On the consistency of PCR6 with the averaging rule and its application to probability estimation

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***Abstract*—Since the development of belief function theory introduced by Shafer in seventies, many combination rules have been proposed in the literature to combine belief functions specially (but not only) in high conflicting situations because the emblematic Dempster's rule generates counter-intuitive and unacceptable results in practical applications. Many attempts have been done during last thirty years to propose better rules of combination based on different frameworks and justifications. Recently in the DSmT (Dezert-Smarandache Theory) framework, two interesting and sophisticate rules (PCR5 and PCR6 rules) have been proposed based on the Proportional Conflict Redistribution (PCR) principle. These two rules coincide for the combination of two basic belief assignments, but they differ in general as soon as three or more sources have to be combined altogether because the PCR used in PCR5 and in PCR6 are different. In this paper we show why PCR6 is better than PCR5 to combine three or more sources of evidence and we prove the coherence of PCR6 with the simple Averaging Rule used classically to estimate the probability based on the frequentist interpretation of the probability measure. We show that such probability estimate cannot be obtained using Dempster-Shafer (DS) rule, nor PCR5 rule.**

Keywords: Information fusion, belief functions, PCR6, PCR5, DSmT, frequentist probability.

I. Introduction

In this paper, we work with belief functions [1] defined from the finite and discrete frame of discernment $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$. In Dempster-Shafer Theory (DST) framework, basic belief assignments (bba's) provided by the distinct sources of evidence are defined on the fusion space $2^\Theta = (\Theta, \cup)$ consisting in the power-set of Θ, that is the set of elements of Θ and those generated from Θ with the union set operator. Such fusion space assumes that the elements of Θ are non-empty, exhaustive and exclusive, which is called Shafer's model of Θ. More generally, in Dezert-Smarandache Theory (DSmT) [2], the fusion space denoted G^Θ can also be either the hyper-power set $D^\Theta = (\Theta, \cup, \cap)$ (Dedekind's lattice), or super-power set[1] $S^\Theta = (\Theta, \cup, \cap, c(.))$ depending on the underlying model of the frame of discernment we choose to fit with the nature of the problem. Details on DSm models are given in [2], Vol. 1.

We assume that $s \geq 2$ basic belief assignments (bba's) $m_i(.)$, $i = 1, 2, \dots, s$ provided by s distinct sources of evidences defined on the fusion space G^Θ are available and we need to combine them for a final decision-making purpose. For doing this, many rules of combination have been proposed in the literature, the most emblematic ones being the simple Averaging Rule, Dempster-Shafer (DS) rule, and more recently the PCR5 and PCR6 fusion rules.

The contribution of this paper is to analyze in deep the behavior of PCR5 and PCR6 fusion rules and to explain why we consider more preferable to use PCR6 rule rather than PCR5 rule for combining several distinct sources of evidence altogether. We will show in details the strong relationship between PCR6 and the averaging fusion rule which is commonly used to estimate the probabilities in the classical frequentist interpretation of probabilities.

This paper is organized as follows. In section II, we briefly recall the background on belief functions and the main fusion rules used in this paper. Section III demonstrates the consistency of PCR6 fusion rule with the Averaging Rule for binary masses in total conflict as well as the ability of PCR6 to discriminate asymmetric fusion cases for the fusion of Bayesian bba's. Section IV shows that PCR6 can also be used to estimate empirical probability in a simple (coin tossing) random experiment. Section V will conclude and open challenging problem about the recursivity of fusion rules formulas that are sought for efficient implementations.

II. Background on belief functions

A. Basic belief assignment

Lets' consider a finite discrete frame of discernment $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$, $n > 1$ of the fusion problem under consideration and its fusion space G^Θ which can be chosen either as 2^Θ, D^Θ or S^Θ depending on the model that fits with the problem. A basic belief assignment (bba) associated with a given source of evidence is defined as the mapping $m(.) : G^\Theta \to [0, 1]$ satisfying $m(\emptyset) = 0$ and $\sum_{A \in G^\Theta} m(A) = 1$. The quantity $m(A)$ is called mass of belief of A committed by the source of evidence. If $m(A) > 0$ then A is called a focal element of the bba $m(.)$. When all focal elements are singletons and $G^\Theta = 2^\Theta$ then $m(.)$ is called a Bayesian bba [1] and it is homogeneous to a (possibly subjective) probability measure. The vacuous bba representing a totally ignorant source is defined as $m_v(\Theta) = 1$. Belief and plausibility functions are defined by

$$\text{Bel}(A) = \sum_{\substack{B \subseteq A \\ B \in G^\Theta}} m(B) \quad \text{and} \quad \text{Pl}(A) = \sum_{\substack{B \cap A \neq \emptyset \\ B \in G^\Theta}} m(B) \tag{1}$$

[1] $\cap$ and $c(.)$ are respectively the set intersection and complement operators.

B. Fusion rules

The main information fusion problem in the belief function frameworks (DST or DSmT) is how to combine efficiently several distinct sources of evidence represented by $m_1(.)$, $m_2(.)$, $\ldots$, $m_s(.)$ ($s \geq 2$) bba's defined on G^Θ. Many rules have been proposed for such task – see [2], Vol. 2, for a detailed list of fusion rules – and we focus here on the following ones: 1) the Averaging Rule because it is the simplest one and it is used to empirically estimate probabilities in random experiment, 2) DS rule because it was historically proposed in DST, and 3) PCR5 and PCR6 rules because they were proposed in DSmT and have shown to provide better results than the DS rule in all applications where they have been tested so far. So we just briefly recall how these rules are mathematically defined.

• Averaging fusion rule $m_{1,2,\ldots,s}^{Average}(.)$

For any X in G^Θ, $m_{1,2,\ldots,s}^{Average}(X)$ is defined by

$$m_{1,2,\ldots,s}^{Average}(X) = Average(m_1, m_2, \ldots, m_s) \triangleq \frac{1}{s}\sum_{i=1}^{s} m_i(X) \tag{2}$$

Note that the vacuous bba $m_v(\Theta) = 1$ is not a neutral element for this rule. This Averaging Rule is commutative but it is not associative because in general

$$m_{1,2,3}^{Average}(X) = \frac{1}{3}[m_1(X) + m_2(X) + m_3(X)]$$

is different from

$$m_{(1,2),3}^{Average}(X) = \frac{1}{2}[\frac{m_1(X) + m_2(X)}{2} + m_3(X)]$$

which is also different from

$$m_{1,(2,3)}^{Average}(X) = \frac{1}{2}[m_1(A) + \frac{m_2(X) + m_3(X)}{2}]$$

and also from

$$m_{(1,3),2}^{Average}(X) = \frac{1}{2}[\frac{m_1(X) + m_3(X)}{2} + m_2(X)]$$

In fact, it is easy to prove that the following recursive formula holds

$$m_{1,2,\ldots,s}^{Average}(X) = \frac{s-1}{s} m_{1,2,\ldots,s-1}^{Average}(X) + \frac{1}{s} m_s(X) \tag{3}$$

This simple averaging fusion rule has been used since more than two centuries for estimating empirically the probability measure in random experiments [3], [4].

• Dempster-Shafer fusion rule $m_{1,2,\ldots,s}^{DS}(.)$

In DST framework, the fusion space G^Θ equals the power-set 2^Θ because Shafer's model of the frame Θ is assumed. The combination of $s \geq 2$ distinct sources of evidences characterized by the bba's $m_i(.)$, $i = 1, 2, \ldots, s$, is done with DS rule as follows [1]: $m_{1,2,\ldots,s}^{DS}(\emptyset) = 0$ and for all $X \neq \emptyset$ in 2^Θ

$$m_{1,2,\ldots,s}^{DS}(X) \triangleq \frac{1}{K_{1,2,\ldots,s}} \sum_{\substack{X_1, X_2, \ldots, X_s \in 2^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} m_i(X_i) \tag{4}$$

where the numerator of (4) is the mass of belief on the conjunctive consensus on X, and where $K_{1,2,\ldots,s}$ is a normalization constant defined by

$$K_{1,2,\ldots,s} = \sum_{\substack{X_1, X_2, \ldots, X_s \in 2^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_s \neq \emptyset}} \prod_{i=1}^{s} m_i(X_i) = 1 - m_{1,2,\ldots,s}(\emptyset)$$

The total degree of conflict between the s sources of evidences is defined by

$$m_{1,2,\ldots,s}(\emptyset) = \sum_{\substack{X_1, X_2, \ldots, X_s \in 2^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_s = \emptyset}} \prod_{i=1}^{s} m_i(X_i)$$

The sources are said in total conflict when $m_{1,2,\ldots,s}(\emptyset) = 1$.

The vacuous bba $m_v(\Theta) = 1$ is a neutral element for DS rule and DS rule is commutative and associative. It remains the milestone fusion rule of DST. The doubts on the validity of such fusion rule has been discussed by Zadeh in 1979 [5]–[7] based on a very simple example with two highly conflicting sources of evidence. Since 1980's, many criticisms have been done about the behavior and justification of such DS rule. More recently, Dezert et al. in [8], [9] have put in light other counter-intuitive behaviors of DS rule even in low conflicting cases and showed serious flaws in logical foundations of DST.

• PCR5 and PCR6 fusion rules

To work in general fusion spaces G^Θ and to provide better fusion results in all (low or high conflicting) situations, several fusion rules have been developed in DSmT framework [2]. Among them, two fusion rules called PCR5 and PCR6 based on the proportional conflict redistribution (PCR) principle have been proved to work efficiently in all different applications where they have been used so far. The PCR principle transfers the conflicting mass only to the elements involved in the conflict and proportionally to their individual masses, so that the specificity of the information is entirely preserved.

The general principle of PCR consists:

1) to apply the conjunctive rule;
2) calculate the total or partial conflicting masses;
3) then redistribute the (total or partial) conflicting mass proportionally on non-empty sets according to the integrity constraints one has for the frame Θ.

Because the proportional transfer can be done in two different ways, this has yielded to two different fusion rules. The PCR5 fusion rule has been proposed by Smarandache and Dezert in [2], Vol. 2, Chap. 1, and PCR6 fusion rule has been proposed by Martin and Osswald in [2], Vol. 2, Chap. 2.

We will not present in deep these two fusion rules since they have already been discussed in details with many examples in the aforementioned references but we only give their expressions for convenience here.

The general formula of PCR5 for the combination of $s \geq 2$ sources is given by $m^{PCR5}_{1,2,\ldots,s}(\emptyset) = 0$ and for $X \neq \emptyset$ in G^Θ

$$m^{PCR5}_{1,2,\ldots,s}(X) = m_{1,2,\ldots,s}(X) + \sum_{\substack{2\leq t\leq s\\ 1\leq r_1,\ldots,r_t\leq s\\ 1\leq r_1<r_2<\ldots<r_{t-1}<(r_t=s)}} \sum_{\substack{X_{j_2},\ldots,X_{j_t}\in G^\Theta\setminus\{X\}\\ \{j_2,\ldots,j_t\}\in\mathcal{P}^{t-1}(\{1,\ldots,n\})\\ X\cap X_{j_2}\cap\ldots\cap X_{j_s}=\emptyset\\ \{i_1,\ldots,i_s\}\in\mathcal{P}^s(\{1,\ldots,s\})}} \frac{(\prod_{k_1=1}^{r_1} m_{i_{k_1}}(X)^2)\cdot[\prod_{l=2}^{t}(\prod_{k_l=r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]}{(\prod_{k_1=1}^{r_1} m_{i_{k_1}}(X)) + [\sum_{l=2}^{t}(\prod_{k_l=r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]} \quad (5)$$

where i, j, k, r, s and t in (5) are integers. $m_{1,2,\ldots,s}(X)$ corresponds to the conjunctive consensus on X between s sources and where all denominators are different from zero. If a denominator is zero, that fraction is discarded; $\mathcal{P}^k(\{1,2,\ldots,n\})$ is the set of all subsets of k elements from $\{1,2,\ldots,n\}$ (permutations of n elements taken by k), the order of elements doesn't count.

The general formula of PCR6 proposed by Martin and Osswald for the combination of $s \geq 2$ sources is given by $m^{PCR6}_{1,2,\ldots,s}(\emptyset) = 0$ and for $X \neq \emptyset$ in G^Θ

$$m^{PCR6}_{1,2,\ldots,s}(X) = m_{1,2,\ldots,s}(X) + \sum_{i=1}^{s} m_i(X)^2 \sum_{\substack{\bigcap_{k=1}^{s-1} Y_{\sigma_i(k)}\cap X\equiv\emptyset\\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(s-1)})\in(G^\Theta)^{s-1}}} \left(\frac{\prod_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(X)+\sum_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}\right) \quad (6)$$

where σ_i counts from 1 to s avoiding i:

$$\begin{cases} \sigma_i(j) = j & \text{if } j < i, \\ \sigma_i(j) = j+1 & \text{if } j \geq i, \end{cases} \quad (7)$$

Since Y_i is a focal element of expert/source i, $m_i(X)+\sum_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \neq 0$.

The general PCR5 and PCR6 formulas (5)–(6) are rather complicate and not very easy to understand. From the implementation point of view, PCR6 is much simple to implement than PCR5. For convenience, very basic (not optimized) Matlab codes of PCR5 and PCR6 fusion rules can be found in [2], [10] and from the toolboxes repository on the web [11]. The PCR5 and PCR6 fusion rules are commutative but not associative, like the averaging fusion rule, but the vacuous belief assignment is a neutral element for these PCR fusion rules.

The PCR5 and PCR6 fusion rules simplify greatly and coincide for the combination of two sources ($s = 2$). In such simplest case, one always gets the resulting bba $m_{PCR5/6}(.) = m^{PCR6}_{1,2}(.) = m^{PCR5}_{1,2}(.)$ expressed as $m_{PCR5/6}(\emptyset) = 0$ and for all $X \neq \emptyset$ in G^Θ

$$m_{PCR5/6}(X) = \sum_{\substack{X_1,X_2\in G^\Theta\\ X_1\cap X_2=X}} m_1(X_1)m_2(X_2) + \sum_{\substack{Y\in G^\Theta\setminus\{X\}\\ X\cap Y=\emptyset}} [\frac{m_1(X)^2 m_2(Y)}{m_1(X)+m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X)+m_1(Y)}] \quad (8)$$

where all denominators in (8) are different from zero. If a denominator is zero, that fraction is discarded. All propositions/sets are in a canonical form.

Example 1: See [2], Vol.2, Chap. 1 for more examples.

Let's consider the frame of discernment $\Theta = \{A, B\}$ of exclusive elements. Here Shafer's model holds so that $G^\Theta = 2^\Theta = \{\emptyset, A, B, A\cup B\}$. We consider two sources of evidences providing the following bba's

$$m_1(A) = 0.6 \quad m_1(B) = 0.3 \quad m_1(A\cup B) = 0.1$$

$$m_2(A) = 0.2 \quad m_2(B) = 0.3 \quad m_2(A\cup B) = 0.5$$

Then the conjunctive consensus yields :

$$m_{12}(A) = 0.44 \quad m_{12}(B) = 0.27 \quad m_{12}(A\cup B) = 0.05$$

with the conflicting mass

$$\begin{aligned} m_{12}(A\cap B = \emptyset) &= m_1(A)m_2(B) + m_1(B)m_2(A) \\ &= 0.18 + 0.06 = 0.24 \end{aligned}$$

One sees that only A and B are involved in the derivation of the conflicting mass, but not $A \cup B$. With PCR5/6, one redistributes the partial conflicting mass 0.18 to A and B proportionally with the masses $m_1(A)$ and $m_2(B)$ assigned to A and B respectively, and also the partial conflicting mass 0.06 to A and B proportionally with the masses $m_2(A)$ and $m_1(B)$ assigned to A and B respectively, thus one gets two weighting factors of the redistribution for each corresponding set A and B respectively. Let x_1 be the conflicting mass to be redistributed to A, and y_1 the conflicting mass redistributed to B from the first partial conflicting mass 0.18. This first partial proportional redistribution is then done according

$$\frac{x_1}{0.6} = \frac{y_1}{0.3} = \frac{x_1+y_1}{0.6+0.3} = \frac{0.18}{0.9} = 0.2$$

whence $x_1 = 0.6 \cdot 0.2 = 0.12$, $y_1 = 0.3 \cdot 0.2 = 0.06$. Now let x_2 be the conflicting mass to be redistributed to A, and y_2 the conflicting mass redistributed to B from the second the partial conflicting mass 0.06. This second partial proportional redistribution is then done according

$$\frac{x_2}{0.2} = \frac{y_2}{0.3} = \frac{x_2+y_2}{0.2+0.3} = \frac{0.06}{0.5} = 0.12$$

whence $x_2 = 0.2 \cdot 0.12 = 0.024$, $y_2 = 0.3 \cdot 0.12 = 0.036$. Thus one finally gets:

$$\begin{aligned} m_{PCR5/6}(A) &= 0.44 + 0.12 + 0.024 = 0.584 \\ m_{PCR5/6}(B) &= 0.27 + 0.06 + 0.036 = 0.366 \\ m_{PCR5/6}(A \cup B) &= 0.05 + 0 = 0.05 \end{aligned}$$

• The difference between PCR5 and PCR6 fusion rules

For the two sources case, PCR5 and PCR6 fusion rules coincide. As soon as three (or more) sources are involved in the fusion process, PCR5 and PCR6 differ in the way the proportional conflict redistribution is done. For example, let's consider three sources with bba's $m_1(.)$, $m_2(.)$ and $m_3(.)$, $A \cap B = \emptyset$ for the model of the frame Θ, and $m_1(A) = 0.6$, $m_2(B) = 0.3$, $m_3(B) = 0.1$.

– With PCR5, the mass $m_1(A)m_2(B)m_3(B) = 0.6 \cdot 0.3 \cdot 0.1 = 0.018$ corresponding to a conflict is redistributed back to A and B only with respect to the following proportions respectively: $x_A^{PCR5} = 0.01714$ and $x_B^{PCR5} = 0.00086$ because the proportionalization requires

$$\frac{x_A^{PCR5}}{m_1(A)} = \frac{x_B^{PCR5}}{m_2(B)m_3(B)} = \frac{m_1(A)m_2(B)m_3(B)}{m_1(A) + m_2(B)m_3(B)}$$

that is

$$\frac{x_A^{PCR5}}{0.6} = \frac{x_B^{PCR5}}{0.03} = \frac{0.018}{0.6 + 0.03} \approx 0.02857$$

Thus

$$\begin{cases} x_A^{PCR5} = 0.60 \cdot 0.02857 \approx 0.01714 \\ x_B^{PCR5} = 0.03 \cdot 0.02857 \approx 0.00086 \end{cases}$$

– With the PCR6 fusion rule, the partial conflicting mass $m_1(A)m_2(B)m_3(B) = 0.6 \cdot 0.3 \cdot 0.1 = 0.018$ is redistributed back to A and B only with respect to the following proportions respectively: $x_A^{PCR6} = 0.0108$ and $x_B^{PCR6} = 0.0072$ because the PCR6 proportionalization is done as follows:

$$\frac{x_A^{PCR6}}{m_1(A)} = \frac{x_B^{PCR6}}{m_2(B) + m_3(B)} = \frac{m_1(A)m_2(B)m_3(B)}{m_1(A) + (m_2(B) + m_3(B))}$$

that is

$$\frac{x_A^{PCR6}}{0.6} = \frac{x_B^{PCR6}}{0.3 + 0.1} = \frac{0.018}{0.6 + (0.3 + 0.1)} = 0.018$$

and therefore with PCR6, one gets finally the following redistributions to A and B:

$$\begin{cases} x_A^{PCR6} = 0.6 \cdot 0.018 = 0.0108 \\ x_B^{PCR6} = (0.3 + 0.1) \cdot 0.018 = 0.0072 \end{cases}$$

In [2], Vol. 2, Chap. 2, Martin and Osswald have proposed PCR6 based on intuitive considerations and the authors have shown through simulations that PCR6 is more stable than PCR5 in term of decision for combining $s > 2$ sources of evidence. Based on these results and the relative "simplicity" of implementation of PCR6 over PCR5, PCR6 has been considered more interesting/efficient than PCR5 for combining 3 (or more) sources of evidences.

III. Consistency of PCR6 with the Averaging Rule

In this section we show why we also consider PCR6 as better than PCR5 for combining bba's. But here, our argumentation is not based on particular simulation results and decision-making as done by Martin and Osswald, but on a theoretical analysis of the structure of PCR6 fusion rule itself. In particular, we show the full consistency of PCR6 rule with the averaging fusion rule used to empirically estimate probabilities in random experiments. For doing this, it is necessary to simplify the original PCR6 fusion formula (6). Such simplification has already been proposed in [12] and the PCR6 fusion rule can be in fact rewritten as

$$\begin{aligned} m_{1,2,\ldots,s}^{PCR6}(X) = m_{1,2,\ldots,s}(X) + \\ \sum_{k=1}^{s-1} \sum_{\substack{X_{i_1}, X_{i_2}, \ldots, X_{i_k} \in G^\Theta \setminus X \\ (\bigcap_{j=1}^{k} X_{i_j}) \cap X = \emptyset}} \sum_{(i_1, i_2, \ldots, i_k) \in \mathcal{P}^s(\{1,\ldots,s\})} \\ [m_{i_1}(X) + m_{i_2}(X) + \ldots + m_{i_k}(X)] \cdot \\ \cdot \frac{m_{i_1}(X) \ldots m_{i_k}(X) m_{i_{k+1}}(X_{i_{k+1}}) \ldots m_{i_s}(X_{i_s})}{m_{i_1}(X) + \ldots + m_{i_k}(X) + m_{i_{k+1}}(X_{i_{k+1}}) + \ldots + m_{i_s}(X_{i_s})} \end{aligned} \quad (9)$$

where $\mathcal{P}^s(\{1, \ldots, s\})$ is the set of all permutations of the elements $\{1, 2, \ldots, s\}$. It should be observed that X_{i_1}, X_{i_2},. . .,X_{i_s} may be different from each other, or some of them equal and others different, etc.

We wrote this PCR6 general formula (9) in the style of PCR5, different from Arnaud Martin & Christophe Oswald's notations, but actually doing the same thing. In order not to complicate the formula of PCR6, we did not use more summations or products after the third Sigma.

We now are able to establish the consistency of general PCR6 formula with the Averaging fusion rule for the case of binary bba's through the following theorem 1.

Theorem 1: When $s \geq 2$ sources of evidences provide binary bba's on G^Θ whose total conflicting mass is 1, then the PCR6 fusion rule coincides with the averaging fusion rule. Otherwise, PCR6 and the averaging fusion rule provide in general different results.

Proof 1: All $s \geq 2$ bba's are assumed binary, i.e. $m(X) = 0$ or 1 (two numerical values 0 and 1 only are allowed) for any bba $m(.)$ and for any set X in the focal elements. A focal element in this case is an element X such that at least one of the s binary sources assigns a mass equals to 1 to X. Let's suppose the focal elements are F_1, F_2,. . . , F_n.. Then the set of bba's to combine can be expressed as in the Table I. where

Table I. List of bba's to combine.

bba's \ Focal elem.	F_1	F_2	…	F_n
$m_1(.)$	$\star$	$\star$	…	$\star$
$m_2(.)$	$\star$	$\star$	…	$\star$
⋮	⋮	⋮	⋮	⋮
$m_s(.)$	$\star$	$\star$	…	$\star$

- all $\star$ are 0's or 1's;
- on each row there is only a 1 (since the sum of all masses of a bba is equal to 1) and all the other elements are 0's;

- also each column has at least an 1 (since all elements are focals; and if there was a column corresponding for example to the set F_p having only 0's, then it would result that the set F_p is not focal, i.e. that all $m(F_p) = 0$).

Using PCR6, we first need to apply the conjunctive rule to all s sources, and the result is a sum of products of the form $m_1(X_1)m_2(X_2)\ldots m_s(X_s)$ where X_1, X_2,…,X_s, are the focal elements F_1, F_2,…,F_n in various permutations, with $s \geq n$. If $s > n$ some focal elements X_i are repeated in the product $m_1(X_1)m_2(X_2)\ldots m_s(X_s)$. But there is only one product of the form $m_1(X_1)m_2(X_2)\ldots m_s(X_s) = 1$ which is not equal to zero, i.e. that product which has each factor equals to "1" (i.e. the product that collects from each row the existing single 1). Since the total conflicting mass is equal to 1, it means that this product represents the total conflict. In this case the PCR6 formula (9) becomes

$$m^{PCR6}_{1,2,\ldots,s}(X) = 0 + \sum_{k=1}^{s-1} \sum_{\substack{X_{i_1},X_{i_2},\ldots,X_{i_k} \in G^\Theta \setminus X \\ (\bigcap_{j=1}^k X_{i_j}) \cap X = \emptyset}} \sum_{(i_1,i_2,\ldots,i_k)\in\mathcal{P}^s(\{1,\ldots,s\})} [1+1+\ldots+1] \cdot \frac{1\cdot 1\cdot \ldots \cdot 1 \cdot 1 \cdot \ldots \cdot 1}{1+1+\ldots+1+1+1+\ldots+1} \quad (10)$$

The previous expression can be rewritten as

$$m^{PCR6}_{1,2,\ldots,s}(X) = \sum_{k=1}^{s-1} \sum_{\substack{X_{i_1},X_{i_2},\ldots,X_{i_k} \in G^\Theta \setminus X \\ (\bigcap_{j=1}^k X_{i_j}) \cap X = \emptyset}} \sum_{\substack{(i_1,i_2,\ldots,i_k) \\ \in\mathcal{P}^s(\{1,\ldots,s\})}} k \cdot \frac{1}{s}$$

which is equal to k/s since there is only one possible non-null product of the form $m_1(X_1)m_2(X_2)\ldots m_s(X_s)$, and all other products are equal to zero. Therefore, we finally get:

$$m^{PCR6}_{1,2,\ldots,s}(X) = \frac{k}{s} \quad (11)$$

where "k" is the number of bba's $m(.)$ which give $m(X) = 1$. Therefore PCR6 in this case reduces to the average of masses, which completes the proof 1 of the theorem.

Proof 2: A second method of proving this theorem can also be done as follows. Let $m_1(.)$, $m_2(.)$, …, $m_s(.)$, for $s \geq 3$, be bba's of the sources of information to combine and denote $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$, for $n \geq 2$, the set of all focal elements. All sources give only binary masses, i.e. $m_k(F_l) = 0$ or $m_k(F_l) = 1$ for any $k \in \{1, 2, \ldots, s\}$ and any $l \in \{1, 2, \ldots, n\}$. Since each F_i, $1 \leq i \leq n$, is a focal element, there exists at least a bba $m_{i_o}(.)$ such that $m_{i_o}(F_i) = 1$, otherwise (i.e. if all sources gave the mass of F_i be equal to zero) F_i would not be focal. Without reducing the generality of the theorem, we can regroup the masses (since we combine all of them at once, so their order doesn't matter), as in Table II. Of course $i_1 + i_2 + \ldots + i_n = s$, since the s bba's are the same but reordered, and $i_1 \geq 1$, $i_2 \geq 1$, …, and $i_n \geq 1$. The total conflicting mass according to the theorem hypothesis $m_{1,2,\ldots,s}(\emptyset)$ is 1. With the PCR6 fusion rule we transfer the conflict mass back to focal elements F_1, F_2, …F_n respectively according to PCR principle such that:

Table II. LIST OF REORDERED BINARY BBA'S.

bba's \ Focal elem.	F_1	F_2	…	F_n	$\emptyset$
$m_{r_1}(.)$	1	0	…	0	0
$m_{r_2}(.)$	1	0	…	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m_{r_{i_1}}(.)$	1	0	…	0	0
$m_{s_1}(.)$	0	1	…	0	0
$m_{s_2}(.)$	0	1	…	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m_{s_{i_2}}(.)$	0	1	…	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m_{u_1}(.)$	0	0	…	1	0
$m_{u_2}(.)$	0	0	…	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m_{u_{i_n}}(.)$	0	0	…	1	0
$m_{1,2,\ldots,s}(.)$	0	0	…	0	1

$$\frac{x_{F_1}}{\underbrace{1+1+\ldots+1}_{i_1 \text{ times}}} = \frac{x_{F_2}}{\underbrace{1+1+\ldots+1}_{i_2 \text{ times}}} = \ldots = \frac{x_{F_n}}{\underbrace{1+1+\ldots+1}_{i_n \text{ times}}} = \frac{m_{1,2,\ldots,s}(\emptyset)}{i_1+i_2+\ldots+i_n} = \frac{1}{s}$$

whence $x_{F_1} = i_1/s$, $x_{F_2} = i_2/s$, …, $x_{F_n} = i_n/s$. Therefore $m^{PCR6}_{1,2,\ldots,s}(F_1) = i_1/s$, $m^{PCR6}_{1,2,\ldots,s}(F_2) = i_2/s$, $\ldots m^{PCR6}_{1,2,\ldots,s}(F_n) = i_n/s$. But averaging the masses $m_1(.)$, $m_2(.)$, …, $m_s(.)$ is equivalent to averaging each column of F_1, F_2, …F_n. Hence average of column F_1 is i_1/s, average of column F_2 is i_2/s, …, average of column F_n is i_n/s. Therefore, in case of binary bba's which are globally totally conflicting, PCR6 rule is equal to the Averaging Rule. This completes the proof 2 of the theorem.

Note that using PCR5 fusion rule, we also transfer the total conflicting mass that is equal to 1 to F_1, F_2, …, F_n respectively, but we replace the addition "+" with the multiplication "·" in the above proportionalizations:

$$\frac{x_{F_1}}{\underbrace{1\cdot 1\cdot\ldots\cdot 1}_{i_1 \text{ times}}} = \frac{x_{F_2}}{\underbrace{1\cdot 1\cdot\ldots\cdot 1}_{i_2 \text{ times}}} = \ldots = \frac{x_{F_n}}{\underbrace{1\cdot 1\cdot\ldots\cdot 1}_{i_n \text{ times}}} = \frac{m_{1,2,\ldots,s}(\emptyset)}{\underbrace{1\cdot 1\cdot\ldots\cdot 1}_{n \text{ times}}} = \frac{1}{n}$$

so that $x_{F_1} = 1/n$, $x_{F_2} = 1/n$, …, $x_{F_n} = 1/n$ and therefore

$$m^{PCR5}_{1,2,\ldots,s}(F_1) = m^{PCR5}_{1,2,\ldots,s}(F_2) = \ldots = m^{PCR5}_{1,2,\ldots,s}(F_n) = 1/n$$

Corollary 1: When $s \geq 2$ sources of evidences provide binary bba's on G^Θ with at least two focal elements, and all focal elements are disjoint two by two, then PCR6 fusion rule coincides with the Averaging Rule.

This Corollary is true because if all focal elements are disjoint two by two then the total conflict is equal to 1.

Examples 2: where PCR6 rule equals the Averaging Rule.

Let's consider the frame $\Theta = \{A, B\}$ with Shafer's model and the bba's to combine as given in Table III.

Table III. LIST OF BBA'S TO COMBINE FOR EXAMPLE 2.

bba's \ Focal elem.	A	B	$A \cup B$	$A \cap B = \emptyset$
$m_1(.)$	1	0	0	
$m_2(.)$	0	1	0	
$m_3(.)$	0	0	1	
$m_{1,2,3}(.)$	0	0	0	1

Since we have binary masses, and their total conflict is 1, we expect getting the same result for PCR6 and the Averaging Rule according to our Theorem 1. The PCR principle gives us

$$\frac{x_A}{1} = \frac{y_B}{1} = \frac{z_{A\cup B}}{1} = \frac{m_{1,2,3}(\emptyset)}{1+1+1} = \frac{1}{3}$$

Hence $x_A = y_B = z_{A\cup B} = \frac{1}{3}$, so that

$$m_{1,2,3}^{PCR6}(A) = m_{1,2,3}(A) + x_A = 0 + \frac{1}{3} = \frac{1}{3}$$
$$m_{1,2,3}^{PCR6}(B) = m_{1,2,3}(B) + y_B = 0 + \frac{1}{3} = \frac{1}{3}$$
$$m_{1,2,3}^{PCR6}(A \cup B) = m_{1,2,3}(A \cup B) + z_{A\cup B} = 0 + \frac{1}{3} = \frac{1}{3}$$

Interestingly, PCR5 gives the same result as PCR6 in this case since one makes the same proportionalizations as for PCR6. Using the Averaging Rule (2), we get

$$m_{1,2,3}^{Average}(A) = \frac{1}{3} \cdot (1+0+0) = \frac{1}{3}$$
$$m_{1,2,3}^{Average}(B) = \frac{1}{3} \cdot (0+1+0) = \frac{1}{3}$$
$$m_{1,2,3}^{Average}(A \cup B) = \frac{1}{3} \cdot (0+0+1) = \frac{1}{3}$$

So we see that PCR6 rule equals the Averaging Rule as proved in the theorem because the bba's are binary and the intersection of all focal elements is empty since $A \cap B \cap (A \cup B) = \emptyset \cap (A \cup B) = \emptyset$ because $A \cap B = \emptyset$ since Shafer's model has been assumed for the frame Θ.

Examples 3: where PCR6 differs from the Averaging Rule.

Let's consider the frame $\Theta = \{A, B, C\}$ with Shafer's model and the bba's to combine as given in Table IV.

Table IV. LIST OF BBA'S TO COMBINE FOR EXAMPLE 3.

bba's \ Focal elem.	A	$A \cup B$	$A \cup B \cup C$	$\emptyset$
$m_1(.)$	1	0	0	
$m_2(.)$	0	1	0	
$m_3(.)$	0	0	1	
$m_{1,2,3}(.)$	1	0	0	

Clearly, in this case the focal elements are nested and the condition on emptiness of intersection of all focal elements is not satisfied because one has $A \cap (A \cup B) \cap (A \cup B \cup C) = A \neq \emptyset$, so that the theorem cannot be applied in such case. The total conflicting mass is not 1. One can verify in such example that PCR6 rule differs from the Averaging Rule because one gets

$$m_{1,2,3}^{PCR6}(A) = m_{1,2,3}(A) = 1$$
$$m_{1,2,3}^{PCR6}(A \cup B) = m_{1,2,3}(A \cup B) = 0$$
$$m_{1,2,3}^{PCR6}(A \cup B \cup C) = m_{1,2,3}(A \cup B \cup C) = 0$$

since there is no conflicting mass to redistribute to apply PCR principle, whereas the averaging fusion rule gives

$$m_{1,2,3}^{Average}(A) = \frac{1}{3} \cdot (1+0+0) = \frac{1}{3}$$
$$m_{1,2,3}^{Average}(A \cup B) = \frac{1}{3} \cdot (0+1+0) = \frac{1}{3}$$
$$m_{1,2,3}^{Average}(A \cup B \cup C) = \frac{1}{3} \cdot (0+0+1) = \frac{1}{3}$$

Examples 4 (Bayesian non-binary bba's): where PCR6 differs from the Averaging Rule.

Let's consider the frame $\Theta = \{A, B\}$ with Shafer's model and the Bayesian bba's to combine as given in Table V.

Table V. LIST OF BBA'S TO COMBINE FOR EXAMPLE 4.

bba's \ Focal elem.	A	B	$A \cap B = \emptyset$
$m_1(.)$	0.2	0.8	0
$m_2(.)$	0.6	0.4	0
$m_3(.)$	0.7	0.3	0
$m_{1,2,3}(.)$	0.084	0.096	0.820

The total conflicting mass $m_{1,2,3}(A\cap B = \emptyset) = 0.82 = 1 - m_1(A)m_2(A)m_3(A) - m_1(B)m_2(B)m_3(B)$ equals the sum of partial conflicting masses that will be redistributed through PCR principle in PCR6

$$\begin{aligned} m_{1,2,3}(A \cap B = \emptyset) &= \underbrace{m_1(A)m_2(B)m_3(B)}_{0.024} \\ &+ \underbrace{m_2(A)m_1(B)m_3(B)}_{0.144} + \underbrace{m_3(A)m_1(B)m_2(B)}_{0.224} \\ &+ \underbrace{m_1(B)m_2(A)m_3(A)}_{0.336} + \underbrace{m_2(B)m_1(A)m_3(A)}_{0.056} \\ &+ \underbrace{m_3(B)m_1(A)m_2(A)}_{0.036} = 0.82 \end{aligned}$$

Applying PCR principle for each of these six partial conflicts, one gets:

- for $m_1(A)m_2(B)m_3(B) = 0.2 \cdot 0.4 \cdot 0.3 = 0.024$
$$\frac{x_1(A)}{0.2} = \frac{y_1(B)}{0.4+0.3} = \frac{0.024}{0.2+0.3+0.4}$$
whence $x_1(A) \approx 0.005333$ and $y_1(B) \approx 0.018667$.
- for $m_2(A)m_1(B)m_3(B) = 0.6 \cdot 0.8 \cdot 0.3 = 0.144$
$$\frac{x_2(A)}{0.6} = \frac{y_2(B)}{0.8+0.3} = \frac{0.144}{0.6+0.8+0.3}$$
whence $x_2(A) \approx 0.050824$ and $y_2(B) \approx 0.093176$.
- for $m_3(A)m_1(B)m_2(B) = 0.7 \cdot 0.8 \cdot 0.4 = 0.224$
$$\frac{x_3(A)}{0.7} = \frac{y_3(B)}{0.8+0.4} = \frac{0.224}{0.7+0.8+0.4}$$
whence $x_3(A) \approx 0.082526$ and $y_3(B) \approx 0.141474$.
- for $m_1(B)m_2(A)m_3(A) = 0.8 \cdot 0.6 \cdot 0.7 = 0.336$
$$\frac{x_4(A)}{0.6+0.7} = \frac{y_4(B)}{0.8} = \frac{0.336}{0.8+0.6+0.7}$$
whence $x_4(A) \approx 0.208000$ and $y_4(B) \approx 0.128000$.

- for $m_2(B)m_1(A)m_3(A) = 0.4 \cdot 0.2 \cdot 0.7 = 0.056$

$$\frac{x_5(A)}{0.2+0.7} = \frac{y_5(B)}{0.4} = \frac{0.056}{0.4+0.2+0.7}$$

whence $x_5(A) \approx 0.038769$ and $y_5(B) \approx 0.017231$.

- for $m_3(B)m_1(A)m_2(A) = 0.3 \cdot 0.2 \cdot 0.6 = 0.036$

$$\frac{x_6(A)}{0.2+0.6} = \frac{y_6(B)}{0.3} = \frac{0.036}{0.3+0.2+0.6}$$

whence $x_6(A) \approx 0.026182$ and $y_6(B) \approx 0.009818$.

Therefore, with PCR6 one finally gets

$$m_{1,2,3}^{PCR6}(A) = \sum_{i=1}^{6} x_i(A) = 0.495634$$
$$m_{1,2,3}^{PCR6}(B) = \sum_{i=1}^{6} y_i(A) = 0.504366$$

whereas the Averaging Rule (2) will give us

$$m_{1,2,3}^{Average}(A) = \frac{1}{3}\cdot(0.2+0.6+0.7) = \frac{1.5}{3} = 0.5$$
$$m_{1,2,3}^{Average}(B) = \frac{1}{3}\cdot(0.8+0.4+0.3) = \frac{1.5}{3} = 0.5$$

In this example, the intersection of focal elements is empty but the bba's to combine are not binary. Therefore the total conflict between sources is not total and the theorem doesn't apply and so PCR6 results differ from the Averaging Rule.

It however can happen that in some very particular symmetric cases PCR6 coincides with the Averaging Rule. For example, if we consider the bba's as given in the Table VI. In such case the opinion of source #1 totally balances opinion of source #3, and the opinion of source #2 cannot support A more than B (and reciprocally), so that the fusion problem is totally symmetrical. In this example, it is expected that the final fusion result should commit an equal mass of belief to A and to B. And indeed, it can be easily verified that one gets in such case

$$m_{1,2,3}^{PCR6}(A) = m_{1,2,3}^{Average}(A) = 0.5$$
$$m_{1,2,3}^{PCR6}(B) = m_{1,2,3}^{Average}(B) = 0.5$$

which makes perfectly sense. Note that the Averaging Rule provides same result on example 4 which is somehow questionable because example 4 doesn't present an inherent symmetrical structure. In our opinion PCR6 presents the advantage to respond more adequately to the change of inherent internal structure (asymmetry) of bba's to combine, which is not well captured by the simple averaging fusion rule.

Table VI. A BAYESIAN NON-BINARY SYMMETRIC EXAMPLE.

bba's \ Focal elem.	A	B	$A \cap B = \emptyset$
$m_1(.)$	0.2	0.8	0
$m_2(.)$	0.5	0.5	0
$m_3(.)$	0.8	0.2	0
$m_{1,2,3}(.)$	0.08	0.08	0.84

IV. APPLICATION TO PROBABILITY ESTIMATION

Let's review a simple coin tossing random experiment. When we flip a coin [13], there are two possible outcomes. The coin could land showing a head (H) or a tail (T). The list of all possible outcomes is called the sample space and correspond to the frame $\Theta = \{H, T\}$. There exist many interpretations of probability [14] that are out of the scope of this paper. We focus here on the estimation of the probability measure $P(H)$ of a given coin (biased or not) based on n outcomes of a coin tossing experiment. The long-run frequentist interpretation of probability [15] considers that the probability of an event A is its relative frequency of occurrence over time after repeating the experiment a large number of times under similar circumstances, that is

$$P(A) = \lim_{n\to\infty} \frac{n(A)}{n} \tag{12}$$

where $n(A)$ denotes the number of occurrences of an event A in $n > 0$ trials. In practice however, we usually estimate the probability of an event A based only on a limited number of data (observations) that are available, and so we estimate the idealistic $P(A)$ defined in (12), by classical Laplace's probability definition

$$\hat{P}(A|n(A), n) = \frac{n(A)}{n} \tag{13}$$

Naturally, $\hat{P}(A) \geq 0$ because $n(A) \geq 0$ and $n > 0$, and $\hat{P}(A) \leq 1$ because we cannot get $n(A) > n$ in a series of n trials. $P(A) + P(\bar{A}) = 1$ because $\frac{n(A)}{n} + \frac{n(\bar{A})}{n} = \frac{n(A)}{n} + \frac{n-n(A)}{n} = 1$ where $\bar{A}$ is the complement of A in the sample space.

It is interesting to note that the classical estimation of the probability measure given by (13) corresponds in fact to the simple averaging fusion rule of distinct pieces of evidence represented by binary masses. For example, let's take a coin and flip it $n = 8$ times and assume for instance that we observe the following series of outcomes $\{o_1 = H, o_2 = H, o_3 = T, o_4 = H, o_5 = T, o_6 = H, o_7 = H, o_8 = T\}$, so that $n(H) = 5$ and $n(T) = 3$. Then these observations can be associated with distinct sources of evidences providing to the following basic (binary) belief assignments:

Table VII. OUTCOMES OF A COIN TOSSING EXPERIMENT.

bba's \ Focal elem.	H	T
$m_1(.)$	1	0
$m_2(.)$	1	0
$m_3(.)$	0	1
$m_4(.)$	1	0
$m_5(.)$	0	1
$m_6(.)$	1	0
$m_7(.)$	1	0
$m_8(.)$	0	1

It is clear that the probability estimate in (13) equals the averaging fusion rule (2) and in such example because

$$\begin{aligned}\hat{P}(H|\{o_1, o_2, \ldots, o_8\}) &= \frac{n(H)}{n} = \frac{5}{8} \qquad \text{by eq. (13)}\\ &= \frac{1}{8}(1+1+0+1+0+1+1+0)\\ &= m_{1,2,\ldots,8}^{Average}(H) \qquad \text{by eq. (2)}\end{aligned}$$

$$
\begin{aligned}
\hat{P}(T|\{o_1,o_2,\ldots,o_8\}) &= \frac{n(T)}{n} = \frac{3}{8} \qquad \text{by eq. (13)}\\
&= \frac{1}{8}(0+0+1+0+1+0+0+1)\\
&= m^{Average}_{1,2,\ldots,8}(T) \qquad \text{by eq. (2)}
\end{aligned}
$$

Because all the bba's to combine here are binary and are in total conflict, our theorem 1 of Section III applies, and PCR6 fusion rule in this case coincides with the averaging fusion rule. Therefore we can use PCR6 to estimate the probabilities that the coin will land on H or T at the next toss given the series of observations. More precisely,

$$
\begin{cases}
m^{PCR6}_{1,2,\ldots,8}(H) = m^{Average}_{1,2,\ldots,8}(H) = \hat{P}(H|\{o_1,o_2,\ldots,o_8\})\\
m^{PCR6}_{1,2,\ldots,8}(T) = m^{Average}_{1,2,\ldots,8}(T) = \hat{P}(T|\{o_1,o_2,\ldots,o_8\})
\end{cases}
$$

We must insist on the fact that Dempster-Shafer (DS) rule of combination (4) cannot be used at all in such very simple case to estimate correctly the probability measure because DS rule doesn't work (because of division by zero) in total conflicting situations. PCR5 rule can be applied to combine these 8 bba's but is unable to provide a consistent result with the classical probability estimates because one will get

$$
\frac{x_H}{1\cdot 1\cdot 1\cdot 1\cdot 1} = \frac{y_T}{1\cdot 1\cdot 1} = \frac{m_{1,2,\ldots,8}(\emptyset)}{(1\cdot 1\cdot 1\cdot 1\cdot 1)+(1\cdot 1\cdot 1)} = \frac{1}{1+1} = 0.5
$$

and therefore the PCR5 fusion result is

$$
\begin{cases}
m^{PCR5}_{1,2,\ldots,8}(H) = x_H = 0.5 \neq (m^{PCR6}_{1,2,\ldots,8}(H) = 5/8)\\
m^{PCR5}_{1,2,\ldots,8}(T) = y_T = 0.5 \neq (m^{PCR6}_{1,2,\ldots,8}(T) = 3/8)
\end{cases}
$$

Remark: The PCR6 fusion result is valid if and only if PCR6 rule is applied globally, and not sequentially. If PCR6 is sequentially applied, it becomes equivalent with PCR5 sequentially applied and it will generate incorrect results for combining $s > 2$ sources of evidence. Because of the ability of PCR6 to estimate frequentist probabilities in a random experiment, we strongly recommend PCR6 rather than PCR5 as soon as $s \geq 2$ bba's have to be combined altogether.

V. Conclusions and Challenge

In this paper, we have proved that PCR6 fusion rule coincides with the Averaging Rule when the bba's to combine are binary and in total conflict. Because of such nice property, PCR6 is able to provide a frequentist probability measure of any event occurring in a random experiment, contrariwise to other fusion rules like DS rule, PCR5 rule, etc. Except the Averaging Rule of course since it is the basis of the frequentist probability interpretation. In a more general context with non-binary bba's, PCR6 is quite complicate to apply to combine globally $s > 2$ sources of evidences, and a general recursive formula of PCR6 would be very convenient. It can be mathematically reformulated as follows: Let R be a fusion rule and assume we have s sources that provide m_1, m_2, $\ldots$, m_{s-1}, m_s respectively on a fusion space G^{Θ}. Find a function (or an operator) T such that: $T(R(m_1,m_2,\ldots m_{s-1}),m_s) = R(m_1,m_2,\ldots,m_{s-1},m_s)$, or by simplifying the notations $T(R_{s-1},m_s) = R_s$, where R_i means the fusion rule R applied to i masses all together. For example, if R equals the Averaging Rule, the function T is defined according to the relation (3) by $T(R_{s-1},m_s) = \frac{s-1}{s}R_{s-1} + \frac{1}{s}m_s = R_s$, and if R equals DS rule one has $T(R_{s-1},m_s) = DS(R_{s-1},m_s)$ because of the associativity of DS rule. What is the T operator associated with PCR6? Such very important open challenging question is left for future research works.

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