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Modeling evidence fusion rules by means of referee functions

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Abstract – *This paper defines a new concept and framework for constructing fusion rules for evidences. This framework is based on a* referee function*, which does a decisional arbitrament conditionally to basic decisions provided by the several sources of information. A simple sampling method is derived from this framework. The purpose of this sampling approach is to avoid the combinatorics which are inherent to the definition of fusion rules of evidences. This definition of the fusion rule by the means of a sampling process makes possible the construction of several rules on the basis of an algorithmic implementation of the referee function, instead of a mathematical formulation. Incidentally, it is a versatile and intuitive way for defining rules. The framework is implemented for various well known evidence rules. On the basis of this framework, new rules for combining evidences are proposed, which takes into account a consensual evaluation of the sources of information.*

Keywords: Evidence, Referee Function, Sampling, Dempster-Shafer rule, PCR6.

Notations

- $I[b]$, function of boolean b, is defined by $I[\textbf{true}] = 1$ and $I[\textbf{false}] = 0$. Typically, $I[x = y]$ has value 1 when $x = y$, and 0 when $x \neq y$,
- Let be given a frame of discernment Θ. Then, the structure G^Θ denotes any *distributive lattice* or *Boolean algebra* generated by Θ and containing $\emptyset$. In particular, G^Θ may be 2^Θ the power set on Θ, or D^Θ the hyperpower set on Θ, or the free Boolean algebra generated by Θ, or S^Θ the *superpower set* on Θ,
- $x_{1:n}$ is an abbreviation for the sequence $x_1, \cdots, x_n$,
- $\max\{x_1, \cdots, x_n\}$, or $\max\{x_{1:n}\}$, is the maximal value of the sequence $x_{1:n}$. Similar notations are used for min,
- $\max_{x\in X}\{f(x)\}$, or $\max\{f(x) \;/\; x \in X\}$, is the maximal value of $f(x)$ when $x \in X$. Similar notations are used for min.

1 Introduction

Evidence theory [1, 2] has often been promoted as an alternative approach for fusing informations, when the hypotheses for a Bayesian approach cannot be precisely stated. While many academic studies have been accomplished, most industrial applications of data fusion still remain based on a probabilistic modeling and fusion of the information. This great success of the Bayesian approach is explained by at least three reasons:

- The underlying logic of the Bayesian inference [3] seems intuitive and obvious at a first glance. It is known however [4] that the logic behinds the Bayesian inference is much more complex,
- The Bayesian rule is entirely compatible with the preeminent theory of Probability and takes advantage of all the probabilistic background,
- Probabilistic computations are tractable, even for reasonably complex problems.

Then, even if evidences allow a more general and subtle manipulation of the information for some case of use, the Bayesian approach still remains the method of choice for most applications. This paper intends to address the three afore mentioned points, by providing a random set interpretation of the fusion rules. This interpretation is based on a *referee function*, which does a decisional arbitrament conditionally to basic decisions provided by the several sources of information. This referee functions will imply a sampling approach for the definition of the rules. Sampling approach is instrumental for the combinatorics avoidance [5].

In the recent literature, there has been a large amount of work devoted to the definition of new fusion rules [6] to [14]. The choice for a rule is often dependent of the

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applications and there is not a systematic approach for this task. The definition of the fusion rule by the means of a sampling process makes possible the construction of several rules on the basis of an algorithmic implementation of the referee function, instead of a mathematical formulation. Incidentally, it is a versatile and intuitive way for defining rules. Subsequently, our approach is illustrated by implementing two well known evidence rules. On the basis of this framework, new rules for combining evidences are also proposed. Typically, these new rules takes into account a consensual evaluation of the sources, by invalidating irrelevant sources of informations on the basis of a majority decision.

Section 2 introduces the notion of referee function and its application to the definition of fusion rules. A sampling method is obtained as a corollary. Section 3 establishes the referee functions for two known rules. Section 4 defines new fusion rules. Section 5 makes some numerical comparisons. Section 6 concludes.

2 Referee functions

Let Θ be a set of propositions, on which the information is represented. Let G^Θ be a distributive lattice or a Boolean algebra generated by Θ and containing $\emptyset$.

2.1 Referee function

Definition. A referee function over G^Θ for s sources of information and with context γ is a mapping $X, Y_{1:s} \mapsto F(X|Y_{1:s};\gamma)$ defined on propositions $X, Y_{1:s} \in G^\Theta$, which satisfies:

- $F(X|Y_{1:s};\gamma) \geq 0$,
- $\displaystyle\sum_{X\in G^\Theta} F(X|Y_{1:s};\gamma) = 1$,

for any $X, Y_{1:s} \in G^\Theta$.

A referee function for s sources of information is also called a s-ary referee function. The quantity $F(X|Y_{1:s};\gamma)$ is called a *conditional arbitrament* between $Y_{1:s}$ in favor of X. Notice that X is not necessary one of the propositions $Y_{1:s}$; typically, it could be a combination of them. The case $X = \emptyset$ is called the *rejection case*.

Fusion rule. Let be given s basic belief assignments (bba) $m_{1:s}$ and a s-ary referee function F with context $m_{1:s}$. Then, the fused bba $m_1 \oplus \cdots \oplus m_s[F]$ based on the referee F is constructed as follows:

$$m_1 \oplus \cdots \oplus m_s[F](X) = \frac{I[X\neq\emptyset]}{1-z} \sum_{Y_{1:s}\in G^\Theta} F(X|Y_{1:s};m_{1:s}) \prod_{i=1}^{s} m_i(Y_i) \ , \tag{1}$$

where $z = \displaystyle\sum_{Y_{1:s}\in G^\Theta} F(\emptyset|Y_{1:s};m_{1:s}) \prod_{i=1}^{s} m_i(Y_i)$.

From now on, the notation $\oplus[m_{1:s}|F] = m_1 \oplus \cdots \oplus m_s[F]$ is used.

The value z is called the *rejection rate*.

Examples. Refer to section 3 and 4.

2.2 Properties

Bba status. The function $\oplus[m_{1:s}|F]$ defined on G^Θ is actually a basic belief assignment.

Proof. It is obvious that $\oplus[m_{1:s}|F] \geq 0$.
Since $I[\emptyset\neq\emptyset] = 0$, it is derived $\oplus[m_{1:s}|F](\emptyset) = 0$.
From $\sum_{X\in G^\Theta} F(X|Y_{1:s};m_{1:s}) = 1$, it is derived:

$$\begin{aligned}
&\sum_{X\in G^\Theta}\sum_{Y_{1:s}\in G^\Theta} F(X|Y_{1:s};m_{1:s}) \prod_{i=1}^{s} m_i(Y_i) \\
&\quad = \sum_{Y_{1:s}\in G^\Theta}\left(\prod_{i=1}^{s} m_i(Y_i)\right)\sum_{X\in G^\Theta} F(X|Y_{1:s};m_{1:s}) \\
&\quad = \sum_{Y_{1:s}\in G^\Theta}\prod_{i=1}^{s} m_i(Y_i) \\
&\quad = \prod_{i=1}^{s}\sum_{Y_i\in G^\Theta} m_i(Y_i) = 1 \ .
\end{aligned}$$

As a consequence:

$$\begin{aligned}
&\sum_{X\in G^\Theta} I[X\neq\emptyset] \sum_{Y_{1:s}\in G^\Theta} F(X|Y_{1:s};m_{1:s}) \prod_{i=1}^{s} m_i(Y_i) \\
&\qquad + \sum_{Y_{1:s}\in G^\Theta} F(\emptyset|Y_{1:s};m_{1:s}) \prod_{i=1}^{s} m_i(Y_i) = 1 \ .
\end{aligned}$$

□□□

2.3 Sampling process

The definition (1) makes apparent a fusion process which is similar to a probabilistic conditional decision on the set of propositions. Notice that the basic belief assignments are not related, in practice, to *physical* probabilities. But the implied mathematics are similar, as well as some concepts. In particular, the fusion could be interpreted as a two stages process. In a first stage, the sources of information generates independent entries according to the respective beliefs. Then, a final decision is done by the referee function conditionally to the entries. As a result, an output is produced or not.

This interpretation has two profitable consequences. First at all, it provides an intuitive background for constructing new rules: in our framework, a new rule is just the design of a new referee. Secondly, our interpretation makes possible sampling methods in order to approximate and accelerate complex fusion processes. Notice that the sampling method is used here as a mathematical tool for approximating the *belief*, not for simulating an individual choice. Indeed, evidence approaches deal with belief on propositions, not with individual propositions.

Sampling algorithm. Samples for the fused basic belief assignment $\oplus[m_{1:s}|F]$ are generated by iterating the following processes:

Entries generation: For each $i \in [\![1, s]\!]$, generates $Y_i \in G^\Theta$ according to the basic belief assignment m_i, considered as a probabilistic distribution over the set G^Θ ,

Conditional arbitrament:

1. Generate $X \in G^\Theta$ according to the referee function $F(X|Y_{1:s}; m_{1:s})$, considered as a probabilistic distribution over the set G^Θ ,
2. In the case $X = \emptyset$, reject the sample. Otherwise, keep the sample.

The performance of the sampling algorithm is at least dependent of two factors. First at all, a fast implementation of the arbitrament is necessary. Secondly, low rejection rate is better. Notice however that the rejection rate is not a true handicap. Indeed, high rejection rate means that the incident bbas are not compatible in regard to the fusion rule: these bba should not be fused. By the way, the ratio of rejected samples will provide an empirical estimate of the rejection rate of the law.

2.4 Algorithmic definition of rules

As seen previously, fusion rules based on referee functions are easily approximated by means of sampling process. This sampling process is double-staged. The first stage computes the samples related to the entry bbas $m_{1:s}$. The second stage computes the fused samples by a conditional arbitrament between the different hypotheses. This arbitrament is formalized by a referee function.

In practice, it is noteworthy that there is no need for a mathematical definition of the referee function. The only important point is to be able to compute the arbitrament. We have here a new approach for defining fusion rules of evidences. Fusion rules may be defined entirely by the means of an algorithm for computing the conditional arbitrament.

Assertion. *There are three equivalent approaches for defining fusion rules in the paradigm of referee:*

- *By defining a formula which maps the entry bbas $m_{1:s}$ to the fused bba $m_1 \oplus \cdots \oplus m_s$ (classical approach),*
- *By defining a referee function F, which makes the conditional arbitrament $F(X|Y_{1:s}; m_{1:s})$,*
- *By constructing an algorithm which actually makes the conditional arbitrament between $Y_{1:s}$ in favor of X .*

It is sometime much easier and more powerful to just construct the algorithm for conditional arbitrament.

The following section illustrates the afore theoretical discussion on well known examples.

3 Examples of referee functions

3.1 Dempster-shafer rule

Classical definition. Let be given s sources of information characterized by their bbas $m_{1:s}$. The fused bba m_{DST} obtained from $m_{1:s}$ by means of *Dempster-Shafer* fusion rule [1, 2] is defined by:

$$\begin{cases} m_{\text{DST}}(\emptyset) = 0 \ , \\ m_{\text{DST}}(X) = \dfrac{m_\wedge(X)}{1 - m_\wedge(\emptyset)} \quad \text{for any } X \in G^\Theta \setminus \{\emptyset\} \ , \end{cases}$$

where $m_\wedge(\cdot)$ corresponds to the conjunctive consensus:

$$m_\wedge(X) \triangleq \sum_{\substack{Y_1 \cap \cdots \cap Y_s = X \\ Y_1, \cdots, Y_s \in G^\Theta}} \prod_{i=1}^{s} m_i(Y_i) \ .$$

Definition by referee function. The definition of a referee function for Dempster-Shafer is immediate:

$$m_{\text{DST}} = \oplus[m_{1:s}|F_{\text{DST}}] \ ,$$

$$\text{where } F_{\text{DST}}(X|Y_{1:s}; m_{1:s}) = I\left[X = \bigcap_{k=1}^{s} Y_k\right] \ .$$

Algorithmic definition. The algorithmic implementation of F_{DST} is described subsequently and typically implies possible conditional rejections:

Conditional arbitrament:

1. Set $X = \bigcap_{k=1}^{s} Y_k$,
2. If $X = \emptyset$, then reject the sample. Otherwise, keep the sample.

3.2 PCR6 rule

The proportional conflic redistribution rules (PCRn) have been introduced By Dezert and Smarandache [12]. The rule PCR6 has been proposed by Martin and Osswald in [10] .

Classical definition. Let be given s sources of information characterized by their bbas $m_{1:s}$. The fused bba m_{PCR6} obtained from $m_{1:s}$ by means of the PCR6 rule is defined by:

$$m_{\text{PCR6}}(\emptyset) = 0 \ ,$$

and, for any $X \in G^\Theta \setminus \{\emptyset\}$, by:

$$m_{\text{PCR6}}(X) = m_\wedge(X) + \sum_{i=1}^{s} m_i(X)^2 \times \sum_{\substack{\bigcap_{k=1}^{s-1} Y_{\sigma_i(k)} \cap X = \emptyset \\ Y_{\sigma_i(1)},\cdots,Y_{\sigma_i(s-1)} \in G^\Theta}} \left(\frac{\prod_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(X) + \sum_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})} \right) , \tag{2}$$

where $m_\wedge(\cdot)$ corresponds to the conjunctive consensus:

$$m_\wedge(X) \triangleq \sum_{\substack{Y_1 \cap \cdots \cap Y_s = X \\ Y_1,\cdots,Y_s \in G^\Theta}} \prod_{i=1}^{s} m_i(Y_i) ,$$

and the function σ_i counts from 1 to s avoiding i:

$$\sigma_i(j) = j \times I[j < i] + (j+1) \times I[j \geq i] .$$

N.B. If the denominator in (2) is zero, then the fraction is discarded.

Definition by referee function. Definition 2 could be reformulated into:

$$m_{\text{PCR6}}(X) = m_\wedge(X) + \sum_{i=1}^{s} \sum_{\substack{\bigcap_{k=1}^{s} Y_k = \emptyset \\ Y_1,\cdots,Y_s \in G^\Theta}} \left(\frac{I[X = Y_i]\; m_i(Y_i) \prod_{j=1}^{s} m_j(Y_j)}{\sum_{j=1}^{s} m_j(Y_j)} \right) ,$$

and then:

$$m_{\text{PCR6}}(X) = m_\wedge(X) + \sum_{\substack{\bigcap_{k=1}^{s} Y_k = \emptyset \\ Y_1,\cdots,Y_s \in G^\Theta}} \prod_{i=1}^{s} m_i(Y_i) \; \frac{\sum_{j=1}^{s} I[X = Y_j]\; m_j(Y_j)}{\sum_{j=1}^{s} m_j(Y_j)} . \tag{3}$$

At last, it is derived a formulation of PCR6 by means of a referee function:

$$m_{\text{PCR6}} = \oplus[m_{1:s} | F_{\text{PCR6}}] ,$$

where the referee function F_{PCR6} is defined by:

$$F_{\text{PCR6}}(X|Y_{1:s}; m_{1:s}) = I\left[X = \bigcap_{k=1}^{s} Y_k \neq \emptyset \right] + I\left[\bigcap_{k=1}^{s} Y_k = \emptyset \right] \frac{\sum_{j=1}^{s} I[X = Y_j]\; m_j(Y_j)}{\sum_{j=1}^{s} m_j(Y_j)} . \tag{4}$$

Algorithmic definition. Again, the algorithmic implementation is immediate:

Conditional arbitrament:

1. If $\bigcap_{k=1}^{s} Y_k \neq \emptyset$, then set $X = \bigcap_{k=1}^{s} Y_k$
2. Otherwise:
   (a) Define the probability P over $[\![1, s]\!]$ by:
   $$P_i = \frac{m_i(Y_i)}{\sum_{j=1}^{s} m_j(Y_j)} \quad \text{for any } i \in [\![1, s]\!] ,$$
   (b) Generate a random integer $k \in [\![1, s]\!]$ accordingly to P ,
   (c) set $X = Y_k$.

It is noticed that this process does not produce any rejection case $X = \emptyset$. As a consequence, the last rejection step has been removed.

Essentially, this algorithm distinguishes two cases:

- there is a consensus; then, answer the consensus,
- there is not a consensus; then choose an entry among all entries proportionally to its belief. It is noteworthy that there is no attempt to transform the entries in this case.

This algorithm is efficient and is not time-consuming. The whole sampling approach should be a good alternative for approximating PCR6, particularly on large frames of discernment.

3.3 Any rule?

Is it possible to construct a referee function for any existing fusion rule?

Actually, the answer to this question is ambiguous. If it is authorized that F depends on $m_{1:s}$ without restriction, then the theoretical answer is trivially yes.

Property. Let be given the fusion rule $m_1 \oplus \cdots \oplus m_s$, applying on the bbas $m_{1:s}$. Define the referee function F by:

$$F(X|Y_{1:s}; m_{1:s}) = m_1 \oplus \cdots \oplus m_s(X) ,$$

for any $X, Y_{1:s} \in G^\Theta$. Then F is actually a referee function and $\oplus[m_{1:s}|F] = m_1 \oplus \cdots \oplus m_s$.

Proof is immediate.

Of course, this result is useless in practice, since such referee function is inefficient. It is inefficient because it does not provide an intuitive interpretation of the rule, and is as difficult to compute as the fusion rule. Then, it is useless to have a sampling approach with such definition.

As a conclusion, referee functions have to be considered together with their efficiency. The efficiency of referee function is not a topic which is studied in this paper.

4 A new rule: PCR♯

Definition. For any $k \in [\![1, s]\!]$, it is defined:

$$C[k|s] = \{\gamma \subset [\![1, s]\!] \,/ \mathrm{card}(\gamma) = k\} \;,$$

the set of k-combinations of $[\![1, s]\!]$. Of course, the cardinal of $C[k|s]$ is $\binom{s}{k}$.

For convenience, the undefined object $C[s+1|s]$ is actually defined by:

$$C[s+1|s] = \{\{\emptyset\}\} \;,$$

so as to ensure:

$$\min_{\gamma \in C[s+1|s]} \left\{ I\left[\bigcap_{i\in\gamma} Y_i = \emptyset \right] \right\} = 1$$

4.1 Limitations of PCR6

The algorithmic interpretation of PCR6 has shown that PCR6 distinguishes two cases:

- The entry informations are compatible; then, the conjunctive consensus is decided,
- The entry informations are not compatible; then, a mean decision is decided, weighted by the relative beliefs of the entries.

In other words, PCR6 only considers consensus or no-consensus cases. But for more than 2 sources, there are many cases of *intermediate consensus*. By construction, PCR6 is not capable to manage intermediate consensus. This is a notable limitation of PCR6.

The new rule PCR♯, which is defined now, extends PCR6 by considering partial consensus in addition to full consensus and absence of consensus. This rule is constructed by specifying the arbitrament algorithm. Then, a referee function is deduced.

4.2 Algorithm

The following algorithm try to reach a maximal consensus. It first tries the full consensus, then consensus of $s-1$ sources, $s-2$ sources, and so on, until a consensus is finally found. When several consensus with k sources is possible, the final answer is chosen randomly, proportionally to the beliefs of the consensus. In the following algorithm, comments are included preceded by // (c++ convention).

Conditional arbitrament:

1. Set $stop = $ `false` and $k = s$,
   // k is the size of the consensus, which are searched. At beginning, it is maximal.
2. For each $\gamma \in C[k|s]$, do:
   // All possible consensus of size k is tested.
   (a) If $\bigcap_{i\in\gamma} Y_i \neq \emptyset$, then set $\omega_\gamma = \prod_{i\in\gamma} m_i(Y_i)$ and $stop = $ `true`,
   // If a consensus of size k is found to be functional, then it is no more necessary to diminish the size of the consensus. This is done by changing the value of boolean stop.
   // Moreover, the functional consensus are weighted by their beliefs.
   (b) Otherwise set $\omega_\gamma = 0$,
   // Non-functional consensus are weighted zero.
3. If $stop = $ `false`, then set $k = k-1$ and go back to 2,
   // If no functional consensus of size k has been found, then it is necessary to test smaller sized consensus. The process is thus repeated for size $k-1$.
4. Choose $\gamma \in C[k|s]$ randomly, according to the probability:
   $$P_\gamma = \frac{\omega_\gamma}{\sum_{\gamma\in C[k|s]} \omega_\gamma} \;,$$
   // Otherwise, choose a functional consensus. Here, the decision is random and proportional to the consensus belief.
5. At last, set $X = \bigcap_{i\in\gamma} Y_i$.
   // Publish the sample related to the consensus.

4.3 Referee function

Historically, PCR♯ has been defined by means of an algorithm, not by means of a formal definition of the referee function. It is however possible to give a formal definition of the referee function which is equivalent to the algorithm:

$$F_{\mathrm{PCR}\sharp}(X|Y_{1:s}; m_{1:s}) = \sum_{k=1}^{s} \min_{\gamma\in C[k+1|s]} \left\{ I\left[\bigcap_{i\in\gamma} Y_i = \emptyset\right]\right\} \times \min\left\{ \max_{\gamma\in C[k|s]} \left\{ I\left[\bigcap_{i\in\gamma} Y_i \neq \emptyset\right]\right\}, \frac{\sum_{\gamma\in C[k|s]} I\left[X = \bigcap_{i\in\gamma} Y_i \neq \emptyset\right] \prod_{i\in\gamma} m_i(Y_i)}{\sum_{\gamma\in C[k|s]} I\left[\bigcap_{i\in\gamma} Y_i \neq \emptyset\right] \prod_{i\in\gamma} m_i(Y_i)} \right\} . \tag{5}$$

Sketch of the proof. The following correspondences are established between the arbitrament algorithm and the referee function:

- The summation $\sum_{k=1}^{s}$ is a formalization of the loop from $k = s$ down to $k = 1$,
- At step k, the component:
$$\min_{\gamma \in C[k+1|s]} \left\{ I\left[\bigcap_{i\in\gamma} Y_i = \emptyset\right]\right\}$$
ensures that there is not a functional consensus of larger size $j > k$. Typically, the component is 0 if a larger sized functional consensus exists, and 1 otherwise. This component is complementary to the summation, as it formalizes the end of the loop, when a functional consensus is actually found,
- At step k, the component:
$$\Omega = \frac{\sum_{\gamma\in C[k|s]} I\left[X = \bigcap_{i\in\gamma} Y_i \neq \emptyset\right] \prod_{i\in\gamma} m_i(Y_i)}{\sum_{\gamma\in C[k|s]} I\left[\bigcap_{i\in\gamma} Y_i \neq \emptyset\right] \prod_{i\in\gamma} m_i(Y_i)}$$
encodes the choice of a functional consensus of size k, proportionally to its belief. The chosen consensus results in the production of the sample X,
- At step k, the component:
$$\max_{\gamma\in C[k|s]} \left\{ I\left[\bigcap_{i\in\gamma} Y_i \neq \emptyset\right]\right\}$$
tests if there is a functional consensus of size k. The component answers 1 if such consensus exists, and 0 otherwise. It is combined with a minimization of the form:
$$\min\left\{ \max_{\gamma\in C[k|s]} \left\{ I\left[\bigcap_{i\in\gamma} Y_i \neq \emptyset\right]\right\}, \Omega \right\},$$
where $\Omega \leq 1$. This is some kind of "`if ... then`" : if a functional consensus of size k exists, then the value Ω is computed. Otherwise, it is the value 0. Since the value Ω encodes a sampling decision, we have here sampling decision, which is conditioned by the fact that a functional consensus exists.

The equivalence is a consequence of these correspondences.
□

4.4 Variants of PCR♯

Actually, $\text{card}(C[k|s]) = \binom{s}{k}$ increases quickly when s is great and k is not near 1 or s. As a consequence, PCR♯ implies hard combinatorics, when used in its general form. On the other hand, it may be interesting to reject samples, when a consensus is not possible with a minimal quorum. In order to address such problems, a slight extension of PCR♯ is proposed now.

Algorithm.

Let $r \in [\![1, s]\!]$ and let $k_{1:r} \in [\![1, s]\!]$ be a decreasing sequence such that:

$$s \geq k_1 > \cdots > k_r \geq 1 .$$

For convenience, the undefined object k_0 is actually defined by:

$$k_0 = s + 1 ,$$

so as to ensure:

$$\min_{\gamma\in C[k_0|s]} \left\{ I\left[\bigcap_{i\in\gamma} Y_i = \emptyset\right]\right\} = 1$$

Then, the rule PCR♯$[k_{1:r}]$ is defined by the following algorithm.

Conditional arbitrament:

1. Set *stop* = `false` and $h = 1$,
2. For each $\gamma \in C[k_h|s]$, do:
   (a) If $\bigcap_{i\in\gamma} Y_i \neq \emptyset$, then set $\omega_\gamma = \prod_{i\in\gamma} m_i(Y_i)$ and *stop* = `true`,
   (b) Otherwise set $\omega_\gamma = 0$,
3. If *stop* = `false`, then:
   (a) set $h = h + 1$,
   (b) If $h \leq r$, go back to 2,
4. If $h > r$, then reject the entries and **end**,
5. Otherwise, choose $\gamma \in C[k_h|s]$ randomly, according to the probability:
$$P_\gamma = \frac{\omega_\gamma}{\sum_{\gamma\in C[k_h|s]} \omega_\gamma},$$
6. Set $X = \bigcap_{i\in\gamma} Y_i$. and **end**.

Referee function

$$F_{\mathrm{PCR}\sharp[k_{1:r}]}(X|Y_{1:s};m_{1:s}) = \min_{\gamma\in C[k_r|s]}\left\{I\left[X=\bigcap_{i\in\gamma}Y_i=\emptyset\right]\right\}$$
$$+\sum_{h=1}^{r}\min_{\gamma\in C[k_{h-1}|s]}\left\{I\left[\bigcap_{i\in\gamma}Y_i=\emptyset\right]\right\}$$
$$\times\min\left\{\begin{array}{c}\max_{\gamma\in C[k_h|s]}\left\{I\left[\bigcap_{i\in\gamma}Y_i\neq\emptyset\right]\right\},\\ \\ \dfrac{\sum_{\gamma\in C[k_h|s]} I\left[X=\bigcap_{i\in\gamma}Y_i\neq\emptyset\right]\prod_{i\in\gamma}m_i(Y_i)}{\sum_{\gamma\in C[k_h|s]} I\left[\bigcap_{i\in\gamma}Y_i\neq\emptyset\right]\prod_{i\in\gamma}m_i(Y_i)}\end{array}\right\}.$$

proof is left to the reader.

PCR6 and PCR♯. Assume that PCR6 is applied to s entries $m_{1:s}$. Then: PCR6=PCR♯$[s,1]$

DST and PCR♯. Assume that DST is applied to s entries $m_{1:s}$. Then: DST=PCR♯$[s]$

5 Numerical examples

It is assumed:

$$G^\Theta = \{\emptyset,\{a\},\{b\},\{c\},\{b,c\},\{c,a\},\{a,b\},\{a,b,c\}\}\ .$$

5.1 Monte-Carlo convergence

The bbas m_1 and m_2 are defined by:

- $m_1(\{a,b,c\})=0.1$, $m_1(\{a,b\})=0.2$, $m_1(\{b,c\})=0.3$, $m_1(\{a,c\})=0.4$
- $m_2(\{a,b,c\})=0.1$, $m_2(\{a,b\})=0.4$, $m_2(\{b,c\})=0.3$, $m_2(\{a,c\})=0.2$

These bbas are fused by means of DST, resulting in $m=m_{\mathrm{DST}}$:

$m(\{a,b,c\})=0.01$, $m(\{a,b\})=0.14$, $m(\{b,c\})=0.15$, $m(\{a,c\})=0.14$, $m(\{a\})=0.2$, $m(\{b\})=0.18$, $m(\{c\})=0.18$.

The following table compares the rounded deviations of the empirical $m=m_{\mathrm{DST}}$, computed by means of sample clouds of different cloud sizes N.

$\log_{10}N$	1	2	3	4	5
$m(\{a,b,c\})$	$3E-2$	$1E-2$	$3E-3$	$1E-3$	$3E-4$
$m(\{a,b\})$	$1E-1$	$3E-2$	$1E-2$	$3E-3$	$1E-3$
$m(\{b,c\})$	$1E-1$	$4E-2$	$1E-2$	$4E-3$	$1E-3$
$m(\{a,c\})$	$1E-1$	$3E-2$	$1E-2$	$3E-3$	$1E-3$
$m(\{a\})$	$1E-1$	$4E-2$	$1E-2$	$4E-3$	$1E-3$
$m(\{b\})$	$1E-1$	$4E-2$	$1E-2$	$4E-3$	$1E-3$
$m(\{c\})$	$1E-1$	$4E-2$	$1E-2$	$4E-3$	$1E-3$

Actually, this table is compliant with the theoretical result: $\sigma(m(X))=\sqrt{\frac{m(X)\cdot(1-m(x))}{N}}$.

5.2 Comparative tests

Example 1. It is assumed 3 bbas $m_{1:3}$ on G^Θ by:

$$m_1(\{a,b\})=m_2(\{a,c\})=m_3(\{c\})=1\ .$$

The bbas m_1 and m_3 are incompatible. However, m_2 is compatible with both m_1 and m_3, which implies that a partial consensus is possible between m_1 and m_2 or between m_2 and m_3. As a consequence, PCR♯ should provide better answers by allowing partial combinations of the bbas. The fusion of the 3 bbas are computed respectively by means of DST, PCR6 and PCR♯, and the results confirm the intuition:

- $z_{\mathrm{DST}}=1$ and m_{DST} is undefined,
- $m_{\mathrm{PCR6}}(\{a,b\})=m_{\mathrm{PCR6}}(\{a,c\})=m_{\mathrm{PCR6}}(\{c\})=\frac{1}{3}$,
- $m_{\mathrm{PCR}\sharp}(\{a\})=m_{\mathrm{PCR}\sharp}(\{c\})=\frac{1}{2}$ derived from the consensus $\{a,b\}\cap\{a,c\}$, $\{a,c\}\cap\{c\}$ and their beliefs $m_1(\{a,b\})m_2(\{a,c\})$, $m_2(\{a,c\})m_3(\{c\})$.

Example 2. It is assumed 3 bbas $m_{1:3}$ on G^Θ by:

- $m_1(\{a,b\})=0.4$, $m_1(\{a\})=0.6$
- $m_2(\{a,c\})=0.7$, $m_2(\{a\})=0.3$
- $m_3(\{a,b,c\})=0.2$, $m_3(\{b\})=0.8$

The computation of PCR♯ is done step by step:

Full consensus. Full functional consensus are:

Y_1	$\{a,b\}$	$\{a,b\}$	$\{a\}$	$\{a\}$
Y_2	$\{a,c\}$	$\{a\}$	$\{a,c\}$	$\{a\}$
Y_3	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\bigcap_i Y_i$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$\prod_i m_i(Y_i)$	0.056	0.024	0.084	0.036

Partial consensus sized 2. The belief ratios for the partial consensus are simplified as follows:

$$\frac{m_1(Y_1)m_2(Y_2)}{m_1(Y_1)m_2(Y_2)+m_3(Y_3)m_1(Y_1)}=\frac{m_2(Y_2)}{m_2(Y_2)+m_3(Y_3)}$$

and similar results are obtained for Y_3,Y_1 and Y_2,Y_3. Then the possible partial consensus are:

Y_1	$\{a,b\}$	$\{a,b\}$	$\{a\}$	$\{a\}$
Y_2	$\{a,c\}$	$\{a\}$	$\{a,c\}$	$\{a\}$
Y_3	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
$Y_1\cap Y_2$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$Y_2\cap Y_3$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$Y_3\cap Y_1$	$\{b\}$	$\{b\}$	$\emptyset$	$\emptyset$
$\frac{m_2(Y_2)}{m_2(Y_2)+m_3(Y_3)}$	0.467	0.273	1	1
$\frac{m_3(Y_3)}{m_2(Y_2)+m_3(Y_3)}$	0.533	0.727	0	0
$\prod_i m_i(Y_i)$	0.224	0.096	0.336	0.144

1-*sized consensus.* There is no remaining 1-sized consensus.

Belief compilation. The different cases resulted in only two propositions, *i.e.* $\{a\}$ and $\{b\}$. By combining the entry beliefs $\prod_i m_i(Y_i)$ and ratio beliefs, the fused bba $m = m_{\text{PCR}\sharp}$ is then deduced:

$$\begin{cases} m(\{a\}) = 0.056 + 0.024 + 0.084 + 0.036 + 0.467 \times 0.224 \\ \quad +0.273 \times 0.096 + 1 \times 0.336 + 1 \times 0.144 = 0.811 \\ m(\{b\}) = 0.533 \times 0.224 + 0.727 \times 0.096 = 0.189 \end{cases}$$

As a conclusion:

$$m_{\text{PCR}\sharp}(\{a\}) = 0.811 \quad \text{and} \quad m_{\text{PCR}\sharp}(\{b\}) = 0.189\,.$$

This result could be compared to DST and PCR6:

- $z_{\text{DST}} = 0.8$ and $m_{\text{DST}}(\{a\}) = 1$,
- $m_{\text{PCR6}}(\{a\}) = 0.391$, $m_{\text{PCR6}}(\{b\}) = 0.341$, $m_{\text{PCR6}}(\{a,b\}) = 0.073$, $m_{\text{PCR6}}(\{a,c\}) = 0.195$,

DST produces highly conflicting results, since source 3 conflicts with the other sources. However, there are some partial consensus which allow the answer $\{b\}$. DST is blind to these partial consensus. On the other hand, PCR6 is able to handle hypothesis $\{b\}$, but is too much optimistic and, still, is unable to fuse partial consensus. Consequently, PCR6 is also unable to diagnose the high inconstancy of belief $m_3(\{b\}) = 0.8$.

6 Conclusion

This paper has investigated a new framework for the definition and interpretation of fusion rule of evidences. This framework is based on the new concept of referee function. A referee function models an arbitrament process conditionally to the contributions of several independent sources of information. It has been shown that fusion rules based on the concept of referee functions have a straightforward sampling-based implementation. As a consequence, a referee function has a natural algorithmic interpretation. Owing to the algorithmic nature of referee functions, the conception of new rules of fusion is made easier and intuitive. Examples of existing fusion rules have been implemented by means of referee functions. Moreover, an example of rule construction has been provided on the basis of an arbitrament algorithm. The new rule is a quite general extension of both PCR6 and Dempster-Shafer rule. This paper also addresses the issue of fusion rule approximation. There are cases for which the fusion computation is prohibitive. The sampling process implied by the referee function provides a natural method for the approximation and the computation speed-up. There are still many questions and improvements to be addressed. For example, samples regularization techniques may reduce possible samples degeneracy thus allowing smaller particles clouds. Some theoretical questions are also pending; especially, the algebraic properties of the referee functions have almost not been studied. However, this preliminary work is certainly promising for future applications.

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