THE MAR REDUCED FORM OF A NATURAL NUMBER

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INTRODUCTION

In this book I define a function which allows the reduction to any non-null positive integer to one of the digits 1, 2, 3, 4, 5, 6, 7, 8 or 9. The utility of this enterprise is well-known in arithmetic; the function defined here differs apparently insignificant but perhaps essentially from the function modulo 9 in that is not defined on 0, also can't have the value 0; essentially, the mar reduced form of a non-null positive integer is the digital root of this number but with the important distinction that is defined as a function such it can be easily used in various applications (divizibility problems, Diophantine equations), a function defined only on the operations of addition and multiplication not on the operations of subtraction and division. Some of the results obtained with this tool are a proof of Fermat's last Theorem, cases n = 3 and n = 4, using just integers, no complex numbers and a Diophantine analysis of perfect numbers.

Note: I understand, in this book, the numbers denoted by "abc" as the numbers where a, b, c are digits, and the numbers denoted by "a*b*c" as the products of the numbers a, b, c.

SUMMARY

1. THE DEFINITION OF THE MAR REDUCED FORM

Let $a = a_1a_2...a_m...a_n$ be a natural number greater or equal to 1.

We denote by mar a the mar reduced form of a, where:

: mar a = a, for a equal to 1, 2, 3, 4, 5, 6, 7, 8 or 9;

: mar a = $(((a_1 \oplus a_2) \oplus ... \oplus a_m) \oplus ... \oplus a_n)$, for $a \geq 10$.

$a_1, a_2, ..., a_m, ..., a_n$ are, obviously, digits so $a_m \oplus a_n$ is a map defined of the Cartesian product:
$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \times \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

We define the composition law $\oplus : \{0, 1, ..., 9\} \times \{0, 1, ..., 9\} \to \{0, 1, ..., 9\}$, $(a_m, a_n) \to a_m \oplus a_n$, by the operation table of $\oplus$:

$\oplus$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	1
2	2	3	4	5	6	7	8	9	1	2
3	3	4	5	6	7	8	9	1	2	3
4	4	5	6	7	8	9	1	2	3	4
5	5	6	7	8	9	1	2	3	4	5
6	6	7	8	9	1	2	3	4	5	6
7	7	8	9	1	2	3	4	5	6	7
8	8	9	1	2	3	4	5	6	7	8
9	9	1	2	3	4	5	6	7	8	9

Thus, we have mar a is equal to 1, 2, 3, 4, 5, 6, 7, 8 or 9, with at least a_1 nonzero.

Example: mar 178523 = $(((((1 \oplus 7) \oplus 8) \oplus 5) \oplus 2) \oplus 3) =$

$= ((((8 \oplus 8) \oplus 5) \oplus 2) \oplus 3) =$

$= ((3 \oplus 2) \oplus 3) = 5 \oplus 3 = 8$

The $\oplus$ composition law is commutative on $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$: $a_m \oplus a_n = a_n \oplus a_m$ (from the operation table:
: $0 \oplus 0 = 0 \oplus 0 = 0$, $1 \oplus 0 = 0 \oplus 1 = 1, ...,$ $1 \oplus 2 = 2 \oplus 1 = 3$
: $1 \oplus 3 = 3 \oplus 1 = 4, ..., 7 \oplus 8 = 8 \oplus 7 = 6$, $7 \oplus 9 = 9 \oplus 7 = 7$
: $9 \oplus 0 = 0 \oplus 9 = 9$, $9 \oplus 1 = 1 \oplus 9 = 1, ...,$ $9 \oplus 9 = 9 \oplus 9 = 9$)

The same composition law has a neutral element, which is 0:

$a_m \oplus 0 = 0 \oplus a_m = a_m$ (also from the operation table)

The $\oplus$ composition law is associative on {0, 1,..., 9}:
$(a_p \oplus a_m) \oplus a_n = a_p \oplus (a_m \oplus a_n)$, for any a_p, a_m, a_n equal to 1, 2, 3, 4, 5, 6, 7, 8 or 9 (also from the table:
: $(1 \oplus 0) \oplus 9 = 1 \oplus (0 \oplus 9) = 1$
: $(0 \oplus 7) \oplus 4 = 0 \oplus (7 \oplus 4) = 2$)

As the $\oplus$ composition law is associative, we may write the mar reduced form without using parentheses:

: mar a = a, for a equal to 1, 2, 3, 4, 5, 6, 7, 8 or 9;
: mar a = $a_1 \oplus a_2 \oplus ... \oplus a_m \oplus ... \oplus a_n$, for $a \geq 10$.

The pair ({0, 1, ..., 9}, $\oplus$) is a commutative monoid (the composition law $\oplus$ on the set {0, 1, ..., 9} satisfies the associability, commutability and neutral element axioms.

2. THE SUM OF THE MAR REDUCED FORM OF TWO NATURAL NUMBERS

The set {1, 2, 3, 4, 5, 6, 7, 8, 9} is a stable part of the set {0, 1, ...,9} with respect to the composition law $\oplus$

Thus we may define the sum mar a $\oplus$ mar b as a map defined on {1, 2, ...,9} × {1, 2,..., 9} → {1, 2, ...,9}
(where b is a natural number, say $b = b_1b_2...b_p$, $b \geq 1$)

The addition table for mar a $\oplus$ mar b will be:

$\oplus$	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	1
2	3	4	5	6	7	8	9	1	2
3	4	5	6	7	8	9	1	2	3
4	5	6	7	8	9	1	2	3	4
5	6	7	8	9	1	2	3	4	5
6	7	8	9	1	2	3	4	5	6
7	8	9	1	2	3	4	5	6	7
8	9	1	2	3	4	5	6	7	8
9	1	2	3	4	5	6	7	8	9

The composition law induced by $\oplus$ on {1, 2, 3, 4, 5, 6, 7, 8, 9}, which means the composition law $\{1, ..., 9\} \times \{1, ..., 9\} \to \{1, ..., 9\}$, (mar a, mar b) $\to$ mar a $\oplus$ mar b, has the following properties:

The composition law $\oplus$ on {1, 2,... , 9} is commutative: mar a $\oplus$ mar b = mar b $\oplus$ mar a (from the operation table).

The composition law $\oplus$ on {1, 2, ..., 9} has a neutral element, and this is 9: mar a $\oplus$ 9 = 9 $\oplus$ mar a = mar a (from the operation table).

Any element mar a $\in$ {1, 2, ..., 9} is invertible with respect to the given composition law (from the table: $1 \oplus 8 = 9 = 8 \oplus 1$; $2 \oplus 7 = 9 = 7 \oplus 2$; $3 \oplus 6 = 9 = 6 \oplus 3$; $4 \oplus 5 = 9 = 5 \oplus 4$).

If we denote by (mar a)′ the inverse of mar a, then for any mar a $\in$ {1, ..., 9} we have an "opposite" (mar a)′ such that mar a $\oplus$ (mar a)′ = (mar a)′ $\oplus$ mar a = 9; also, (mar a $\oplus$ mar b)′ = (mar a)′ $\oplus$ (mar b)′.

The composition law $\oplus$ on {1, ..., 9} is associative:
(mar a $\oplus$ mar b) $\oplus$ mar c = mar a $\oplus$ (mar b $\oplus$ mar c)
(where c is a natural number greater or equal to 1).
Example:
(mar 17 $\oplus$ mar 130) $\oplus$ mar 9 = mar 17 $\oplus$ (mar 130 $\oplus$ mar 9)
$\Leftrightarrow (8 \oplus 4) \oplus 9 = 8 \oplus (4 \oplus 9) \Leftrightarrow 3 \oplus 9 = 8 \oplus 4 = 3$.

The pair ({1, 2, 3, 4, 5, 6, 7, 8, 9}, $\oplus$) is a commutative field (the composition law $\oplus$ on the set {1, ..., 9} satisfies the associability, commutability, neutral element, and invertible elements).

It follows that the simplification rules apply:

: mar a $\oplus$ mar b = mar a $\oplus$ mar c $\Rightarrow$ mar b = mar c (to the left);
: mar a $\oplus$ mar b = mar c $\oplus$ mar b $\Rightarrow$ mar a = mar c (to the right).

Also (as we may see from the table too), the equations
: mar a $\oplus$ mar x = mar b and mar y $\oplus$ mar a = mar b
have unique solutions in {1, ...,9}, which are:
: mar x = (mar a)′ $\oplus$ mar b, and respectively
: mar y = mar b $\oplus$(mar a)′,
where (mar a)′ is the reverse of mar a.

3. THE PRODUCT OF THE MAR REDUCED FORM OF TWO NATURAL NUMBERS

We define the product mar a ⊗ mar b, as a map defined on {1, ..., 9} × {1, ..., 9} with values in {1, ..., 9}.

We define the composition law ⊗ on the set {1, ..., 9} through the following table:

⊗	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	1	3	5	7	9
3	3	6	9	3	6	9	3	6	9
4	4	8	3	7	2	6	1	5	9
5	5	1	6	2	7	3	8	4	9
6	6	3	9	6	3	9	6	3	9
7	7	5	3	1	8	6	4	2	9
8	8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9	9

The composition law {1, ...,9} × {1, ..., 9} → {1, ..., 9},
(mar a, mar b) → mar a ⊗ mar b is commutative:
mar a ⊗ mar b = mar b ⊗ mar a, for any a, b nonzero naturals.

Ex.: 5 ⊗ 6 = 6 ⊗ 5 = 3, ..., 7 ⊗ 4 = 4 ⊗ 7 = 1, (...)
(all other combinations may be verified from the table).

The given composition law is associative:
(mar a ⊗ mar b) ⊗ mar c = mar a ⊗ (mar b ⊗ mar c), for any natural c, c ≥ 1.

Example:
(mar 15 ⊗ mar 3) ⊗ mar 113 = mar 15 ⊗ (mar 3 ⊗ mar 113) ⇔
⇔ (6 ⊗ 3) ⊗ 5 = 6 ⊗(3 ⊗ 5) ⇔ 9 ⊗ 5 = 6 ⊗ 6 ⇔ 9 = 9.

The composition law ⊗ on {1, ..., 9} has a neutral element and this is 1: mar a ⊗ 1 = 1 ⊗ mar a = mar a (from the table: 2 ⊗ 1 = 1 ⊗ 2 = 2, ..., 7 ⊗ 1 = 1 ⊗ 7 = 7, ...)

The pair ({1, ..., 9}, ⊗) is a commutative monoid (it satisfies the associability, commutability, and neutral element properties) in which the following computation rules apply:

: (mar a)^0 = 1; (mar a)^1 = mar a; (mar a)^2 = mar a ⊗ mar a;
(mar a)^3 = (mar a)^2 ⊗ mar a = mar a ⊗ mar a ⊗ mar a;
(...); (mar a)^n = (mar a)^(n − 1) ⊗ mar a.

Also, for any natural numbers m and n, we have:
: $(\text{mar } a)^n \otimes (\text{mar } a)^m = (\text{mar } a)^{(n+m)}$ and $((\text{mar } a)^n)^m = (\text{mar } a)^{(n*m)}$.

The operation ⊗ (multiplication) is distributive with respect to the operation ⊕ (addition); we have:
: mar a ⊗ (mar b ⊕ mar c) = mar a ⊗ mar b ⊕ mar a ⊗ mar c and
: (mar a ⊕ mar b) ⊗ mar c = mar a ⊗ mar c ⊕ mar b ⊗ mar c

Example: mar 131 ⊗ (mar 22 ⊕ mar 7141) =
= mar 131 ⊗ mar 22 ⊕ mar 131 ⊗ mar 7141 ⇔
5 ⊗(4 ⊕ 4 = 5 ⊗ 4 ⊕ 5 ⊗ 4 ⇔ 5 ⊗ 8 = 2 ⊗ 2 ⇔ 4 = 4
(all possible combinations of the type mar a ⊗ (mar b ⊕ mar c): 1 ⊗ (2 ⊕ 3), ..., 9 ⊗ (9 ⊕ 9) may be verified by using the tables of the operations ⊕ and ⊗).

The set {1, 2, 3, 4, 5, 6, 7, 8, 9}, together with the composition laws ⊕ and ⊗, make up a commutative ring because:
: ({1, ..., 9}, ⊕) is a commutative field;
: ({1, ..., 9}, ⊗) is a commutative monoid;
: The multiplication (⊗) is distributive with respect to addition (⊕)

In the ring ({1, ...,9}, ⊕, ⊗) we have the following computation rules:
: (mar a)′ ⊗ (mar b)′ = mar a ⊗ mar b, for any a, b nonzero naturals, where (mar a)′ and (mar b)′ are the inverse of mar a and mar b respectively. Example: (mar 15)′ ⊗ (mar 221)′ = mar 15 ⊗ mar 221 ⇔ 3 ⊗ 4 = 6 ⊗ 5 ⇔ 3 = 3;
: mar a ⊗ 9 = 9 ⊗ mar a = 9, for any natural a, $a \geq 1$.

4. MAR REDUCED FORM PROPERTIES AND MAR REDUCED FORM CLASSES

We are highlighting now the following obvious properties of the mar reduced form:

Let $a = a_1a_2...a_n$, $b = b_1b_2...b_p$, $c = c_1c_2...c_r$, where a, b, c nonzero naturals; we have mar a = $a_1 \oplus a_2 \oplus ...\oplus a_n$, mar b = $b_1 \oplus b_2 \oplus ...\oplus b_p$, mar c = $c_1 \oplus c_2 \oplus ...\oplus c_r$. Then:

: a = b ⇒ mar a = mar b (a = b ⇒ $a_1 = b_1$, $a_2 = b_2$, ...,$a_n = b_p$, and n = p);
: mar (mar a) = mar a;
: a + b = c ⇒ mar (a + b) = mar c

We may divide the set of all nonzero natural numbers in nine classes, after the value of their mar reduced form like this:

: M_1 = {a natural, a > 0/ mar a = 1}
: M_2 = {a natural, a > 0/ mar a = 2}
(...)
: M_9 = {a natural, a > 0/ mar a = 9}

Any nonzero natural number belongs to one and only one of these classes (the mar reduced form of a natural number being obviously unique).

The mar reduced form classes are thus disjoint.

We may call the numbers a and b congruent if they have the same mar reduced form; we write $a \equiv b$ if mar a = mar b, if a and b are part of the same mar reduced form class M_1, M_2 ,..., M_8 or M_9
We have: $a \equiv a$; $a \equiv b \Rightarrow b \equiv a$; $a \equiv b$, $b \equiv c \Rightarrow a \equiv c$

Obviously, all the numbers belonging to the same class are congruent between them. Example: mar 17 = mar 224 = 8 $\Rightarrow 17 \equiv 224 \equiv 8 \Rightarrow 8, 17, 224 \in M_8$

We may define the addition and multiplication of the mar reduced form classes by:

: $M_x + M_y = M_z$ and $M_x{*}M_y = M_w$, where $x, y, z, w \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, such that: mar a $\oplus$ mar b = c and mar a $\otimes$ mar b = d, where $a \in M_x$, $b \in M_y$, $c \in M_z$ and $d \in M_w$

Obviously, mar x $\oplus$ mar y = mar z and mar x $\otimes$ mar y = mar w; x, y, z, w being the representatives of the classes M_x, M_y, M_z and M_w (of course $x \in M_x$ etc.)

Example:
$M_5 + M_7 = M_3$ and $M_5{*}M_7 = M_8$, because mar 5 $\oplus$ mar 7 = 5 $\oplus$ 7 = 3 and mar 5 $\otimes$ mar 7 = 5 $\otimes$ 7 = 8; (3, 5, 7 and 8 are the representatives of the classes M_3, M_5, M_7 and M_8)

We won’t be writing any tables for the addition and multiplication of the mar reduced form classes, as these are identical to the tables for the addition and multiplication of the representatives of the classes.

The set $\{M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9\}$ of the classes of mar reduced forms, together with the addition and multiplication of the mar reduced form classes make up a commutative ring, the ring of the mar reduced form classes.

5. TWO THEOREMS

In order to effectively use the mar reduced form in arithmetic problems, the following two theorems, which I named simply, The Sum Theorem and The Product Theorem are vital:

The Sum Theorem: The mar reduced form of two nonzero natural numbers equals the sum of the mar reduced form of the two numbers:

$$\boxed{\text{mar } (a + b) = \text{mar } a \oplus \text{mar } b}$$

The Product Theorem: The mar reduced form of the product of two nonzero natural numbers equals the product of the mar reduced forms of the two numbers:

$$\boxed{\text{mar } (a*b) = \text{mar } a \otimes \text{mar } b}$$

Proof of The Sum Theorem:

We initially take a particular case, which we shall extrapolate afterwards:

Let: $a = a_1a_2a_3$; $a_1, a_2, a_3 \in \{0, ..., 9\}$; $a_1 > 0$;
$b = b_1b_2b_3$; $b_1, b_2, b_3 \in \{0, ..., 9\}$; $b_1 > 0$;
$c = c_0c_1c_2c_3$; $c_0, c_1, c_2, c_3 \in \{0, ..., 9\}$; $c_0 \geq 0$,
such that $a + b = c$. We shall prove that mar c = mar $a \oplus$ mar b:

Case I. : $a_3 + b_3 < 10 \Rightarrow a_3 + b_3 = c_3$

Case I.A. : $a_2 + b_2 < 10 \Rightarrow a_2 + b_2 = c_2$

Case I.A.1. : $a_1 + b_1 < 10 \Rightarrow a_1 + b_1 = c_1 \Rightarrow c_0 = 0$

Thus, we have $c_1c_2c_3 = (a_1 + b_1)(a_2 + b_2)(a_3 + b_3) \Rightarrow c_1c_2c_3 = (a_1 \oplus b_1)(a_2 \oplus b_2)(a_3 \oplus b_3)$, because $a + b = a \oplus b$ for $a + b < 10$.

But mar $c_1c_2c_3$ = mar $(a_1 + b_1)(a_2 + b_2)(a_3 + b_3) \Leftrightarrow$ mar $c_1c_2c_3$ = mar $(a_1 \oplus b_1)(a_2 \oplus b_2)(a_3 \oplus b \Leftrightarrow c_1 \oplus c_2 \oplus c_3 = a_1 \oplus b_1 \oplus a_2 \oplus b_2 \oplus a_3 \oplus b_3 \Leftrightarrow$ mar $c = (a_1 \oplus a_2 \oplus a_3) \oplus (b_1 \oplus b_2 \oplus b_3)$ $\Leftrightarrow$ mar $(a + b)$ = mar $a \oplus$ mar b

Case I.A.2. : $a_1 + b_1 \geq 10 \Rightarrow a_1 + b_1 = c_1 + 10 \Rightarrow c_0 = 1$

We have $c_0c_1c_2c_3 = 1(a_1 + b_1 - 10)(a_2 + b_2)(a_3 + b_3) \Rightarrow$ mar $c_0c_1c_2c_3 =$ mar $1(a_1 + b_1 - 10)(a_2 + b_2)(a_3 + b_3) \Rightarrow$
$c_0 \oplus c_1 \oplus c_2 \oplus c_3 = 1 \oplus (a_1 + b_1 - 10) \oplus a_2 \oplus b_2 \oplus a_3 \oplus b_3$

But $1 \oplus (a_1 + b_1 - 10) = a_1 \oplus b_1$ (from the table) for any a_1 and b_1 such that $a_1 + b_1 \geq 10$

Case I.B. : $a_2 + b_2 \geq 10 \Rightarrow c_2 = a_2 + b_2 - 10$

Case I.B.1. : $a_1 + b_1 \geq 9 \Rightarrow c_1 = a_1 + b_1 - 9 \Rightarrow c_0 = 1$

So, we have $c_0c_1c_2c_3 = 1(a_1 + b_1 - 9)(a_2 + b_2 - 10)(a_3 + b_3)$ $\Rightarrow$ mar $c_0c_1c_2c_3 = 1 \oplus (a_2 + b_2 - 10) \oplus (a_1 + b_1 - 9) \oplus a_3 \oplus b_3 \Rightarrow c_0 \oplus c_1 \oplus c_2 \oplus c_3 = a_2 \oplus b_2 \oplus (a_1 + b_1 - 9) \oplus a_3 \oplus b_3$

But $a_1 + b_1 - 9 = a_1 \oplus b_1$ for any a_1 and b_1 such that $a_1 + b_1 > 9$, and $a_1 + b_1 - 9 = 0$ for a_1 and b_1 such that $a_1 + b_1 = 9$

Thus, we have $c_0 \oplus c_1 \oplus c_2 \oplus c_3 = a_2 \oplus b_2 \oplus a_1 \oplus b_1 \oplus a_3 \oplus b_3 \Rightarrow$ mar c = mar a $\oplus$ mar b $\Leftrightarrow$ mar (a + b) = mar a $\oplus$ mar b, or we have $c_0 \oplus c_1 \oplus c_2 \oplus c_3 = a_2 \oplus b_2 \oplus 0 \oplus a_3 \oplus b_3$, which is equivalent with $c_0 \oplus c_1 \oplus c_2 \oplus c_3 = a_2 \oplus b_2 \oplus a_3 \oplus b_3 \oplus 9$, because $e = (a_2 \oplus b_2 \oplus a_3 \oplus b_3) > 0$, and $e \oplus 0 = e \oplus 9 = e$, for any $e \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. So $c_0 \oplus c_1 \oplus c_2 \oplus c_3 = a_2 \oplus b_2 \oplus a_3 \oplus b_3 \oplus 9 = a_2 \oplus b_2 \oplus a_3 \oplus b_3 \oplus a_1 \oplus b_1 \Leftrightarrow$ mar c = mar a $\oplus$ mar b $\Leftrightarrow$ mar (a + b) = mar a $\oplus$ mar b

Case I.B.2. : $a_1 + b_1 < 9 \Rightarrow c_1 = a_1 + b_1 + 1 \Rightarrow c_0 = 0$

We have $c_1c_2c_3 = 1(a_1 + b_1 + 1)(a_2 + b_2 - 10)(a_3 + b_3) \Rightarrow$ mar $c_1c_2c_3 = (a_1 + b_1 + 1) \oplus (a_2 + b_2 - 10) \oplus a_3 \oplus b_3$

But $(a_1 + b_1 + 1) \oplus (a_2 + b_2 - 10) = a_1 \oplus a_2 \oplus b_1 \oplus b_2$ for $a_1 + b_1 < 9$ and $a_2 + b_2 \geq 10$ (from the table)

So $c_1 \oplus c_2 \oplus c_3 = a_1 \oplus a_2 \oplus a_3 \oplus b_1 \oplus b_2 \oplus b_3 \Rightarrow$ mar c = mar a $\oplus$ mar b $\Leftrightarrow$ mar(a + b) = mar a $\oplus$ mar b

Case II. : $a_3 + b_3 \geq 10 \Rightarrow a_3 + b_3 = c_3 + 10 \Leftrightarrow c_3 = a_3 + b_3 - 10$

Case II.A. : $a_2 + b_2 \geq 9 \Rightarrow c_2 = a_2 + b_2 - 9$

Case II.A.1. : $a_1 + b_1 \geq 9 \Rightarrow c_1 = a_1 + b_1 - 9 \Rightarrow c_0 = 1$

We have $c_0c_1c_2c_3 = 1(a_1 + b_1 - 9)(a_2 + b_2 - 9)(a_3 + b_3 - 10)$ $\Rightarrow c_0 \oplus c_1 \oplus c_2 \oplus c_3 = 1 \oplus (a_3 + b_3 - 10) \oplus (a_1 + b_1 - 9) \oplus (a_2 + b_2 - 9)$

But $1 \oplus (a_3 + b_3 - 10) = a_3 \oplus b_3$ for $a_3 + b_3 \geq 10$;
$(a_1 + b_1 - 9) = a_1 \oplus b_1$ for $a_1 + b_1 > 9$ and
$(a_1 + b_1 - 9) = 0$ for $a_1 + b_1 = 9$

So $c_0 \oplus c_1 \oplus c_2 \oplus c_3 = a_1 \oplus b_1 \oplus a_2 \oplus b_2 \oplus a_3 \oplus b_3$ for $a_1 + b_1 > 9$, $a_2 + b_2 > 9 \Leftrightarrow \text{mar } (a + b) = \text{mar } a \oplus \text{mar } b$

Or $c_0 \oplus c_1 \oplus c_2 \oplus c_3 = a_3 \oplus b_3 \oplus 0 \oplus 0$ for $a_1 + b_1 = 9$; $a_2 + b_2 = 9 \Leftrightarrow c_0 \oplus c_1 \oplus c_2 \oplus c_3 = a_3 \oplus b_3 \oplus 9 \oplus 9 \Leftrightarrow c_0 \oplus c_1 \oplus c_2 \oplus c_3 = a_3 \oplus b_3 \oplus 9 \oplus 9$ ($a_3 > 0$ and $b_3 > 0$, so $a_3 + 9 = a_3 + 0 = a_3$ and $b_3 + 9 = b_3 + 0 = b_3$)

So $c_0 \oplus c_1 \oplus c_2 \oplus c_3 = a_3 \oplus b_3 \oplus (a_1 + b_1) \oplus (a_2 + b_2) = a_3 \oplus b_3 \oplus a_1 \oplus b_1 \oplus a_2 \oplus b_2 \Leftrightarrow \text{mar } (a + b) = \text{mar } a \oplus \text{mar } b$ ($a_1 + b_1 = a_1 \oplus b_1$ for $a_1 + b_1 = 9$, actually for any $a_1 + b_1 < 10$)

Case II.A.2. : $a_1 + b_1 < 9 \Rightarrow c_1 = a_1 + b_1 + 1 \Rightarrow c_0 = 0$

We have $c_1c_2c_3 = (a_1 + b_1 + 1)(a_2 + b_2 - 9)(a_3 + b_3 - 10) \Leftrightarrow c_1 \oplus c_2 \oplus c_3 = (a_1 + b_1 + 1) \oplus (a_3 + b_3 - 10) \oplus (a_2 + b_2 - 9)$

But $a_2 + b_2 - 9 = a_2 \oplus b_2$ for $a_2 + b_2 > 9$ or $a_2 + b_2 - 9 = 0$ for $a_2 + b_2 = 9$

and $(a_1 + b_1 + 1) \oplus (a_3 + b_3 - 10) = a_1 \oplus b_1 \oplus a_3 \oplus b_3$ for $a_1 + b_1 < 9$ and $a_3 + b_3 \geq 10$

So $\text{mar } c = \text{mar } a \oplus \text{mar } b \Leftrightarrow \text{mar } (a + b) = \text{mar } a \oplus \text{mar } b$

Case II.B : $a_2 + b_2 < 9 \Rightarrow c_2 = a_2 + b_2 + 1$

Case II.B.1. : $a_1 + b_1 \geq 10 \Rightarrow c_1 = a_1 + b_1 - 10 \Rightarrow c_0 = 1$

We have $c_0c_1c_2c_3 = 1(a_1 + b_1 - 10)(a_2 + b_2 + 1)(a_3 + b_3 - 10)$

But $1 \oplus (a_1 + b_1 - 10) = a_1 \oplus b_1$ (from the table for $a_1 + b_1 \geq 10$)
and $(a_2 + b_2 + 1) \oplus (a_3 + b_3 - 10) = a_2 \oplus b_2 \oplus a_3 \oplus b_3$ for $a_2 + b_2 < 9$ and $a_3 + b_3 \geq 10$

So $\text{mar } (a + b) = \text{mar } a \oplus \text{mar } b$

Case II.B.2. : $a_1 + b_1 < 10 \Rightarrow c_1 = a_1 + b_1 \Rightarrow c_0 = 0$

We have $c_1c_2c_3 = (a_1 + b_1)(a_2 + b_2 + 1)(a_3 + b_3 - 10)$

But $(a_2 + b_2 + 1) \oplus (a_3 + b_3 - 10) = a_2 \oplus b_2 \oplus a_3 \oplus b_3$ for $a_2 + b_2 < 9$ and $a_3 + b_3 \geq 10$

Conclusion:

We proved that mar $(a + b)$ = mar $a \oplus$ mar b for any a and b, $a = a_1a_2a_3$ and $b = b_1b_2b_3$. We had these cases:

: $c_0 = 0$:

$c_1c_2c_3 = (a_1 + b_1)(a_2 + b_2)(a_3 + b_3)$, $a_1 + b_1 < 10$, $a_2 + b_2 < 10$, $a_3 + b_3 < 10$;

$c_1c_2c_3 = (a_1 + b_1 + 1)(a_2 + b_2 - 10)(a_3 + b_3)$, $a_1 + b_1 < 9$, $a_2 + b_2 \geq 10$, $a_3 + b_3 < 10$;

$c_1c_2c_3 = (a_1 + b_1 + 1)(a_2 + b_2 - 9)(a_3 + b_3 - 10)$, $a_1 + b_1 < 9$, $a_2 + b_2 \geq 9$, $a_3 + b_3 \geq 10$;

$c_1c_2c_3 = (a_1 + b_1)(a_2 + b_2 + 1)(a_3 + b_3 - 10)$, $a_1 + b_1 < 10$, $a_2 + b_2 < 9$, $a_3 + b_3 \geq 10$;

: $c_0 = 1$:

$c_0c_1c_2c_3 = 1(a_1 + b_1 - 10)(a_2 + b_2)(a_3 + b_3)$, $a_1 + b_1 \geq 10$, $a_2 + b_2 < 10$, $a_3 + b_3 < 10$;

$c_0c_1c_2c_3 = 1(a_1 + b_1 - 9)(a_2 + b_2 - 10)(a_3 + b_3)$, $a_1 + b_1 \geq 9$, $a_2 + b_2 \geq 10$, $a_3 + b_3 < 10$;

$c_0c_1c_2c_3 = 1(a_1 + b_1 - 9)(a_2 + b_2 - 9)(a_3 + b_3 - 10)$, $a_1 + b_1 \geq 9$, $a_2 + b_2 \geq 9$, $a_3 + b_3 \geq 10$;

$c_0c_1c_2c_3 = 1(a_1 + b_1 - 10)(a_2 + b_2 + 1)(a_3 + b_3 - 10)$, $a_1 + b_1 \geq 10$, $a_2 + b_2 < 9$, $a_3 + b_3 \geq 10$;

Let $a = a_1a_2...a_n...a_m$, $b = b_1b_2...b_n...b_m$, $c = c_1c_2...c_n...c_m$, where $a + b = c$.

Take the case $c_0 = 0$:

We have $a_1a_2...a_n...a_m + b_1b_2...b_n...b_m = c_1c_2...c_n...c_m$;

The mar reduced form of the natural number $c_1c_2...c_n...c_m$ is a mar sum of "bulks" of the type:

(1) $(a_n + b_n)$, $a_n + b_n < 10$

(2) $(a_n + b_n + 1) \oplus (a_{n+1} + b_{n+1} - 10)$, $a_n + b_n < 9$, $a_{n+1} + b_{n+1} \geq 10$ and

(3) $(a_n + b_n + 1) \oplus (a_{n+1} + b_{n+1} - 9) \oplus \ldots \oplus (a_{n+p-1} + b_{n+p-1} - 9) \oplus (a_{n+p} + b_{n+p} - 10)$, $a_n + b_n < 9$, $a_{n+1} + b_{n+1} \geq 9$, ..., $a_{n+p-1} + b_{n+p-1} \geq 9$, $a_{n+p} + b_{n+p} \geq 10$

The mar sum of the "bulk" (i) is $a_n \oplus b_n$

The mar sum of the "bulk" (ii) is $a_n \oplus b_n \oplus a_{n+1} \oplus b_{n+1}$ and

The mar sum of the "bulk" (iii) is $a_n \oplus b_n \oplus a_{n+1} \oplus b_{n+1} \oplus \ldots \oplus a_{n+p-1} \oplus b_{n+p-1} \oplus a_{n+p} \oplus b_{n+p}$

We proved that mar c = mar a $\oplus$ mar b $\Leftrightarrow$ mar (a + b) = mar a $\oplus$ mar b for any a + b = c of the given type: a, b and c have the same number of digits.

Take the case $c_0 = 1$:

We have $a_1a_2\ldots a_n\ldots a_m + b_1b_2\ldots b_n\ldots b_m = c_0c_1c_2\ldots c_n\ldots c_m$;

The mar reduced form of the natural number $c_0c_1c_2\ldots c_n\ldots c_m$ is a mar sum of "bulks" of the type:

(1) $(a_n + b_n)$, $a_n + b_n < 10$

(2) $(a_n + b_n + 1) \oplus (a_{n+1} + b_{n+1} - 10)$, $a_n + b_n < 9$, $a_{n+1} + b_{n+1} \geq 10$

(3) $(a_n + b_n + 1) \oplus (a_{n+1} + b_{n+1} - 9) \oplus \ldots \oplus (a_{n+p-1} + b_{n+p-1} - 9) \oplus (a_{n+p} + b_{n+p} - 10)$, $a_n + b_n < 9$, $a_{n+1} + b_{n+1} \geq 9$, ..., $a_{n+p-1} + b_{n+p-1} \geq 9$, $a_{n+p} + b_{n+p} \geq 10$

(4) $1 \oplus (a_n + b_n - 10)$, $a_n + b_n \geq 10$
and

(5) $1 \oplus (a_n + b_n - 9) \oplus \ldots \oplus (a_{n+p-1} + b_{n+p-1} - 9) \oplus (a_{n+p} + b_{n+p} - 10)$, $a_n + b_n \geq 9$, ..., $a_{n+p-1} + b_{n+p-1} \geq 9$, $a_{n+p} + b_{n+p} \geq 10$

The mar sum of the "bulk" (4) is $a_n \oplus b_n$

The mar sum of the "bulk" (5) is $a_n \oplus b_n \oplus \ldots \oplus a_{n+p-1} \oplus b_{n+p-1} \oplus a_{n+p} \oplus b_{n+p}$

We proved that mar c = mar a $\oplus$ mar b $\Leftrightarrow$ mar (a + b) = mar a $\oplus$ mar b for any a + b = c of the given type.

It can be shown easily that, in the case when a and b have a different number of digits, the mar sum of a and b contain the same type of “bulks”, and additional digits a_n or b_n (depending on which of the two numbers is greater and thus has more digits). Thus, mar (a + b) = mar a ⊕ mar b in this case also.

We mention that in the proof of The Sum Theorem we used the following properties of the mar sum (from the operation table):

: a + b = a ⊕ b, for a + b < 10

: a + b − 9 = a ⊕ b, for a + b > 9

: 1 ⊕ (a + b − 10) = a ⊕ b, for a + b ≥ 10 and

: $(a_1 + b_1 + 1) \oplus (a_2 + b_2 - 10) = a_1 \oplus b_1 \oplus a_2 \oplus b_2$, for $a_1 + b_1 < 9$ and $a_2 + b_2 \geq 10$

Proof of The Product Theorem: mar (a*b) = mar a ⊗ mar b, where a and b are nonzero naturals, so mar a and mar b are nonzero.

We have: mar (a*b) = mar (a + a + ... + a) [b times] ⇒ according to the Sum Theorem ⇒ mar (a*b) = mar a ⊕ mar a ⊕ ... ⊕ mar a [b times]

But mar a ∊ {1, , 3 , 4, 5, 6, 7, 8, 9} ⇒ we have these cases:

: mar a = 1 ⇒ mar (a*b) = 1 ⊕ 1 ⊕ ... ⊕ 1 [b times];
: mar a = 2 ⇒ mar (a*b) = 2 ⊕ 2 ⊕ ... ⊕ 2 [b times];
(...)
: mar a = 9 ⇒ mar (a*b) = 9 ⊕ 9 ⊕ ... ⊕ 9 [b times].

Take the case mar a = 1:

: Let b = 1 ⇒ mar b = 1 ⇒ mar (a*b) = 1 = 1 ⊗ 1 = mar a ⊗ mar b;

: Let b = 2 ⇒ mar b = 2 ⇒ mar (a*b) = 1 ⊕ 1 = 1 ⊗ 2 = mar a ⊗ mar b;

: Let b = 3 ⇒ mar b = 3 ⇒ mar (a*b) = 1 ⊕ 1 ⊕ 1 = 1 ⊗ 3 = mar a ⊗ mar b
(...)

: Let $b = 9 \Rightarrow$ mar $b = 9 \Rightarrow$ mar $(a*b) = 1 \oplus 1 \oplus \ldots \oplus 1$ [9 times] $= 1 \otimes 9 =$ mar $a \otimes$ mar b.

The same proof for the cases mar $a = 2$, mar $a = 3$, ..., mar $a = 9$, for $b \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

For $b > 9$ we use in the proof this helping theorem:

$$\boxed{\text{mar } (9*n) = \text{mar } 9 = 9}$$

for any nonzero natural n.

Proof of the helping theorem:

: mar $9*n$ = mar $(n + n + \ldots + n)$[9 times] = mar $a \oplus$ mar $n \oplus \ldots \oplus$ mar n [9 times]. But mar $a \oplus$ mar $n \oplus \ldots \oplus$ mar n [9 times] is always reduced to the following sums, which are always 9: $1 \oplus 1 \oplus \ldots \oplus 1$ [9 times], $2 \oplus 2 \oplus \ldots \oplus 2$ [9 times],..., $9 \oplus 9 \oplus \ldots \oplus 9$ [9 times].

A consequence of the helping theorem:

: Any natural number greater or equal to 9 may be written as one of these forms: $9*n$, $9*n + 1$, $9*n + 2$, ..., $9*n + 8$, where $n \geq 1$. And also, mar $9*n$ = mar $9 = 9$; mar $(9*n + 1)$ = mar $9*n \oplus$ mar $1 = 9 \oplus$ mar 1 = mar $1 = 1$; mar $(9*n + 2)$ = mar $9*n \oplus$ mar $2 = 9 \oplus$ mar 2 = mar $2 = 2$; ...; mar $(9*n + 8)$ = mar $9*n \oplus$ mar $8 = 9 \oplus$ mar 8 = mar $8 = 8$.

Take the case mar $a = 1$, $b \geq 10$;

: Let b of the type $9*n + 1 \Rightarrow$ mar $b = 1 \Rightarrow$ mar $(a*b) = 1 \oplus 1 \oplus \ldots \oplus 1$ [$(9*n + 1)$ times] $= 1 \oplus 1 \oplus \ldots \oplus 1$ [$9*n$ times] $\oplus 1 = (1 \oplus 1 \oplus \ldots \oplus 1$ [$9*n$ times] $\oplus \ldots \oplus 1 \oplus 1 \oplus \ldots \oplus 1$ [$9*n$ times]$) \oplus 1 = 9 \oplus 9 \oplus \ldots \oplus 9$ [n times] $\oplus 1 = 9 \oplus 1 = 1$ = mar $a \otimes$ mar b;

: Let b of the type $9*n + 2 \Rightarrow$ mar $b = 2 \Rightarrow$ mar $(a*b) = 1 \oplus 1 \oplus \ldots \oplus 1$ [$(9*n + 2)$ times] $= 1 \oplus 1 \oplus \ldots \oplus 1$ [$9*n$ times] $\oplus 1 \oplus 1 = 9 \oplus 9 \oplus \ldots \oplus 9$ [n times] $\oplus 2 = 9 \oplus 2 = 2$ = mar $a \otimes$ mar b.

We have an analogue proof for b of the type $9*n + 3$, $9*n + 4$, ..., $9n + 8$, and then for mar $a = 2$, mar $a = 3$, ..., mar $a = 9$.

Thus we proven that mar $(a*b)$ = mar $a \otimes$ mar b.

6. THE MAR REDUCED FORM OF A NATURAL NUMBER RAISED TO A NATURAL POWER

From the product table of mar a ⊗ mar b, we see that mar a^2 = mar(a*a) = mar a ⊗ mar a may only be 1, 4, 7 or 9.

⊗	1	2	3	4	5	6	7	8	9
1	1								
2		4							
3			9						
4				7					
5					7				
6						9			
7							4		
8								1	
9									9

Precisely, we have:

mar a	1	2	3	4	5	6	7	8	9
mar a^2	1	4	9	7	7	9	4	1	9

Now let's see what values mar a^3, mar a^4, ..., mar a^8 may take:

⊗	1	2	3	4	5	6	7	8	9
1	1								
4		8							
9			9						
7				1					
7					8				
9						9			
4							1		
1								8	
9									9

⊗	1	2	3	4	5	6	7	8	9
1	1								
8		7							
9			9						
1				4					
8					4				
9						9			
1							7		
8								1	
9									9

mar a^3 = mar a^2 ⊗ mar a and mar a^4 = mar a^3 ⊗ mar a

⊗	1	2	3	4	5	6	7	8	9
1	1								
4		8							
9			9						
7				1					
7					8				
9						9			
4							1		
1								8	
9									9

⊗	1	2	3	4	5	6	7	8	9
1	1								
8		7							
9			9						
1				4					
8					4				
9						9			
1							7		
8								1	
9									9

mar a^5 = mar a^4 ⊗ mar a and mar a^6 = mar a^5 ⊗ mar a

⊗	1	2	3	4	5	6	7	8	9
1	1								
1		2							
9			9						
1				4					
1					5				
9						9			
1							7		
1								8	
9									9

⊗	1	2	3	4	5	6	7	8	9
1	1								
2		4							
9			9						
4				7					
5					7				
9						9			
7							4		
8								1	
9									9

mar a^7 = mar a^6 ⊗ mar a and mar a^8 = mar a^7 ⊗ mar a

We see that mar a^8 = mar a^2 ⇒ mar a^9 = mar (a*a^8) = mar a^8 ⊗ mar a = mar a^2 ⊗ mar a = mar a^3;

Analogously, mar a^10 = mar a^4; mar a^11 = mar a^5; mar a^12 = mar a^6; mar a^13 = mar a^7; mar a^14 = mar a^8 ⊗ mar a^6 = mar a^2 ⊗ mar a^6 = mar a^8 = mar a^2 etc.

Let n and m nonzero naturals. We have mar a^(6*n) = mar (a^6*a^6*...*a^6)[n times] = (mar a^6 ⊗ mar a^6 ⊗ ... ⊗ mar a^6) [n times]

But mar a^6 may only be (we see from the table) 1 or 9 ⇒

⇒ mar a^(6*n) = 1 ⊗ 1 ⊗ ... ⊗ 1 [n times] = 1 or 9 ⊗ 9 ⊗ ... ⊗ 9 [n times] = 9 ⇒

⇒ mar a^(6*n) = 1 = mar a^6 for mar a = 1 or mar a^(6*n) = 9 = mar a^6 for mar a = 9 ⇒

mar a^(6*n) = mar a^6

Also,

mar a^(6*n + 1) = mar (a*a^(6*n)) = mar a^(6*n) ⊗ mar a = mar a^6 ⊗ mar a = mar a^7;

mar a^(6*n + 2) = mar a^8 = mar a^2,

mar a^(6*n + 3) = mar a^9 = mar a^3,

mar a^(6*n + 4) = mar a^10 = mar a^4,

mar a^(6*n + 5) = mar a^11 = mar a^5,

♦ On the other hand we have:

mar 9*m = mar 9 ⊗ mar m = 9 ⊗ mar m = 9 ,

mar (9*m + 1) = mar 9*m ⊕ mar 1 = 9 ⊕ 1 = 1,

mar (9*m + 2) = mar 9*m ⊕ mar 2 = 9 ⊕ 2 = 2,

Analogously, mar(9*m + 3) = 3, mar (9*m + 4) = 4, mar (9*m + 5) = 5, mar (9*m + 6) = 6, mar (9*m + 7) = 7, mar (9*m + 8) = 8

♦ Thus, we may write the following powers tables:

mar a	1	2	3	4	5	6	7	8	9
mar a^2	1	4	9	7	7	9	4	1	9
mar a^3	1	8	9	1	8	9	1	8	9
mar a^4	1	7	9	4	4	9	7	1	9
mar a^5	1	5	9	7	2	9	4	8	9
mar a^6	1	1	9	1	1	9	1	1	9
mar a^7	1	2	9	4	5	9	7	8	9

	a= 9m +1	a= 9m +2	a= 9m +3	a= 9m +4	a= 9m +5	a= 9m +6	a= 9m +7	a= 9m +8	a= 9m
mar a^(6*n + 2)	1	4	9	7	7	9	4	1	9
mar a^(6*n + 3)	1	8	9	1	8	9	1	8	9
mar a^(6*n + 4)	1	7	9	4	4	9	7	1	9
mar a^(6*n + 5)	1	5	9	7	2	9	4	8	9
mar a^(6*n)	1	1	9	1	1	9	1	1	9
mar a^(6*n + 1)	1	2	9	4	5	9	7	8	9

Take, for example, the mar reduced form of a natural number of the type a = 9*m + 7, raised to the power 6*n + 5 (with m and n naturals), is: mar (9*m + 7)^(6*n + 5) = 4.

Let's compute, for example, the mar reduced form of 4413^5678; we have: 4413 = 490*9 + 3 and 5678 = 946*6 + 2; then mar 4413^5678 = mar (9*m + 3)^(6*n + 2) = 9.

7. APPLICATIONS OF THE MAR REDUCED FORM IN THE ARITHMETHICS OF NATURAL NUMBERS

The most obvious and reasonable arithmetic applications of the mar reduced form are in problems concerning squares and cubes, in divisibility problems and especially in Diophantine equations. But before we begin solving these sorts of problems, we state a very important consequence of The Sum Theorem:

The mar reduced form of the sum of the digits of a number equals the mar reduced form of the number

Proof:

Let $a_1a_2 \ldots a_n$ be a nonzero natural number and S the sum of its digits; we have: $S = a_1 + a_2 + \ldots + a_n \Rightarrow$ mar S = mar $(a_1 + a_2 + \ldots + a_n) = a_1 \oplus a_2 \oplus \ldots \oplus a_n$ = mar $a_1a_2 \ldots a_n$

Example: Take the number 789342. We have mar 789342 = 7 ⊕ 8 ⊕ 9 ⊕ 3 ⊕ 4 ⊕ 2 = 6. But the sum of the digits of 789342 is S = 7 + 8 + 9 + 3 + 4 + 2, so mar S = mar (7 + 8 + 9 + 3 + 4 + 2) = 7 ⊕ 8 ⊕ 9 ⊕ 3 ⊕ 4 ⊕ 2 = 6

In exercises, we may compute the mar reduced form of the sum of the digits of the number instead of the mar reduced form of that number: mar 789342 = mar S = mar (7 + 8 + 9 + 3 + 4 + 2) = mar 33 = 3 ⊕ 3 = 6.

8. SQUARES AND CUBES

(1) A number is a square. Prove that the sum of its digits, either is divisible by 9, or by division by 3 we get modulus 1.

We have x^2 square ⇒ mar x^2 is 1, 4, 7 or 9 (this can be seen from the powers table), for any natural nonzero x.

(We will denote mar $x^2 = \{1/4/7/9\} \Leftrightarrow$ mar x^2 is equal to one from the values 1, 4, 7 or 9)

From mar $x^2 = \{1/4/7/9\} \Rightarrow x^2$ is of the type $9*k + 1$ or $9*k + 4$ or $9*k + 7$ or $9*k$.

But we know that the mar reduced form of the sum of the digits of a number equals the mar reduced form of the number $\Rightarrow$ the mar reduced form of the sum of the digits of the number x^2 is 1, 4, 7 or 9 $\Rightarrow$ the sum of the digits of the number x^2 is also a number of the type $9*h + 1$ or $9*h + 4$ or $9*h + 7$ or $9*h$.

Denote the sum of the digits of x^2 by S and we take these cases :

: S is of the type $9*k + 1$: obviously, S modulo 3 is 1;

: S is of the type $9*k + 4 = 3*(3*k + 1) + 1 \Rightarrow$

$\Rightarrow$ S modulo 3 is 1;

: S is of the type $9*k + 7 = 3*(3*k + 2) + 1 \Rightarrow$

$\Rightarrow$ S modulo 3 is 1;

: S is of the type $9*k \Leftrightarrow$ S is divisible by 9.

(2) Prove that the sum of the cubes of any 3 consecutive natural numbers is divisible by 9.

We have: $(n - 1)^3 + n^3 + (n + 1)^3 = n^3 - 3*n^2 + 3*n - 1 + n^3 + 3*n^2 + 3*n + 1 = 3*n^3 + 6*n$ for natural n, $n > 1$

That leaves to prove $3*n^3 + 6*n$ is divisible by 9.

Denote $3*n^3 + 6*n = m$, m natural, $m > 0 \Rightarrow$ mar $(3*n^3 + 6*n) =$ mar m $\Rightarrow$ mar $3*n^3 \oplus$ mar $6*n =$ mar m $\Rightarrow 3 \otimes$ mar $n^3 \oplus 6 \otimes$ mar n = mar m $\Leftrightarrow 3 \otimes \{1/8/9\} \oplus 6 \otimes$ mar n = mar m.

(As I mentioned above I denote $x = \{1/8/9\}$ when x can have one from the values 1, 8 or 9).

We have these cases:

:mar $n^3 = 1$ $\Rightarrow$ mar n = 1, 4 or 7 $\Rightarrow 3 \otimes 1 \oplus 6 \otimes \{1/4/7\} =$ mar m $\Rightarrow$ mar m $= 3 \oplus 6 = 9 \Rightarrow$ m is of the type 9*k, k natural, k > 0 ⇒ 3*n^3 + 6*n is divisible by 9;

:mar n^3 = 8 ⇒ mar n = 2, 5 or 8 ⇒ 3 ⊗ 8 ⊕ 6 ⊗ {2/5/8} = mar m ⇒ mar m = 6 ⊕ 3 = 9 ⇒ m is of the type 9*k, k natural, k > 0 ⇒ 3*n^3 + 6*n is divisible by 9;

:mar n^3 = 9 ⇒ mar n = 3, 6 or 9 ⇒ 3 ⊗ 9 ⊕ 6 ⊗ {3/6/9} = mar m ⇒ mar m = 9 ⊕ 9 = 9 ⇒ m is of the type 9*k, k natural, k > 0 ⇒ 3*n^3 + 6*n is divisible by 9;

(3) We have three square natural numbers. If the sum of these three numbers is divisible by 9, then we can choose two of them whose difference is divisible by 9.

We have x^2, y^2, z^2 naturals such that x^2 + y^2 + z^2 = 9*k, k natural, k ≠ 0 ⇒ mar (x^2 + y^2 + z^2) = mar 9*k ⇒ mar x^2 ⊕ mar y^2 ⊕ mar z^2 = 9 ⇒ {1/4/7/9} ⊕ {1/4/7/9} ⊕ {1/4/7/9} = 9.

We have these cases:

A: 1 ⊕ 1 ⊕ 7 = 9 ⇒ mar x^2 = mar y^2 = 1; mar z^2 = 7
B: 1 ⊕ 4 ⊕ 4 = 9 ⇒ mar y^2 = mar z^2 = 4; mar x^2 = 1
C: 7 ⊕ 7 ⊕ 4 = 9 ⇒ mar x^2 = mar y^2 = 7; mar z^2 = 4
D: 9 ⊕ 9 ⊕ 9 = 9 ⇒ mar x^2 = mar y^2 = mar z^2 = 9

Take case A: we have: mar x^2 = mar y^2 = 1 ⇒

: mar x^2 = mar y^2 ⊕ 9 ⇒ x^2 - y^2 = 9*k ⇒ x^2 - y^2 is divisible by 9 or
: mar y^2 = mar x^2 ⊕ 9 ⇒ y^2 - x^2 = 9*k ⇒ y^2 - x^2 is divisible by 9.

Take case B: we have: mar y^2 = mar z^2 = 4 ⇒

: mar y^2 = mar z^2 ⊕ 9 ⇒ y^2 - z^2 = 9*h ⇒ y^2 - z^2 is divisible by 9 or
: mar z^2 = mar y^2 ⊕ 9 ⇒ z^2 - y^2 = 9*h ⇒ z^2 - y^2 is divisible by 9.

The same proof is for case C.

Take case D: mar x^2 = mar y^2 = mar z^2 = 9 ⇒ x^2, y^2, z^2 are divisible by 9 ⇒ the absolute value of the difference between any two of these numbers is divisible by 9.

(4) Prove that the number N = 1978^1 + 1978^2 + 1978^3 + 1978^4 + 1978^5 is not a square.

We have: mar N = mar 1978^1 ⊕ mar 1978^2 ⊕ mar 1978^3 ⊕ mar 1978^4 ⊕ mar 1978^5

But:
mar 1978^1 = mar (219*9 + 7) = mar (9*k + 7) = 7;

mar 1978^2 = mar (9*k + 7)^2 = 4;

mar 1978^3 = mar (9*k + 7)^3 = 1;

mar 1978^4 = mar (9*k + 7)^4 = 7;

mar 1978^5 = mar (9*k + 7)^5 = 4.

(From the powers table).

So, we have: mar N = 7 ⊕ 4 ⊕ 1 ⊕ 7 ⊕ 4 = 5 ⇒ N is not a square (we know that the mar reduced form of a square may only be 1, 4, 7 or 9).

(5) Prove that the sum of the digits of a square number can't be 5*k ≠ 0.

Let n natural, n ≠ 0, n square and S the sum of the digits of n.

We saw that the sum of the digits of a number has the same mar reduced form as that number. But mar n may only be 1, 4, 7 or 9 (from the powers table for any n = x^2, x ≠ 0) ⇒ mar S may only be 1, 4, 7 or 9 ⇒ S is of the type 9*k + 1, 9*k + 4, 9*k + 7 or 9*k ⇒ S ≠ 5.

(6) Let N = 5*n^2 – 20*n + 23; show that N can't be a square.

We have: N + 20*n = 5*n^2 + 23 ⇒ mar N ⊕ 2 ⊗ mar n = 5 ⊗ mar n^2 ⊕ 5

Take mar n = 1 ⇒ mar n^2 = 1 ⇒ mar N ⊕ 2 ⊗ 1 = 5 ⊗ 1 ⊕ 5 ⇔ mar N ⊕ 2 = 1 ⇒ mar N = 8

Take mar n = 2 ⇒ mar n^2 = 4 ⇒ mar N ⊕ 2 ⊗ 2 = 5 ⊗ 4 ⊕ 5 ⇔ mar N ⊕ 4 = 7 ⇒ mar N = 3

Take mar n = 3 ⇒ mar n^2 = 9 ⇒ mar N ⊕ 2 ⊗ 3 = 5 ⊗ 9 ⊕ 5 ⇔ mar N ⊕ 6 = 5 ⇒ mar N = 8

Take mar n = 4 ⇒ mar n^2 = 7 ⇒ mar N ⊕ 2 ⊗ 4 = 5 ⊗ 7 ⊕ 5 ⇔ mar N ⊕ 8 = 4 ⇒ mar N = 5

Take mar n = 5 ⇒ mar n^2 = 7 ⇒ mar N ⊕ 2 ⊗ 5 = 5 ⊗ 7 ⊕ 5 ⇔ mar N ⊕ 1 = 4 ⇒ mar N = 3

Take mar n = 6 ⇒ mar n^2 = 9 ⇒ mar N ⊕ 2 ⊗ 62 = 5 ⊗ 9 ⊕ 5 ⇔ mar N ⊕ 3 = 5 ⇒ mar N = 2

Take mar n = 7 ⇒ mar n^2 = 4 ⇒ mar N ⊕ 2 ⊗ 7 = 5 ⊗ 4 ⊕ 5 ⇔ mar N ⊕ 5 = 7 ⇒ mar N = 2

Take mar n = 8 ⇒ mar n^2 = 1 ⇒ mar N ⊕ 2 ⊗ 8 = 5 ⊗ 1 ⊕ 5 ⇔ mar N ⊕ 7 = 1 ⇒ mar N = 3

Take mar n = 9 ⇒ mar n^2 = 9 ⇒ mar N ⊕ 2 ⊗ 9 = 5 ⊗ 9 ⊕ 5 ⇔ mar N ⊕ 9 = 5 ⇒ mar N = 5

We obtained mar N = 2, 3, 5 or 8 ⇒ mar N ≠ 1, 4, 7 or 9 ⇒ N can't be a square.

(7) Prove that the square of any natural number is of the type 3*m or 3*m + 1.

Let n = x^2 natural, n ≠ 0: mar x^2 may be 1, 4, 7 or 9 ⇒ n = x^2 is of the type 9*h + 1. 9*h + 4, 9*h + 7 or 9*h ⇔ 3*(3*h) + 1, 3*(3*h + 1) + 1, 3*(3*h + 2) + 1 or 3*(3*h) ⇒ n is of the type 3*m or 3*m + 1.

(8) Prove that the sum of the squares of three consecutive natural numbers can't be a square.

Let n natural, n > 1 and S = (n – 1)^2 + n^2 + (n + 1)^2 = n^2 – 2*n + 1 + n^2 + n^2 + 2*n + 1 = 3*n^ + 2

But mar S = mar (3*n^2 + 2) = mar 3*n^2 ⊕ mar 2 = 3 ⊗ mar n^2 ⊕ 2 = 3 ⊗ {1/4/7/9} ⊕ 2 = {3/3/3/9} ⊕ 2 = {5/5/5/2} ≠ {1/4/7/9}

We proved that mar S = mar (3*n^2 + 2) ≠ 1, 4, 7 or 9 for any natural n, n > 1 ⇒ S, so the sum of the squares of three consecutive natural numbers can't be a square (we know that the mar reduced form of a square may only be 1, 4, 7 or 9).

(9) Prove that: E = (9*a + 1)^4 +(9*a + 2)^4 +(9*a + 3)^4 + (9*a + 4)^4 +(9*a + 5)^4 +(9*a + 6)^4 + (9*a + 7)^4 + (9*a + 8)^4 can't be square for n natural.

We have, according with the powers table: mar (9*a + 1)^4 = 1; mar (9*a + 2)^4 = 7; mar (9*a + 3)^4 = 9; mar (9*a + 4)^4 = 4; mar (9*a + 5)^4 = 4; mar (9*a + 6)^4 = 9; mar (9*a + 7)^4 = 7; mar (9*a + 8)^4 = 1.

But E = (9*a + 1)^4 +(9*a + 2)^4 +(9*a + 3)^4 + (9*a + 4)^4 +(9*a + 5)^4 +(9*a + 6)^4 + (9*a + 7)^4 + (9*a + 8)^4 implies mar E = mar ((9*a + 1)^4 + ... + (9*a + 8)^4)) ⇔ mar E = mar (9*a + 1)^4 + ... + mar (9*a + 8)^4) ⇔ mar E = 1 ⊕ 7 ⊕ 9 ⊕ 4 ⊕ 4 ⊕ 9 ⊕ 7 ⊕ 1 = 6 ⇒ mar E ≠ 1, 4, 7 or 9 ⇒ E can't be a square.

(10) Prove that the number 1*2*3*...*n + 5 can't be a square, for any natural n.

Let N = 1*2*3*...*n + 5. We have N = n! + 5 ⇒ mar N = mar (n! + 5) ⇔ mar N = mar n! ⊕ mar 5 ⇔ mar N = mar n! ⊕ 5

But we know that mar n! is 1 for n = 1, 2 for n = 2, 3 for n = 5, 6 for n = 3, 6 for n = 4 respectively 9 for n ≥ 6.

For n = 1, n = 2, n = 3, n = 4, n = 5 we have: N = 6, N = 7, N = 11, N = 29, N = 125, N is not square. For n ≥ 6 we have mar N = 9 ⊕ 5 = 5 ⇒ mar N ≠ 1, 4, 7 or 9 ⇒ N can't be a square.

9. DIVISIBILITY PROBLEMS

(1) Prove that n^3 – n is divisible by 3 for any natural n, n ≠ 0.

Denote: n^3 – n = m ⇔ mar n^3 = mar (n + m) = mar n ⊕ mar m. But mar n^3 may only be 1, 8 or 9, for any natural nonzero n.

Take the case when mar n^3 = 1
⇒ mar n = 1, 4 or 7 (from the powers table). We have the following cases:
: mar n = 1 ⇒ 1 = 1 ⊕ mar m ⇒ mar m = 9
: mar n = 4 ⇒ 1 = 4 ⊕ mar m ⇒ mar m = 6
: mar n = 7 ⇒ 1 = 7 ⊕ mar m ⇒ mar m = 3

Take the case mar n^3 = 8

$\Rightarrow$ mar n = 2, 5 or 8. We have:

: mar n = 2 $\Rightarrow$ 8 = 2 $\oplus$ mar m $\Rightarrow$ mar m = 6

: mar n = 5 $\Rightarrow$ 8 = 5 $\oplus$ mar m $\Rightarrow$ mar m = 3

: mar n = 8 $\Rightarrow$ 8 = 8 $\oplus$ mar m $\Rightarrow$ mar m = 9

Take the case mar n^3 = 9

$\Rightarrow$ mar n = 3, 6 or 9. We have:

: mar n = 3 $\Rightarrow$ 9 = 3 $\oplus$ mar m $\Rightarrow$ mar m = 6

: mar n = 6 $\Rightarrow$ 9 = 6 $\oplus$ mar m $\Rightarrow$ mar m = 3

: mar n = 9 $\Rightarrow$ 9 = 9 $\oplus$ mar m $\Rightarrow$ mar m = 9

We get that mar m may be 3, 6 or 9 $\Rightarrow$ m is of the type 9*k + 3, 9*k + 6 or 9*k $\Rightarrow$ m = n^3 – n is divisible by 3.

(2) Prove that n^3 + 11*n is divisible by 6 for any natural n, n $\neq$ 0.

It's obvious that n^3 + 11*n is divisible by 2 (from n even it follows that n^3 + 11*n is even; for n odd, we have n^3 and 11*n odd $\Rightarrow$ n^3 + 11*n is even).

That only leaves us to prove that n^3 + 11*n is divisible by 3.

Denote: n^3 + 11*n = m (m natural, m > 0) $\Leftrightarrow$ mar (n^3 + 11*n) = mar m $\Leftrightarrow$ mar n^3 $\oplus$ mar 11*n = mar m $\Leftrightarrow$ mar n^3 $\oplus$ mar 11 $\otimes$ mar n = mar m $\Leftrightarrow$ mar n^3 $\oplus$ 2 $\otimes$ mar n = mar m

But mar n^3 may only be 1, 8 or 9, for any n natural

Take the case mar n^3 = 1 $\Rightarrow$ mar n = 1, 4 or 7. We have these cases:

: mar n = 1 $\Rightarrow$ mar n^3 $\oplus$ 2 $\otimes$ mar n = mar m $\Leftrightarrow$ 1 $\oplus$ 2 $\otimes$ 1 = mar m $\Rightarrow$ mar m = 3

: mar n = 4 $\Rightarrow$ 1 $\oplus$ 2 $\otimes$ 4 = mar m $\Rightarrow$ mar m = 9

: mar n = 7 $\Rightarrow$ 1 $\oplus$ 2 $\otimes$ 7 = mar m $\Rightarrow$ mar m = 6

We obtained mar m = 3, 6 or 9 $\Rightarrow$ m = n^3 + 11*n is divisible by 3.

Take the case mar n^3 = 8 $\Rightarrow$ mar n = 2, 5 or 8. We have:

: mar n = 2 $\Rightarrow$ 8 $\oplus$ 2 $\otimes$ 2 = mar m $\Rightarrow$ mar m = 3

: mar n = 5 $\Rightarrow$ 8 $\oplus$ 2 $\otimes$ 5 = mar m $\Rightarrow$ mar m = 9

: mar n = 8 $\Rightarrow$ 8 $\oplus$ 2 $\otimes$ 8 = mar m $\Rightarrow$ mar m = 6

Take the case mar n^3 = 9 $\Rightarrow$ mar n = 3, 6 or 9. We have:

: mar n = 3 $\Rightarrow$ 9 $\oplus$ 2 $\otimes$ 3 = mar m $\Rightarrow$ mar m = 6

: mar n = 6 ⇒ 9 ⊕ 2 ⊗ 6 = mar m ⇒ mar m = 3
: mar n = 9 ⇒ 9 ⊕ 2 ⊗ 9 = mar m ⇒ mar m = 9

We obtained mar m = 3, 6 or 9, for any n natural, n ≠ 0 ⇒ m is divisible by 3 ⇒ We proved what we needed to prove.

(3) Prove that 7^n – 1 is divisible by 6 for any natural n ≠ 0.

Obviously 7^n – 1 is divisible by 2 (7^n odd ⇒ 7^n – 1 even). It remains to prove that 7^n – 1 is divisible by 3.

Denote: 7^n – 1 = m, m natural, m > 0 ⇔ 7^n = m + 1 ⇒ mar 7^n = mar (m + 1) ⇔ mar 7^n = mar m ⊕ mar 1 = mar m ⊕ 1. But mar 7^n may only be 1, 4 or 7 (from the powers table). Thus, we have:

mar 7^n = mar m ⊕ 1 ⇔ mar m ⊕ 1 = 1, 4 or 7 ⇒ mar m = {3/6/9} ⇒ m is of the type 9*k + 3, 9*k + 6 or 9*k, k natural, k > 0 ⇒ m = 7^n - 1 is divisible by 3

(denote mar m = {3/6/9} ⇔ mar m is equal to 3, 6 or 9, as I mentioned few times above)

(4) Prove that 4^n + 15*n – 1 is divisible by 9, for any n natural, n ≠ 0 .

Denote 4^n + 15*n – 1 = m ⇔ 4^n + 15*n = m + 1 ⇒ mar (4^n +15*n) = mar (m + 1) ⇔ mar 4^n ⊕ 6 ⊗ mar n = mar m ⊕ 1

But mar 4^n may only be 1, 4 or 7.

Take the case mar 4^n = 1
⇒ n may only be of the type 6*k + 3 or 6*k (from the powers table) ⇒ n = 3*h, h natural, h > 0. We have 1 ⊕ 6 ⊗ mar 3*h = mar m ⊕ 1 ⇔ 6 ⊗ 3 ⊗ mar h = mar m ⇔ 9 ⊗ mar h = mar m ⇒ mar m = 9 ⇒ m is divisible by 9.

Take the case mar 4^n = 4
⇒ n may only be of the type 6*k + 4 or 6*k + 1. We have:
: 4 ⊕ 6 ⊗ mar (6*k + 4) = mar m ⊕ 1 ⇔ 4 ⊕ 6 ⊗ (6 ⊗ mar k ⊕ 4) = mar m ⊕ 1 ⇔ mar m = 3 ⊕ 6 ⊗ ({3/6/9} ⊕ 4) or, respectively,

: $4 \oplus 6 \otimes$ mar $(6*k + 1)$ = mar m $\oplus 1 \Leftrightarrow 4 \oplus 6 \otimes (6 \otimes$ mar k $\oplus 1)$ = mar m $\oplus 1 \Leftrightarrow$ mar m = $3 \oplus 6 \otimes (\{3/6/9\} \oplus 1)$,

from both cases above resulting that mar m = $3 \oplus 6 \otimes \{1/4/7\} \Rightarrow$ mar m = 9 $\Rightarrow$ m = 4^n + 15*n – 1 is divisible by 9

Take the case mar 4^n = 7

$\Rightarrow$ n is of the type 6*k + 2 or 6*k + 5 (from the powers table). We have:

: $7 \oplus 6 \otimes$ mar $(6*k + 2)$ = mar m $\oplus 1 \Leftrightarrow 7 \oplus 6 \otimes \{3/6/9\} \oplus 6 \otimes \{2/5\}$ = mar m $\oplus$ 1, or, respectively,

: $7 \oplus 6 \otimes$ mar $(6*k + 5)$ = mar m $\oplus 1 \Leftrightarrow 7 \oplus 6 \otimes \{3/6/9\} \oplus 6 \otimes \{2/5\}$ = mar m $\oplus$ 1,

resulting that mar m = $6 \oplus 9 \oplus 3 = 9 \Rightarrow$ m is divisible by 9.

(5) Prove that 7^n + 30*n – 1 is divisible by 18, for any n natural, $n \neq 0$.

Denote 7^n + 30*n – 1 = m, m natural, m > 0; we have 7^n + 30*n = m + 1 $\Rightarrow$ mar 7^n $\oplus$ mar 30*n = mar m $\oplus$ mar 1 $\Leftrightarrow$ mar 7^n $\oplus 3 \otimes$ mar n = mar m $\oplus$ mar 1

We have these cases:

: n = 1 $\Rightarrow 7 \oplus 3 \otimes 1$ = mar m $\oplus 1 \Leftrightarrow$ mar m $\oplus 1 = 1 \Rightarrow$ mar m = 9

: n = 2 $\Rightarrow 4 \oplus 3 \otimes 2$ = mar m $\oplus 1 \Leftrightarrow$ mar m $\oplus 1 = 1 \Rightarrow$ mar m = 9

: n = 3 $\Rightarrow 1 \oplus 3 \otimes 3$ = mar m $\oplus 1 \Leftrightarrow$ mar m $\oplus 1 = 1 \Rightarrow$ mar m = 9

: n = 4 $\Rightarrow 7 \oplus 3 \otimes 4$ = mar m $\oplus 1 \Leftrightarrow$ mar m $\oplus 1 = 1 \Rightarrow$ mar m = 9

: n = 5 $\Rightarrow 4 \oplus 3 \otimes 5$ = mar m $\oplus 1 \Leftrightarrow$ mar m $\oplus 1 = 1 \Rightarrow$ mar m = 9

: n = 6 $\Rightarrow 1 \oplus 3 \otimes 6$ = mar m $\oplus 1 \Leftrightarrow$ mar m $\oplus 1 = 1 \Rightarrow$ mar m = 9

: n = 7 $\Rightarrow 7 \oplus 3 \otimes 7$ = mar m $\oplus 1 \Leftrightarrow$ mar m $\oplus 1 = 1 \Rightarrow$ mar m = 9

: n = 8 $\Rightarrow 4 \oplus 3 \otimes 8$ = mar m $\oplus 1 \Leftrightarrow$ mar m $\oplus 1 = 1 \Rightarrow$ mar m = 9

: n = 9 $\Rightarrow 1 \oplus 3 \otimes 9$ = mar m $\oplus 1 \Leftrightarrow$ mar m $\oplus 1 = 1 \Rightarrow$ mar m = 9

For n > 9, all cases are reduced to one of the cases n < 9, according to the powers table ⇒ mar m = 9 ⇒ m = 7^n + 30*n – 1 is divisible by 9, for any nonzero natural n.

It’s easy to prove that m is divisible by 2 ⇒ m is divisible by 18.

(6) Prove that 2^(2^1959) – 1 is divisible by 3.

We have: 2^1959 = 2^(326*6 + 3) = 2^(6*k + 3), where k natural, k ≠ 0 ⇒ mar 2^1959 = 8 (from the powers table) ⇒ 2^1959 is of the type 9*h + 8, h natural, h different from zero. We have: 2^(2^1959) – 1 = 2^(9*h + 8) – 1. But h is even (9*h = 2^1959 – 8 is equal to the difference of two even numbers) ⇒ h = 2*r, with r natural, r ≠ 0 ⇒ 9*h + 8 = 18*r + 8 = 6*m + 2 (with m natural, m ≠ 0). So 2^(2^1959) – 1 = 2^(6*m + 2) – 1 = n, n natural, n ≠ 0 ⇔ 2^(6*m + 2) = n + 1 ⇒ mar 2^(6*m + 2) = mar n ⊕ 1 ⇒ 4 = mar n ⊕ 1 ⇒ mar n = 3 ⇒ n is divisible by 3.

(7) Prove that, if a + b + c is divisible by 6, then a^3 + b^3 + c^3 is divisible by 6 (a, b, c naturals).

We have a + b + c = 6*k, k natural, k ≠ 0. But a^3 + b^3 + c^3 = (a + b + c)^3 – 6*a*b*c – 3 (a*b^2 + a^2*b + a*c^2 + a^2*c + b*c^2 + b^2*c) ⇒ (a^3 + b^3 + c^3) + 3*(2*a*b*c + a*b^2 + a^2*b + a*c^2 + a^2*c + b*c^2 + b^2*c) = (a + b + c)^3 = (6*k)^3 = 9*n, with n natural, n ≠ 0. We have (a^3 + b^3 + c^3) + 3*(2*a*b*c + a*b^2 + a^2*b + a*c^2 + a^2*c + b*c^2 + b^2*c) = 9*n ⇔ (a^3 + b^3 + c^3) + 3*m = 9*n , with m natural, m ≠ 0 ⇒ mar (a^3 + b^3 + c^3) ⊕ 3 ⊗ mar m = 9 ⊗ mar m ⇒ mar (a^3 + b^3 + c^3) ⊕ {3/6/9} = 9 ⇒ mar (a^3 + b^3 + c^3) = 3, 6 or 9 ⇒ a^3 + b^3 + c^3 is divisible by 3. It’s easy to prove that a^3 + b^3 + c^3 is divisible by 2 ⇒ a^3 + b^3 + c^3 is divisible by 6.

(8) Using the digits 1, 2, 3, 4, 5, 6, 7 one takes all the 7 digit numbers which contain these digits exactly once. Prove that the sum of all these numbers is divisible by 9.

Let N the number of 7 digit numbers that are obtained by arranging the digits. We have N = A_7^7 = 1*2*3*4*5*6*7 = 5040.

On the other hand, the mar reduced form of any of these N numbers is 1 ⊕ 2 ⊕ 3 ⊕ 4 ⊕ 5 ⊕ 6 ⊕ 7 = mar (1 + 2 + 3 + 4 + 5 + 6 + 7) = mar 28 = 1

Let S be the sum of the N numbers. The mar reduced form of S equals the sum of the mar reduced form of the N numbers:

Mar S = 1 ⊕ 1 ⊕ ... ⊕ 1 [N times] = 1 ⊕ 1 ⊕ ... ⊕ 1 [5040 times] = mar 5040 ⊗ 1= 9 ⊗ 1 = 9

We proved that mar S = 9 ⇒ the sum of all the considered 7 digit numbers is divisible by 9.

(9) Show that 1971^5 + 1972^4 + 1973^3 is a multiple of 9.

We have E = 1971^5 + 1972^4 + 1973^3 ⇒ mar E = mar 1971^5 ⊕ mar 1972^4 ⊕ mar 1973^3. But:

: mar 1971^5 = mar (9*219)^5 = mar (9*k)^5 = 9
: mar 1971^4 = mar (9*219 + 1)^4 = mar (9*k + 1)^4 = 1
: mar 1971^3 = mar (9*219 + 2)^3 = mar (9*k + 2)^3 = 8

So mar E = 9 ⊕ 1 ⊕ 8 = 9 ⇒ E is of the type 9*h, h natural, h > 0 ⇒ E is a multiple of 9.

(10) Find S such that S = 1980 + 19a8b is divisible by 18 and a ≠ b. We have S is divisible by 9 ⇒ mar S = 9. So S = 1980 + 19a8b ⇒ mar S = mar 1980 ⊕ mar 19a8b ⇔ 9 = 9 ⊕ mar 19a8b ⇒ mar 19a8b = 9. But mar19a8b = 1 ⊕ 9 ⊕ a ⊕ 8 ⊕ b = a ⊕ b ⊕ 9. So a ⊕ b ⊕ 9 = 9 ⇒ a ⊕ b = 9.

If a and b are different, we have these possibilities:

{(1 ⊕ 8 = 9)/(2 ⊕ 7 = 9)/(3 ⊕ 6 = 9)/(4 ⊕ 5 = 9)} ⇒
{(1 ⊕ 8 = 9)/(2 ⊕ 7 = 9)/(3 ⊕ 6 = 9)/(4 ⊕ 5 = 9)}

(I denote by this way of writing, with paranthesis, that one of the values from the left term of the equality above implies one of the values from the right term)

The possibilities are: [a, b] = [1, 8]; [a, b] = [8, 1]; [a, b] = [2, 7]; [a, b] = [7, 2]; [a, b] = [3, 6]; [a, b] = [6, 3]; [a, b] = [4, 5]; [a, b] = [5, 4].

The solutions are: S = 1980 + 19188; S = 1980 + 19881; S = 1980 + 19287; S = 1980 + 19782; S = 1980 + 19386; S = 1980 + 19683; S = 1980 + 19485; S = 1980 + 19584.

Accounting that S is also divisible by 2 we have these final solutions: S = 1980 + 19782 = 21762; S = 1980 + 19386 = 21366; S = 1980 + 19584 = 21564.

10. DIOPHANTINE EQUATIONS

(1) Show that the equation x^2 + 3*y^2 = 1976 has no natural solutions.

We have x^2 + 3*y^2 = 1976 ⇒ mar (x^2 + 3*y^2) = mar 1976 ⇔ mar x^2 ⊕ mar 3*y^2 = mar 1976 ⇔ mar x^2 ⊕ 3 ⊗ mar y^2 = 1 ⊕ 9 ⊕ 7 ⊕ 6 ⇔ mar x^2 ⊕ 3 ⊗ mar y^2 = 5

But mar x^2 and mar y^2 may only be 1, 4, 7 or 9 (from the powers table). Thus, we obtain: {1/4/7/9} ⊕ 3 ⊗ {1/4/7/9} = 5 ⇔ {1/4/7/9} ⊕ {3/9} = 5 ⇔ {1/3/4/7/9} = 5, which is impossible ⇒ the given equation has no natural solutions.

(2) Show that the Diophantine equations:

(A) x^3 + y^3 + z^3 = 1578964
(B) x^3 + y^3 + z^3 = 3277463
have no natural solutions.

(A) We have x^3 + y^3 + z^3 = 1578964 ⇒ mar (x^3 + y^3 + z^3) = mar 1578964 ⇔ mar x^3 ⊕ mar y^3 ⊕ mar z^3 = 4. But mar x^3, mar y^3 and mar z^3 may only be 1, 8 or 9 (from the powers table). So mar x^3 ⊕ mar y^3 ⊕ mar z^3 = {1/8/9} ⊕ {1/8/9} ⊕ {1/8/9} = {1/2/3/6/7/8/9} ≠ 4 ⇒ x^3 + y^3 + z^3 ≠ 1578964

We've considered the following combinations:

: 1 ⊕ 1 ⊕ 1 = 3; 1 ⊕ 1 ⊕ 8 = 1; 1 ⊕ 8 ⊕ 8 = 8; 1 ⊕ 1 ⊕ 9 = 1; 1 ⊕ 9 ⊕ 9 = 1; 1 ⊕ 8 ⊕ 9 = 9; 8 ⊕ 8 ⊕ 8 = 6; 8 ⊕ 8 ⊕ 9 = 7; 8 ⊕ 9 ⊕ 9 = 8; 9 ⊕ 9 ⊕ 9 = 9 .

(B) From the proof at point (A) we see that mar x^3 ⊕ mar y^3 ⊕ mar z^3 ≠ 5.

(3) Solve the equation (1 + x!)*(1 + y!) = (x + y)! in the set of natural numbers

Let's see what is the value of the mar reduced form of n!. We have:

: 1! = 1 ⇒ mar 1! = mar 1 = 1;
: 2! = 1*2 ⇒ mar 2! = mar 2 = 2;
: 3! = 1*2*3 ⇒ mar 3! = mar 6 = 6;
: 4! = 1*2*3*4 ⇒ mar 4! = mar 24 = 6;
: 5! = 1*2*3*4*5 ⇒ mar 5! = mar 120 = 3;
: 6! = 1*2*3*4*5*6 ⇒ mar 6! = mar 720 = 9.

As $n! = (n - 1)!*n \Rightarrow 7! = 6!*7 \Rightarrow mar\ 7! = mar\ 6! \otimes mar\ 7 = 9 \otimes 7 = 9$; $8! = 7!*8 \Rightarrow mar\ 8! = mar\ 7! \otimes mar\ 8 = 9 \otimes 8 = 9$. Obviously, mar n! will be 9 for any $n \geq 6$.

Thus, we have: $(1 + x!)*(1 + y!) = (x + y)! \Leftrightarrow 1 + x!+ y! + x!*y! = (x + y)! \Rightarrow 1 \oplus mar\ x! \oplus mar\ y! \oplus mar\ x!*y! = mar\ (x + y)!$. For $x \geq 6$ and $y \geq 6$, we have: $1 \oplus mar\ x! \oplus mar\ y! \oplus mar\ x!*y! = mar\ (x + y)! \Leftrightarrow 1 \oplus 9 \oplus 9 \oplus 9 = 9 \Leftrightarrow 1 = 9$, which is impossible $\Rightarrow$ for $x \geq 6$ and $y \geq 6$ the given equation has no natural solutions.

For $x < 6$ and $y < 6$ we have the solutions $(x = 1, y = 2)$ and $(x = 2, y = 1)$.

(4) Find x such that $2^5*9^x = 259x$

We have $2^5*9^x = 259x \Rightarrow mar\ 2^5*9^x = mar\ 259x \Leftrightarrow mar\ 2^5 \otimes mar\ 9^x = mar\ 259x \Leftrightarrow 5 \otimes 9 = 2 \oplus 5 \oplus 9 \oplus x \Leftrightarrow 9 = 7 \oplus x \Rightarrow x = 2$. Indeed, $2^5*9^2 = 2592$.

(5) Find a and b nonzero naturals, such that:

$9ab = 909 + a^2 + b^2$

We have $9ab = 909 + a^2 + b^2 \Rightarrow mar\ 9ab = mar\ (909 + a^2 + b^2) \Leftrightarrow mar\ 9ab = mar\ 909 \oplus mar\ a^2 \oplus mar\ b^2 \Leftrightarrow 9 \oplus a \oplus b = 9 \oplus a^2 \oplus b^2 \Rightarrow a \oplus b = a^2 \oplus b^2$.

We look in the powers table and we see that the only combinations that satisfy the equality are:

: $1^2 \oplus 9^2 = 1 \oplus 9 \Rightarrow (a = 1, b = 9)$ or $(a = 9, b = 1)$
: $3^2 \oplus 4^2 = 3 \oplus 4 \Rightarrow (a = 3, b = 4)$ or $(a = 4, b = 3)$
: $3^2 \oplus 6^2 = 3 \oplus 6 \Rightarrow (a = 3, b = 6)$ or $(a = 6, b = 3)$
: $4^2 \oplus 7^2 = 4 \oplus 7 \Rightarrow (a = 4, b = 7)$ or $(a = 7, b = 4)$
: $6^2 \oplus 7^2 = 6 \oplus 7 \Rightarrow (a = 6, b = 7)$ or $(a = 7, b = 6)$

We go back to the initial equation and we have the following possibilities:

: 919 = 909 + 1 + 81; 991 = 909 + 1 + 81;
: 934 = 909 + 9 + 16; 943 = 909 + 9 + 16;
: 936 = 909 + 9 + 36; 963 = 909 + 9 + 36;
: 947 = 909 + 16 + 49; 974 = 909 + 16 + 49;
: 967 = 909 + 36 + 49; 976 = 909 + 36 + 49.

From these, the only valid possibilities are:

: 991 = 909 + 1 + 81;
: 934 = 909 + 9 + 16;
: 974 = 909 + 16 + 49.

The solutions of the equation are: (a = 9, b = 1); (a = 3, b = 4); (a = 7, b = 4).

(6) Prove that the equation 1! − 2! + 3! − 4! + ... + (−1)^(n − 1)*n! = k^2 has no natural nonzero solutions.

We know that mar n! = 9 for n ≥ 6.

We have 1! − 2! + 3! − 4! + ... + (−1)^(n − 1)*n! = k^2 ⇔ 1! + 3! + 5! + 7! + (...) = k^2 + 2! + 4! + 6!+ (...) ⇒ mar (1! + 3! + 5! + 7! + ...) = mar (k^2 + 2! + 4! + 6! + ...) ⇔ mar 1! ⊕ mar 3! ⊕ mar 5! ⊕ mar 7! ⊕ mar (...) mar k^2 ⊕ mar 2! ⊕ mar 4! ⊕ mar 6! ⊕ mar (...) ⇔ 1 ⊕ 6 ⊕ 3 ⊕ 9 ⊕ (9 ⊕ ... ⊕ 9) = {1/4/7/9} ⊕ 2 ⊕ 6 ⊕ 9 ⊕ (9 ⊕ ... ⊕ 9) ⇔ 1 = {1/4/7/9} ⊕ 8 ⇔ 1 = 3, 6, 8 or 9, which is impossible ⇒ the given equation has no nonzero natural solutions.

(7) Solve this equation in the set of natural numbers:
2^x + 7 = 3^y

For y = 1, the equation hasn't any solutions.

For y ≥ 2, we have 2^x + 7 = 3^y ⇒ mar (2^x + 7) = mar 3^y ⇔ mar 2^x ⊕ mar 7 = mar 3^y ⇔ mar 2^x ⊕ 7 = 9 ⇒ mar 2^x = 2 ⇒ x = 1 or x = 6*k + 1, k natural, k > 0.

For x = 1 we have 3^y = 9 ⇒ (x = 1, y = 2) is a solution for the equation.

For y > 2, the equation becomes 2^(6*k + 1) + 7 = 3^y.

: We take the case when k is even, y is even, k = 2*h, y = 2*z, h > 0, z > 0. We have 2^(12*h + 2) + 7 = 3^(2*z) ⇔ (2^(6*h + 1))^2 − 3^(2*z) = 7 ⇒ (3^z − 2^(6*h + 1))*(3^z + 2^(6*h + 1)) = 7. Obviously, in this case we have no natural solutions.

: We take the case when k is even, y is odd, k = 2*h, y = 2*z + 1, h > 0, z > 0. We have 2^(12*h + 2) + 7 = 3^(2*z + 1) ⇒ mar (2^(12*h + 2) + 7) = mar 3^(2*z + 1) ⇔ mar 2^(12*h + 2) ⊕ mar 7 = mar 3^(2*z + 1) ⇒ 4 ⊕ 7 = 9 ⇔ 2 = 9, which is impossible ⇒ the case has no natural solutions.

: We take the case when k is odd, y is even, k = 2*h + 1, y = 2*z, h > 0, z > 0. We have 2^(12*h + 7) + 7 = 3^(2*z) ⇔ 2^(12*h)*2^7 + 7 = 3^(2*z) ⇔ 2^(12*h)*2^7 − 2 = 3^(2*z) − 9 ⇔ 2*(2^(12*h)*2^6 − 1) = 3^(2*z) − 3^2 ⇔ 2*(2^(6*h)*2^3 − 1)*(2^(6*h)*2^3 + 1) = (3^z − 3)*(3^z + 3). But (3^z − 3) is divisible by 2 and (3^z + 3) is also divisible by 2, it follows that (3^z − 3)*(3^z + 3) is divisible by 4, while 2*(2^(6*h)*2^3 − 1)*(2^(6*h)*2^3 + 1) is divisible only by 2 ⇒ in this case we have no natural solutions.

: We take the case when k is odd, y is odd, k = 2*h + 1, y = 2*z + 1, h > 0, z > 0. We have 2^(12*h + 7) + 7 = 3^(2*z + 1) ⇔ 2^(12*h)*2^7 + 4 = 3^(2*z)*3 − 3 ⇔ 4*(2^(12*h)*2^5 + 1) = 3*(3^(2*z) − 1)*(3^(2*z) + 1). But (3^(2*z) − 1) is divisible by 2 while (3^(2*z) + 1) is divisible by 4, or (3^(2*z) − 1) is divisible by 4 while (3^(2*z) + 1) is divisible by 2. It follows that 3*(3^(2*z) − 1)*(3^(2*z) + 1) is divisible by 8, while 4*(2^(12*h)*2^5 + 1) is divisible only by 4 ⇒ the equation hasn't any natural solutions in this case either.

(8) Solve in the set of natural numbers the equation:
2^x − 7 = 3^y

We have 2^x − 7 = 3^y ⇔ 2^x = 3^y + 7

For y = 0 we have 2^x = 8, so (x = 3, y = 0) a solution of the equation.

For y = 1 we have 2^x = 10, so there are no solutions.

For y > 1 we have 2^x = 7 + 3^y ⇒ mar 2^x = mar (7 + 3^y) ⇔ mar 2^x = 7 ⊕ 9 = 7 ⇒ x = 4 or x = 6*k + 4, k natural, k > 0 ⇔ mar 2^x = 7 ⊕ 9 = 7 ⇒ x = 4 or x = 6*k + 4, k natural, k > 0.

For x = 4 we have 16 = 7 + 9 ⇒ (x = 4, y = 2) is a solution of the equation.

For y > 1, x ≠ 4, the initial equation becomes 2^(6*k + 4) = 3^y + 7.

: Take the case when k is even, y is even, k = 2*h, y = 2*z, h > 0, z > 0. We have 2^(12*h + 4) = 3^(2*z) + 7 ⇔ (2^(6*h + 2))^2 − 3^(2*z) = 7 ⇔

(2^(6*h + 2) − 3^z)*(2^(6*h + 2) + 3^z) = 7. Obviously, in this case there are no natural solutions.

: Take the case when k is even, y is odd, k = 2*h, y = 2*z + 1, h > 0, z > 0. We have 2^(12*h + 4) = 3^(2*z + 1) + 7 ⇔ 2^(12*h + 4) − 6 = 3^(2*z + 1) + 1 ⇔ 2*(2^(12*h + 3) − 3) = (3 + 1)*(3^(2*z) − 3^(2*z − 1) + ... − 3 + 1) ⇔ 2*(2^(12*h + 3) − 3) = 4*(3^(2*z) − 3^(2*z − 1) + ... − 3 + 1). But (2^(12*h + 3) − 3) is not divisible by 2 ⇒ the equality is impossible.

: Take the case when k is odd, y is even, k = 2*h + 1, y = 2*z, h > 0, z > 0. We have 2^(12*h + 10) = 3^(2*z) + 7 ⇔ (2^(6*h + 5) − 3^z)*(2^(6*h + 5) + 3^z) = 7, which is impossible ⇒ in this case we have no natural solutions.

: Take the case when k is odd, y is even, k = 2*h + 1, y = 2*z, h > 0, z > 0. We have 2^(12*h + 10) = 3^(2*z + 1) + 7 ⇔ 2^(12*h + 10) − 6 = 3^(2*z + 1) + 1 ⇔ 2*(2^(12*h + 9) − 3) = (3 + 1)*(3^(2*z) − 3^(2*z − 1) + ... − 3 + 1) ⇔ 2*(2^(12*h + 9) − 3) = 4*(3^(2*z) − 3^(2*z − 1) + ... + 1). But (2^(12*h + 9) − 3) is not divisible by 2 ⇒ the equality is impossible.

(9) Solve the equation in the set of natural numbers:
x^2 − 6*x*y + y^2 = 1

We have x^2 + y^2 = 6*x*y + 1 ⇒ mar (x^2 + y^2) = mar (6*x*y + 1) ⇔ mar x^2 ⊕ mar y^2 = 6 ⊗ mar x*y ⊕ 1· But mar x^2 and mar y^2 may only be 1, 4, 7 or 9, and 6 ⊗ mar n is 3, 6 or 9 for any natural n, n ≠ 0. So: {1/4/7/9} ⊕ {1/4/7/9} = {3/6/9} ⊕ 1 ⇔ {2/5/8/9} = {1/4/7}, which is impossible ⇒ the equation has no natural solutions.

(10) Solve this equation in the set of natural numbers:
(x + y)^5 = x^4 + y^4

For x = 0 we have (x = 0, y = 0) trivial solution of the equation.

For x > 0, y > 0 we have (x + y)^5 = x^4 + y^4 ⇒ mar (x + y)^5 = mar (x^4 + y^4) ⇔ mar (x + y)^5 = mar x^4 ⊕ mar y^4. But mar x^4 and mar y^4 may only be 1, 4, 7 or 9 $\Rightarrow$ mar (x + y)^5 = {1/4/7/9} $\oplus$ {1/4/7/9} = {2/5/8/9}

We have the following cases:

(1) mar (x + y)^5 = 1 $\oplus$ 1 = 2
(2) mar (x + y)^5 = 4 $\oplus$ 7 = 2
(3) mar (x + y)^5 = 1 $\oplus$ 4 = 5
(4) mar (x + y)^5 = 7 $\oplus$ 7 = 5
(5) mar (x + y)^5 = 1 $\oplus$ 7 = 8
(6) mar (x + y)^5 = 4 $\oplus$ 4 = 8
(7) mar (x + y)^5 = 9 $\oplus$ 9 = 2

Take the case 1: we have mar (x + y)^5 = 2 $\Rightarrow$ mar (x + y) = 5. But mar x^4 = mar y^4 = 1 $\Rightarrow$ mar x and mar y may only be 1 or 8. So mar (x + y) = mar x $\oplus$ mar y = {1/8} $\oplus$ {1/8} = {2/7/9} $\neq$ 5 $\Rightarrow$ the case is impossible.

Take the case 2: we have mar (x + y)^5 = 2 $\Rightarrow$ mar (x + y) = 5. But mar x^4 = 4 $\Rightarrow$ mar x may only be 4 or 5, and mar y^4 = 7 $\Rightarrow$ mar y may only be 2 or 7. So mar (x + y) = mar x $\oplus$ mar y = {4/5} $\oplus$ {2/7} = {2/3/6/7} $\neq$ 5 $\Rightarrow$ the case is impossible.

A similar proof is for cases (3), (4), (5) and (6). The case (7) is reduced to one of the previous cases (after simplification by 9).

11. COMPARED SOLUTIONS

We've seen a few of the applications of the mar reduced form in problems which are usually solved by means of induction, modulo n classes, or by unsystematic, somewhat empirical but standard methods, such as the value of the last digit of a number, the sum of the digits of a number etc. By choosing and solving the exercises until now, I didn't press for showing of the mar reduced form uses in contrast with the traditional methods, I just presented a working alternative for them. In fact, most of the presented exercises could have been solved easier using the traditional methods. In spite of that, compared with each of those methods, using mar reduced form has its advantages: it is simpler than some of the methods, more synthetic and less arbitrary than the others. Actually, the mar reduced form is an intrinsic, invariant and easy to compute characteristic of a natural number. It offers a fixed starting point, at least for the basic approach of arithmetic problems: it supplies easy results which we can process in secondary steps by other methods, or it may often lead us to a result. In the chapter COMPARED SOLUTIONS we shall solve the same type of problems that we did in the previous applications chapters, but this time every exercise will also have a traditional solution. The title of this chapter is a little wrong, as the comparisons won't be made explicitly, for each solution, at least not by the author. I just put together one solution after another, hoping that the readers will make the called for comparisons. The problems and the traditional solutions were picked from problem books and Math journals, published in my country from 1978 to 1998. Most of the problems and their solutions are classical, and don't belong to specific authors. But even if I'm wrong, I take the liberty and responsibility of not mentioning anyone, considering the comparison with the traditional methods, in general, and not with someone's particular methods.

(1) Show that the equation $x^3 - 2*y^3 - 4*z^3 = 0$ has no natural solutions except $x = y = z = 0$.

Proof using the mar reduced form:

We have $x^3 - 2*y^3 - 4*z^3 = 0 \Leftrightarrow x^3 = 2*y^3 + 4*z^3 \Rightarrow$ $\text{mar } x^3 = \text{mar } (2*y^3 + 4*z^3) \Leftrightarrow \text{mar } x^3 = \text{mar } 2*y^3 \oplus \text{mar } 4*z^3 \Leftrightarrow \text{mar } x^3 = 2 \otimes \text{mar } y^3 \oplus 4 \otimes \text{mar } z^3$

But mar x^3, mar y^3 and mar z^3 may only be 1, 8 or 9 $\Rightarrow$ $\{1/8/9\} = 2 \otimes \{1/8/9\} \oplus 4 \otimes \{1/8/9\} \Leftrightarrow \{1/8/9\} = \{2/7/9\} \oplus \{4/5/9\}$

The only possible combination is $9 = 9 \oplus 9 \Leftrightarrow \text{mar } x^3 = \text{mar } y^3 = \text{mar } z^3 = 9 \Rightarrow$ x^3, y^3 and z^3 are divisible by 9, which leads to two cases: either by simplifying the equation by 9, we end up in another combination, or x^3, y^3 and z^3 are same order powers of 9. Obviously, both cases are impossible $\Rightarrow$ the given equation has no nonzero natural solutions.

Classical proof:

The equation may be written as: $x^3 = 2*(y^3 + 2*z^3)$, so it follows that 2 is a divisor of x.

Denote $x = 2*x_1$ and we substitute; we get: $8*x_1^3 = 2*(y^3 + 2*z^3) \Leftrightarrow y^3 = 2*(2*x_1^3 - z^3) \Rightarrow$ 2 is a divisor of y.

Denote $y = 2*y_1$ and we substitute; we get: $4*y_1^3 = 2*x_1^3 - z^3 \Leftrightarrow z^3 = 2*(x_1^3 - 2*y_1^3) \Rightarrow$ 2 is a divisor of z.

We obtained that x, y and z are even numbers. The same proof is for x/2, y/2 and z/2 even and so on. It follows that x, y and z are divisible by any power of 2, which is possible only if x = y = z = 0.

(2) Prove that N = 1^t + 2^t + 3^t + ... + 9^t − 3*(1^t + 6^t + 8^t) is divisible by 18 for any natural t.

Proof using the mar reduced form:

We have, according to the power table:

: mar (1^t + 2^t + 3^t + ... + 9^t) = 1 ⊕ 2 ⊕ 4 ⊕ 5 ⊕ 7 ⊕ 8 ⊕ 9 ⊕ 9 ⊕ 9 = 9, for t of the type 6*n + 1 or 6*n + 5, n natural;
: mar (1^t + 2^t + 3^t + ... + 9^t) = 1 ⊕ 1 ⊕ 4 ⊕ 4 ⊕ 7 ⊕ 7 ⊕ 9 ⊕ 9 ⊕ 9 = 6, for t of the type 6*n + 2 or 6*n + 4, n natural;
: mar (1^t + 2^t + 3^t + ... + 9^t) = 1 ⊕ 1 ⊕ 1 ⊕ 8 ⊕ 8 ⊕ 8 ⊕ 9 ⊕ 9 ⊕ 9 = 9, for t of the type 6*n + 3, n natural;
: mar (1^t + 2^t + 3^t + ... + 9^t) = 1 ⊕ 1 ⊕ 1 ⊕ 1 ⊕ 1 ⊕ 1 ⊕ 9 ⊕ 9 ⊕ 9 = 6, for t = 6 or t of the type 6*n, n natural, n ≠ 0.

On the other hand, we have:

: mar (1^t + 6^t + 8^t) = 1 ⊕ 1 ⊕ 9 = 2, for t of the type 6*n + 2, 6*n + 4 or 6*n;
: mar (1^t + 6^t + 8^t) = 1 ⊕ 8 ⊕ 9 = 9, for t of the type 6*n + 1, 6*n + 3 or 6*n + 5.

So N + 3*(1^t + 6^t + 8^t) = 1^t + 2^t + 3^t + ... + 9^t which means:

: mar N ⊕ 3 ⊗ 2 = 6 ⇔ mar N = 9 for t of the type 6*n + 2, 6*n + 4, 6*n;
: mar N ⊕ 3 ⊗ 9 = 9 ⇔ mar N = 9 for t of the type 6*n + 1, 6*n + 3 or 6*n + 5.

But mar N = 9 ⇔ N is divisible by 9. As N is divisible by 2 also (is the difference of two odd numbers) ⇒ N is divisible by 18.

Classical proof:

In the sum 1^t + 2^t + 3^t + ... + 9^t there are 5 odd numbers and 4 even numbers, so the sum is odd. The number

3*(1^t + 6^t + 8^t) is odd, so N is even. Let's prove that 9 is a divisor of N.

Consider the cases:

(A) t is odd. For t = 1 we get N = 0 so 9 is a divisor of N.

We may assume that t ≥ 3. Then 3^t, 6^t, 9^t are divisible by 9, and N ≡ ((1^t + 8^t) + (2^t + 7^t) + (4^t + 5^t) − 3 (1^t + 8^t))(mod 9)

As t is odd, each parenthesis is divisible by 9 ⇒ N is divisible by 9.

(B) t is even, t = 2*p, p ≥ 1, N ≡ 1^p + 2*(4^p + 7^p) + 1^p − 3*(1^p + 1^p) = 2*(4^p + 7^p − 2) ≡ 0(mod 9)

To prove the last identity we use induction.

For p = 1 we have 4^1 + 7^1 − 2 = 9 ≡ 0(mod 9).

Suppose that 4^p + 7^p − 2 ≡ 0(mod 9). Then: 4^(p + 1) + 7^(p + 1) − 2 = 4*(4^p + 7^p − 2) + 3*(7^p + 2) ≡ 0(mod 9).

The first parenthesis is divisible by 9 according to the induction hypothesis, and the second is divisible by 3 because 7^p ≡ 1(mod 3).

So 9 is a divisor of N and, as 2 is a divisor of N, it follows that 18 is a divisor of N.

(3) Prove that the equation x^3 − 3*x*y^2 + y^3 = 2891 has no natural solutions.

Proof using the mar reduced form:

We have x^3 + y^3 = 3*x*y^2 + 2891 ⇒ mar (x^3 + y^3) = mar (3*x*y^2 + 2891) ⇒ mar x^3 ⊕ mar y^3 = 3 ⊗ mar x*y^2 ⊕ mar 2891. But mar x^3 and mar y^3 may only be 1, 8 or 9, for any x and y naturals, and 3 ⊗ mar z may only be 3, 6 or 9, for any z natural. So mar x^3 ⊕ mar y^3 = 3 ⊗ mar x*y^2 ⊕ mar 2891 ⇔ {1/8/9} ⊕ {1/8/9} = {3/6/9} ⊕ 2.

The only combinations that satisfy the equality are:

: (i) $1 \oplus 1 = 9 \oplus 2$
: (ii) $8 \oplus 9 = 6 \oplus 2$
: (iii) $9 \oplus 8 = 6 \oplus 2$

Take the case (i): we have mar x^3 = mar y^3 = 1 $\Rightarrow$ mar x is 1, 4 or 7; mar y is 1, 4 or 7 $\Rightarrow$ mar 3*x*y^2 = 3 $\otimes$ {1/4/7} $\otimes$ {1/4/7} = 3 $\otimes$ {1/4/7} = 3 $\neq$ 9 $\Rightarrow$ the combination is impossible.

Take the cases (ii) and (iii): we have mar x^3 = 9 and mar y^3 = 8 or mar x^3 = 8 and mar y^3 = 9, so mar x = 3, 6 or 9 and mar y^2 = 2, 5 or 8 or mar x = 2, 5 or 8 and mar y^2 = 9. We get:

: mar 3*x*y^2 = 3 $\otimes$ {3/6/9} $\otimes$ {2/5/8} = 9 $\neq$ 6
or
: mar 3*x*y^2 = 3 $\otimes$ 9 $\otimes$ {2/5/8} = 9 $\neq$ 6

$\Rightarrow$ these combinations are impossible $\Rightarrow$ the equation has no natural solutions.

Classical proof:

The proof is done by using “modulo something”. By using modulo 2 we don’t get anywhere, but by using modulo 3, we eventually obtain the desired result. For any modulo 3 class, denoted by u, we have u^3 = u, and as 2891 equals 2(mod 3), we get x + y = 2(mod 3).

So there are three possible cases:

: (i) x = y = 1(mod 3)
: (ii) x = 0(mod 3); y = 2(mod 3)
: (iii) x = 2(mod 3); y = 0(mod 3)

We see that, if the given equation has a solution, it will have it in the second case, so x = 3*m, y = 3*k − 1, with m and k naturals. By substituting in the equation we obtain 9*r − 1 = 2891, which contradicts the hypothesis that 2891 = 9*s + 2 (r and s naturals).

(4) Let A be the sum of the digits of 4444^{4444}, and B the sum of the digits of A. Find the sum of the digits of B.

Solution using the mar reduced form:

Because the sum of the digits of B is not bigger than 9, the unknown is the mar reduced form of B. But we know that the mar reduced form of the sum of the digits of a number is equal to the mar reduced form of the number, so the unknown is really the mar reduced form of A.

We have:

mar A = mar 4444^4444 = mar (493*9 + 7)^(740*6 + 4) = mar (9*k + 7)^(6*n + 4), k = 493 and n = 740 naturals.

From the powers table we see: mar (9*k + 7)^(6*n + 4) = 7, so 7 is the solution.

Classical solution:

We see that because 4444^4444 < 10000^4444, the number of digits of 4444^4444 doesn’t exceed 4444*4 + 1 < 20000.

It follows that A < 9*20000 ⇔ A < 180000, and B < 9*5 = 45.

If we denote by C the sum of the digits of B, we get C < 4 + 9 = 13.

We see that the by dividing sum of the digits of a number by 9 we get the same modulus as for dividing the number itself. It follows that 4444^4444 ≡ C(mod 9).

On the other hand, 4444 ≡ 7(mod 9) ⇒ 4444^4444 ≡ 7^4444(mod 9) ⇒ 4444^4444 ≡ (–2)^(3*1481)*7 ≡ (–8)^1481*7 ≡ 7(mod 9).

Considering the previous relations, we obtain C = 7.

(5) Prove that the equality x^2 + y^2 + z^2 = 2*x*y*z is possible for natural numbers only if x = y = z = 0.

Proof using the mar reduced form:

For x > 0, y > 0, z > 0 we have x^2 + y^2 + z^2 = 2*x*y*z ⇒ mar x^2 ⊕ mar y^2 ⊕ mar z^2 = mar 2*x*y*z

But mar x^2, mar y^2, mar z^2 may only be 1, 4, 7 or 9 ⇒ {1/4/7/9} ⊕ {1/4/7/9} ⊕ {1/4/7/9} = 2 ⊗ mar x ⊗ mar y ⊗ mar z

Take the case mar x^2, mar y^2, mar z^2 are 1, 4 or 7.

We have: {1/4/7} ⊕ {1/4/7} ⊕ {1/4/7} = {1/2/4/5/7/8}

(In this case 2 ⊗ mar x ⊗ mar y ⊗ mar z ≠ 3, 6 or 9)

But this is not possible, as $\{1/4/7\} \oplus \{1/4/7\} \oplus \{1/4/7\} = \{3/6/9\}$

Take the case when one or two of mar x^2, mar y^2, mar z^2 are 9 (the case when all are 9 is reduced to one of the cases discussed after simplifying the equation by 3).

We get: $9 \oplus \{1/4/7\} \oplus \{1/4/7\} = \{3/6/9\}$ or $9 \oplus 9 \oplus \{1/4/7\} = \{3/6/9\}$

Both cases are impossible $\Rightarrow$ the equality is impossible $\Leftrightarrow$ the given equation has no nonzero natural solutions.

Classical solution:

x = y = z = 0 verifies the equality. One of the numbers zero implies that all numbers are zero.

Let x > 0, y > 0, z > 0. As the right side is an even number, the left side should also be even. We have the following cases:

(i) x, y odd; z even
(ii) x, y, z even

In the first case the right side is a multiple of 4 and the left side is a multiple of 4 plus 2; the equality is not possible.

So, let $x = 2^a*h_1$, $y = 2^b*h_2$, $z = 2^c*h_3$; h_1, h_2, h_3 odd; a, b, c $\geq$ 1. By substituting in the equation we get: $2^{(2*a)}*h_1^2 + 2^{(2*b)}*h_2^2 + 2^{(2*c)}*h_3^2 = 2^{(a + b + c + 1)}*h_1*h_2*h_3$. Let a = Min (a, b, c); we have: $2^{(2*a)}*(h_1^2 +2^{(2*(b - a))}*h_2^2 + 2^{(2*(c - a))}*h_3^2) = 2^{(a + b + c + 1)}*h_1*h_2*h_3$

If b > a and c > a it follows that a + b + c + 1 > 2 and, as the parenthesis is an odd number, the equality is impossible.

We can't have a = b = c because, after simplifying by $2^{(2*a)}$, the left side is odd and the right side is even.

If b = a and c > a, we can extract 2^1 as common divisor and nothing more. But in this case a + b + c + 1 > 2*a + 1

It means that the equality is possible only if x = y = z = 0.

(6) Prove that the equation $x^3 + y^3 + z^3 = 1969^2$ has no natural solutions.

Proof using the mar reduced form:

We have $x^3 + y^3 + z^3 = 1969^2 \Rightarrow \text{mar } x^3 \oplus \text{mar } y^3 \oplus \text{mar } z^3 = \text{mar } 1969^2$

But mar x^3, mar y^3, mar z^3 may only be 1, 8 or 9 (from the table of powers), and mar 1969^2 = mar $(218*9 + 7)^2 = 4$

So, we have $\{1/8/9\} \oplus \{1/8/9\} \oplus \{1/8/9\} = 4$, which is, as you may see, impossible as the left side can only have the values 1, 2, 3, 6, 7, 8 and 9.

Classical solution:

Let x, y and z integers such that $x^3 + y^3 + z^3 = 1969^2$

The modulus of the division of 1969^2 by 9 is 4. We will analyze the modulus obtained from division by 9 of numbers of the type x^3

The case (i):
: $x = 3*k$. The modulus of the division of x^3 by 9 will be $r = 0$

The case (ii):
: $x = 3*k + 1$. Then $x^3 = 3^3*k^3 + 3*3^2*k^2 + 3*3*k + 1$. The considered modulus is $r = 1$

The case (iii) :
: $x = 3*k + 2$. Then $x^3 = 3^3*k^3 - 3*3^2*k^2 + 3*3*k - 1$. The considered modulus is $r = 8$

By dividing x^3, y^3, z^3 by 9 we obtain the modulus 0, 1 or 8. If we divide the sum $r_1 + r_2 + r_3$ by 9, the three terms having the values 0, 1 or 8, we can't obtain the modulus 4, which proves the statement.

(7) Show that if 9 is a divisor of $a^3 + b^3 + c^3$, with a, b and c naturals, that at least one of the numbers a, b or c is divisible by 3.

Proof using the mar reduced form:

Let $E = a^3 + b^3 + c^3$; E is divisible by 9 $\Leftrightarrow$ mar $E = 9 \Rightarrow$ mar E = mar $(a^3 + b^3 + c^3) = 9 \Rightarrow$ mar $a^3 \oplus$ mar b^3 mar $c^3 = 9 \Rightarrow \{1/8/9\} \oplus \{1/8/9\} \oplus \{1/8/9\} = 9$ (we know that mar x^3 may only be 1, 8 or 9, for any x natural).

The possible combinations are (having in mind that the equation is symmetrical):

: $1 \oplus 1 \oplus 1 = 3 \neq 9$;
: $8 \oplus 8 \oplus 8 = 6 \neq 9$;
: $1 \oplus 1 \oplus 8 = 1 \neq 9$;
: $8 \oplus 8 \oplus 1 = 8 \neq 9$;
: $1 \oplus 1 \oplus 9 = 2 \neq 9$;
: $8 \oplus 8 \oplus 9 = 7 \neq 9$;
: $1 \oplus 9 \oplus 9 = 1 \neq 9$;
: $8 \oplus 9 \oplus 9 = 8 \neq 9$;
: $1 \oplus 8 \oplus 9 = 9$.

We see that only the last combination satisfies the equality, and this is equivalent with mar a^3 = 9, mar b^3 = 9 or mar c^3 = 9 $\Leftrightarrow$ mar a, mar b or mar c is 3, 6 or 9 $\Leftrightarrow$ at least one of the numbers a, b or c is divisible by 3.

Classical solution:

The cube of a natural number which is not divisible by 9 is of the type 9*k + 1.

If none of the numbers a, b or c is divisible by 9, then a^3 + b^3 + c^3 is of the type:
: 9*k′ + 1 + 1 + 1 = 9*k′ + 3;
: 9*k′ − 1 − 1 − 1 = 9*k′ − 3;
: 9*k′ + 1 − 1 − 1 = 9*k′ − 1;
: 9*k′ + 1 + 1 − 1 = 9*k′ + 1 .

For none of these combinations of signs a^3 + b^3 + c^3 is a multiple of 9.

(8) Prove that, if n ≥ 2, then 3 is not a divisor of C_n^2 + 1.

Proof using the mar reduced form:

We have C_n^2 = n!/(2!*(n − 2)!) = n*(n − 1)/2
Denote N = C_n^2 + 1
We have 2*N = (n − 1)*n + 2 $\Leftrightarrow$ 2*N = n^2 − n + 2 $\Rightarrow$ 2 $\otimes$ mar N $\oplus$ mar n = mar n^2 $\oplus$ 2

We will take three cases:

(i) mar n = 1, 4 or 7 (N is of the type 9*k + 1, 9*k + 4 or 9*k + 7)
(ii) mar n = 2, 5 or 8 (N is of the type 9*k + 2, 9*k + 5 or 9*k + 8)

(iii) mar n = 3, 6 or 9 (N is of the type 9*k + 3, 9*k + 6 or 9*k)

Take the case (i):

We have: 2 ⊗ mar N ⊕ {1/4/7} = {1/4/7} ⊕ 2 = {2/5/8} ⇒ mar N = 2, 5 or 8 ⇒ mar N ≠ 3, 6 or 9 ⇒ N is not divisible by 3.

Take the case (ii):

We have: 2 ⊗ mar N ⊕ {2/5/8} = {1/4/7} ⊕ 2 = {3/6/9} ⇒ mar N = 2, 5 or 8 ⇒ mar N ≠ 3, 6 or 9 ⇒ N is not divisible by 3.

Take the case (iii):

We have: 2 ⊗ mar N ⊕ {3/6/9} = 9 ⊕ 2 = 2 ⇒ 2 ⊗ mar N = 2, 5 or 8 ⇒ mar N = {1/4/7} ⇒ mar N ≠ 3, 6 or 9 ⇒ N is not divisible by 3.

Classical solution:

We have: 2*N = n^2 – n + 2; let n = 6*k + r; we take the cases:

: r = 1 :
we have 2*N = (6*k + 1)^2 – (6*k + 1) + 2 ⇔ 2*N = 36*k^2 + 6*k + 2 ⇔ N = 18*k^2 + 3*k + 1 ⇒ 3 is not a divisor of N

: r = 2 :
we have 2*N = (6*k + 2)^2 – (6*k + 2) + 2 ⇔ 2*N = 36*k^2 + 18*k + 4 ⇔ N = 18*k^2 + 9*k + 2 ⇒ 3 is not a divisor of N

: r = 3 :
we have 2*N = (6*k + 3)^2 – (6*k + 3) + 2 ⇔ 2*N = 36*k^2 + 30*k + 8 ⇔ N = 18*k^2 + 15*k + 4 ⇒ 3 is not a divisor of N

: r = 4 :
we have 2*N = (6*k + 4)^2 – (6*k + 4) + 2 ⇔ 2*N = 36*k^2 + 42*k + 14 ⇔ N = 18*k^2 + 21*k + 7 ⇒ 3 is not a divisor of N

: r = 5 :
we have 2*N = (6*k + 5)^2 – (6*k + 5) + 2 ⇔ 2*N = 36*k^2 + 60*k + 22 ⇔ N = 18*k^2 + 30*k + 11 ⇒ 3 is not a divisor of N

: r = 0 :
we have 2*N = (6*k)^2 – (6*k) + 2 ⇔ 2*N = 36*k^2 – 6*k + 2 ⇒ 3 is not a divisor of N

We proved that N is not divisible by 3, for any n natural

(9) Prove that, if a and b are natural numbers which are not divisible by 3, then either a - b or a + b is divisible by 3.

Proof using the mar reduced form:

a and b are not divisible by 3 $\Rightarrow$ mar a and mar b $\neq$ 3, 6 or 9.

Take the cases:

(i) mar a = 1, 4 or 7; mar b = 1, 4 or 7; we have: {1/4/7} = {1/4/7} $\oplus$ {3/6/9} $\Leftrightarrow$ mar a = mar b $\oplus$ {3/6/9} $\Rightarrow$ mar (a − b) = 3, 6 or 9 $\Rightarrow$ (a − b) is divisible by 3

(ii) mar a = 2, 5 or 8; mar b = 2, 5 or 8; we have: {2/5/8} = {2/5/8} $\oplus$ {3/6/9} $\Leftrightarrow$ mar a = mar b $\oplus$ {3/6/9} $\Rightarrow$ mar (a − b) = 3, 6 or 9 $\Rightarrow$ (a − b) is divisible by 3

(iii) mar a = 1, 4 or 7; mar b = 2, 5 or 8; we have: {1/4/7} $\oplus$ {2/5/8} = {3/6/9} $\Leftrightarrow$ mar a $\oplus$ mar b = {3/6/9} $\Rightarrow$ mar (a + b) = 3, 6 or 9 $\Rightarrow$ (a + b) is divisible by 3

Classical solution:

Take a = 3*m + p and b = 3*n + r; m and n naturals, p, r = 1 or 2

We have:

(i) p = 1 and r = 1 $\Rightarrow$ a = 3*m + 1 and b = 3*n + 1 $\Rightarrow$ (a − b) = 3*(m − n) $\Rightarrow$ (a − b) is divisible by 3

(ii) p = 1 and r = 2 $\Rightarrow$ a = 3*m + 1 and b = 3*n + 2 $\Rightarrow$ (a + b) = 3*(m + n + 1) $\Rightarrow$ (a + b) is divisible by 3

(iii) p = 2 and r = 1 (equivalent with the case (ii), because the equation is symmetrical) $\Rightarrow$ (a + b) is divisible by 3

(iv) p = 2 and r = 2 $\Rightarrow$ a = 3*m + 2 and b = 3*n + 2 $\Rightarrow$ (a − b) = 3*(m − n) $\Rightarrow$ (a − b) is divisible by 3.

12. A SPECIAL DIOPHANTINE EQUATION

We want to solve separately a Diophantine equation for two reasons: first, because it is famous, belonging to Fermat (1601-1665), and, second, because the well-known proof is not elementary, but involves concepts as complex numbers, Euclidian rings etc. The equation is proved (in the mentioned manner) in the book "Elemente de aritmetică" by Mariana Vraciu and Constantin Vraciu, published in Romania by the publishing house "B.I.C. ALL" in 1998. Finally, this equation is: y^2 + 2 = x^3. We shall treat this Diophantine equation from the point of view of natural solutions only. As you have already seen, generalizing the definition of the mar reduced form on the set of integers is not our purpose here.

Prove that the only natural solution of the equation y^2 + 2 = x^3 is [x, y] = [3, 5].

Proof:

For x = 0 and y = 0 the equation has no natural solutions.

For x > 0 and y > 0 we have y^2 + 2 = x^3 ⇒ mar (y^2 + 2) = mar x^3 ⇔ mar y^2 ⊕ 2 = mar x^3

We know that mar y^2 may only be 1, 4, 7 or 9, for any y natural, y > 0 and mar x^3 may only be 1, 8 or 9, for any x natural, x > 0.

Thus, we have: {1/4/7/9} ⊕ 2 = {1/8/9}

The only combination that satisfies the equality is 7 ⊕ 2 = 9 ⇒ mar y^2 = 7 and mar x^3 = 9 ⇒ mar y is equal to 4 or 5 and mar x is equal to 3, 6 or 9. It follows that y is of the type 9*k + 4 or 9*k + 5, k natural and x is of the type 3*m, m non-null natural.

Take the case y = 9*k + 4, x = 3*m; we have:
y^2 + 2 = x^3 ⇔ (9*k + 4)^2 + 2 = 27*m^3

For k = 0 we get 18 = 27*m^3 ⇒ no natural solutions.

Take the case when k is even, k = 2*h, h > 0; m even, m = 2*n, n > 0. We have: (18*h + 4)^2 + 2 = 27*8*n^3 ⇔ 4*(9*h + 2)^2 + 2 = 27*8*n^3 ⇔ 2*(9*h + 2)^2 + 1 = 27*4*n^3. But the right side of the equality is divisible by 2, and the left side is not divisible by 2 ⇒ the equality is impossible.

Take the case when k is even, k = 2*h, h > 0; m odd, m = 2*n + 1. We have: (18*h + 4)^2 + 2 = 27*(2*n + 1)^3. This time, the left side of the equality is divisible by 2, while the right side isn't ⇒ the equality is impossible.

Take the case when k is odd, k = 2*h + 1; m even, m = 2*n, n > 0. We have: (18*h + 13)^2 + 2 = 27*8*n^3. The right side of the equality is divisible by 2, and the left side is not divisible by 2 ⇒ the equality is impossible.

The case when k is odd, k = 2*h + 1; m odd, m = 2*n + 1 we leave it for last (it requires special treatment).

Take the case y = 9*k + 5, x = 3*m; we have:
y^2 + 2 = x^3 ⇔ (9*k + 5)^2 + 2 = 27*m^3

For k = 0 we get 25 + 2 = 27*1 ⇒ (x = 3, y = 5) is indeed a solution of the equation.

Take the case when k is even, k = 2*h, h > 0; m even, m = 2*n, n > 0. We have: (18*h + 5)^2 + 2 = 27*8*n^3 . It's easy to see that the left side of the equality is not divisible by 2, and the right side is ⇒ the equality is impossible.

The case when k is even, k = 2*h, h > 0; m odd, m = 2*n + 1 we leave it for last (for reasons mentioned above).

Take the case when k is odd, k = 2*h + 1; m even, m = 2*n, n > 0. We have: (18*h + 14)^2 + 2 = 27*8*n^3 ⇔ 4*(9*h + 7)^2 + 2 = 27*8*n^3 ⇔ 2*(9*h + 7)^2 + 1 = 27*4*n^3. It's easy to see that the right side of the equality is divisible by 2, and the left side isn't ⇒ the equality is impossible.

Take the case when k is even, k = 2*h + 1; m odd, m = 2*n + 1. We have: (18*h + 14)^2 + 2 = 27*(2*n + 1)^3 ⇔ 4*(9*h + 7)^2 + 2 = 27*(2*n + 1). But the left side of the equality is divisible by 2, and the right side isn't.

We now consider the two remaining cases; we have:

(18*h + 13)^2 + 2 = 27*(2*n + 1)^3
and
(18*h + 5)^2 + 2 = 27*(2*n + 1)^3

But (18*h + 13) may be written as (18*h – 5) ⇒ the cases are:

(18*h - 5)^2 + 2 = 27*(2*n + 1)^3
and
(18*h + 5)^2 + 2 = 27*(2*n + 1)^3

We go on in parallel with them; we have:
18^2*h^2 ± 18*10*h + 25 + 2 = 27*(2*n + 1)^3, equivalent with:
18^2*h^2 ± 18*10*h = 27*(2*n + 1)^3 - 27

We simplify both equalities by 9; we have: 18*2*h^2 ± 20*h = 3*(2*n + 1)^3 – 3. So 3 should be a divisor of h; denote h = 3*r, r natural, r > 0; we have:
18*2*9*r^2 ± 20*3*r = 3*(2*n + 1)^3 – 3

We simplify the equalities by 3; we have:
6*2*9*r^2 ± 20*r = (2*n + 1)^3 – 1 = 8*n^3 + 12*n^2 + 6*n

We simplify the equalities by 2; we have:
6*9*r^2 ± 10*r = 4*n^3 + 6*n^2 + 3*n

It's easy to see that n must be divisible by 2; denote n = 2*p, p natural, p > 0;
54*r^2 ± 10*r = 4*8*p^3 + 6*4*p^2 + 3*2*p

We simplify the equalities by 2; we have:
27*r^2 ± 5*r = 16*p^3 + 12*p^2 + 3

Take the second degree equation with unknown r:
27*r^2 ± 5*r – (16*p^3 + 12*p^2 + 3*p) = 0

For this equation to have natural roots ⇔ in order to have an r such that the equalities are true, the discriminant of the equation must be the square of a natural number; let D be the discriminant of the equation. We have:

D = 25 + 4*27*(16*p^3 + 12*p^2 + 3*p) = z^2, z natural

We obtained this third degree equation with unknown p:

4*27*16*p^3 + 4*27*12 p^2 + 4*27*3*p + 25 – z^2 = 0

Denote by c_1, c_2, c_3, c_4 the coefficients of p; we solve the equation using Cardano's formulas:

We have c_1 = 64*27, c_2 = 48*27, c_3 = 12*27 and c_4 = 25*z^2, also 3*u = (3*c_1*c_3 - c_2^2)/(3*c_1^2) and 2*v = (2*c_2^3)/(27*c_1^3) - (c_2*c_3)/(3*c_1^2) + c_4/c_1 which is equivalent with 3*u = (3*64*27*12*27*48^2*27^2).

We have $D = v^2 + u^3 > 0 \Rightarrow$ the equation has a real root (and two complex conjugate roots). The only real root of the equation is $w = w_1 + w_2$, where $w_1 = (-v + D^{1/2})^{1/3}$ and $w_2 = (-v - D^{1/2})^{1/3}$

But $w_1 = (-v + v)^{1/3} = 0$ and $w_2 = (-v - v)^{1/3}$

We have $w = w_1 + w_2 = (-2 \cdot v)^{1/3} \Rightarrow w^3 = -2 \cdot v = (z^2 + 2)/(27 \cdot 4^3)$ which is equivalent with $z^2 + 2 = w^3 \cdot 3^3 \cdot 4^3$

Subtract 6 from each side of the obtained equality; we have: $Z^2 - 4 = w^3 \cdot 3^3 \cdot 4^3 - 6 \Leftrightarrow (z + 2) \cdot (z - 2) = 6 \cdot (w^3 \cdot 3^2 \cdot 2^5 - 1)$

We notice that the left side of the equality either isn't divisible by 2, or is at least divisible by 4, while the right side is always divisible with two and at most with two.

We have proven what we wanted, which is that the only natural solution of the equation $y^2 + 2 = x^3$ is ($x = 3$, $y = 5$).

13. INTRODUCTION TO FERMAT'S LAST THEOREM

Well-known to all mathematicians and not only to them, rightfully the most famous and most discussed Diophantine equation of all times is the so-called "Fermat's last Theorem", which states that there aren't any nonzero integers x, y and z for which $x^n + y^n = z^n$, where n integer, $n > 2$. Intriguing by the simplicity of its statement, along with the systematic failure of all attempts to solve it that have spanned along four centuries (the Theorem was proved in 1995 by Andrew Wiles), Fermat's last Theorem has always been in the twilight zone of mathematics, defying, like the Egyptian Pyramids, all the evolutionistic theories that state that with piling of years and concepts, science moves closer to the truth.

It is also very well known the interest of a series of standing mathematicians – Euler and Gauss, to nominate just a few – towards this Theorem. Many others have tried and succeeded in proving partially Fermat's Theorem: for $n = 2$, $n = 3$, $n < 100$, $n < 100000$ and so on. The history of mathematics mentions all and all their results, no matter how modest. Anyway, if they hadn't succeeded in giving a general proof of Fermat's Theorem, they succeeded in return in creating new and more and more powerful methods and instruments. We mention here Ernst Kummel (1810-1893), whose results in proving the Theorem are the basis of The Algebraic Theory of Numbers.

Also not unimportant to the legend aura surrounding Fermat's Theorem are other few thousands anonymous (the so-called fermatists), idealists, less respected, less learned, who dreamed of the glory and – why not? – of the money they could earn by proving this Theorem (indeed, at the beginning of the twentieth century, a German millionaire had offered a DM 100000 prize to whom would had proved Fermat's Theorem).

We wouldn't rush into despising those people, with respect to their perhaps debatable and inadequate, but surely simplistic methods, but would rather compare them with the characters of the famous American Gold Rush. Armed with just a pickaxe and determination you have a chance (no matter how slim) to find a vein. Just the same you may use diamond-head drills and may plough the mud in vain.

As far as we're concerned, we confess that the dream of proving Fermat's last Theorem made us create this arithmetic instrument, the mar reduced form.

For the moment, the mar reduced form helped us proved Fermat's Theorem just for the n = 3 and n = 4 cases (and, implicitly those which may be reduced to these cases). Better used, by us or by someone else, this instrument will surely have more to say in the elementary approach (and why not, in the proof) of Fermat's last Theorem.

14. PROOF OF FERMAT'S LAST THEOREM: CASE N = 3

We have a^3 + b^3 = c^3 ⇒ mar (a^3 + b^3) = mar c^3 ⇔ mar a^3 ⊕ mar b^3 = mar c^3. From the mar a^n table we see that mar a^3, mar b^3, mar c^3 may only take the values 1, 8 and 9: mar a^3 ∈ {1, 8, 9}, mar b^3 ∈ {1, 8, 9}, mar c^3 ∈ {1, 8, 9}.

The equation a^3 + b^3 = c^3 may take natural solutions only if mar a^3 ⊕ mar b^3 = mar c^3, which is possible only in one of these cases:

(A) 1 ⊕ 9 = 1 (mar a^3 = mar c^3 = 1, mar b^3 = 9)
(B) 8 ⊕ 9 = 8 (mar a^3 = mar c^3 = 8, mar b^3 = 9)
(C) 9 ⊕ 9 = 9 (mar a^3 = mar b^3 = mar c^3 = 9)
(D) 1 ⊕ 8 = 9 (mar a^3 = 1, mar b^3 = 8, mar c^3 = 9)

We take the cases (A) and (B):

We have: mar b^3 = 9 ⇒ b is of the type 9*k + 3, 9*k + 6 or 9*k ⇒ b is divisible by 3 ⇒ b^3 is divisible by 3^3 = 27.

Denote b = 3*p (p natural, p > 0). We have b^3 = 27*p^3

On the other hand, b^3 = c^3 – a^3 = (c – a)*(c^2 + a*c + a^2) = 27*p^3

We have the following possibilities:

c – a	c^2 + a*c + a^2
	divisible with 27
divisible with 3	divisible with 9
divisible with 9	divisible with 3
divisible with 27	

Suppose that c^2 + a*c + a^2 is divisible by 27.

Denote c^2 + a*c + a^2 = 27*r, where r nonzero natural.

Thus, we have: c^2 + a*c + a^2 = (c – a)^2 + 3*a*c =
= 27*r ⇒ (c – a)^2 = 27*r – 3*a*c = 3*(9*r – a*c) ⇒
c – a = (3*(9*r – a*c))^(1/2)

(we shall use the notation a^(1/2) for sqrt a and a^(1/3) for third root of a)

If c – a is a natural number ⇒ (3*(9*r – a*c))^(1/2) is natural ⇒ (9*r – a*c) is divisible by 3 ⇒ a*c is divisible by 3 ⇒ mar (a*c) is 3, 6 or 9. But mar (a*c) = mar a ⊗ mar c, and mar a^3 may be:

: 1 ⇒ mar a is 1, 4 or 7
or
: 8 ⇒ mar a is 2, 5 or 8.

We have (in the considered cases) mar a^3 = mar c^3 ⇒

: mar (a*c) = mar a ⊗ mar c = {1/4/7} ⊗ {1/4/7} for mar a^3 = mar c^3 = 1
and
: mar (a*c) = mar a ⊗ mar c = {2/5/8} ⊗ {2/5/8} for mar a^3 = mar c^3 = 8

(we mention again that we denote mar x = {1/4/7}, for instance, when mar x is equal to 1 or 4 or 7)

Finally, mar (a*c) may be 1 ⊗ 1 = 1, 1 ⊗ 4 = 4, 1 ⊗ 7 = 7, 4 ⊗ 4 = 7, 4 ⊗ 7 = 1, 7 ⊗ 7 = 4, respectively

$2 \otimes 2 = 4$, $2 \otimes 5 = 1$, $2 \otimes 8 = 7$, $5 \otimes 5 = 7$, $5 \otimes 8 = 4$, $8 \otimes 8 = 1$.

In all of these cases, mar (a*c) is not 3, 6 or 9 ⇒ 3 is not a divisor of a*c ⇒ the initial assumption that 27 is a divisor of (c^2 + a*c + a^2) is false.

Suppose that c^2 + a*c + a^2 is divisible by 9

Denote c^2 + a*c + a^2 = 9*r, where r nonzero natural.

We have: c^2 + a*c + a^2 = (c – a)^2 + 3*a*c = 9*r

But c – a is divisible by 3 because 27*p^3 =(c – a)*(c^2 + a*c + a^2)

Denote c – a = 3*m (m nonzero natural)

We have: (c – a)^2 + 3*a*c = 9*r ⇔ 9*m^2 + 3*a*c = 9*r ⇔ 3*m^2 + a*c = 3*r ⇒ a*c = 3*(r – m^2) ⇒ 3 is a divisor of a c ⇒ a or c is divisible by 3

But mar a^3 = mar c^3 = 1 or mar a^3 = mar c^3 = 8. So, we have mar a = 1, 2, 4, 5, 7 or 8 and mar c = 1, 2, 4, 5, 7 or 8 ⇒ a and c are of the type 9*k + 1, 9*k + 2, 9*k + 4, 9*k + 5, 9*k + 7 or 9*k + 8 ⇔ neither a, nor c are divisible by 3 ⇒ 3 is not a divisor of a c ⇒ the initial assumption that c^2 + a*c + a^2 is divisible by 9 is false.

We proved that c^2 + a*c + a^2 is not divisible by 27, not even by 9 ⇒ c^2 + a*c + a^2 is divisible, at most, by 3. We'll prove that, actually, c^2 + a*c + a^2 is always divisible by 3:

We have: c^2 + a*c + a^2 is divisible by 3 ⇔ mar (c^2 + a*c + a^2) = 3, 6 or 9 ⇔ mar c^2 ⊕ mar (a*c) ⊕ mar a^2 = 3, 6 or 9 ⇔ mar c^2 ⊕ mar a ⊗ mar c ⊕ mar a^2 = 3, 6 or 9. But mar a^3 = mar c^3 = 1 or mar a^3 = mar c^3 = 8 ⇒

: mar a^2 = 1, 4 or 7, mar a = 1, 4 or 7; mar c^2 = 1, 4 or 7, mar c = 1, 4 or 7

or

: mar a^2 = 1, 4 or 7, mar a = 2, 5 or 8; mar c^2 = 1, 4 or 7, mar c = 2, 5 or 8

Then,

:   mar $(c^2 + a*c + a^2)$ = mar c^2 ⊕ mar a ⊗ mar c ⊕ mar a^2 = {1/4/7} ⊕ {1/4/7} ⊗ {1/4/7} ⊕ {1/4/7} = {1/4/7} ⊕ {1/4/7} ⊕ {1/4/7} = {3/6/9}

or

:   mar $(c^2 + a*c + a^2)$ = mar c^2 ⊕ mar a ⊗ mar c ⊕ mar a^2 = {1/4/7} ⊕ {2/5/8} ⊗ {2/5/8} ⊕ {1/4/7} = {1/4/7} ⊕ {1/4/7} ⊕ {1/4/7} = {3/6/9}

Indeed, mar $(c^2 + a*c + a^2)$ may be equal with:

:   1 ⊕ 4 ⊕ 7 = 3
:   1 ⊕ 1 ⊕ 1 = 3; 4 ⊕ 4 ⊕ 4 = 3; 7 ⊕ 7 ⊕ 7 = 3
:   1 ⊕ 1 ⊕ 4 = 6; 4 ⊕ 4 ⊕ 7 = 6; 7 ⊕ 7 ⊕ 1 = 6
:   1 ⊕ 1 ⊕ 7 = 9; 4 ⊕ 4 ⊕ 1 = 9; 7 ⊕ 7 ⊕ 4 = 9

So mar $(c^2 + a*c + a^2)$ = 3, 6 or 9 ⇒ $c^2 + a*c + a^2$ is divisible by 3.

Thus, we have $b^3 = c^3 - a^3 = (c - a)*(c^2 + a*c + a^2) = 27*p^3$, and $c^2 + a*c + a^2$ is divisible by 3, but isn’t divisible by 9. It follows that $c - a$ is divisible by 9.

Denote $c - a = 9*m$, where m nonzero natural. We have $b = 3*p$ and $c - a = 9*m \Rightarrow c = a + 9*m$

We have: $a^3 + b^3 = c^3 \Rightarrow a^3 + 27*p^3 = (a + 9*m)^3$
$\Leftrightarrow a^3 + 27*p^3 = a^3 + 27*a^2*m + 243*a*m^2 + 729*m^3$
$\Leftrightarrow 27*a^2*m + 243*a*m^2 + 729*m^3 - 27*p^3 = 0$
$\Leftrightarrow a^2*m + 9*a*m^2 + 27*m^3 - p^3 = 0$

We have the third degree equation with unknown m: $27*m^3 + 9*a*m^2 + a^2*m - p^3 = 0$

Denote by c_1, c_2, c_3, c_4 the coefficients of m; we solve the equation:

We have $c_1 = 27$, $c_2 = 9*a$, $c_3 = a^2$, $c_4 = -p^3$ and
$3*x = (3*c_1*c_3 - c_2^2)/(3*c_1^2)$ and
$2*y = (2*c_2^3)/(27*c_1^3) - (c_2*c_3)/(3*c_1^2) + c_4/c_1$
which eventually gives us the solutions $[x, y] = [0, -(a^3 + 27*p^3)/(2*27^2)$

We have $D = y^2 + x^3 = y^2 > 0 \Rightarrow$ the equation has a real root (and two complex conjugate roots).

The only real root of the equation is: $m = u + v$, where $u = (-y + D^{(1/2)})^{(1/3)}$ and $v = (-y -$

D^(1/2))^(1/3) which gives us u = 0 and v = (-2*y)^(1/3)

We have: m = u + v = (-2*y)^(1/3) ⇒ m^3 = −2*y = (a^3 + 27*p^3)/27^2 ⇒ 27^2*m^3 = a^3 + 27*p^3

But 27^2*m^3 = a^3 + 27*p^3 ⇒ mar(27^2*m^3) = mar (a^3 + 27*p^3) ⇒ mar 27^2 ⊗ mar m^3 = mar a^3 ⊕ mar 27 ⊗ mar p^3 ⇒ 9 ⊗ mar m^3 = mar a^3 ⊕ 9 ⊗ mar p^3 ⇒ 9 = mar a^3 ⊕ 9

But mar a^3 = 1 or mar a^3 = 8 ⇒ the equality mar a^3 ⊕ 9 = 9 is impossible ⇒ there aren’t any a, b, c naturals which satisfy the equation a^3 + b^3 = c^3, cases (A) and (B).

We take the case (C):

We have mar a^3 = mar b^3 = mar c^3 = 9 ⇒ a, b and c are of the type 9*k + 3, 9*k + 6 or 9*k (k nonzero natural) ⇔ a, b and c are divisible by 3, so they may be written as a = 3*a′, b = 3*b′, c = 3*c′.

The equation a^3 + b^3 = c^3 ⇔ (3*a′)^3 + (3*b′)^3 = (3*c′)^3 ⇔ 27*a′^3 + 27*b′^3 = 27*c′^3 ⇔ a′^3 + b′^3 = c′^3

If a′, b′ and c′ are all divisible by 3, we repeat the simplification; eventually, we will obtain an equation a″^3 + b″^3 = c″^3, where a″, b″ and c″ are not all divisible by 3 (it’s obvious that a, b, and c can’t all be powers of 3). Solving the equation a″^3 + b″^3 = c″^3 is reduced to one of the cases (A), (B) or (D).

We take the case (D):

So a^3 + b^3 = c^3 ⇒ mar a^3 ⊕ mar b^3 = mar c^3 ⇔ 1 ⊕ 8 = 9 (mar a^3 = 1, mar b^3 = 8, mar c^3 = 9).

We have mar c^3 = 9 ⇒ mar c = 3, 6 or 9 ⇒ c is of the type 9*k + 3, 9*k + 6 or 9*k (k nonzero natural) ⇒ c is divisible by 3, so it may be written as c = 3*m, where m nonzero natural.

So, we have c^3 = a^3 + b^3 = (a + b)*(a^2 – a*b + b^2) = 27*m^3.

Let’s see if a^2 – a*b + b^2 is divisible by 3.

Let $a^2 - a*b + b^2 = r \Rightarrow a^2 + b^2 = r + a*b \Rightarrow$ mar $(a^2 + b^2) =$ mar $(r + a*b) \Rightarrow$ mar $a^2 \oplus$ mar $b^2 =$ mar $r \oplus$ mar $a \otimes$ mar b. We have mar $a^3 = 1 \Rightarrow$ mar $a =$ 1, 4 or 7 $\Rightarrow$ mar $a^2 =$ 1, 4 or 7 and mar $b^3 = 8 \Rightarrow$ mar $b =$ 2, 5 or 8 $\Rightarrow$ mar $b^2 =$ 1, 4 or 7.

So mar $a^2 \oplus$ mar $b^2 =$ mar $r \oplus$ mar $a \otimes$ mar $b \Leftrightarrow$
$\Leftrightarrow \{1/4/7\} \oplus \{1/4/7\} =$ mar $r \oplus \{1/4/7\} \otimes \{2/5/8\} \Leftrightarrow$
$\Leftrightarrow \{2/5/8\} =$ mar $r \oplus \{2/5/8\} \Rightarrow$ mar $r = \{3/6/9\}$

So r is of the type $9*k + 3$, $9*k + 6$ or $9*k$ (k nonzero natural) $\Rightarrow r = a^2 - a*b + b^2$ is divisible by 3.

Let's see if $r = a^2 - a*b + b^2$ is divisible by 9.

So r divisible by 9 $\Leftrightarrow$ mar $r =$ mar $(a^2 - a*b + b^2) =$ 9. So, we have $2 \oplus 9 = 2$, $5 \oplus 9 = 5$ and $8 \oplus 9 = 8$, where mar $a^2 \oplus$ mar $b^2 =$ mar $a \otimes$ mar $b = \{2/5/8\}$, the only combinations that comply with the condition mar $r = 9$.

Take the case $2 \oplus 9 = 2 \Leftrightarrow$
$\Leftrightarrow$ mar $a^2 \oplus$ mar $b^2 =$ mar $a \otimes$ mar $b = 2$. We have $\{1/4/7\} \oplus \{1/4/7\} = \{1/4/7\} \otimes \{2/5/8\} = 2$. The combination isn't satisfied by any of the possibilities:

: $1 \oplus 1 = 1 \otimes 2 = 2 \Rightarrow$ mar $a^2 =$ mar $b^2 = 1 \Rightarrow$ mar $a \neq 2$, mar $b \neq 2$
: $4 \oplus 7 = 1 \otimes 2 = 2 \Rightarrow$ mar $a^2 = 4$, mar $b^2 = 7 \Rightarrow$ mar $a \neq 1$, mar $b \neq 1$
: $7 \oplus 4 = 1 \otimes 2 = 2 \Rightarrow$ mar $a^2 = 7$, mar $b^2 = 4 \Rightarrow$ mar $a \neq 1$, mar $b \neq 1$
: $1 \oplus 1 = 4 \otimes 5 = 2 \Rightarrow$ mar $a^2 =$ mar $b^2 = 1 \Rightarrow$ mar $a \neq 4$ or 5, mar $b \neq 4$ or 5
: $4 \oplus 7 = 4 \otimes 5 = 2 \Rightarrow$ mar $a^2 = 4$, mar $b^2 = 7 \Rightarrow$ mar $a \neq 4$, mar $a \neq 5$
: $7 \oplus 4 = 4 \otimes 5 = 2 \Rightarrow$ mar $a^2 = 7$, mar $b^2 = 4 \Rightarrow$ mar $b \neq 4$, mar $b \neq 5$
: $1 \oplus 1 = 7 \otimes 8 = 2 \Rightarrow$ mar $a^2 =$ mar $b^2 = 1 \Rightarrow$ mar $a \neq 7$, mar $b \neq 7$
: $4 \oplus 7 = 7 \otimes 8 = 2 \Rightarrow$ mar $a^2 = 4$, mar $b^2 = 7 \Rightarrow$ mar $a \neq 8$, mar $b \neq 8$
: $7 \oplus 4 = 7 \otimes 8 = 2 \Rightarrow$ mar $a^2 = 7$, mar $b^2 = 4 \Rightarrow$ mar $a \neq 8$, mar $b \neq 8$

Take the case 5 ⊕ 9 = 5 ⇔

⇔ mar a^2 ⊕ mar b^2 = mar a ⊗ mar b = 5. We have {1/4/7} ⊕ {1/4/7} = {1/4/7} ⊗ {2/5/8} = 5. The combination isn’t satisfied by any of the possibilities:

: 1 ⊕ 4 = 1 ⊗ 5 = 5 ⇒ mar a^2 = 1, mar b^2 = 4 ⇒ mar a ≠ 5, mar b ≠ 5

: 7 ⊕ 7 = 1 ⊗ 5 = 5 ⇒ mar a^2 = mar b^2 = 7 ⇒ mar a ≠ 1, mar b ≠ 1

: 1 ⊕ 4 = 4 ⊗ 8 = 5 ⇒ mar a^2 = 1, mar b^2 = 1 ⇒ mar a ≠ 4, mar b ≠ 4

: 7 ⊕ 7 = 4 ⊗ 8 = 5 ⇒ mar a^2 = mar b^2 = 7 ⇒ mar a ≠ 8, mar b ≠ 8

: 1 ⊕ 4 = 7 ⊗ 2 = 5 ⇒ mar a^2 = 1, mar b^2 = 4 ⇒ mar a ≠ 7, mar b ≠ 2

: 7 ⊕ 7 = 7 ⊗ 2 = 5 ⇒ mar a^2 = mar b^2 = 7 ⇒ mar a ≠ 2 or 7, mar b ≠ 2 or 7

Take the case 8 ⊕ 9 = 8 ⇔

⇔ mar a^2 ⊕ mar b^2 = mar a ⊗ mar b = 8. We have {1/4/7} ⊕ {1/4/7} = {1/4/7} ⊗ {2/5/8} = 8. The combination isn’t satisfied by any of the possibilities:

: 1 ⊕ 7 = 1 ⊗ 8 = 8 ⇒ mar a^2 = 1, mar b^2 = 7 ⇒ mar b ≠ 1, mar b ≠ 8

: 1 ⊕ 7 = 4 ⊗ 2 = 8 ⇒ mar a^2 = 1,mar b^2 = 7 ⇒ mar a ≠ 2, mar b ≠ 2

: 4 ⊕ 4 = 1 ⊗ 8 = 8 ⇒ mar a^2 = mar b^2 = 4 ⇒ mar a ≠ 1 or 8, mar b ≠ 1 or 8

: 4 ⊕ 4 = 4 ⊗ 2 = 8 ⇒ mar a^2 = mar b^2 = 4 ⇒ mar a ≠ 4, mar b ≠ 4

: 1 ⊕ 7 = 7 ⊗ 5 = 8 ⇒ mar a^2 = 1, mar b^2 = 7 ⇒ mar a ≠ 7, mar b ≠ 7

: 4 ⊕ 4 = 7 ⊗ 5 = 8 ⇒ mar a^2 = mar b^2 = 4 ⇒ mar a ≠ 5, mar b ≠ 5

We have proved that mar r = mar (a^2 – a*b + b^2) ≠ 9 ⇒ a^2 – a*b + b^2 isn’t divisible by 9 ⇒ a + b is divisible by 9, as c^3 = (a + b)*(a^2 – a*b + b^2) = 27*m^3.

We have c = 3*m and a + b = 9*p ⇒ b = 9*p - a

We have a^3 + b^3 = c^3 ⇔ a^3 +(9*p – a)^3 = 27*m^3 ⇔

$\Leftrightarrow$ $a^3 + 729*p^3 - 243*a*p^2 + 27*a^2*p - a^3 = 27*m^3$ $\Leftrightarrow$ $729*p^3 - 243*a*p^2 + 27*a^2*p = 27*m^3$ $\Leftrightarrow$ $27*p^3 - 9*a*p^2 + a^2*p = m^3$ $\Leftrightarrow$ $27*p^3 - 9*a*p^2 + a^2*p - m^3 = 0$

We have a third degree equation with the unknown p, which we'll solve using Cardano's formulas. Denote by c_1, c_2, c_3, c_4 the coefficients of p; we have:

: $c_1 = 27$, $c_2 = -9*a$, $c_3 = a^2$, $c_4 = -m^3$;
: $3*x = (3*c_1*c_3 - c_2^2)/(3*c_1^2)$ and
$2*y = (2*c_2^3)/(27*c_1^3) - (c_2*c_3)/(3*c_1^2) + c_4/c_1$

which eventually gives us the solutions $[x, y] = [0, (a^3 - 27*m^3)/(2*27^2)$

We have $D = y^2 + x^3 = y^2 > 0 \Rightarrow$ the equation has a real root (and two complex conjugate roots).

The only real root of the equation is: $p = u + v$, where $u = (-y + D^{(1/2)})^{(1/3)}$ and $v = (-y - D^{(1/2)})^{(1/3)}$ which gives us $u = 0$ and $v = (-2*y)^{(1/3)}$

We have: $p = u + v = (-2*y)^{(1/3)} \Rightarrow m^3 = -2*y = (27*p^3 - a^3)/27^2 \Rightarrow 27^2*p^3 = 27*m^3 - a^3 \Rightarrow a^3 + 27^2*p^3 = 27*m^3 \Rightarrow$ mar $a^3 \oplus 9 = 9$, which is impossible because mar $a^3 = 1$.

We have proved Fermat's last Theorem for n = 3.

15. PROOF OF THE FERMAT'S LAST THEOREM: CASE n = 4

We have $a^4 + b^4 = c^4$ where a, b, c natural. But $a^4 + b^4 = c^4 \Rightarrow$ mar $(a^4 + b^4)$ = mar c^4 $\Leftrightarrow$ mar $a^4 \oplus$ mar b^4 = mar c^4. We've seen from the table that mar a^4, mar b^4 and mar c^4 can only have the values 1, 4, 7 or 9.

The equation $a^4 + b^4 = c^4$ has natural solutions only if mar $a^4 \oplus$ mar b^4 = mar c^4, which is only possible in one of the cases:

(A) $1 \oplus 9 = 1$ (mar a^4= mar c^4= 1, mar b^4= 9)
(B) $4 \oplus 9 = 4$ (mar a^4 = mar c^4 = 4, mar b^4 = 9)
(C) $7 \oplus 9 = 7$ (mar a^4 = mar c^4 = 7, mar b^4 = 9)
(D) $9 \oplus 9 = 9$ (mar a^4 = mar b^4 = mar c^4 = 9)

It's obvious that the last combination, $9 \oplus 9 = 9$, redresses to one of the other combinations, as a, b, c can't all be powers of 3.

We take the cases (A), (B) and (C):

We have mar b^4 = 9 ⇒ b^4 is divisible by 9 ⇒ b is divisible by 3 ⇒ b^4 is divisible by 3^4. On the other hand, we have mar a^4 = mar c^4 = {1/4/7} ⇒ mar a ≠ 3, 6 or 9 and mar c ≠ 3, 6 or 9.

We denote b^4 = 3^4*p^4 and b = 3*p

We have a^4 + b^4 = c^4 ⇔ b^4 = c^4 - a^4 ⇔ b^4 = (c^2 – a^2)*(c^2 + a^2) = 3^4*p^4.

From mar a ≠ 3, 6 or 9 and mar c ≠ 3, 6 or 9 results that mar a^2 = 1, 4 or 7 and mar c^2 = 1, 4 or 7 ⇒ mar (a^2 + c^2) = mar a^2 ⊕ mar c^2 = {1/4/7} ⊕ {1/4/7} = {2/5/8} ⇒ mar (a^2 + c^2) ≠ 3, 6 or 9 ⇒ a^2 + c^2 isn't divisible by 3 ⇒ the only possibility is that c^2 – a^2 is divisible by 3 .

But as b^4 = 3^4*p^4 = (c^2 – a^2)*(c^2 + a^2) ⇒ c^2 – a^2 is divisible by 3^4; thus, we have c^2 – a^2 = (c – a)*(c + a) is divisible by 3^4.

Let's prove now that (c – a) isn't divisible by 3^4 either;

As a and c aren't divisible by 3, it's obvious that (c + a) and (c - a) can't be simultaneously divisible by 3. The only possibility left is that (c + a) is divisible by 3.

So, we suppose that (c - a) is divisible by 3^4, which is equivalent with c – a = 3^4*r, where r is nonzero natural.

We have:

: c – a = 3^4*r
: c + a = 3^4*r + 2*a
: c^2 + a^2 = (c + a)^2 – 2*a*c = (3^4*r + 2*a)^2 – 2*a*(a + 3^4*r)

So b^4 = c^4 – a^4 ⇔ 3^4*p^4 = (c – a)*(c + a)*(c^2 + a^2) ⇔
⇔ p^4 = (3^4*r^2 + 2*a*r)*(3^8*r^2 + 2*a^2 + 2*3^4*a*r*) ⇔
⇔ p^4 = 3^12*r^4 + 2*3^4*a^2*r^2 + 2*3^8*a*r^3 + 2*3^8*a*r^3 + 4*a^3*r + 4*3^4*a^2*r^2 ⇔ p^4 = 3^12*r^4 + 4*3^8*a*r^3 + 6*3^4*a^2*r^2 + 4*a^3*r

We have this third degree equation with unknown a:

4*r*a^3 + 6*3^4*r^2*a^2 + 4*3^8*r^3*a + 3^12*r^4 – p^4 = 0

Denote by c_1, c_2, c_3, c_4 the coefficients of a; we solve the equation by using Cardano's formulas:

: $c_1 = 4*r$, $c_2 = 6*3\text{^}4*r\text{^}2$, $c_3 = 4*3\text{^}8*r\text{^}3$, $c_4 = 3\text{^}12*r\text{^}4 - p\text{^}4$;
: $3*x = (3*c_1*c_3 - c_2\text{^}2)/(3*c_1\text{^}2)$ and
: $2*y = (2*c_2\text{^}3)/(27*c_1\text{^}3) - (c_2*c_3)/(3*c_1\text{^}2) + c_4/c_1$

which eventually gives us the solutions [x, y] = [(3^7*r^2)/4, (-p^4)/(8*r)

We have D = y^2 + x^3 > 0 $\Rightarrow$ the equation has only one real root, a = (-y + D^(1/2))^(1/3) + (-y - D^(1/2))^(1/3)

Denote by E_1 and E_2 the expressions (-y + D^(1/2))^(1/3) and (-y - D^(1/2))^(1/3)

We have a = E_1 + E_2; raise to the third and we have a^3 = (E_1 + E_2)^3 = E_1^3 + E_2^3 + 3*E_1*E_2*(E_1 + E_2) = E_1^3 + E_2^3 + 3*a*E_1*E_2

$\Leftrightarrow$ a^3 = - y + D^(1/2) - y - D^(1/2) + 3*a*(-(D^(1/2) + y)*(D^(1/2) - y))^(1/3) $\Leftrightarrow$
$\Leftrightarrow$ a^3 = -2*y - 3*a*(D - y^2)^(1/3) $\Leftrightarrow$
$\Leftrightarrow$ a^3 = (-p^4)/(4*r) - (3^8*a*r^2)/4 $\Leftrightarrow$
$\Leftrightarrow$ 4*r*a^3 = p^4 - 3^8*a*r^3 $\Leftrightarrow$
$\Leftrightarrow$ p^4 = a*r*(4*a^2 + 3^8*r^2) $\Rightarrow$

$\Rightarrow$ p^4 is divisible by a $\Rightarrow$
$\Rightarrow$ b^4 is divisible by a $\Rightarrow$
$\Rightarrow$ c^4 is divisible by a $\Rightarrow$

$\Rightarrow$ a, b and c aren't and can't be relatively prime $\Rightarrow$ nonsense $\Rightarrow$ there aren't any naturals a, b and c which satisfy the equation a^4 + b^4 = c^4 in the considered case ((c – a) divisible by 3^4) $\Rightarrow$

$\Rightarrow$ We have proved Fermat's last Theorem for n = 4.

(The case (c + a) is divisible by 3^4 has an analogue solution)

16. INTRODUCTION TO PERFECT NUMBERS

We know that a natural number n is called perfect if f(n) = 2*n, where f(n) is the sum of the natural divisors of n. Examples of such perfect numbers are 6, 28, 496. Indeed:

f(6) = 1 + 2 + 3 + 6 = 12 = 2*6

f(28) = 1 + 2 + 4 + 7 + 14 + 28 = 56 = 2*28

f(496) = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 + 496 = 992 = 2*496

As you may see, the given examples are even natural numbers: at this time we don't know if there are any odd perfect numbers or not.

For the even perfect natural numbers we have a reference formula and this is:

: An even number n is perfect if and only if there is a natural number m such that n = 2^m*(2^(m + 1) − 1) and 2^(m + 1) − 1 are prime numbers.

We won't insist on this formula, we'll just mention that it was obtained by expressing the sum of the different divisors of n with respect to the decomposition of n in prime factors.

The classical Diophantine analysis of perfect numbers is thus based on prime numbers (a connection between the set of even perfect numbers and the set of prime numbers, i.e. Mersenne primes).

Next, we will try to obtain, using the mar reduced form, some interesting conclusions about the characteristics of perfect numbers.

17. PERFECT NUMBERS: A DIOPHANTINE ANALISYS

(A) We take the perfect numbers with the mar reduced form equal to 2, so the numbers of the type n = 9*m + 2, m natural.

We have f(9*m + 2) = 2*(9*m + 2) and the following cases:

(i) f(9*m + 2) = 1 + a_1 + a_2 + ... + a_p + b_p + ... + b_2 + b_1 + (9*m + 2)
(ii) f(9*m + 2) = 1 + a_1 + a_2 + ... + a_{p-1} + a_p + b_{p-1} + ... + b_2 + b_1 + (9*m + 2)

We denoted by a_1 and b_1, a_2 and b_2, ..., a_p and b_p, the complementary divisors of the considered number (n = 9*m + 2), such that we have, obviously, for
(i) a_1*b_1 = a*b_2 = ... = a_p*b_p = 9*m + 2, and for
(ii) a_1*b_1 = a*b_2 = ... = a_{p-1}*b_{p-1} = a_p^2 = 9*m + 2

Take the case (i); we have:

$f(9*m + 2) = 1 + a_1 + a_2 + ... + a_p + b_p + ... + b_2 + b_1 + (9*m + 2) = 2*(9*m + 2)$

From $a_1*b_1 = a*b_2 = ... = a_p*b_p = 9*m + 2 \Rightarrow$ mar (a_1*b_1) = mar $(a_2*b_2) = ... =$ mar $(a_p*b_p) = 2$

But from $f(9*m + 2) = 2*(9*m + 2) \Rightarrow$
$\Rightarrow$ mar $f(9*m + 2)$ = mar $(2*(9*m + 2))$. So we have $1 \oplus$ mar $a_1 \oplus$ mar $a_2 \oplus ... \oplus$ mar $b_2 \oplus$ mar $b_1 \oplus 2 = 2 \otimes 2 = 4 \Leftrightarrow$
$3 \oplus$ (mar $a_1 \oplus$ mar b_1) $\oplus ... \oplus$ (mar $a_p \oplus$ mar b_p) = 4

As mar $a_1 \otimes$ mar $b_1 = ... =$ mar $a_p \otimes$ mar $b_p = 2$, the only possible combination is: $\{1/4/7\} \otimes \{2/5/8\} = ...\{1/4/7\} \otimes \{2/5/8\} = 2$, where mar $a_1 = \{1/4/7\}$ and mar $b_1 = \{2/5/8\}$ or the opposite, ..., mar $a_p = \{1/4/7\}$ and mar $b_p = \{2/5/8\}$ or the opposite.

Anyway, mar $a_1 \oplus$ mar $b_1 = ... =$ mar $a_p \oplus$ mar $b_p = \{3/6/9\}$ so mar $f(9*m + 2)$ = mar $(2*(9*m + 2))$ becomes mar $f(9*m + 2) = 3 \oplus \{3/6/9\} \oplus ... \oplus \{3/6/9\} = 4 \Leftrightarrow \{3/6/9\} = 4$, which is obviously impossible $\Rightarrow$ in the case (i) there is no perfect number of the type $9*m + 2$

In order not to break the reasoning, we haven't considered separately the case when n has only 2 different divisors. In this case, we will have $1 \oplus 2 = 4$, so $3 = 4$, which is obviously impossible.

Take the case (ii); we have:

$f(9*m + 2) = 1 + a_1 + a_2 + ... + a_{p-1} + a_p + b_{p-1} + ... + b_2 + b_1 + (9*m + 2) = 2*(9*m + 2)$

From $a_1*b_1 = a*b_2 = ... = a_{p-1}*b_{p-1} = a_p\hat{\ }2 = 9*m + 2 \Rightarrow$ mar (a_1*b_1) = mar $(a_2*b_2) = ... =$ mar $(a_{p-1}*b_{p-1})$ = mar $a_p\hat{\ }2 = 2$

But mar $a_p\hat{\ }2 = 2$ is impossible (mar x^2 may only be 1, 4, 7 or 9 for any natural nonzero x) $\Rightarrow$ there aren't any perfect numbers of the type $9*m + 2$ in the case (ii) either.

We have thus reached an interesting conclusion, which is that there are no perfect numbers of the type $9*m + 2$, m natural.

In a similar manner can be shown that there are no perfect numbers of the type $9*m + 5$ respectively $9*m + 8$.