Translational and Rotational Properties of Tensor Fields in Relativistic Quantum Mechanics

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Abstract. Recently, several discussions on the possible observability of 4-vector fields have been published in literature. Furthermore, several authors recently claimed existence of the helicity=0 fundamental field. We re-examine the theory of antisymmetric tensor fields and 4-vector potentials. We study the massless limits. In fact, a theoretical motivation for this venture is the old papers of Ogievetskiĭ and Polubarinov, Hayashi, and Kalb and Ramond. They proposed the concept of the *notoph*, whose helicity properties are complementary to those of the *photon*. We analyze the quantum field theory with taking into account mass dimensions of the notoph and the photon. We also proceed to derive equations for the symmetric tensor of the second rank on the basis of the Bargmann-Wigner formalism They are consistent with the general relativity. Particular attention has been paid to the correct definitions of the energy-momentum tensor and other Nöther currents. We estimate possible interactions, fermion-notoph, graviton-notoph, photon-notoph. PACS number: 03.65.Pm , 04.50.-h , 11.30.Cp

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1. Introduction

In this presentation we re-examine the theory of the 4-vector field, the antisymmetric tensor fields of the second ranks and the spin-2 fields coming from the modified Bargmann-Wigner formalism. In the series of the papers [1, 2, 3, 4] we tried to find connection between the theory of the quantized antisymmetric tensor (AST) field of the second rank (and that of the corresponding 4-vector field) with the $2(2s+1)$ Weinberg-Tucker-Hammer formalism [5, 6]. Several previously published works [7, 8, 9] introduced the concept of the notoph (the Kalb-Ramond field) which is constructed on the basis of the antisymmetric tensor "potentials", cf. [10, 11, 12, 13]. It represents itself the non-trivial spin-0 field. We posed the problems, whether the massless *quantized* AST potential and the *quantized* 4-vector field are transverse or longitudinal fields (in the sense if the helicity $h=\pm 1$ or $h=0$)? can the electromagnetic potential be a 4-vector in a quantized theory? contradictions with the Weinberg theorem "that no symmetric tensor field of rank s can be constructed from the creation and annihilation operators of massless particles of spin s"? how should the massless limit be taken?

First of all, we note that 1) "...In natural units ($c=\hbar=1$) ... a lagrangian density, since the action is dimensionless, has dimension of [energy]^4"; 2) One can always renormalize the lagrangian density and "one can obtain the same equations of motion... by substituting $L \rightarrow (1/M^N)L$, where M is an arbitrary energy scale", cf. [3]; 3) the right physical dimension of the field strength tensor $F^{\mu\nu}$ is [energy]^2; "the transformation $F^{\mu\nu} \rightarrow (1/2m)F^{\mu\nu}$ [which was regarded in Ref. [14, 15]] ... requires a more detailed study ... [because] the transformation above changes its physical dimension: it is not a simple normalization transformation". Furthermore, in the first papers on the notoph the authors used the normalization of the 4-vector F^μ field, which is related to a third-rank antisymmetric field tensor, to [energy]^2 and, hence, the antisymmetric tensor "potentials" $A^{\mu\nu}$, to [energy]^1. We discuss these problems on the basis of the generalized Bargmann-Wigner formalism [16]. The Proca and Maxwell formalisms are generalized, Ref. [4]. In the next Section we consider the spin-2 equations. A field of the rest mass m and the spin $s \geq \frac{1}{2}$ is represented by a completely symmetric multispinor of rank $2s$. The particular cases $s=1$ and $s=\frac{3}{2}$ have been considered in the textbooks, e. g., Ref. [17]. Nevertheless, questions of the redundant components of the higher-spin relativistic equations are not yet understood in detail [18]. In the last Sections we discuss the questions of quantization, interactions and relations between various higher-spin theories.

2. 4-potentials and Antisymmetric Tensor Fields. Normalization.

The spin-0 and spin-1 particles can be constructed by taking the direct product of 4-spinors [16, 17]. The set of basic equations for $s=0$ and $s=1$ are written:

$$\left[i\gamma^\mu\partial_\mu - m\right]_{\alpha\beta}\Psi_{\beta\gamma}(x) = 0 \quad , \tag{2.1}$$

$$\left[i\gamma^\mu\partial_\mu - m\right]_{\gamma\beta}\Psi_{\alpha\beta}(x) = 0 \quad . \tag{2.2}$$

We expand the 4×4 matrix field function into the antisymmetric and symmetric parts in the standard way

$$\Psi_{[\alpha\beta]} = R_{\alpha\beta}\phi + \gamma^5_{\alpha\delta}R_{\delta\beta}\widetilde{\phi} + \gamma^5_{\alpha\delta}\gamma^\mu_{\delta\tau}R_{\tau\beta}\widetilde{A}_\mu \quad , \tag{2.3}$$

$$\Psi_{\{\alpha\beta\}} = \gamma^\mu_{\alpha\delta}R_{\delta\beta}A_\mu + \sigma^{\mu\nu}_{\alpha\delta}R_{\delta\beta}F_{\mu\nu} \quad , \tag{2.4}$$

where $R = CP$. The explicit form of this matrix can be chosen: $R = \begin{pmatrix} i\Theta & 0 \\ 0 & -i\Theta \end{pmatrix}$, $\Theta = -i\sigma_2$, provided that γ^μ matrices are in the Weyl representation. The equations (2.1,2.2) lead to the Kemmer set of the $s=0$ equations:

$$m\phi = 0 \quad , \tag{2.5}$$

$$m\widetilde{\phi} = -i\partial_\mu \widetilde{A}^\mu \quad , \tag{2.6}$$

$$m\widetilde{A}^\mu = -i\partial^\mu \widetilde{\phi} \quad , \tag{2.7}$$

and to the Proca-Duffin-Kemmer set of the equations in the $s=1$ case: [1]

$$\partial_\alpha F^{\alpha\mu} + \frac{m}{2} A^\mu = 0 \quad , \tag{2.10}$$

$$2mF^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad , \tag{2.11}$$

In the meantime, the textbooks equations are obtained from (2.10,2.11) after the normalization change $A_\mu \to 2mA_\mu$ or $F_{\mu\nu} \to \frac{1}{2m}F_{\mu\nu}$. Of course, one can investigate other sets of equations with different normalization of the $F_{\mu\nu}$ and A_μ fields. Are all these systems of equations equivalent? As we shall see, to answer this question is not trivial. We want to relate it to the question of the good behaviour in the massless limit, which must be taken in the end of all calculations only, *i. e.*, for physical quantities.

In order to be able to answer the question about the behaviour of eigenvalues of the spin operator $\mathbf{J}^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$ in the massless limit one should know the behaviour of the fields $F_{\mu\nu}$ and/or A_μ in the massless limit. We choose the usual definitions (p. 209 of [19]) for polarization vectors $\epsilon^\mu(\mathbf{0},\sigma)$:

$$\epsilon^\mu(\mathbf{0},+1) = -\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}, \epsilon^\mu(\mathbf{0},0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \epsilon^\mu(\mathbf{0},-1) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}. \tag{2.12}$$

Then, ($\widehat{p}_i = p_i/\mid \mathbf{p}\mid,\ \gamma = E_p/m$), p. 68 of ref. [19],

$$\epsilon^\mu(\mathbf{p},\sigma) = L^\mu{}_\nu(\mathbf{p})\epsilon^\nu(\mathbf{0},\sigma)\,, \tag{2.13}$$

$$L^0{}_0(\mathbf{p}) = \gamma\,, \quad L^i{}_0(\mathbf{p}) = L^0{}_i(\mathbf{p}) = \widehat{p}_i\sqrt{\gamma^2-1}\,, \tag{2.14}$$

$$L^i{}_k(\mathbf{p}) = \delta_{ik} + (\gamma-1)\widehat{p}_i\widehat{p}_k \tag{2.15}$$

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[1] We could use another symmetric matrix $\gamma^5\sigma^{\mu\nu}R$ in the expansion of the symmetric spinor of the second rank [15]. In this case the equations will read

$$i\partial_\alpha \widetilde{F}^{\alpha\mu} + \frac{m}{2}B^\mu = 0 \quad , \tag{2.8}$$

$$2im\widetilde{F}^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu \quad . \tag{2.9}$$

in which the dual tensor $\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ presents, because we used that in the Weyl representation $\gamma^5\sigma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$; B^μ is the corresponding vector potential.

for the 4-vector potential field,

$$A^\mu(x^\mu) = \sum_{\sigma=0,\pm1} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} \left[\epsilon^\mu(\mathbf{p},\sigma) a(\mathbf{p},\sigma) e^{-ip\cdot x} + (\epsilon^\mu(\mathbf{p},\sigma))^c b^\dagger(\mathbf{p},\sigma) e^{+ip\cdot x}\right] . \tag{2.16}$$

The normalization of the field functions in the momentum representation is thus chosen to the negative unit, $\epsilon^*_\mu(\mathbf{p},\sigma)\epsilon^\mu(\mathbf{p},\sigma) = -1$. We observe that in the massless limit all the defined polarization vectors of the momentum space do not have good behaviour; the functions describing spin-1 particles tend to infinity.[2] Nevertheless, after renormalizing the polarization vectors, *e. g.*, $\epsilon^\mu \to u^\mu \equiv m\epsilon^\mu$ we come to the field functions in the momentum representation:

$$u^\mu(\mathbf{p},+1) = -\frac{N}{\sqrt{2}m} \begin{pmatrix} p_r \\ m + \frac{p_1 p_r}{E_p+m} \\ im + \frac{p_2 p_r}{E_p+m} \\ \frac{p_3 p_r}{E_p+m} \end{pmatrix} , \quad u^\mu(\mathbf{p},-1) = \frac{N}{\sqrt{2}m} \begin{pmatrix} p_l \\ m + \frac{p_1 p_l}{E_p+m} \\ -im + \frac{p_2 p_l}{E_p+m} \\ \frac{p_3 p_l}{E_p+m} \end{pmatrix} \tag{2.17}$$

$$u^\mu(\mathbf{p},0) = \frac{N}{m} \begin{pmatrix} p_3 \\ \frac{p_1 p_3}{E_p+m} \\ \frac{p_2 p_3}{E_p+m} \\ m + \frac{p_3^2}{E_p+m} \end{pmatrix} , \tag{2.18}$$

($N = m$ and $p_{r,l} = p_1 \pm ip_2$) which do not diverge in the massless limit. Two of the massless functions (with $\sigma = \pm 1$) are equal to zero when a particle, described by this field, is moving along the third axis ($p_1 = p_2 = 0,\ p_3 \neq 0$). The third one ($\sigma = 0$) is

$$u^\mu(p_3,0) \mid_{m\to 0} = \begin{pmatrix} p_3 \\ 0 \\ 0 \\ \frac{p_3^2}{E_p} \end{pmatrix} \equiv \begin{pmatrix} E_p \\ 0 \\ 0 \\ E_p \end{pmatrix} , \tag{2.19}$$

and at the rest ($E_p = p_3 \to 0$) also vanishes. Thus, such a field operator describes the "longitudinal photons" what is in the complete accordance with the Weinberg theorem $B - A = h$ for massless particles (we use the $D(1/2,1/2)$ representation). The change of the normalization can lead to the change of physical content described by the classical field. In the quantum case one should somehow fix the form of commutation relations by some physical principles. They may be fixed by requirements of the dimensionless of the action in the natural unit system (apart from the requirements of the

[2] It is interesting to remind that some authors tries to inforce the Stueckelberg's Lagrangian in order to overcome the difficulties related to the $m \to 0$ limit (or the Proca theory $\to$ Quantum Electrodynamics). The Stueckelberg's Lagrangian is well known to contain an additional term which may be put in correspondence to some scalar (longitudinal) field (cf. also [20]).

translational and rotational invariancies; the accustomed behaviour of the Feynman-Dyson propagator, etc.).

Furthermore, it is easy to find the properties of the physical fields $F^{\mu\nu}$ (defined as in (2.10,2.11), for instance) in the massless zero-momentum limit. It is straightforward to find $\mathbf{B}^{(+)}(\mathbf{p},\sigma) = \frac{i}{2m}\mathbf{p}\times\mathbf{u}(\mathbf{p},\sigma)$, $\mathbf{E}^{(+)}(\mathbf{p},\sigma) = \frac{i}{2m}p_0\mathbf{u}(\mathbf{p},\sigma) - \frac{i}{2m}\mathbf{p}u^0(\mathbf{p},\sigma)$ and the corresponding negative-energy strengths for the field operator (in general, the complex-valued one).

For the sake of completeness let us present the vector corresponding to the "time-like" polarization:

$$u^\mu(\mathbf{p},0_t) = \frac{N}{m}\begin{pmatrix}E_p\\ p_1\\ p_2\\ p_3\end{pmatrix} \quad , \quad \mathbf{B}^{(\pm)}(\mathbf{p},0_t) = \mathbf{0} \quad , \quad \mathbf{E}^{(\pm)}(\mathbf{p},0_t) = \mathbf{0}\ . \tag{2.20}$$

The polarization vector $u^\mu(\mathbf{p},0_t)$ has good behaviour in $m\rightarrow 0$, $N = m$ (and also in the subsequent limit $\mathbf{p}\rightarrow\mathbf{0}$) and it may correspond to some field (particle). As one can see the field operator composed of the states of longitudinal polazrizations (e.g., as the "positive-energy" solution) and time-like (e.g., as the "negative-energy" solution) may describe a situation when a particle and an antiparticle have *opposite* intrinsic parities.

3. Lagrangian, Energy-Momentum Tensor and Angular Momentum. Photon-Notoph Equations.

We begin with the Lagrangian, including, in general, mass term:[3]

$$L = \frac{1}{4}(\partial_\mu A_{\nu\alpha})(\partial^\mu A^{\nu\alpha}) - \frac{1}{2}(\partial_\mu A^{\mu\alpha})(\partial^\nu A_{\nu\alpha}) - \frac{1}{2}(\partial_\mu A_{\nu\alpha})(\partial^\nu A^{\mu\alpha}) + \frac{1}{4}m^2 A_{\mu\nu}A^{\mu\nu}\ . \tag{3.2}$$

The Lagrangian leads to the equation of motion in the following form:

$$\frac{1}{2}(\square + m^2)A_{\mu\nu} + (\partial_\mu A_{\alpha\nu}{}^{,\alpha} - \partial_\nu A_{\alpha\mu}{}^{,\alpha}) = 0 \quad , \tag{3.3}$$

It is this equation for antisymmetric-tensor-field components that follows from the Proca-Duffin-Kemmer consideration provided that $m\neq 0$ and in the final expression one takes into account the Klein-Gordon equation.

[3]Here we use the notation $A_{\mu\nu}$ for the AST due to possible different "mass dimensions" of the fields. The massless ($m=0$) Lagrangian is connected with the Lagrangians used in other theories by adding the total derivative:

$$L_{CFT} = L + \frac{1}{2}\partial_\mu\left(A_{\nu\alpha}\partial^\nu A^{\mu\alpha} - A^{\mu\alpha}\partial^\nu A_{\nu\alpha}\right)\ . \tag{3.1}$$

The Kalb-Ramond gauge-invariant form (with respect to the "gauge" transformations $A_{\mu\nu}\rightarrow A_{\mu\nu} + \partial_\nu\Lambda_\mu - \partial_\mu\Lambda_\nu$), Ref. [7, 8, 9], is obtained only if one uses the Fermi procedure *mutatis mutandis* by removing the additional "phase" field $\lambda(\partial_\mu A^{\mu\nu})^2$ from the Lagrangian. This has certain analogy with the QED, where the question, whether the Lagrangian is gauge-invariant or not, is solved depending on the presence of the term $\lambda(\partial_\mu A^\mu)^2$.

Following the variation procedure one can obtain the energy-momentum tensor:

$$\Theta^{\lambda\beta} = \frac{1}{2}\left[(\partial^\lambda A_{\mu\alpha})(\partial^\beta A^{\mu\alpha}) - 2(\partial_\mu A^{\mu\alpha})(\partial^\beta A^{\lambda}{}_{\alpha}) - 2(\partial^\mu A^{\lambda\alpha})(\partial^\beta A_{\mu\alpha})\right] - Lg^{\lambda\beta}\,. \tag{3.4}$$

One can also obtain that for rotations $x^{\mu'} = x^\mu + \omega^{\mu\nu}x_\nu$ the corresponding variation of the wave function is found from the formula: $\delta A^{\alpha\beta} = \frac{1}{2}\omega^{\kappa\tau}T_{\kappa\tau}^{\alpha\beta,\mu\nu}A_{\mu\nu}$. The generators of infinitesimal transformations are defined as

$$\begin{aligned} T_{\kappa\tau}^{\alpha\beta,\mu\nu} &= \frac{1}{2}g^{\alpha\mu}(\delta_\kappa^\beta\,\delta_\tau^\nu - \delta_\tau^\beta\,\delta_\kappa^\nu) + \frac{1}{2}g^{\beta\mu}(\delta_\kappa^\nu\delta_\tau^\alpha - \delta_\tau^\nu\,\delta_\kappa^\alpha) + \\ &+ \frac{1}{2}g^{\alpha\nu}(\delta_\kappa^\mu\,\delta_\tau^\beta - \delta_\tau^\mu\,\delta_\kappa^\beta) + \frac{1}{2}g^{\beta\nu}(\delta_\kappa^\alpha\,\delta_\tau^\mu - \delta_\tau^\alpha\,\delta_\kappa^\mu)\,. \end{aligned} \tag{3.5}$$

It is $T_{\kappa\tau}^{\alpha\beta,\mu\nu}$, the generators of infinitesimal transformations, that enter in the formula for the relativistic spin tensor:

$$J_{\kappa\tau} = \int d^3\mathbf{x}\left[\frac{\partial L}{\partial(\partial A^{\alpha\beta}/\partial t)}\,T_{\kappa\tau}^{\alpha\beta,\mu\nu}A_{\mu\nu}\right]\quad . \tag{3.6}$$

As a result one obtains:

$$\begin{aligned} J_{\kappa\tau} &= \int d^3\mathbf{x}\,[(\partial_\mu A^{\mu\nu})(g_{0\kappa}A_{\nu\tau} - g_{0\tau}A_{\nu\kappa}) - (\partial_\mu A^\mu{}_\kappa)A_{0\tau} + (\partial_\mu A^\mu{}_\tau)A_{0\kappa} + \\ &+ A^\mu{}_\kappa(\partial_0 A_{\tau\mu} + \partial_\mu A_{0\tau} + \partial_\tau A_{\mu 0}) - A^\mu{}_\tau(\partial_0 A_{\kappa\mu} + \partial_\mu A_{0\kappa} + \partial_\kappa A_{\mu 0})]\,. \end{aligned} \tag{3.7}$$

Furthermore, one should choose space-like normalized vector $n^\mu n_\mu = -1$, for example $n_0 = 0$, $\mathbf{n} = \widehat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$. One can find the explicit form of the relativistic spin after lengthy calculations:

$$(W_\mu \cdot n^\mu) = -(\mathbf{W}\cdot\mathbf{n}) = -\frac{1}{2}\epsilon^{ijk}n^k J^{ij}p^0\,, \tag{3.8}$$

$$\mathbf{J}^k = \epsilon^{ijk}\int d^3\mathbf{x}\left[A^{0i}(\partial_\mu A^{\mu j}) + A_\mu{}^j(\partial^0 A^{\mu i} + \partial^\mu A^{i0} + \partial^i A^{0\mu})\right]\,. \tag{3.9}$$

Now it becomes obvious that the application of the generalized Lorentz conditions leads in such a formulation to the fact that the resulting Kalb-Ramond field is longitudinal (helicity $h = 0$). All the components of the angular momentum tensor for this case are identically equated to zero.

According to [7, Eqs.(9,10)] we proceed in the construction of the "potentials" for the notoph (by taking, in fact, the 4-cross product of polarization vectors):

$$\tilde{F}_{\mu\nu}(\mathbf{p}) \sim A_{\mu\nu}(\mathbf{p}) = N\left[\epsilon_\mu^{(1)}(\mathbf{p})\epsilon_\nu^{(2)}(\mathbf{p}) - \epsilon_\nu^{(1)}(\mathbf{p})\epsilon_\mu^{(2)}(\mathbf{p})\right] \tag{3.10}$$

On using explicit forms for the polarization vectors in the momentum space one obtains

$$A^{\mu\nu}(\mathbf{p}) = \frac{iN^2}{m}\begin{pmatrix} 0 & -p_2 & p_1 & 0 \\ p_2 & 0 & m + \frac{p_r p_l}{p_0+m} & \frac{p_2 p_3}{p_0+m} \\ -p_1 & -m - \frac{p_r p_l}{p_0+m} & 0 & -\frac{p_1 p_3}{p_0+m} \\ 0 & -\frac{p_2 p_3}{p_0+m} & \frac{p_1 p_3}{p_0+m} & 0 \end{pmatrix}\,, \tag{3.11}$$

i.e., it coincides with the longitudinal components of the antisymmetric tensor obtained in Refs. [1, Eqs.(2.14,2.17)] and [14, Eqs.(17b,18b)] within the normalization and different choice of the spin basis. The longitudinal states reduce to zero in the massless case under appropriate normalization. and only if the $s=1$ particle moves along with the third axis.

Finally, we agree with the previous authors, e. g., ref. [21], see Eq. (4) therein, about the gauge *non*-invariance of the division of the angular momentum of the electromagnetic field into the "orbital" and "spin" part, Eq. (3.9). We proved again that for the antisymmetric tensor field $\mathbf{J} \sim \int d^3\mathbf{x}\,(\mathbf{E}\times\mathbf{A})$. So, what people actually do (when speaking on the Ogievetskiĭ-Polubarinov-Kalb-Ramond field) is: When $N=m$ they considered the gauge part of the 4-vector field functions. Then, they gauged $\mathbf{A}$ of the transverse modes on choosing $p_r = p_l = 0$ in the massless limit (see formulas (2.17)). The reader, of course, can consider this procedure as the usual gauge transformation, $A^\mu \to A^\mu + \partial^\mu \chi$. Under this choice the $\mathbf{E}(\mathbf{p},0)$ and $\mathbf{B}(\mathbf{p},0)$ are equal to zero in massless limit. But, the gauge part of $u^\mu(\mathbf{p},0)$ is not. The spin angular momentum can still be zero.

From the other hand, for the spin 1 one can start from

$$[\gamma_{\alpha\beta}p^\alpha p^\beta - Ap^\alpha p_\alpha + Bm^2]\Psi = 0\,, \tag{3.12}$$

where $p_\mu = -i\partial_\mu$ and $\gamma_{\alpha\beta}$ are the Barut-Muzinich-Williams covariantly defined 6×6 matrices. Then, the corresponding equations follow straightforwardly for the AST fields of different parities [3].

Bargmann and Wigner claimed explicitly that they constructed $(2s+1)$ states. Meanwhile, the Weinberg-Tucker-Hammer theory has essentially $2(2s+1)$ components. Therefore, we now apply

$$\Psi_{\{\alpha\beta\}} = (\gamma^\mu R)_{\alpha\beta}(c_a m A_\mu + c_f F_\mu) + (\sigma^{\mu\nu}R)_{\alpha\rho}(c_A m(\gamma^5)_{\rho\beta}A_{\mu\nu} + c_F I_{\rho\beta}F_{\mu\nu})\,. \tag{3.13}$$

Thus, A_μ, $A_{\mu\nu}$ and F_μ, $F_{\mu\nu}$ have different mass dimension. The constants c_i are some numerical dimensionless coefficients. The substitution of the above expansion into the Bargmann-Wigner set, Ref. [17], gives us new Proca-like equations:

$$c_a m(\partial_\mu A_\nu - \partial_\nu A_\mu) + c_f(\partial_\mu F_\nu - \partial_\nu F_\mu) = ic_A m^2 \epsilon_{\alpha\beta\mu\nu}A^{\alpha\beta} + 2mc_F F_{\mu\nu}, \tag{3.14}$$

$$c_a m^2 A_\mu + c_f m F_\mu = ic_A m \epsilon_{\mu\nu\alpha\beta}\partial^\nu A^{\alpha\beta} + 2c_F \partial^\nu F_{\mu\nu}\,. \tag{3.15}$$

In the case $c_a = 1$, $c_F = \frac{1}{2}$ and $c_f = c_A = 0$ they are reduced to the ordinary Proca equations. In the general case we obtain dynamical equations which connect the photon, the notoph and their potentials. The divergent (in $m\to 0$) parts of field functions and those of dynamical variables should be removed by the corresponding gauge (or the Kalb-Ramond gauge) transformations. Apart from these dynamical equations we can obtain a number of constraints by means of the subtraction of the equations of the Bargmann-Wigner system (instead of the addition as for (3.14,3.15)). In fact, they give

$\widetilde{F}^{\mu\nu} \sim imA^{\mu\nu}$ and $F^\mu \sim mA^\mu$. Thus, after the suitable choice of the dimensionless coefficients c_i the Lagrangian density for the photon-notoph field can be proposed:

$$L = L^{Proca} + L^{Notoph} = -\frac{1}{8}F_\mu F^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu + \frac{m^2}{4}A_{\mu\nu}A^{\mu\nu}\,, \tag{3.16}$$

The limit $m \to 0$ may be taken for dynamical variables, in the end of calculations only. Furthermore, it is logical to introduce the normalization scalar field $\varphi(x)$ and consider the expansion:

$$\Psi_{\{\alpha\beta\}} = (\gamma^\mu R)_{\alpha\beta}(\varphi A_\mu) + (\sigma^{\mu\nu}R)_{\alpha\beta}F_{\mu\nu}\,. \tag{3.17}$$

Then, we arrive at the set of equations, which in the case of the constant scalar field $\varphi = 2m$, can again be reduced to the system of the Proca equations.

Next, the Tam-Happer experiments [22] on two laser-beams interaction did not find satisfactory explanation in the framework of the ordinary QED. In Refs. [23, 24] a very interesting model has been proposed. It is based on gauging the Dirac field on using the coordinate-dependent parameters $\alpha_{\mu\nu}(x)$. Thus, the second "photon" was introduced. The compensating 24-component (in general) field $B_{\mu,\nu\lambda}$ reduces to the 4-vector field as follows: $B_{\mu,\nu\lambda} = \frac{1}{4}\epsilon_{\mu\nu\lambda\sigma}a^\sigma(x)$. As readily seen after comparison of these formulas with those of Refs. [7, 8, 9], the second photon is nothing more than the Ogievetskiĭ-Polubarinov *notoph* within the normalization.

4. The Bargmann-Wigner Formalism for Spin 2.

In this Section we begin with the equations for the 4-rank symmetric spinor:

$$\left[i\gamma^\mu\partial_\mu - m\right]_{\alpha\alpha'}\Psi_{\alpha'\beta\gamma\delta} = 0\,, \tag{4.1}$$

$$\left[i\gamma^\mu\partial_\mu - m\right]_{\beta\beta'}\Psi_{\alpha\beta'\gamma\delta} = 0\,, \tag{4.2}$$

$$\left[i\gamma^\mu\partial_\mu - m\right]_{\gamma\gamma'}\Psi_{\alpha\beta\gamma'\delta} = 0\,, \tag{4.3}$$

$$\left[i\gamma^\mu\partial_\mu - m\right]_{\delta\delta'}\Psi_{\alpha\beta\gamma\delta'} = 0\,. \tag{4.4}$$

We proceed expanding the field function in the complete set of symmetric matrices. In the beginning let us use the first two indices:

$$\Psi_{\{\alpha\beta\}\gamma\delta} = (\gamma_\mu R)_{\alpha\beta}\Psi^\mu_{\gamma\delta} + (\sigma_{\mu\nu}R)_{\alpha\beta}\Psi^{\mu\nu}_{\gamma\delta}\,. \tag{4.5}$$

Next, we present the vector-spinor and tensor-spinor functions as

$$\Psi^\mu_{\{\gamma\delta\}} = (\gamma^\kappa R)_{\gamma\delta}G_\kappa{}^{\;\mu} + (\sigma^{\kappa\tau}R)_{\gamma\delta}F_{\kappa\tau}{}^{\mu}\,, \tag{4.6}$$

$$\Psi^{\mu\nu}_{\{\gamma\delta\}} = (\gamma^\kappa R)_{\gamma\delta}T_\kappa{}^{\;\mu\nu} + (\sigma^{\kappa\tau}R)_{\gamma\delta}R_{\kappa\tau}{}^{\mu\nu}\,, \tag{4.7}$$

i. e., using the symmetric matrix coefficients in indices γ and δ. Hence, the resulting tensor equations coincide with the equations obtained in Ref. [25].

However, we need to make symmetrization over two sets of indices $\{\alpha\beta\}$ and $\{\gamma\delta\}$. The total symmetry can be ensured if one contracts the function $\Psi_{\{\alpha\beta\}\{\gamma\delta\}}$ with the *antisymmetric* matrices $R^{-1}_{\beta\gamma}$, $(R^{-1}\gamma^5)_{\beta\gamma}$ and $(R^{-1}\gamma^5\gamma^\lambda)_{\beta\gamma}$, and equate all these contractions to zero. We obtain additional constraints on the tensor field functions. We explicitly showed that all field functions become to be equal to zero. Such a situation cannot be considered as a satisfactory one because it does not give us any physical information.

We shall modify the formalism in the spirit of Ref. [15]. The field functions take now into account $\gamma^5\sigma^{\mu\nu}R$ terms. Hence, the function $\Psi_{\{\alpha\beta\}\{\gamma\delta\}}$ can be expressed as a sum of nine terms. The corresponding dynamical equations are given in the following form:

$$\frac{2\alpha_2\beta_4}{m}\partial_\nu T_{\kappa}{}^{\mu\nu} + \frac{i\alpha_3\beta_7}{m}\epsilon^{\mu\nu\alpha\beta}\partial_\nu \widetilde{T}_{\kappa,\alpha\beta} = \alpha_1\beta_1 G_{\kappa}{}^{\mu}\,, \tag{4.8}$$

$$\frac{2\alpha_2\beta_5}{m}\partial_\nu R_{\kappa\tau}{}^{\mu\nu} + \frac{i\alpha_2\beta_6}{m}\epsilon_{\alpha\beta\kappa\tau}\partial_\nu \widetilde{R}^{\alpha\beta,\mu\nu} + \frac{i\alpha_3\beta_8}{m}\epsilon^{\mu\nu\alpha\beta}\partial_\nu \widetilde{D}_{\kappa\tau,\alpha\beta} -$$
$$- \frac{\alpha_3\beta_9}{2}\epsilon^{\mu\nu\alpha\beta}\epsilon_{\lambda\delta\kappa\tau}D^{\lambda\delta}{}_{\alpha\beta} = \alpha_1\beta_2 F_{\kappa\tau}{}^{\mu} + \frac{i\alpha_1\beta_3}{2}\epsilon_{\alpha\beta\kappa\tau}\widetilde{F}^{\alpha\beta,\mu}\,, \tag{4.9}$$

$$2\alpha_2\beta_4 T_{\kappa}{}^{\mu\nu} + i\alpha_3\beta_7\epsilon^{\alpha\beta\mu\nu}\widetilde{T}_{\kappa,\alpha\beta} = \frac{\alpha_1\beta_1}{m}(\partial^\mu G_{\kappa}{}^{\nu} - \partial^\nu G_{\kappa}{}^{\mu})\,, \tag{4.10}$$

$$2\alpha_2\beta_5 R_{\kappa\tau}{}^{\mu\nu} + i\alpha_3\beta_8\epsilon^{\alpha\beta\mu\nu}\widetilde{D}_{\kappa\tau,\alpha\beta} + i\alpha_2\beta_6\epsilon_{\alpha\beta\kappa\tau}\widetilde{R}^{\alpha\beta,\mu\nu} -$$
$$-\frac{\alpha_3\beta_9}{2}\epsilon^{\alpha\beta\mu\nu}\epsilon_{\lambda\delta\kappa\tau}D^{\lambda\delta}{}_{\alpha\beta} = \frac{\alpha_1\beta_2}{m}(\partial^\mu F_{\kappa\tau}{}^{\nu} - \partial^\nu F_{\kappa\tau}{}^{\mu}) +$$
$$+\frac{i\alpha_1\beta_3}{2m}\epsilon_{\alpha\beta\kappa\tau}(\partial^\mu \widetilde{F}^{\alpha\beta,\nu} - \partial^\nu \widetilde{F}^{\alpha\beta,\mu})\,. \tag{4.11}$$

In general, the coefficients α_i and β_i may now carry some dimension. The essential constraints can be found in Ref. [26]. They are the results of contractions of the field function with six antisymmetric matrices as above. As a discussion, we note that in such a framework we, already, have physical content because only certain combinations of field functions can be equal to zero. In general, the fields $F_{\kappa\tau}{}^{\mu}$, $\widetilde{F}_{\kappa\tau}{}^{\mu}$, $T_{\kappa}{}^{\mu\nu}$, $\widetilde{T}_{\kappa}{}^{\mu\nu}$, and $R_{\kappa\tau}{}^{\mu\nu}$, $\widetilde{R}_{\kappa\tau}{}^{\mu\nu}$, $D_{\kappa\tau}{}^{\mu\nu}$, $\widetilde{D}_{\kappa\tau}{}^{\mu\nu}$ can correspond to different physical states and the equations describe couplings one state to another.

Furthermore, from the set of equations (4.8-4.11) one obtains the *second*-order equation for the symmetric traceless tensor of the second rank ($\alpha_1 \neq 0$, $\beta_1 \neq 0$):

$$\frac{1}{m^2}\left[\partial_\nu\partial^\mu G_{\kappa}{}^{\nu} - \partial_\nu\partial^\nu G_{\kappa}{}^{\mu}\right] = G_{\kappa}{}^{\mu}\,. \tag{4.12}$$

After the contraction in indices κ and μ this equation is reduced to:

$$\partial_\alpha G^{\alpha}{}_{\beta} = F_\beta\,, \quad \frac{1}{m^2}\partial_\beta F^\beta = 0\,, \tag{4.13}$$

i. e., the equations which connect the analogue of the energy-momentum tensor and the analogue of the 4-vector field.

5. Interactions.

The possibility of terms such as $\sigma\cdot[\mathbf{A}\times\mathbf{A}^*]$ appears to be related to the matters of chiral interactions [27, 28]. The Dirac field operator can always be presented as a superposition of the self- and anti-self charge conjugate field operators. The anti-self charge conjugate part can give the self charge conjugate part after multiplying by the γ^5 matrix, and *vice versa*. We derived

$$[i\gamma^\mu D^*_\mu - m]\psi_1^s = 0\,, \tag{5.1}$$

$$[i\gamma^\mu D_\mu - m]\psi_2^a = 0\,. \tag{5.2}$$

Both equations lead to the terms of interaction such as $\sigma\cdot[\mathbf{A}\times\mathbf{A}^*]$ provided that the 4-vector potential is considered as a complex function(al). In fact, from (5.1) we have:

$$i\sigma^\mu\nabla_\mu\chi_1 - m\phi_1 = 0\,,\quad i\tilde\sigma^\mu\nabla^*_\mu\phi_1 - m\chi_1 = 0\,. \tag{5.3}$$

And, from (5.2) we have

$$i\sigma^\mu\nabla^*_\mu\chi_2 - m\phi_2 = 0\,,\quad i\tilde\sigma^\mu\nabla_\mu\phi_2 - m\chi_2 = 0\,. \tag{5.4}$$

The meanings of σ^μ and $\tilde\sigma^\mu$ are obvious from the definition of γ matrices. The derivatives are defined:

$$D_\mu = \partial_\mu - ie\gamma^5 C_\mu + eB_\mu\,,\quad \nabla_\mu = \partial_\mu - ieA_\mu\,, \tag{5.5}$$

and $A_\mu = C_\mu + iB_\mu$. Thus, relations with the magnetic monopoles can also be established. From the above systems we extract the terms as $\pm e^2\sigma^i\sigma^j A_i A_j^*$, which lead to the discussed terms [27, 28]. We would like to note that the terms of the type $\sigma\cdot[\mathbf{A}\times\mathbf{A}^*]$ can be reduced to $(\sigma\cdot\nabla)V$, where V is the scalar potential.

Furthermore, one can come to the same conclusions *not* applying the constraints on the creation/annihilation operators. It is possible to work with self/anti-self charge conjugate fields and the Majorana *anzatzen*. Thus, it is the γ^5 transformation which distinguishes various field configurations (helicity, self/anti-self charge conjugate properties etc) in the coordinate representation in the considered cases.

The most general relativistic-invariant Lagrangian for the symmetric 2nd-rank tensor is

$$\begin{aligned} L = &-\alpha_1(\partial^\alpha G_{\alpha\lambda})(\partial_\beta G^{\beta\lambda}) - \alpha_2(\partial_\alpha G^{\beta\lambda})(\partial^\alpha G_{\beta\lambda}) - \\ &-\alpha_3(\partial^\alpha G^{\beta\lambda})(\partial_\beta G_{\alpha\lambda}) + m^2 G_{\alpha\beta}G^{\alpha\beta}\,. \end{aligned} \tag{5.6}$$

It leads to the equation

$$\left[\alpha_2(\partial_\alpha\partial^\alpha) + m^2\right] G^{\{\mu\nu\}} + (\alpha_1+\alpha_3)\partial^{\{\mu|}(\partial_\alpha G^{\alpha|\nu\}}) = 0\,. \tag{5.7}$$

In the case $\alpha_2 = 1{>}0$ and $\alpha_1+\alpha_3 = -1$ it coincides with Eq. (4.12). There is no any problem to obtain the dynamical invariants for the fields of the spin 2 from the above Lagrangian. The mass dimension of $G^{\mu\nu}$ is $[energy]^1$. We now present possible relativistic interactions of the symemtric 2-rank tensor. The simplest ones should be the the following ones: $L^{int}_{(1)} \sim G_{\mu\nu}F^\mu F^\nu$, $L^{int}_{(2)} \sim (\partial^\mu G_{\mu\nu})F^\nu$, $L^{int}_{(3)} \sim G_{\mu\nu}(\partial^\mu F^\nu)$. The term $(\partial_\mu G^\alpha{}_\alpha)F^\mu$ vanishes due to the constraint of tracelessness.

It is also interesting to note that thanks to the possible terms

$$V(F) = \lambda_1(F_\mu F^\mu) + \lambda_2(F_\mu F^\mu)(F_\nu F^\nu) \tag{5.8}$$

we can give the mass to the G_{00} component of the spin-2 field. This is due to the possibility of the Higgs spontaneous symmetry breaking:

$$F^{\mu}(x) = \begin{pmatrix} v + \partial_0 \chi(x) \\ g^1 \\ g^2 \\ g^3 \end{pmatrix} , \tag{5.9}$$

with v being the vacuum expectation value, $v^2 = (F_\mu F^\mu) = -\lambda_1/2\lambda_2 > 0$. Other degrees of freedom of the 4-vector field are removed since they are the Goldstone bosons. As one can readily seen, this expression does not permit an arbitrary phase for F^μ, which is only possible if the 4-vector would be the complex one.

Next, since the interaction of fermions with notoph, for instance, are that of the order $\sim e^2$ in the initial Lagrangian, it is more difficult to observe it. However, as far as I know the theoretical precision calculus in QED (the Landé factor, the anomalous magnetic moment, the hyperfine splittings in positronium and muonium, and the decay rates of *o-Ps* and *p-Ps*) are near the order corresponding to the 4th-5th loops, where the difference may appear with the experiments, cf. [29].

6. Conclusions.

We considered the Bargmann-Wigner formalism in order to derive the equations for the AST fields, and for the symmetric tensor of the 2nd rank. We introduced the additional normalization scalar field in the Bargmann-Wigner formalism in order to account for possible physical significance of the Ogievetskii-Polubarinov–Kalb-Ramond modes. Both the antisymmetric tensor fields and the 4-vector fields may have third helicity state in the massless limits. This problem is connected with the problem of the observability of the gauge [20]. We introduced the additional symmetric matrix in the Bargmann-Wigner expansion ($\gamma^5 \sigma^{\mu\nu} R$) in order to account for the dual fields. The consideration was similar to Ref. [30]. The problem was discussed, what are the the correct definitions of the energy-momentum tensor and other Nöther currents in the electromagnetic theory, the relativistic theory of gravitation, the general relativity, and their generalizations. Furthermore, we discussed the interactions of notoph, photon and graviton. Probably, the notivarg should also be taken into account. In order to analize its dynamical invariants and interactions one should construct the Lagrangian from the analogs of the Riemann tensor, such as $\widetilde{D}^{\mu\nu,\alpha\beta}$. The notoph-graviton interaction may give the mass to spin-2 particles in the way similar to the spontaneous-symmetry-breaking Higgs formalism.

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References

[1] D. V. Ahluwalia and D. J. Ernst, Int. J. Mod. Phys. E**2** (1993) 397.

[2] D. V. Ahluwalia and M. Sawicki, Phys. Rev. D**47** (1993) 5161.

[3] V. V. Dvoeglazov, Int. J. Theor. Phys. **37** (1998), 1915; hep-th/9611068.

[4] V. V. Dvoeglazov, Hadronic J. Suppl. **12** (1997) 221.

[5] S. Weinberg, Phys. Rev. B**133** (1964) 1318; ibid. B**134** (1964) 882; ibid. **181** (1969) 1893.

[6] R. H. Tucker and C. L. Hammer, Phys. Rev. D**3** (1971) 2448.

[7] V. I. Ogievetskiĭ and I. V. Polubarinov, Yadern. Fiz. **4** (1966) 216 [English translation: Sov. J. Nucl. Phys. **4** (1967) 156].

[8] K. Hayashi, Phys. Lett. B**44** (1973) 497.

[9] M. Kalb and P. Ramond, Phys. Rev. D**9** (1974) 2273.

[10] E. Durandin and A. Erschow, Phys. Z. Sowjet. **12** (1937) 466.

[11] F. Belinfante, Physica **6** (1939) 887.

[12] L. V. Avdeev and M. V. Chizhov, Phys. Lett. B**321** (1994) 212; hep-th/9407067.

[13] M. V. Chizhov, Phys. Part. Nucl. **42** (2011) 93.

[14] V. V. Dvoeglazov, Czech. J. Phys. **50** (2000) 225, hep-th/9712036.

[15] V. V. Dvoeglazov, Phys. Scripta **64** (2001) 201, physics/9804010.

[16] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. (USA) **34** (1948) 211.

[17] D. Luriè, *Particles and Fields* (Interscience Publishers, 1968). Ch. 1.

[18] M. Kirchbach, Mod. Phys. Lett. A**12** (1997) 2373.

[19] S. Weinberg, *The Quantum Theory of Fields. Vol. I. Foundations.* (Cambridge University Press, 1995), Ch. 5.

[20] A. Staruszkiewicz, Acta Phys. Polon. B**13** (1982) 617; ibid. **14** (1983) 63, 67, 903; ibid. **15** (1984) 225; ibid **23** (1992) 591.

[21] H. C. Ohanian, Am. J. Phys. **54** (1986) 500.

[22] A. C. Tam and W. Happer, Phys. Rev. Lett. **38** (1977) 278.

[23] P. C. Naik and T. Pradhan, J. Phys. A**14** (1981) 2795.

[24] T. Pradhan, R. P. Malik and P. C. Naik, Pramana J. Phys. **24** (1985) 77.

[25] G. Marques and D. Spehler, Mod. Phys. Lett. A**13** (1998) 553.

[26] V. V. Dvoeglazov, Adv. Appl. Clifford Algebras **10** (2000) 7, math-ph/9805017.

[27] V. V. Dvoeglazov, Electromagnetic Phenomena, **2**, No. 1 (5) (2001) 3, hep-ph/9801287.

[28] V. V. Dvoeglazov, Acta Phys. Slovaca **50** (2000) 629.

[29] V. V. Dvoeglazov, R. N. Faustov and Yu. N. Tyukhtyaev, Phys. Part. Nucl. **25** (1994) 58.

[30] V. V. Dvoeglazov, Int. J. Mod. Phys. CS **3** (2011) 121.

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