{\pi d x}{D\lambda}\right)$ (Ref. eq. (2.13))	$I = I_{max} Cos^2\left(\frac{\pi}{\sqrt{2}\lambda}\sqrt{\left(d^2 + 4{x_P}^2 + 4D^2\right) - \sqrt{\left(d^2 + 4{x_P}^2 + 4D^2\right)^2 - 16{x_P}^2 d^2}}\right)$ (Ref. eq. (4.25))

6. Results of a Simulation Study (Refer Appendix)

The following numerical values are adopted for simulating the principal formulae of the Old and New Analysis, summarized in §5: $D = 1.2$ *meters*, $d = 3x10^{-5}$ *meters*, $\lambda = 560x10^{-9}$ *meters*, $n = \{0,1,2,\ldots,50\}$.

6.1. Fringe Positions

6.1.1. According to the Old Analysis

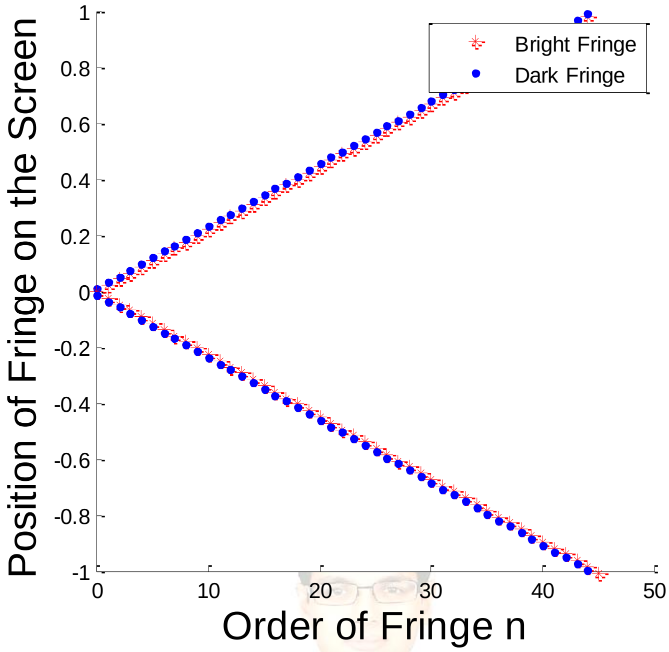


Figure 6.1.1

6.1.2. According to the New Analysis

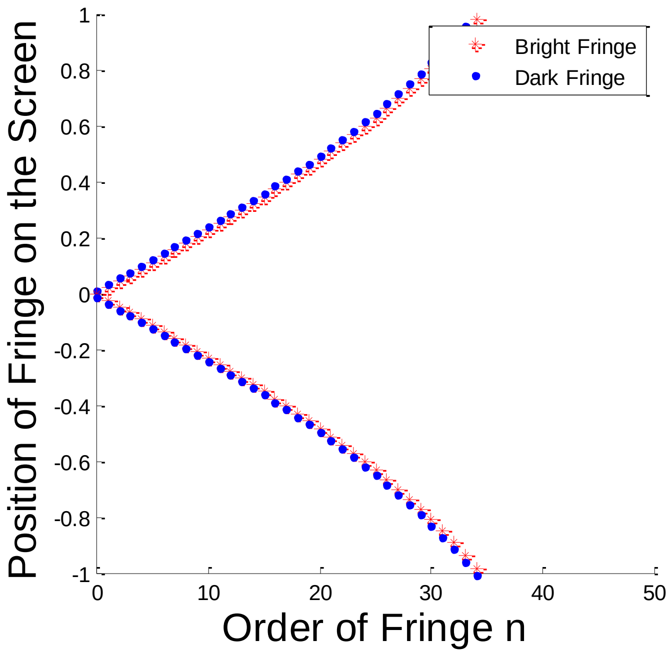


Figure 6.1.2

6.2 Fringe Widths

6.2.1. According to the Old Analysis

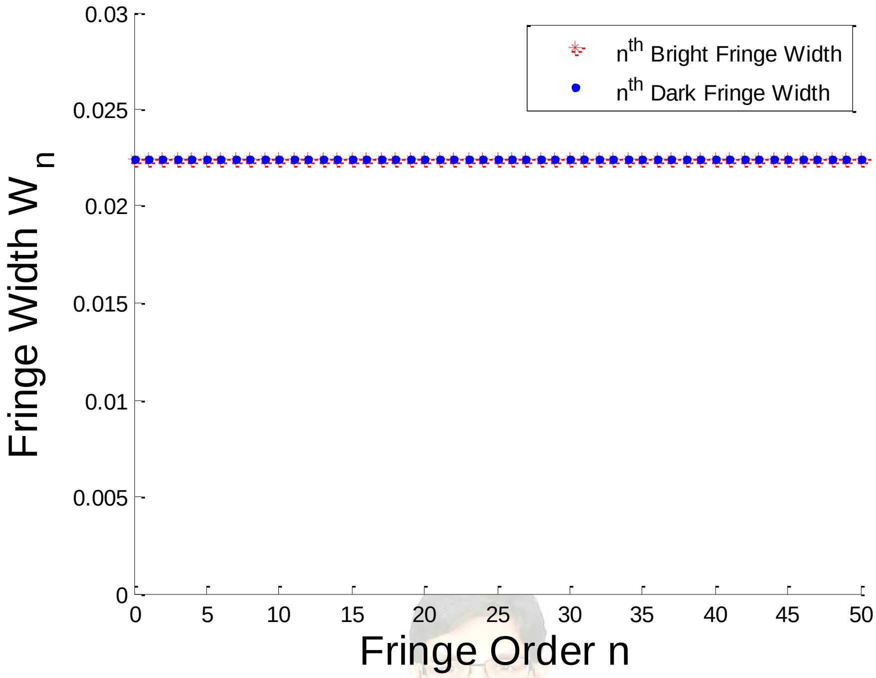


Figure 6.2.1

6.2.2. According to the New Analysis

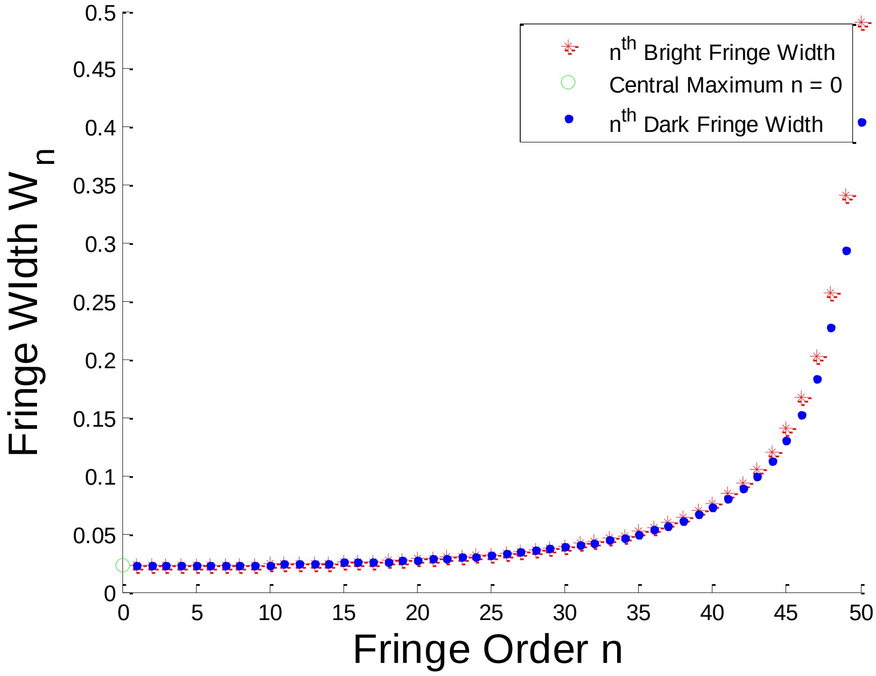


Figure 6.2.2

6.3 Variation of Light Intensity on the Screen

6.3.1. According to the Old Analysis

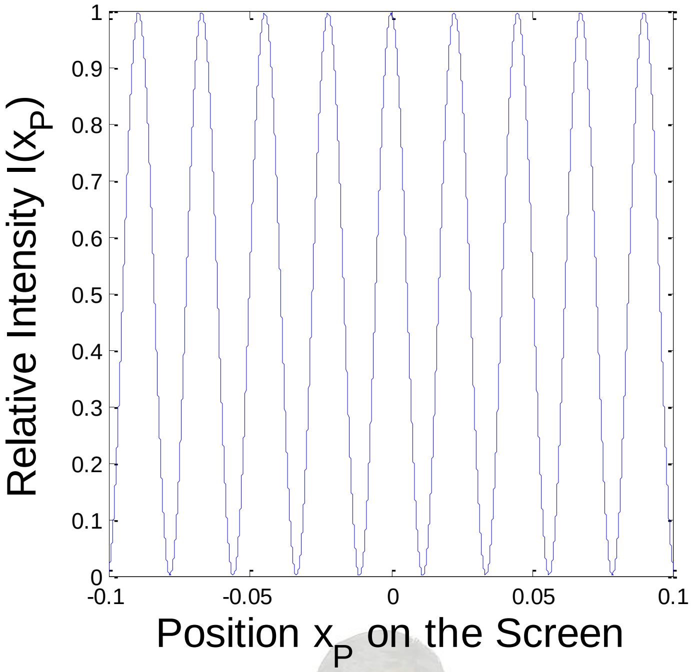


Figure 6.3.1a

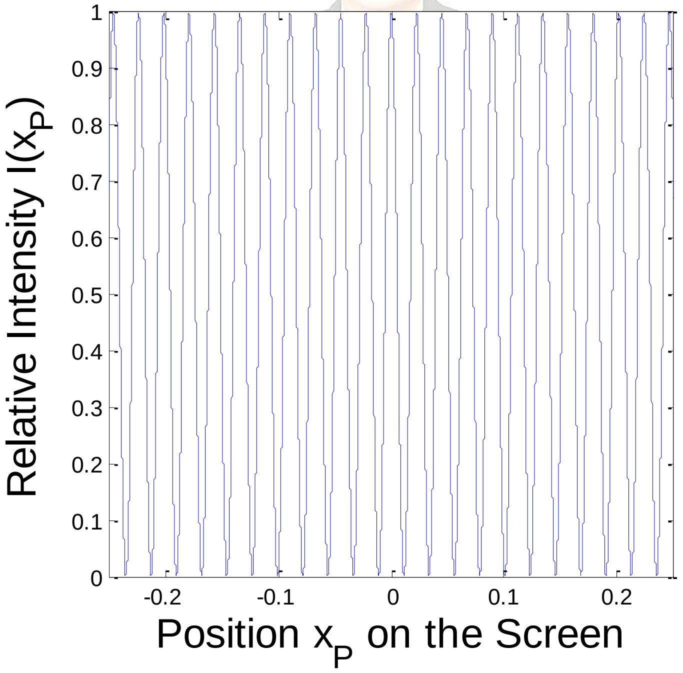


Figure 6.3.1b

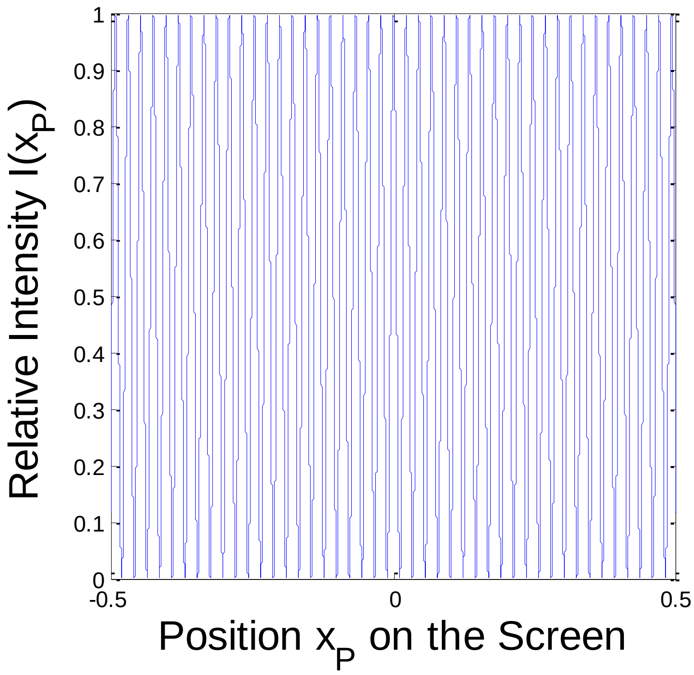


Figure 6.3.1c

6.3.2. According to the New Analysis

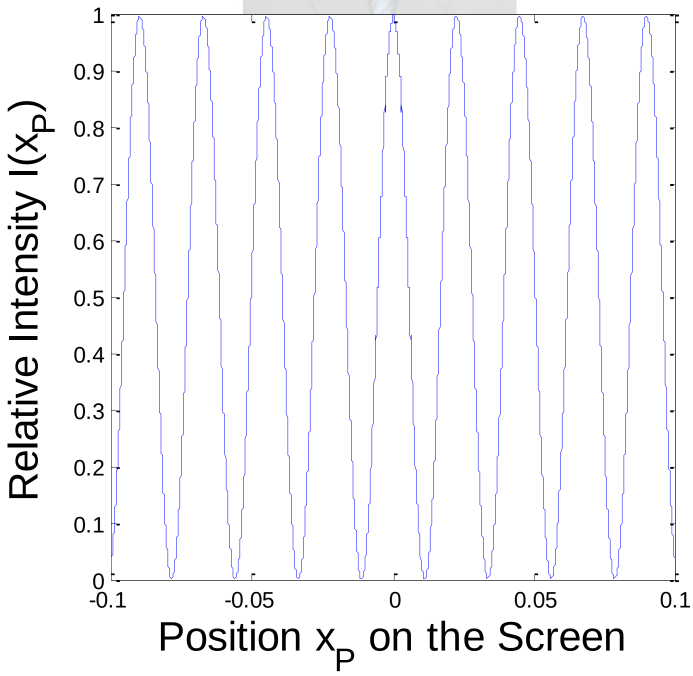


Figure 6.3.2a

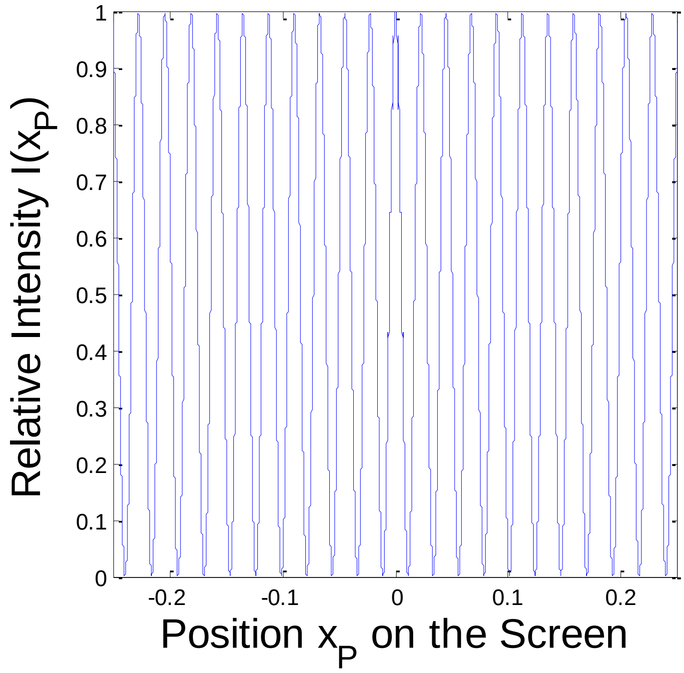


Figure 6.3.2b

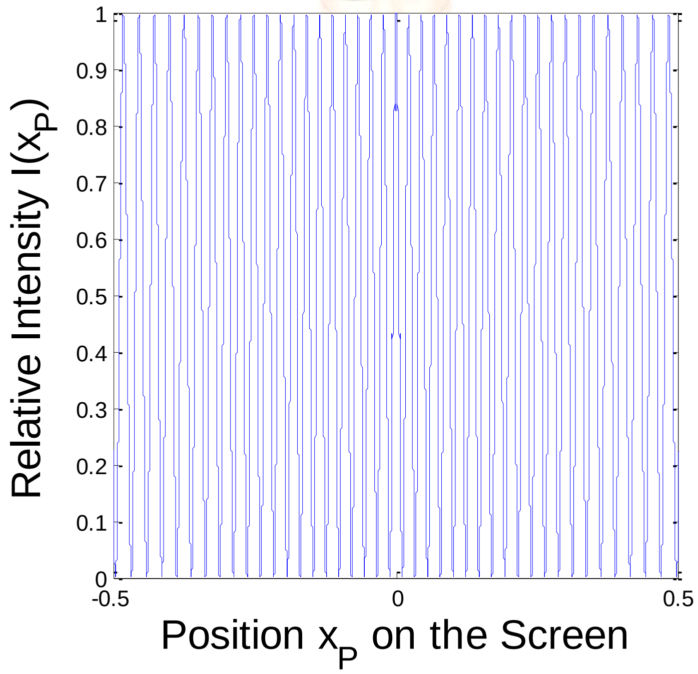


Figure 6.3.2c

7. Three Thought Experiments to illustrate the Predictions of the New Analysis

In the thought experiments described below, different values of the parameters D, d and λ are chosen depending on the nature of the waves and the apparatus design employed. The relative intensity of the regions of interference are plotted against the position of an arbitrary point on the distant screen, with respect to the center Q. The equation used for the purpose of simulation is (4.25). It predicts that as the distance from the screen center Q increases, both the spacing and the widths of the regions of constructive interference increases. Here, spacing refers to the distance between the centers of two consecutive maxima and width refers to the distance between the centers of two consecutive minima.

7.1 Using Light Waves

The apparatus consists of two barriers. The first barrier has a single slit and the second barrier has two slits. Light emanating from a source placed behind the first barrier emerges as two circular wavefronts at the second barrier. A screen is located at a certain distance from the second barrier, which has a number of parallel electrical circuits looping its surfaces. Each of these circuits consists of two parts. The first part is a very thin strip of photoelectric material that faces the second barrier. The second part is a very low voltage light source (e.g. LED) lying on the opposite side of the screen, which is directly linked to the photoelectric material. The positions of these circuits can be adjusted by means of sliding movements over the screen. When the photoelectric material is positioned over the screen at a point where waves from both the slits interfere constructively, a current of sufficient strength begins to flow through the circuit so as to turn the LED on. The placement positions of the circuits can be found using equation (4.25).

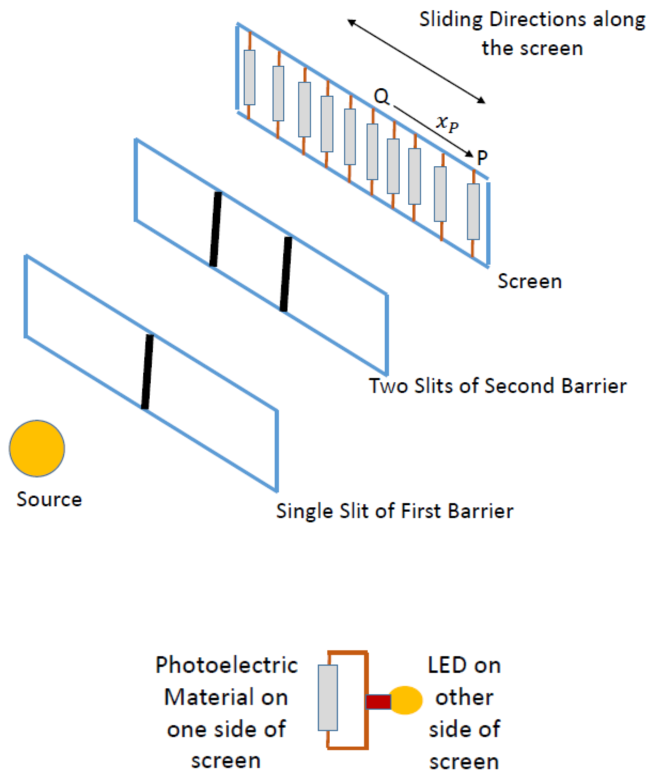


Figure 7.1a

For the purpose of a simulation the following values for D, d and λ are chosen:

$$D = 0.1\, meters, \;\; d = 3 \times 10^{-5}\, meters, \;\; \lambda = \;\; 560 \times 10^{-9}\, meters$$

The position of the circuits on the screen at which the LEDs turn on, correspond to unit relative intensity. Near the screen center, the circuits seem to be nearly uniformly spaced. However, towards the periphery their locations are more spread out from each other.

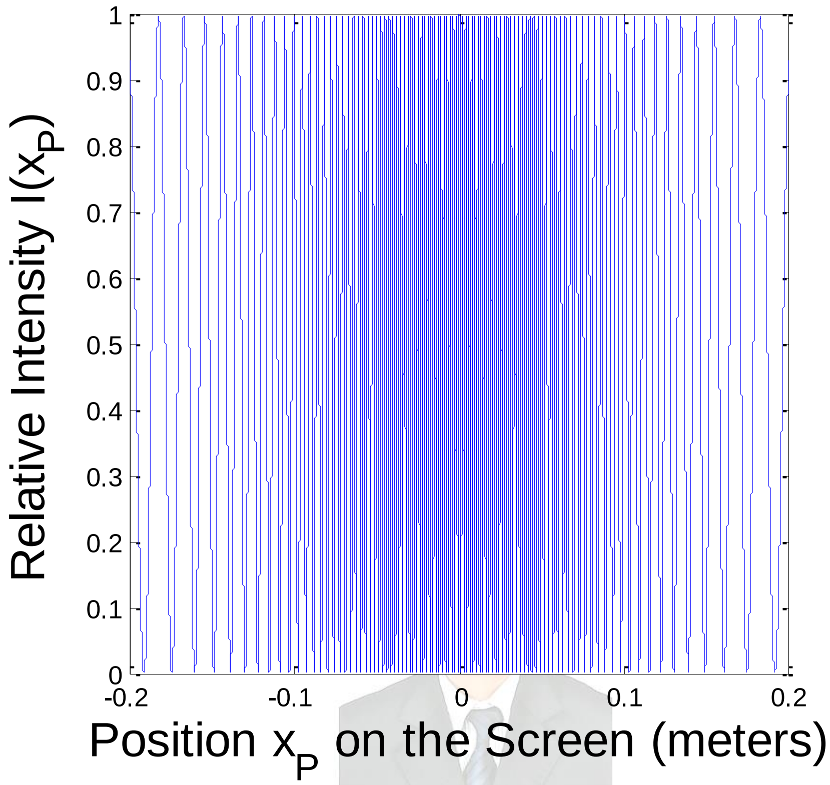


Figure 7.1b

7.2 Using Sound Waves

The construction of the apparatus used is mostly identical to that described in §7.1, except for three main differences. Firstly, the sliding circuits are replaced by sliding taut strings. The frequency of vibrations of these strings can be changed using rotatable knobs that increase or decrease the tension in them. Secondly, the two slit sources are replaced by two acoustic speakers that emit sound waves of the same frequency. Thirdly, the screen is replaced by a rail or slider, over which the strings can be slid to the desired positions. If the strings are so positioned in the regions where constructive interference of sound waves occur, they will vibrate with maximal amplitude (since intensity is proportional to the square of the amplitude). The placement positions of the strings can be found using equation (4.25).

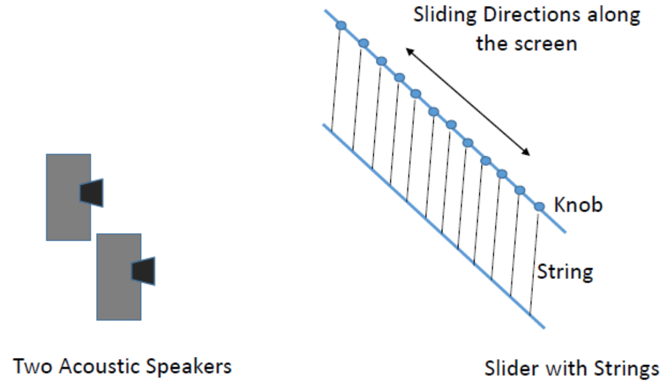


Figure 7.2a

For the purpose of a simulation the following values for D, d and λ are chosen:

$$D = 10\,meters, \quad d = 5\,meters, \quad \lambda = 1\,meter$$

The position of the strings on the slider at which they vibrate with the maximum amplitude, correspond to unit relative intensity. Near the center of the slider, the strings seem to be nearly uniformly spaced. However, towards the periphery their locations are more spread out from each other.

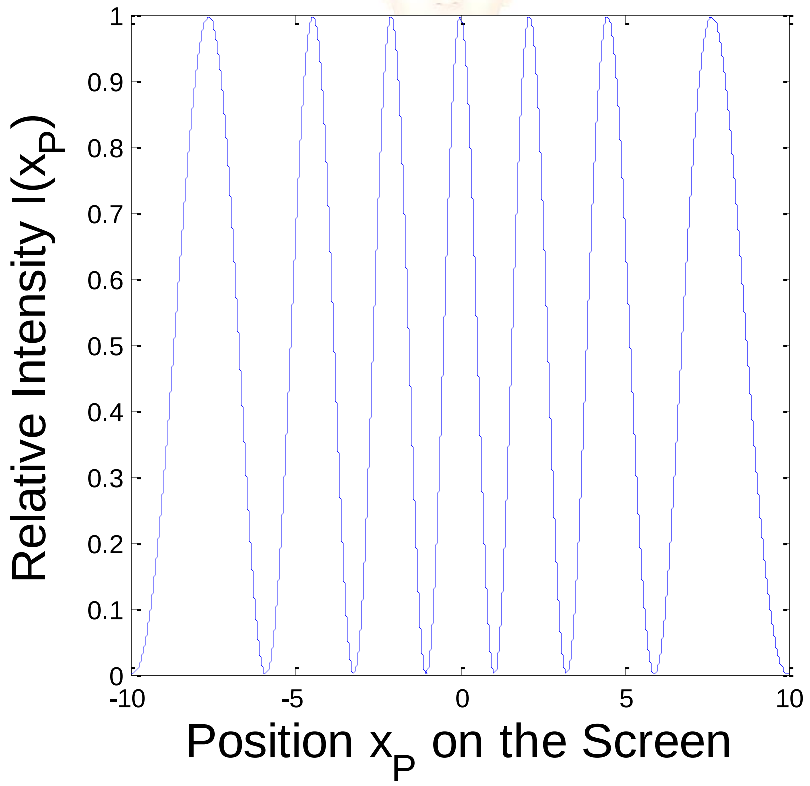


Figure 7.2b

7.3 Using Water Waves

Consider a large basin or tank filled with a very low viscosity fluid like water which is colored, say red. The basin is partitioned into two compartments by a barrier that has two narrow slits. A white paper is stuck to one extreme wall of the basin and acts as a screen. This paper acts as the screen to capture the impression of the interference pattern. On the other side of the barrier, water drops are released from a height at a steady rate so that evenly spaced ripples are generated, which then pass through the two slits. Each of the two slits act as a source of circular wavefronts, that go on to strike the screen. The red wavy pattern impressed upon the screen, should follow the same trace dictated by equation (4.25).

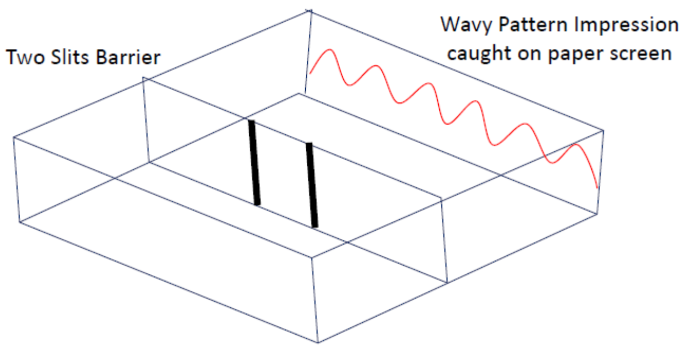


Figure 7.3a

For the purpose of a simulation the following values for D, d and λ are chosen:

$$D = 0.5\,meters, \quad d = 0.1\,meters, \quad \lambda = 0.01\,meter$$

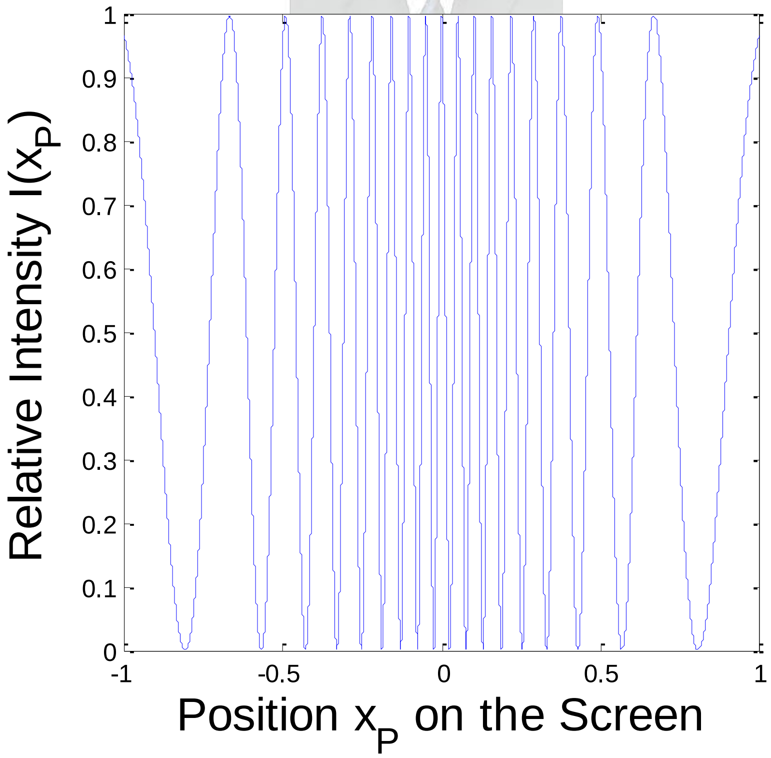


Figure 7.3b

8. Conclusions

In §6.1, the Old Analysis predicts that the fringe position on the screen has a linear dependence on fringe order, for all $n > 0$. But in the New Analysis, a non-linear relationship becomes evident by $n > 20$.
In §6.2, the Old Analysis predicts the fringe widths to be independent of fringe order. That is, all the fringes are of equal widths regardless of the magnitude of n. In the New Analysis, however, the fringe widths increase non-linearly with increasing fringe order. Also, the rate at which the fringe widths increase, is greater for the bright fringes than for the dark fringes.
In §6.3, the Old Analysis predicts that the spacing pattern of intensity variation is uniform across the entire screen. In the New Analysis, only those bright fringes located in the immediate vicinity of the screen center show uniform spacing. For fringes located more towards the periphery, a non-uniform spacing pattern of intensity variation emerges.
The above listed differences in the simulation results for the Old and New Analysis, become visibly more pronounced when the numerical values adopted are such that $D \rightarrow d$ and $d \rightarrow \lambda$. It can therefore be concluded, that the New Analysis is simply a geometric generalization of the Old Analysis, since the former approach bypasses the need for the Parallel Ray Approximation.
As a final note, any one of the three thought experiments described in §7 offers an empirical test bed for the proposed reformulation of the classical double slit analysis.

References

1. The details regarding interference phenomenon, briefly touched upon in §1 and §2, can be found in any Advanced Physics Textbook that covers Wave Optics.
2. A Mathematical Treatise on Polychronous Wavefront Computation and its Application into Modeling Neurosensory Systems, Joseph I. Thomas, 12 Mar 2014: http://vixra.org/abs/1408.0104
3. A Novel Trilateration Algorithm for Localization of a Transmitter/Receiver Station in a 2D plane using Analytical Geometry, Joseph I. Thomas, 1 Sep 2014: http://vixra.org/abs/1409.0022

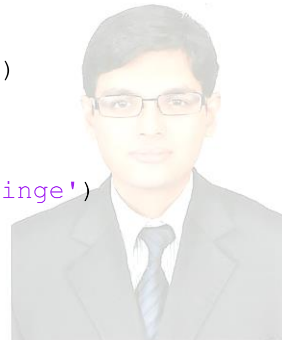
Appendix

MATLAB Coding

§ 6.1.1.

```
% Old Analysis - for combined bright and dark fringe positions
D = 1.2 ;                % distance of screen in meters
d = 3.*10.^-5    ;       % inter-slit distance in meters
lambda = 560.*10.^-9 ;   % wavelength in meters
n = linspace(0,50,51) ;
x_n_plus_bright = (n.*D.*lambda)./d  ;     % bright finges on one side of center
Q of screen
x_n_minus_bright = -(n.*D.*lambda)./d  ;   % bright fringes on other side of
center Q of screen
x_n_plus_dark = (((2.*n) + 1).*D.*lambda)./(2.*d)  ;     % dark fringes on one
side of center Q of screen
x_n_minus_dark = -(((2.*n) + 1).*D.*lambda)./(2.*d)  ;   % dark fringes on other
side of center Q of screen

scatter(n,x_n_plus_bright,'r*')
xlabel('Order of Fringe n','FontSize',18)
ylabel('Position of Fringe on the Screen','FontSize',18)
hold on
scatter(n,x_n_plus_dark,'b.')
scatter(n,x_n_minus_bright,'r*')
scatter(n,x_n_minus_dark,'b.')
axis square
axis([0 50 -1 1])
legend('Bright Fringe','Dark Fringe')
hold off
```



§ 6.1.2.

```
% New Analysis - for combined bright and dark fringe positions

D = 1.2 ;                % distance of screen in meters
d = 3.*10.^-5   ;        % inter-slit distance in meters
lambda = 560.*10.^-9 ;   % wavelength in meters
n = linspace(0,50,51) ;
A = ((n.^2).*(lambda).^2)./4 ;
B = ((D.^2).*(n.^2).*(lambda.^2))./((d.^2) - (n.^2).*(lambda.^2))    ;
C = sqrt(A + B) ;
F = ((((2.*n) + 1).^2).*(lambda).^2)./16 ;
G = ((D.^2).*(((2.*n) + 1).^2).*(lambda.^2))./(4.*(d.^2) - (((2.*n) +
1).^2).*(lambda.^2))    ;
H = sqrt(F + G) ;

x_n_plus_bright = C  ;      % bright finges on one side of center Q of screen
x_n_minus_bright = -C  ;    % bright fringes on other side of center Q of screen
x_n_plus_dark = H  ;        % dark finges on one side of center Q of screen
x_n_minus_dark = -H  ;      % dark fringes on other side of center Q of screen

scatter(n,x_n_plus_bright,'r*')
xlabel('Order of Fringe n','FontSize',18)
ylabel('Position of Fringe on the Screen','FontSize',18)
hold on
scatter(n,x_n_plus_dark,'b.')
scatter(n,x_n_minus_bright,'r*')
scatter(n,x_n_minus_dark,'b.')
axis square
axis([0 50 -1 1])
legend('Bright Fringe','Dark Fringe')
hold off
```

§ 6.2.1.

```
% Old Analysis - Bright and Dark Fringe Widths plotted together

D = 1.2 ;                 % distance of screen in meters
d = 3.*10.^-5   ;         % inter-slit distance in meters
lambda = 560.*10.^-9 ;    % wavelength in meters

n = linspace(0,50,51) ;

W_n_bright = ((n.^0).*(D.*lambda))./d ;  % Width of nth Bright Fringe
W_n_dark = ((n.^0).*(D.*lambda))./d ;    % Width of nth Dark Fringe

scatter(n,W_n_bright,'r*')
xlabel('Fringe Order n','FontSize',18)
ylabel('Fringe Width W_{n}','FontSize',18)
hold on
scatter(n,W_n_dark,'b.')
axis([0 50 0 0.03])
legend('n^{th} Bright Fringe Width','n^{th} Dark Fringe Width')
hold off
```

§ 6.2.2.

```
% New Analysis - Bright and Dark Fringe Widths plotted together

D = 1.2 ;                % distance of screen in meters
d = 3.*10.^-5   ;        % inter-slit distance in meters
lambda = 560.*10.^-9 ;   % wavelength in meters

n = linspace(1,50,50) ;

A = ((((2.*n) + 1).^2).*(lambda).^2)./16 ;
B = ((D.^2).*(((2.*n) + 1).^2).*(lambda.^2))./(4.*(d.^2) - (((2.*n) +
1).^2).*(lambda.^2))   ;
C = sqrt(A + B) ;

E = ((((2.*n) - 1).^2).*(lambda).^2)./16 ;
F = ((D.^2).*(((2.*n) - 1).^2).*(lambda.^2))./(4.*(d.^2) - (((2.*n) -
1).^2).*(lambda.^2))   ;
G = sqrt(E + F) ;

J = ((n.^2).*(lambda).^2)./4 ;
K = ((D.^2).*(n.^2).*(lambda.^2))./((d.^2) - (n.^2).*(lambda.^2))   ;
L = sqrt(J + K) ;

T = (((n-1).^2).*(lambda).^2)./4 ;
U = ((D.^2).*((n-1).^2).*(lambda.^2))./((d.^2) - ((n-1).^2).*(lambda.^2))   ;
V = sqrt(T + U) ;

P = (lambda.^2)./16 ;
Q = ((D.^2).*(lambda.^2))./(4.*(d.^2) - lambda.^2) ;

W_n_bright = C - G ;      % Width of nth Bright Fringe (where 'n' is not equal
to zero)
W_n_dark = L - V ;        % Width of nth Dark Fringe (where 'n' is not equal to
zero)
W_cent_max = 2.*sqrt(P + Q) ;   % Width of central maximum (n = 0)

scatter(n,W_n_bright,'r*')
xlabel('Fringe Order n','FontSize',18)
ylabel('Fringe WIdth W_{n}','FontSize',18)
axis([0 50 0 0.5])
hold on
scatter(0,W_cent_max,'go')
scatter(n,W_n_dark,'b.')
hold off
legend('n^{th} Bright Fringe Width','Central Maximum n = 0','n^{th} Dark Fringe
Width')
```

§ 6.3.1.

```
% Old Analysis - Intensity versus Position

D = 1.2 ;                 % distance of screen in meters
d = 3.*10.^-5    ;        % inter-slit distance in meters
lambda = 560.*10.^-9 ;    % wavelength in meters

x_p = linspace(-1.5,1.5,100000) ;

A = (pi.*d.*x_p)./(D.*lambda) ;

I_relative = (cos(A)).^2    ;

plot(x_p,I_relative)
axis square
axis([-0.25 0.25 0 1])
xlabel('Position x_{P} on the Screen','Fontsize',18)
ylabel('Relative Intensity I(x_{P})','FontSize',18)
```

§ 6.3.2.

```
% New Analysis - Intensity versus Position

D = 1.2 ;                 % distance of screen in meters
d = 3.*10.^-5    ;        % inter-slit distance in meters
lambda = 560.*10.^-9 ;    % wavelength in meters

x_p = linspace(-1.5,1.5,100000) ;

A = d.^2 + 4.*(x_p).^2 + 4.*D.^2 ;
B = A.^2 - 16.*((x_p).^2).*d.^2 ;
C = pi./(sqrt(2).*lambda)    ;

I_relative = (cos(C.*sqrt(A - sqrt(B)))).^2    ;

plot(x_p,I_relative)
axis square
axis([-0.25 0.25 0 1])
xlabel('Position x_{P} on the Screen','Fontsize',18)
ylabel('Relative Intensity I(x_{P})','FontSize',18)
```