A proof of strong Goldbach conjecture and twin prime conjecture

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Abstract

In this paper we give a proof of the strong Goldbach conjecture by studying limit status of original continuous Godbach natural number sequence {3, 4, …, G_{NL}} generated by original continuous odd prime number sequence {3, 5, …, p} when $p \to \infty$, that is, $G_{NL} = p$ when $p \to \infty$. It implies the weak Goldbach conjecture. If a prime p is defined as Goldbach prime when $G_{NL} = p$ then Goldbach prime is the higher member of a twin prime pair (p–2, p), from which we will give a proof of the twin prime conjecture.

Keywords: prime; Goldbach integer; central Goldbach integer; Goldbach natural number; strong Goldbach conjecture; weak Goldbach conjecture; Goldbach prime; twin prime conjecture.

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1. Introduction

Goldbach's conjecture has been one of best-known unsolved problems in mathematics for many years, which was listed as a subproblem of Hilbert's 8th problem at 1900 ICM[1]. The conjecture includes strong and weak statements. The strong Goldbach conjecture states that every even number greater than 2 is the sum of two primes, which is equivalent to the statement that every even number greater than 4 is the sum of two odd primes. The weak Goldbach conjecture states that every odd number greater than 7 is the sum of three odd primes. It is generally considered that if the strong Goldbach conjecture is true then the weak Goldbach conjecture is true. Thus the strong Goldbach conjecture has been the research topic of many mathematicians. In 1973 Chen showed that every sufficiently large even integer is the sum of either two primes, or a prime and the product of two primes[2]. In this paper we present a mathematical framework in which every original continuous odd prime number sequence {3, 5,…, p} will generate corresponding original continuous Goldbach natural number sequence {3, 4,…, G_{NL}} in the set of Goldbach integers G_I = $(q_1+q_2)/2$ arising from all sums of any two same or distinct odd primes not greater than p for {3, 5,…, p}. Basing such a framework, we will find a method, from finite cases to infinite case, to give a proof of the strong Goldbach conjecture.

2. Status of { 3, 4, …, G_{NL} } generated by { 3, 5, …, p }

Definition 2.1 Let p be prime greater than 3. For a continuous odd prime number sequence, if its first term is 3 then the sequence is called an original continuous odd prime number sequence and written as {3, 5, …, p}, where p is the last term of the sequence.

Obviously, any given {3, 5, …, p} contains all odd primes from 3 to p.

Definition 2.2 Let Q be natural number greater than 3. For a continuous natural number sequence, if its first term is 3 then the sequence is called an original continuous natural number sequence and written as {3, 4, …, Q}, where Q is the last term of the sequence.

Obviously, any given {3, 4, …, Q} contains all natural numbers from 3 to Q.

Definition 2.3 For a given {3, 5, …, p}, if q_1 and q_2 are two same or distinct odd primes not greater than p then $G_I = (q_1+q_2)/2$ is called a Goldbach integer for the given {3, 5, …, p}.

Definition 2.4 In the set of Goldbach integers G_I arising from all sums of any two same or distinct odd primes not greater than p for a given {3, 5, …, p}, if all terms of {3, 4, …, Q} are Goldbach integers G_I in the set then {3, 4, …, Q} is called original continuous Goldbach natural number sequence and written as {3, 4, …, G_{NL}}, in which all terms (from 3 to $G_{NL} = Q$) are called Goldbach natural numbers and written as G_N, and G_{NL} is the largest Goldbach natural number generated by p with its sequence {3, 5, …, p}.

Therefore, by Definition 2.3 and Definition 2.4 we see every term G_N in original continuous Goldbach natural number sequence {3, 4, …, G_{NL}} must be also a Goldbach integer G_I for a given {3, 5, …, p} but any Goldbach integer G_I outside {3, 4, …, G_{NL}} is not a Goldbach natural number G_N.

Definition 2.5 For a given {3, 5, …, p}, $G_C = (p+3)/2$ is called central Goldbach integer for the given {3, 5, …, p}.

Lemma 2.6 There is an original continuous Godbach natural number sequence {3, 4, …, G_{NL}} in the set of Goldbach integers G_I arising from all sums of any two same or distinct odd primes not greater than *p* for any given {3, 5, …, *p*}.

Proof. For the first {3, 5, …, *p*} i.e. {3, 5}, there is a original continuous Godbach natural number sequence {3, 4, 5} and G_{NL} = 5 is the last term of the sequence. We see {3, 4, …, G_{NL}} generated by every {3, 5, …, *p*} will remain in the set of Goldbach integers G_I arising from all sums of any two same or distinct odd primes not greater than *p* for the next {3, 5, …, *p*} as the first part (including complete sequence) of {3, 4, …, G_{NL}} for the next {3, 5, …, *p*}. Hence the lemma holds. Put simply, the lemma means a prime *p* greater than 3 will generate a corresponding G_{NL}.

Observation 2.7 Status of {3, 4, …, G_{NL}} generated by {3, 5, …, *p*} for *p* less than 500.

In the Observation, G_N are Goldbach natural numbers in original continuous Goldbach natural number sequence {3, 4, …, G_{NL}} with the last term being G_{NL} generated by a given {3, 5, …, *p*}, G_I are Goldbach integers outside the sequence {3, 4, …, G_{NL}}, $G_C = (p+3)/2$ is central Goldbach integer for a given {3, 5,…, *p*}, $\varDelta = G_{NL}-G_C$, $\xi(p) = (G_{NL}-G_C)/G_C = \varDelta/G_C$. Considering $G_{NL} = p$ in some known cases, we get asymptotic function of $\xi(p)$ i. e. $\varphi(p) = (p-3)/(p+3)$. In order to understand origin of data in the observation table, we give two suitable examples as follows

Status of {3, 4, 5, 6, 7, 8, 9} generated by {3, 5, 7, 11}
(3+3)/2 = 3, (3+5)/2 = 4, (3+7)/2 = 5, (3+11)/2 = 7, (5+5)/2 = 5; (5+7)/2 = 6, (5+11)/2 = 8, (7+7)/2 = 7, (7+11)/2 = 9, (11+11)/2 = 11.
G_N : 3, 4, 5, 6, 7, 8, 9. G_I : 11.
$p = 11$, $G_{NL} = 9$, $G_C = (11+3)/2 = 7$, $\varDelta = 2$, $\xi(p) = 2/7 = 0.28571$, $\varphi(p) = 8/14 = 0.57143$.

Status of {3, 4, 5, …, 17, 18, 19} generated by {3, 5, 7, 11, 13, 17, 19}
G_N : 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19. G_I : None
$p = 19$, $G_{NL} = 19$, $G_C = (19+3)/2 = 11$, $\varDelta = 8$, $\xi(p) = 8/11 = 0.72727$, $\varphi(p) = 16/22 = 0.72727$.

p	G_{NL}	G_C	$\varDelta$	$\xi(p)$	$\varphi(p)$	G_I
5	5	4	1	0.25000	0.25000	None
7	7	5	2	0.40000	0.40000	None
11	9	7	2	0.28571	0.57143	11
13	13	8	5	0.62500	0.62500	None
17	15	10	5	0.50000	0.70000	17
19	19	11	8	0.72727	0.72727	None
23	21	13	8	0.61538	0.76923	23
29	21	16	5	0.31250	0.81250	23, 24, 26, 29

31	27	17	10	0.58824	0.82353	29, 30, 31
37	31	20	11	0.55000	0.85000	33, 34, 37
41	37	22	15	0.68182	0.86364	39, 41
43	37	23	14	0.60870	0.86957	39, 40, 41, 42, 43
47	45	25	20	0.80000	0.88000	47
53	45	28	17	0.60714	0.89286	47, 48, 50, 53
59	45	31	14	0.45161	0.90323	47, 48, 50, 51, 53, 56, 59
61	54	32	22	0.68750	0.90625	56, 57, 59, 60, 61
67	57	35	22	0.62857	0.91429	59, 60, 61, 63, 64, 67
71	57	37	20	0.54054	0.91892	59, 60, 61, 62, 63, 64, 65, 66, 67, 69, 71
73	67	38	29	0.76316	0.92105	69, 70, 71, 72, 73
79	67	41	26	0.63415	0.92683	69, 70, 71, 72, 73, 75, 76, 79
83	73	43	30	0.69767	0.93023	75, 76, 77, 78, 79, 81, 83
89	81	46	35	0.76087	0.93478	83, 84, 86, 89
97	86	50	36	0.72000	0.94000	88, 89, 90, 93, 97
101	90	52	38	0.73077	0.94231	92, 93, 95, 97, 99, 101
103	93	53	40	0.75472	0.94340	95, 96, 97, 99, 100, 101, 102, 103
107	93	55	38	0.69091	0.94545	95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 107
109	109	56	53	0.94643	0.94643	None
113	111	58	53	0.91379	0.94828	113
127	115	65	50	0.76923	0.95385	117, 118, 120, 127
131	120	67	53	0.79104	0.95522	122, 127, 129, 131
137	120	70	50	0.71429	0.95714	122, 123, 125, 127, 129, 131, 132, 134, 137
139	127	71	56	0.78873	0.95775	129, 131, 132, 133, 134, 135, 137, 138, 139
149	129	76	53	0.69737	0.96053	131, 132, 133, 134, 135, 137, 138, 139, 140, 143, 144, 149
151	135	77	58	0.75325	0.96104	137, 138, 139, 140, 141, 143, 144, 145, 149, 150, 151
157	135	80	55	0.68750	0.96250	137, 138, 139, 140, 141, 142, 143, 144, 145, 147, 148, 149, 150, 151, 153, 154, 157
163	145	83	62	0.74699	0.96386	147, 148, 149, 150, 151, 153, 154, 156, 157, 160, 163
167	145	85	60	0.70588	0.96471	147, 148, 149, 150, 151, 152, 153, 154, 156, 157, 158, 159, 160, 162, 163, 165, 167
173	145	88	57	0.64773	0.96591	147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 165, 167, 168, 170, 173
179	165	91	74	0.81319	0.96703	167, 168, 170, 171, 173, 176, 179

181	174	92	82	0.89130	0.96739	176, 177, 179, 180, 181
191	174	97	77	0.79381	0.96907	176, 177, 179, 180, 181, 182, 185, 186, 191
193	183	98	85	0.86735	0.96939	185, 186, 187, 191, 192, 193
197	183	100	83	0.83000	0.97000	185, 186, 187, 188, 189, 191, 192, 193, 194, 195, 197
199	183	101	82	0.81188	0.97030	185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199
211	199	107	92	0.85981	0.97196	201, 202, 204, 205, 211
223	199	113	86	0.76106	0.97345	201, 202, 204, 205, 207, 208, 210, 211, 217, 223
227	205	115	90	0.78261	0.97391	207, 208, 209, 210, 211, 212, 213, 217, 219, 223, 225, 227
229	205	116	89	0.76724	0.97414	207, 208, 209, 210, 211, 212, 213, 214, 217, 219, 220, 223, 225, 226, 227, 228, 229
233	217	118	99	0.83898	0.97458	219, 220, 222, 223, 225, 226, 227, 228, 229, 230, 231, 233
239	220	121	99	0.81818	0.97521	222, 223, 225, 226, 227, 228, 229, 230, 231, 233, 234, 236, 239
241	220	122	98	0.80328	0.97541	222, 223, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 239, 240, 241
251	237	127	110	0.86614	0.97638	239, 240, 241, 242, 245, 246, 251
257	237	130	107	0.82308	0.97692	239, 240, 241, 242, 243, 245, 246, 248, 249, 251, 254, 257
263	237	133	104	0.78195	0.97744	239, 240, 241, 242, 243, 245, 246, 248, 249, 251, 252, 254, 257, 260, 263
269	237	136	101	0.74265	0.97794	239, 240, 241, 242, 243, 245, 246, 248, 249, 251, 252, 254, 255, 257, 260, 263, 266, 269
271	237	137	100	0.72993	0.97810	239, 240, 241, 242, 243, 245, 246, 247, 248, 249, 250, 251, 252, 254, 255, 256, 257, 260, 261, 263, 264, 266, 267, 269, 270, 271
277	261	140	121	0.86429	0.97857	263, 264, 266, 267, 269, 270, 271, 273, 274, 277
281	261	142	119	0.83803	0.97887	263, 264, 266, 267, 269, 270, 271, 272, 273, 274, 275, 276, 277, 279, 281
283	264	143	121	0.84615	0.97902	266, 267, 269, 270, 271, 272, 273, 274, 275, 276, 277, 279, 280, 281, 282, 283
293	264	148	116	0.78378	0.97973	266, 267, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 285, 287, 288, 293

307	283	155	128	0.82581	0.98065	285, 287, 288, 289, 292, 293, 294, 295, 300, 307
311	285	157	128	0.81529	0.98089	287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 300, 302, 307, 309, 311
313	285	158	127	0.80380	0.98101	287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 300, 302, 303, 307, 309, 310, 311, 312, 313
317	285	160	125	0.78125	0.98125	287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 397, 298, 299, 300, 302, 303, 305, 307, 309, 310, 311, 312, 313, 314, 315, 317
331	307	167	140	0.83832	0.98204	309, 310, 311, 312, 313, 314, 315, 317, 319, 321, 322, 324, 331
337	307	170	137	0.80588	0.98235	309, 310, 311, 312, 313, 314, 315, 317, 319, 321, 322, 324, 325, 327, 331, 334, 337
347	315	175	140	0.80000	0.98286	317, 319, 320, 321, 322, 324, 325, 327, 329, 330, 331, 332, 334, 337, 339, 342, 347
349	317	176	141	0.80114	0.98295	319, 320, 321, 322, 324, 325, 327, 328, 329, 330, 331, 332, 333, 334, 337, 339, 340, 342, 343, 347, 348, 349
353	325	178	147	0.82584	0.98315	327, 328, 329, 330, 331, 332, 333, 334, 335, 337, 339, 340, 342, 343, 345, 347, 348, 349, 350, 351, 353
359	340	181	159	0.87845	0.98343	342, 343, 345, 347, 348, 349, 350, 351, 353, 354, 356, 359
367	340	185	155	0.83784	0.98378	342, 343, 345, 347, 348, 349, 350, 351, 352, 353, 354, 356, 357, 358, 359, 360, 363, 367
373	340	188	152	0.80851	0.98404	342, 343, 345, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 363, 366, 367, 370, 373
379	340	191	149	0.78010	0.98429	342, 343, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 363, 364, 366, 367, 369, 370, 373, 376, 379
383	340	193	147	0.76166	0.98446	342, 343, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 363, 364, 365, 366, 367, 368, 369, 370, 371, 373, 375, 376, 378, 379, 381, 383
389	343	196	147	0.75000	0.98469	345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360,

						361, 363, 364, 365, 366, 367, 368, 369, 370, 371, 373, 374, 375, 376, 378, 379, 381, 383, 384, 386, 389
397	343	200	143	0.71500	0.98500	345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 378, 379, 381, 382, 383, 384, 385, 386, 388, 389, 390, 393, 397
401	343	202	141	0.69802	0.98515	345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 392, 393, 395, 397, 399, 401
409	343	206	137	0.66505	0.98544	345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 363, 364, 365, 366, 367, 368, 369, 370, 371 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 399, 401, 403, 405, 409
419	361	211	150	0.71090	0.98578	363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 399, 401, 403, 404, 405, 408, 409, 410, 414, 419
421	361	212	149	0.70283	0.98585	363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 399, 400, 401, 402, 403, 404, 405, 408, 409, 410, 411, 414, 415, 419, 420, 421
431	397	217	180	0.82949	0.98618	399, 400, 401, 402, 403, 404, 405, 407, 408, 409, 410, 411, 414, 415, 416, 419, 420, 421, 425, 426, 431
433	397	218	179	0.82110	0.98624	399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 414, 415, 416, 417, 419, 420, 421, 425, 426, 427, 431, 432, 433

439	397	221	176	0.79638	0.98643	399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 414, 415, 416, 417, 418, 419, 420, 421, 424, 425, 426, 427, 429, 430, 431, 432, 433, 435, 436, 439
443	411	223	188	0.84305	0.98655	413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 424, 425, 426, 427, 429, 430, 431, 432, 433, 435, 436, 437, 438, 439, 441, 443
449	411	226	185	0.81858	0.98673	413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 443, 444, 446, 449
457	427	230	197	0.85652	0.98696	429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 443, 444, 445, 446, 448, 449, 450, 453, 457
461	427	232	195	0.84052	0.98707	429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 443, 444, 445, 446, 447, 448, 449, 450, 452, 453, 455, 457, 459, 461
463	427	233	194	0.83262	0.98712	429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 455, 456, 457, 459, 460, 461, 462, 463
467	453	235	218	0.92766	0.98723	455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 467
479	453	241	212	0.87967	0.98755	455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 467, 468, 470, 471, 473, 479
487	465	245	220	0.89796	0.98776	467, 468, 470, 471, 472, 473, 474, 475, 477, 479, 483, 487
491	465	247	218	0.88259	0.98785	467, 468, 470, 471, 472, 473, 474, 475, 476, 477, 479, 483,485, 487, 489, 491
499	481	251	230	0.91633	0.98805	483, 485, 487, 489, 491, 493, 495, 499

In above observation table we see there is always $G_{NL} > G_C$ in {3, 4, …, G_{NL}} generated by all {3, 5, …, p} for p less than 500, which means G_C is also G_N less than G_{NL}. In other words, $\xi(p) > 0$ for p less than 500. When $G_{NL} = p$, $\xi(p) = \varphi(p)$ as status for p = 5, 7, 13, 19, 109 shows. It means there is always $G_{NL} \leq p$ for p less than 500.

Let σ_1 be the average value of $\xi(p)$ for $3 < p < 2m$ ($2m$ is even number greater than 4) but σ_2 be the average value of $\xi(p)$ for $2m < p < 4m$, we have the following comparison for $2m \leq 250$.

$2m$	$3 < p < 2m$	σ_1	$4m$	$2m < p < 4m$	σ_2
6	5	0.25000	12	7, 11	0.34286
8	5, 7	0.32500	16	11, 13	0.45536
10	5, 7	0.32500	20	11, 13, 17, 19	0.53450
12	5, 7, 11	0.31190	24	13, 17, 19, 23	0.61703
14	5, 7, 11, 13	0.39018	28	17, 19, 23	0.61437
16	5, 7, 11, 13	0.39018	32	17, 19, 23, 29, 31	0.54877
18	5, 7, 11, 13, 17	0.41214	36	19, 23, 29, 31	0.56096
20	5, 7, 11, 13, 17, 19	0.46466	40	23, 29, 31, 37	0.51664
22	5, 7, 11, 13, 17, 19	0.46466	44	23, 29, 31, 37, 41, 43	0.55952
24	5, 7, 11, 13, …, 23	0.48626	48	29, 31, 37, 41, …, 47	0.59021
26	5, 7, 11, 13, …, 23	0.48626	52	29, 31, 37, 41, …, 47	0.59021
28	5, 7, 11, 13, …, 23	0.48626	56	29, 31, 37, 41, …, 53	0.59263
30	5, 7, 11, 13, …, 29	0.46454	60	31, 37, 41, 43, …, 59	0.61250
32	5, 7, 11, 13, …, 31	0.47828	64	37, 41, 43, 47, …, 61	0.62668
34	5, 7, 11, 13, …, 31	0.47828	68	37, 41, 43, 47, …, 67	0.62692
36	5, 7, 11, 13, …, 31	0.47828	72	37, 41, 43, 47, …, 71	0.61732
38	5, 7, 11, 13, …, 37	0.48546	76	41, 43, 47, 53, …, 73	0.64100
40	5, 7, 11, 13, …, 37	0.48546	80	41, 43, 47, 53, …, 79	0.64032
42	5, 7, 11, 13, …, 41	0.50331	84	43, 47, 53, 59, …, 83	0.64190
44	5, 7, 11, 13, …, 43	0.51209	88	47, 53, 59, 61, …, 83	0.64559
46	5, 7, 11, 13, …, 43	0.51209	92	47, 53, 59, 61, …, 89	0.65712
48	5, 7, 11, 13, …, 47	0.53424	96	53, 59, 61, 67, …, 89	0.64125
50	5, 7, 11, 13, …, 47	0.53424	100	53, 59, 61, 67, …, 97	0.64912
52	5, 7, 11, 13, …, 47	0.53424	104	53, 59, 61, 67, …, 103	0.66473
54	5, 7, 11, 13, …, 53	0.53944	108	59, 61, 67, 71, …, 107	0.67171
56	5, 7, 11, 13, …, 53	0.53944	112	59, 61, 67, 71, …, 109	0.69284
58	5, 7, 11, 13, …, 53	0.53944	116	59, 61, 67, 71, …, 113	0.70862
60	5, 7, 11, 13, …, 59	0.53359	120	61, 67, 71, 73, …, 113	0.72839
62	5, 7, 11, 13, …, 61	0.54321	124	67, 71, 73, 79, …, 113	0.73180
64	5, 7, 11, 13, …, 61	0.54321	128	67, 71, 73, 79, …, 127	0.73468
66	5, 7, 11, 13, …, 61	0.54321	132	67, 71, 73, 79, …, 131	0.73870
68	5, 7, 11, 13, …, 67	0.54823	136	71, 73, 79, 83, …, 131	0.74718
70	5, 7, 11, 13, …, 67	0.54823	140	71, 73, 79, 83, …, 139	0.74775
72	5, 7, 11, 13, …, 71	0.54780	144	73, 79, 83, 89, …, 139	0.76255
74	5, 7, 11, 13, …, 73	0.55914	148	79, 83, 89, 97, …, 139	0.76251
76	5, 7, 11, 13, …, 73	0.55914	152	79, 83, 89, 97, …, 151	0.75755
78	5, 7, 11, 13, …, 73	0.55914	156	79, 83, 89, 97, …, 151	0.75755
80	5, 7, 11, 13, …, 79	0.56289	160	83, 89, 97, 101, …, 157	0.76110
82	5, 7, 11, 13, …, 79	0.56289	164	83, 89, 97, 101, …, 163	0.76022
84	5, 7, 11, 13, …, 83	0.56931	168	89, 97, 101, 103, …, 167	0.76074
86	5, 7, 11, 13, …, 83	0.56931	172	89, 97, 101, 103, …, 167	0.76074
88	5, 7, 11, 13, …, 83	0.56931	176	89, 97, 101, 103, …, 173	0.75409
90	5, 7, 11, 13, …, 89	0.57801	180	97, 101, 103, 107, …, 179	0.75717

92	5, 7, 11, 13, …, 89	0.57801	184	97, 101, 103, 107, …, 181	0.76462
94	5, 7, 11, 13, …, 89	0.57801	188	97, 101, 103, 107, …, 181	0.76462
96	5, 7, 11, 13, …, 89	0.57801	192	97, 101, 103, 107, …, 191	0.76615
98	5, 7, 11, 13, …, 97	0.58419	196	101, 103, 107, 109, …, 193	0.77391
100	5, 7, 11, 13, …, 97	0.58419	200	101, 103, 107, 109, …, 199	0.77839
102	5, 7, 11, 13, …, 101	0.59029	204	103, 107, 109, 113, …, 199	0.78077
104	5, 7, 11, 13, …, 103	0.59687	208	107, 109, 113, 127, …, 199	0.78214
106	5, 7, 11, 13, …, 103	0.59687	212	107, 109, 113, 127, …, 211	0.78602
108	5, 7, 11, 13, …, 107	0.60049	216	109, 113, 127, 131, …, 211	0.79103
110	5, 7, 11, 13, …, 109	0.61330	220	113, 127, 131, 137, …, 211	0.78240
112	5, 7, 11, 13, …, 109	0.61330	224	113, 127, 131, 137, …, 223	0.78127
114	5, 7, 11, 13, …, 113	0.62403	228	127, 131, 137, 139, …, 227	0.77437
116	5, 7, 11, 13, …, 113	0.62403	232	127, 131, 137, 139, …, 229	0.77401
118	5, 7, 11, 13, …, 113	0.62403	236	127, 131, 137, 139, …, 233	0.77711
120	5, 7, 11, 13, …, 113	0.62403	240	127, 131, 137, 139, …, 239	0.77897
122	5, 7, 11, 13, …, 113	0.62403	244	127, 131, 137, 139, …, 241	0.78003
124	5, 7, 11, 13, …, 113	0.62403	248	127, 131, 137, 139, …, 241	0.78003
126	5, 7, 11, 13, …, 113	0.62403	252	127, 131, 137, 139, …, 251	0.78362
128	5, 7, 11, 13, …, 127	0.62904	256	131, 137, 139, 149, …, 251	0.78424
130	5, 7, 11, 13, …, 127	0.62904	260	131, 137, 139, 149, …, 257	0.78586
132	5, 7, 11, 13, …, 131	0.63444	264	137, 139, 149, 151, …, 263	0.78548
134	5, 7, 11, 13, …, 131	0.63444	268	137, 139, 149, 151, …, 263	0.78548
136	5, 7, 11, 13, …, 131	0.63444	272	137, 139, 149, 151, …, 271	0.78170
138	5, 7, 11, 13, …, 137	0.63701	276	139, 149, 151, 157, …, 271	0.78440
140	5, 7, 11, 13, …, 139	0.64176	280	149, 151, 157, 163, …, 277	0.78742
142	5, 7, 11, 13, …, 139	0.64176	284	149, 151, 157, 163, …, 283	0.79147
144	5, 7, 11, 13, …, 139	0.64176	288	149, 151, 157, 163, …, 283	0.79147
146	5, 7, 11, 13, …, 139	0.64176	292	149, 151, 157, 163, …, 283	0.79147
148	5, 7, 11, 13, …, 139	0.64176	296	149, 151, 157, 163, …, 293	0.79119
150	5, 7, 11, 13, …, 149	0.64344	300	151, 157, 163, 167, …, 293	0.79467
152	5, 7, 11, 13, …, 151	0.64667	304	157, 163, 167, 173, …, 293	0.79626
154	5, 7, 11, 13, …, 151	0.64667	308	157, 163, 167, 173, …, 307	0.79736
156	5, 7, 11, 13, …, 151	0.64667	312	157, 163, 167, 173, …, 311	0.79800
158	5, 7, 11, 13, …, 157	0.64784	316	163, 167, 173, 179, …, 313	0.80215
160	5, 7, 11, 13, …, 157	0.64784	320	163, 167, 173, 179, …, 317	0.80143
162	5, 7, 11, 13, …, 157	0.64784	324	163, 167, 173, 179, …, 317	0.80143
164	5, 7, 11, 13, …, 163	0.65059	328	167, 173, 179, 181, …, 317	0.80337
166	5, 7, 11, 13, …, 163	0.65059	332	167, 173, 179, 181, …, 331	0.80458
168	5, 7, 11, 13, …, 167	0.65209	336	173, 179, 181, 191, …, 331	0.80810
170	5, 7, 11, 13, …, 167	0.65209	340	173, 179, 181, 191, …, 337	0.80803
172	5, 7, 11, 13, …, 167	0.65209	344	173, 179, 181, 191, …, 337	0.80803
174	5, 7, 11, 13, …, 173	0.65197	348	179, 181, 191, 193, …, 347	0.81328
176	5, 7, 11, 13, …, 173	0.65197	352	179, 181, 191, 193, …, 349	0.81287
178	5, 7, 11, 13, …, 173	0.65197	356	179, 181, 191, 193, …, 353	0.81329

180	5, 7, 11, 13, …, 179	0.65611	360	181, 191, 193, 197, …, 359	0.81540
182	5, 7, 11, 13, …, 181	0.66199	364	191, 193, 197, 199, …, 359	0.81287
184	5, 7, 11, 13, …, 181	0.66199	368	191, 193, 197, 199, …, 367	0.81367
186	5, 7, 11, 13, …, 181	0.66199	372	191, 193, 197, 199, …, 367	0.81367
188	5, 7, 11, 13, …, 181	0.66199	376	191, 193, 197, 199, …, 373	0.81351
190	5, 7, 11, 13, …, 181	0.66199	380	191, 193, 197, 199, …, 379	0.81250
192	5, 7, 11, 13, …, 191	0.66520	384	193, 197, 199, 211, …, 383	0.81152
194	5, 7, 11, 13, …, 193	0.67001	388	197, 199, 211, 223, …, 383	0.80978
196	5, 7, 11, 13, …, 193	0.67001	392	197, 199, 211, 223, …, 389	0.80797
198	5, 7, 11, 13, …, 197	0.67373	396	199, 211, 223, 227, …, 389	0.80728
200	5, 7, 11, 13, …, 199	0.67687	400	211, 223, 227, 229, …, 397	0.80425
202	5, 7, 11, 13, …, 199	0.67687	404	211, 223, 227, 229, …, 401	0.80103
204	5, 7, 11, 13, …, 199	0.67687	408	211, 223, 227, 229, …, 401	0.80103
206	5, 7, 11, 13, …, 199	0.67687	412	211, 223, 227, 229, …, 409	0.79703
208	5, 7, 11, 13, …, 199	0.67687	416	211, 223, 227, 229, …, 409	0.79703
210	5, 7, 11, 13, …, 199	0.67687	420	211, 223, 227, 229, …, 419	0.79457
212	5, 7, 11, 13, …, 211	0.68094	424	223, 227, 229, 233, …, 421	0.79009
214	5, 7, 11, 13, …, 211	0.68094	428	223, 227, 229, 233, …, 421	0.79009
216	5, 7, 11, 13, …, 211	0.68094	432	223, 227, 229, 233, …, 431	0.79118
218	5, 7, 11, 13, …, 211	0.68094	436	223, 227, 229, 233, …, 433	0.79199
220	5, 7, 11, 13, …, 211	0.68094	440	223, 227, 229, 233, …, 439	0.79210
222	5, 7, 11, 13, …, 211	0.68094	444	223, 227, 229, 233, …, 443	0.79341
224	5, 7, 11, 13, …, 223	0.68268	448	227, 229, 233, 239, …, 443	0.79426
226	5, 7, 11, 13, …, 223	0.68268	452	227, 229, 233, 239, …, 449	0.79488
228	5, 7, 11, 13, …, 227	0.68481	456	229, 233, 239, 241, …, 449	0.79521
230	5, 7, 11, 13, …, 229	0.68652	460	233, 239, 241, 251, …, 457	0.79755
232	5, 7, 11, 13, …, 229	0.68652	464	233, 239, 241, 251, …, 463	0.79951
234	5, 7, 11, 13, …, 233	0.68964	468	239, 241, 251, 257, …, 467	0.80172
236	5, 7, 11, 13, …, 233	0.68964	472	239, 241, 251, 257, …, 467	0.80172
238	5, 7, 11, 13, …, 233	0.68964	476	239, 241, 251, 257, …, 467	0.80172
240	5, 7, 11, 13, …, 239	0.69221	480	241, 251, 257, 263, …, 479	0.80326
242	5, 7, 11, 13, …, 241	0.69438	484	251, 257, 263, 269, …, 479	0.80326
244	5, 7, 11, 13, …, 241	0.69438	488	251, 257, 263, 269, …, 487	0.80563
246	5, 7, 11, 13, …, 241	0.69438	492	251, 257, 263, 269, …, 491	0.80750
248	5, 7, 11, 13, …, 241	0.69438	496	251, 257, 263, 269, …, 491	0.80750
250	5, 7, 11, 13, …, 241	0.69438	500	251, 257, 263, 269, …, 499	0.81009

From above comparison we see there is always $\sigma_2 > \sigma_1$ for any given even number $2m \leq 250$.

3. The proof of strong Goldbach conjecture

Lemma 3.1 For any given prime p greater than 3, there is $G_{NL} \leq p$.

Proof. By Definition 2.3 and Definition 2.4 $p = (p+p)/2$ is the largest Goldbach integer G_I for a given prime p greater than 3. Since there exist some known cases in which $G_{NL} = p$ as Observation 2.7 shows. Hence there is $G_{NL} \leq p$ for any given prime p greater than 3 and the lemma holds.

Lemma 3.2 For a given prime p greater than 3, if $G_{NL} = p$ then all integers N from 3 to p can generate corresponding even numbers $2N$ to be the sum of two odd primes not greater than p.

Proof. When $G_{NL} = p$, because of G_{NL} being the largest Goldbach natural number, all integers N from 3 to p are Goldbach natural numbers G_N and every even number $2N = 2G_N$ is the sum of two odd primes not greater than p by Definition 2.4. For example, $G_{NL} = p$ for $p = 109$ as Observation 2.7 shows. From it we have a continuous even number sequence $2·3$, $2·4$, $2·5$, …, $2·107$, $2·108$, $2·109$. In the sequence every even number $2N$ is the sum of two odd primes not greater than prime 109, for example, $2·3 = 6 = 3+3$ and $2·109 = 218 = 109+109$. Hence the lemma holds.

Lemma 3.3 For a given prime p greater than 3 and integer N greater than 2, if $G_{NL} = p$ then every even number $2N$ greater than 4 is the sum of two odd primes less than $2N$ and all pairs of same or distinct odd primes less than $2N$ with the sum being $2N$ can be completely found for $2N \leq p–1$.

Proof. When studying G_{NL} for a given p, all odd primes not greater than p have been considered and these odd primes have contained all odd primes less than $2N$ for $2N \leq p–1$. Hence by Definition 2.4 every even number $2N$ greater than 4 is the sum of two odd primes less than $2N$ and all pairs of same or distinct odd primes less than $2N$ with the sum being $2N$ can be completely found for $2N \leq G_{NL} –1 = p–1$ because of $G_{NL} = p$. For example, $G_{NL} = p$ for $p = 19$ as Observation 2.7 shows. From it we get the result that 6 = 3+3 and 3 is odd prime less than 6; 8 = 5+3 and 3, 5 are odd primes less than 8; 10 = 7+3 = 5+5 and 3, 5, 7 are odd primes less than 10; 12 = 7+5 and 5, 7 are odd primes less than 12; 14 = 11+3 = 7+7 and 3, 7, 11 are odd primes less than 14; 16 = 13+3 = 11+5 and 3, 5, 11, 13 are odd primes less than 16; 18 = 13+5 = 11+7 and 5, 7, 11, 13 are odd primes less than 18. Hence the lemma holds.

Conjecture 3.4（strong Goldbach conjecture）Every even number greater than 4 is the sum of two odd primes.

Proof. It has been proven by many methods that primes are infinite (such as Euclid’s proof, Euler’s analytical proof, Goldbach’s proof on Fermat numbers, Furstenberg’s proof using general topology and Kummer’s elegant proof).

Let $p \to \infty$, we have

$\varphi(p) = (p–3)/(p+3) = 1$.

By Definition 2.5 $G_C = (p+3)/2$, we obtain

$$\xi(p) = (G_{NL}-G_C)/G_C$$
$$= (2G_{NL}-p-3)/(p+3).$$

Considering $\varphi(p)$ to be asymptotic function of $\xi(p)$, the limit of function $\xi(p)$ should be same as the limit of function $\varphi(p)$ when $p \to \infty$. Thus we get

$$\xi(p) = (2G_{NL}-p-3)/(p+3) = 1 \text{ when } p \to \infty.$$

From above results we have

$$(2G_{NL}-p-3)/(p+3) = (p-3)/(p+3) \text{ when } p \to \infty.$$

Hence we obtain

$$G_{NL} = p \text{ when } p \to \infty.$$

1. From $G_{NL} = p$ when $p \to \infty$ we have $G_{NL} \to \infty$ when $p \to \infty$. Since G_{NL} is the largest Goldbach natural number G_N generated by prime p by Definition 2.4, we discover original continuous Goldbach natural number sequence $\{3, 4, \ldots, G_{NL} = p\}$ will become an infinite sequence when $p \to \infty$. By Lemma 3.2 all integers N from 3 to p will become Goldbach natural numbers G_N to generate even numbers $2N = 2G_N$ greater than 4, in which every even number $2N$ is the sum of two odd primes not greater than $G_{NL} = p$ when $p \to \infty$. In other words, we have an infinite continuous even number sequence $2\cdot3, 2\cdot4, 2\cdot5, \ldots, 2(p-2), 2(p-1), 2p$ when $p \to \infty$, in which every even number $2N$ is the sum of two odd primes not greater than p. For example, taking $N = 3$ and $N = p$, we get $2N = 2\cdot3 = 6 = 3+3$ and $2N = 2p = p+p$ when $p \to \infty$. It means every even number greater than 4 is the sum of two odd primes. Hence the strong Goldbach conjecture is true.

2. Let N be integer greater than 2, from $G_{NL} = p$ when $p \to \infty$ we have $2N = G_{NL} - 1 = p-1 \to \infty$ when $p \to \infty$. Thus we have an infinite continuous even number sequence $2\cdot3, 2\cdot4, 2\cdot5, \ldots, p-5, p-3, p-1$ when $p \to \infty$, in which every even number $2N$ greater than 4 is the sum of two odd primes less than $2N$ and all pairs of same or distinct odd primes less than $2N$ with the sum being $2N$ can be completely found for $2N \leq p-1$ by Lemma 3.3. Therefore, every even number $2N$ greater than 4 is the sum of two odd primes and all pairs of odd primes less than $2N$ with the sum being $2N$ can be completely found. Hence the strong Goldbach conjecture is true.

Conjecture 3.5（weak Goldbach conjecture）Every odd number greater than 7 is the sum of three odd primes.

Proof. It is very clear that weak Goldbach conjecture is implied by strong Goldbach conjecture, which means if strong Goldbach conjecture is true then weak Goldbach conjecture is true by tanking one odd prime as 3[3].

In our framework, every odd number greater than 7 will be generated from $2N+3$ when $p \to \infty$ because every even number $2N$ greater than 4 is the sum of two odd primes i. e. $2N = p_1+p_2$ as our proof of the strong Goldbach conjecture shows. Let $p_3 = 3$, we have $2N+3 = p_1+p_2+p_3$. For example, if $N = 3$ then $2N+3 = 9 = 3+3+3$.

It means every odd number greater than 7 is the sum of three odd primes. Hence the weak Goldbach conjecture is true.

4. Goldbach prime and twin prime conjecture

The twin prime conjecture is also a subproblem of Hilbert's 8th problem and remains unsolved. Chen showed that there are infinitely many primes p with $p+2$ being a prime or the product of two primes and such primes p are called Chen prime[4]. In another research direction, Zhang proved that for some integers N less than 70000000, there is an infinite number of pairs of primes that differ by N, and according to the Polymath project, the bound has been reduced to 246 (see Bounded gaps between primes in The On-Line PolyMath). However, in our mathematical framework the twin prime conjecture can be directly proven by introducing Goldbach prime.

In Observation 2.7 we see some sequences {3, 5, …, p} for p less than 500, including {3, 5}, {3, 5, 7}, {3, 5, …, 13}, {3, 5, …, 19} and {3, 5, …, 109}, generate corresponding $G_{NL} = p$. It means all integers N from 3 to p are Goldbach natural numbers G_N and also Goldbach integers G_I for p = 5, 7, 13, 19, 109 by Definition 2.4. Therefore, we have the following definition.

Definition 4.1 A prime p greater than 3 is called Goldbach prime if and only if $G_{NL} = p$.

By Definition 4.1 there are 5 known Goldbach primes i. e. 5, 7, 13, 19, 109 for p less than 500. It is worth noting that 5, 7, 13, 19, 109 all are the higher member of twin prime pairs, that is, 5 is the higher member of twin prime pair (3, 5), 7 is the higher member of twin prime pair (5, 7), 13 is the higher member of twin prime pair (11, 13), 19 is the higher member of twin prime pair (17, 19), 109 is the higher member of twin prime pair (107, 109). Hence we have the following lemma.

Lemma 4.2 If p is Goldbach prime, then p is the higher member of a twin prime pair.

Proof. If prime p is Goldbach prime then $G_{NL} = p$ by Definition 4.1, so that all integers N from 3 to p must be Goldbach natural numbers G_N and also Goldbach integers G_I, specially, including p–2, p–1 and p by Definition 2.3 and Definition 2.4. It means any prime greater than p will not be considered. It is known that p is a prime and $p = (p+p)/2$ is a Goldbach integer G_I but p–1 is a composite number, so we can not consider $p-1 = [(p-1)+(p-1)]/2$ to be a Goldbach integers G_I. Thus we have the following results.

If p–2 is a composite number and q is odd prime less than p–2 then it is not true that $q+p = 2(p-1)$ to make $p-1 = (q+p)/2$ become Goldbach integer G_I, because we have $q+p < 2(p-1)$ from $2(p-1) = p+(p-2)$ and $q < p-2$. If q_1, q_2 are two same or distinct odd primes less than p–2 then we have $q_1+q_2 < 2(p-1)$ from $2(p-1) = p+(p-2)$ and $q_1 < p-2$, $q_2 < p-2$ so that p–1 is not Goldbach integer G_I when q_1 and q_2 are odd primes less than p–2. However, if p–2 is a prime then $p-2 = [(p-2)+(p-2)]/2$ is a Goldbach integer G_I and $p-1 = [p+(p-2)]/2$ is also a Goldbach integer G_I by Definition 2.3. For example, $G_{NL} = p$ for p = 109 as Observation 2.7 shows. From it we see 109 = (109+109)/2 and 107 = (107+107)/2 all are Goldbach integers G_I and

108 = (109+107)/2 is also Goldbach integer G_I because of 109 and 107 being all prime. Hence if p is Goldbach prime then p–2 is a prime so that (p–2, p) is a twin prime pair and p is the higher member of the twin prime pair.

Lemma 4.2 means there is no isolated prime p greater than 3 to be Goldbach prime and if p is Goldbach prime then (p–2, p) is a twin prime pair so that the lower member of the twin prime pair i. e. p–2 is Chen prime. Hence Lemma 4.2 implies if p is Goldbach prime then p–2 is Chen prime.

Conjecture 4.3 (twin prime conjecture) There is an infinite number of twin prime pairs.

Proof. In our proof of the strong Goldbach conjecture, we get the result as follows

$G_{NL} = p$ when $p \to \infty$.

By Definition 4.1 p is a Goldbach prime when $p \to \infty$.

By Lemma 4.2 (p–2, p) is a twin prime pair when $p \to \infty$.

There is a finite number of known twin prime pairs such as (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73), (101, 103), (107, 109) and so on and the largest known twin prime pair is $3756801695685 \cdot 2^{666669} \pm 1$ discovered in 2011 (see Twin prime in The On-Line Wikipedia). Since there is a finite number of known twin prime pairs and (p–2, p) is a twin prime pair when $p \to \infty$. Hence there is an infinite number of twin prime pairs. It means the twin prime conjecture is true.

5. Conclusion

In our mathematical framework, original continuous odd prime number sequence {3, 5,…, p} is the starting point in studying anything, for example, more and more sets of odd primes not greater than p are considered up to $p \to \infty$, which brings us all results. The limit status of original continuous Goldbach natural number sequence {3, 4, …, G_{NL}} generated by {3, 5,…, p} when $p \to \infty$, that is, $G_{NL} = p$ when $p \to \infty$, is the base for proving the strong Goldbach conjecture and also that for proving the twin prime conjecture. It is existence of $G_{NL} = p$ when $p \to \infty$ that makes it become possible to link the twin prime conjecture with the strong Goldbach conjecture such closely. In this paper we also see it is really true that the strong Goldbach conjecture implies the weak Goldbach conjecture.

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