IMPROVEMENT ON OPERATOR AXIOMS AND FUNDAMENTAL OPERATOR FUNCTIONS

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Abstract. The Operator axioms have been contructed to deduce number systems. In this paper, we slightly improve on the syntax of the Operator axioms and construct a semantics of the Operator axioms. Then on the basis of the improved Operator axioms, we define two fundamental operator functions to study the analytic properties of the Operator axioms. Finally, we prove two theorems about the fundamental operator functions and pose some conjectures. Real operators can give new equations and inequalities so as to precisely describe the relation of mathematical objects or scientific objects.

1. Introduction

In [1], we distinguish the limit from the infinite sequence. Then in [2], we define the Operator axioms to extend the traditional real number system. the Operator axioms have shown typical features as follows:

1. The logical calculus provides a uniform frame for arithmetic axioms. Based on the same logical calculus, small number system can import new axioms to produce big number systems.

2. Consistent binary relation are the nature of number systems. The order relation and equivalence relation of each number system is consistent, so all numbers of each number system are layed in fixed positions of number line.

3. The number systems equate number with operation. The number '1' and various operators compose all numbers through operation. Any number except '1' is also an opertion. For example, the number "$[[1+[1+1]]----[1+1]]$" derives from the operation of the number "$[1+[1+1]]$", the number "$[1+1]$" and the real operator "$----$". So operator distinguishes different number systems and is the foremost component of number system.

4. The Operator axioms produces new real numbers with new operators. While producing new real numbers in arithmetic, the new operators certainly produce new equations and inequalities in algebra. So the Operator axioms not only extends real number system, but also extends equations and inequalities.

In conclusion, the Operator axioms forms a new arithmetic axiom. The [2, Definition 2.2] defines 'number' on the basis of the logical calculus $\{\Phi, \Psi\}$. The [2, TABLE 2] defines new operators according to the definition of number systems. Real operators naturally produce new equations such as $y = [x++++[1+1]]$, $y = [[1+1]----x]$, $y = [x////[1+1]]$ and so on. In other words, real operators extend the traditional mathematical models which are selected to describe various scientific rules.

Thus, real operators exhibit potential value as follows:

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1. Real operators can give new equations and inequalities so as to precisely describe the relation of mathematical objects.

2. Real operators can give new equations and inequalities so as to precisely describe the relation of scientific objects.

So real operators help to describe complex scientific rules which are difficult described by traditional equations and have an enormous application potential.

The paper is organized as follows. In Section 2, we construct the syntax and semantics of the improved Operator axioms. In Section 3, we define two fundamental operator functions, prove two theorems about the fundamental operator functions and pose some conjectures.

2. Operator Axioms

The Operator axioms divides into syntax and semantics. The syntax aims at logical deduction, while the semantics aims at the objects mapped from the syntax deduction.

2.1. **Syntax Of The Operator Axioms[2].** The syntax of the Operator axioms derives from [2] and is revised a little in this section. In mathematical logic, logical calculus is a formal system to abstract and analyze the induction and deduction apart from specific meanings. In this section, however, we construct a logical calculus by virtue of formal language and deduce numbers to intuitively and logically denote number systems. The logical calculus not only denotes real numbers, but also allows them to join in algebraical operations.

The introduction of formal language aims to use computer fast execute real number operations. For clarity, we will explain the logical calculus with natural language.

In [3], the producer "$\to$" substitutes the right permutations for the left permutations to produce new permutations. In [4], the connectives "$\neg$", "$\wedge$", "$\vee$", "$\Rightarrow$" and "$\Leftrightarrow$" stand for "not", "and", "or", "implies" and "if and only if" respectively. Here, the producer "$\to$" is considered as a predicate symbol and embedded into the logical calculus.

Table 1 extends the [2, TABLE 2] and translates the simple part of the syntax to natural language. The complex part of the syntax is difficult to be translated to natural language.

Table 1. Translation From Syntax To Natural Language.

Syntax	**Natural Language**
$\wedge$	and
$\vee$	or
$\neg$	not
$\rightarrow$	replaced by
$\Rightarrow$	imply
$\Leftrightarrow$	symmetrical imply
$\{$	punctuation
$\}$	punctuation
,	punctuation
(	punctuation
)	punctuation
$\emptyset$	emptiness
$\cdots$	omission
$\Phi\{\cdots\}$	Denote Φ as a set of notations and particular axioms between $\{$ and $\}$. Different logical calculus correspond to different notations and particular axioms.
$V\{\cdots\}$	Denote V as a set of variables between $\{$ and $\}$.
$C\{\cdots\}$	Denote C as a set of constants between $\{$ and $\}$.
$P\{\cdots\}$	Denote P as a set of predicate symbols between $\{$ and $\}$.
$V \circ C\{\cdots\}$	Denote $V \circ C$ as a set of concatenations between V and C.
$C \circ C\{\cdots\}$	Denote $C \circ C$ as a set of concatenations between C and C.
$V \circ C \circ P\{\cdots\}$	Denote $V \circ C \circ P$ as a set of concatenations among V, C and P.
$(\cdots \in \cdots) \cdots \Leftrightarrow \cdots$	Define the binary predicate symbol $\in$.
$(\cdots \in (V \circ C))$	Define a variable ranging over $V \circ C$.
$(\cdots \in (V \circ C \circ P))$	Define a variable ranging over $V \circ C \circ P$.
$(\cdots \subseteq \cdots) \cdots \Leftrightarrow \cdots$	Define the binary predicate symbol $\subseteq$.
$\Psi\{\cdots\}$	Denote Ψ as a set of general axioms between $\{$ and $\}$. Different logical calculus correspond to the same general axioms.
$(\cdots \rightarrow \cdots) \wedge (\cdots \rightarrow \cdots) \Rightarrow \cdots$	Define the binary predicate symbol $\rightarrow$.
$(\cdots \mid \cdots) \cdots \Rightarrow \cdots$	Define the binary predicate symbol $\mid$.
$(\cdots < \cdots) \cdots \Rightarrow \cdots$	Define the binary predicate symbol $<$.
$(\cdots = \cdots)$	Define the binary predicate symbol $=$.
$(\cdots = \cdots) \cdots \Rightarrow \cdots$	Define the binary predicate symbol $=$.
$(\cdots \leq \cdots) \cdots \Leftrightarrow \cdots$	Define the binary predicate symbol $\leq$.
$(\cdots \leq \cdots) \cdots \Rightarrow \cdots$	Define the binary predicate symbol $\leq$.
$(\cdots \rightarrow \cdots)\Vert(\cdots \rightarrow \cdots)$	Define the binary predicate symbol $\Vert$.

Definition 2.1. [2, Definition 2.2] In a logical calculus $\{\Phi, \Psi\}$, if $\bar{a} \equiv true$, then $\bar{a}$ is a number.

Definition 2.2. the Operator axioms is a logical calculus $R\{\Phi, \Psi\}$ such that:

$$\Phi\{$$

(OA.1) $V\{\emptyset, \dot{a}, \dot{b}, \dot{c}, \dot{d}, \dot{e}, \dot{f}, \dot{g}, \dot{h}, \dot{i}, \dot{j}, \dot{k}\},$

(OA.2) $C\{\emptyset, 1, +, [,], -, /\},$

(OA.3) $P\{\emptyset, \in, \subseteq, \rightarrow, |, =, <, \leq, \|\},$

(OA.4) $V \circ C\{\emptyset, \dot{a}, \dot{b}\cdots, 1, +\cdots, \dot{a}\dot{a}, \dot{a}\dot{b}\cdots, \dot{a}1, \dot{a}+\cdots, \dot{b}\dot{a}, \dot{b}\dot{b}\cdots, \dot{b}1, \dot{b}+\cdots,$
$\dot{a}\dot{a}\dot{a}, \dot{a}\dot{a}\dot{b}\cdots, \dot{a}\dot{a}1, \dot{a}\dot{a}+\cdots, \dot{b}\dot{a}\dot{a}, \dot{b}\dot{a}\dot{b}\cdots, \dot{b}\dot{a}1, \dot{b}\dot{a}+\cdots\},$

(OA.5) $C \circ C\{\emptyset, 1, +\cdots, 11, 1+\cdots, 111, 11+\cdots\},$

(OA.6) $V \circ C \circ P\{\emptyset, \dot{a}, \dot{b}\cdots, 1, +\cdots, \in, \subseteq\cdots, \dot{a}\dot{a}, \dot{a}\dot{b}\cdots, \dot{a}1, \dot{a}+\cdots, \dot{a}\in, \dot{a}\subseteq\cdots,$
$\dot{b}\dot{a}, \dot{b}\dot{b}\cdots, \dot{b}1, \dot{b}+\cdots, \dot{b}\in, \dot{b}\subseteq\cdots, \dot{a}\dot{a}\dot{a}, \dot{a}\dot{a}\dot{b}\cdots, \dot{a}\dot{a}1, \dot{a}\dot{a}+\cdots, \dot{a}\dot{a}\in, \dot{a}\dot{a}\subseteq\cdots,$
$\dot{b}\dot{a}\dot{a}, \dot{b}\dot{a}\dot{b}\cdots, \dot{b}\dot{a}1, \dot{b}\dot{a}+\cdots, \dot{b}\dot{a}\in, \dot{b}\dot{a}\subseteq\cdots\},$

(OA.7) $(\hat{a} \in V) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv \dot{a}) \vee (\hat{a} \equiv \dot{b})\cdots),$

(OA.8) $(\hat{a} \in C) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +)\cdots),$

(OA.9) $(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv \dot{a}) \vee (\hat{a} \equiv \dot{b})\cdots \vee (\hat{a} \equiv 1)\cdots \vee (\hat{a} \equiv \dot{a}\dot{a})$
$\vee(\hat{a} \equiv \dot{a}\dot{b})\cdots \vee (\hat{a} \equiv \dot{a}1)\cdots \vee (\hat{a} \equiv \dot{a}\dot{a}\dot{a}) \vee (\hat{a} \equiv \dot{a}\dot{a}\dot{b})\cdots \vee (\hat{a} \equiv \dot{a}\dot{a}1)\cdots),$

(OA.10) $(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +)\cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+)\cdots$
$\vee(\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+)\cdots),$

(OA.11) $(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv \dot{a}) \vee (\hat{a} \equiv \dot{b})\cdots \vee (\hat{a} \equiv \in)\cdots \vee (\hat{a} \equiv \dot{a}\dot{a})$
$\vee(\hat{a} \equiv \dot{a}\dot{b})\cdots \vee (\hat{a} \equiv \dot{a}\in)\cdots \vee (\hat{a} \equiv \dot{a}\dot{a}\dot{a}) \vee (\hat{a} \equiv \dot{a}\dot{a}\dot{b})\cdots \vee (\hat{a} \equiv \dot{a}\dot{a}\in)\cdots),$

(OA.12) $(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C))$
$\wedge(\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge$
$(\bar{j} \in (V \circ C)) \wedge (\bar{k} \in (V \circ C)) \wedge (\bar{l} \in (V \circ C)) \wedge (\bar{\bar{a}} \in (V \circ C \circ P)) \wedge$
$(\bar{\bar{b}} \in (V \circ C \circ P)) \wedge (\bar{\bar{c}} \in (V \circ C \circ P)),$

(OA.13) $((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee(\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))),$

(OA.14) $(\bar{a}\bar{b}\bar{c} = \bar{d}\bar{e}\bar{f}\bar{e}\bar{g}\bar{e}\bar{h}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \wedge \neg(\bar{e} \subseteq \{\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{f}, \bar{g}, \bar{h}\}) \wedge$
$((\bar{b} \rightarrow \bar{i}) \| (\bar{e} \rightarrow \bar{j})) \Rightarrow (\bar{a}\bar{i}\bar{c} = \bar{d}\bar{j}\bar{f}\bar{j}\bar{g}\bar{j}\bar{h}),$

(OA.15) $(\bar{a}\bar{b}\bar{c}\bar{d}\bar{e} = \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \wedge \neg(\bar{d} \subseteq \{\bar{a}, \bar{b}, \bar{c}, \bar{e}, \bar{f}\}) \wedge ((\bar{b} \rightarrow \bar{g}) \| (\bar{d} \rightarrow \bar{h}))$

$\Rightarrow (\bar{a}\bar{g}\bar{c}\bar{h}\bar{e} = \bar{f})$,

$(OA.16)$ $\dot{a} \to 1|[\dot{a}\dot{b}\dot{a}]$,

$(OA.17)$ $\dot{b} \to +|-$,

$(OA.18)$ $\dot{c}|\dot{d} \to \dot{e}|\dot{f}|\dot{g}$,

$(OA.19)$ $\dot{e} \to +|+\dot{e}$,

$(OA.20)$ $\dot{f} \to -|-\dot{f}$,

$(OA.21)$ $\dot{g} \to /|/\dot{g}$,

$(OA.22)$ $(\dot{h} \to +)\|(\dot{i} \to -)$,

$(OA.23)$ $(\dot{h} \to +\dot{h})\|(\dot{i} \to -\dot{i})$,

$(OA.24)$ $(\dot{i} \to -)\|(\dot{h} \to +)$,

$(OA.25)$ $(\dot{i} \to -\dot{i})\|(\dot{h} \to +\dot{h})$,

$(OA.26)$ $(\dot{h} \to +)\|(\dot{j} \to /)$,

$(OA.27)$ $(\dot{h} \to +\dot{h})\|(\dot{j} \to /\dot{j})$,

$(OA.28)$ $(\dot{h} \to +)\|(\dot{k} \to 1)$,

$(OA.29)$ $(\dot{h} \to +\dot{h})\|(\dot{k} \to [1+\dot{k}])$,

$(OA.30)$ $\dot{a} < [1+\dot{a}]$,

$(OA.31)$ $(\bar{a} < \bar{b}) \wedge \bar{c} \Rightarrow (([\bar{a}+\bar{c}] < [\bar{b}+\bar{c}]) \wedge ([\bar{a}-\bar{c}] < [\bar{b}-\bar{c}]) \wedge ([\bar{c}-\bar{b}] < [\bar{c}-\bar{a}]))$,

$(OA.32)$ $([1-1] \leq \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1-1] < \bar{c}) \Rightarrow ([\bar{a}--\bar{c}] < [\bar{b}--\bar{c}])$,

$(OA.33)$ $([1-1] < \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1-1] < \bar{c}) \Rightarrow ([\bar{c}--\bar{b}] < [\bar{c}--\bar{a}])$,

$(OA.34)$ $(1 < \bar{a}) \wedge ([1-1] < \bar{b}) \Rightarrow ((1 < [\bar{a}--\dot{f}\bar{b}]) \wedge (1 < [[1--\bar{a}]--\dot{f}[[1-1]-\bar{b}]]))$,

$(OA.35)$ $([1-1] < \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([\bar{a}/\dot{g}\bar{b}])$,

$(OA.36)$ $(1 < \bar{a}) \wedge (\bar{a} < \bar{b}) \Rightarrow (1 < [\bar{b}/\dot{g}\bar{a}])$,

$(OA.37)$ $(1 \leq \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1-1] < \bar{c}) \Rightarrow ([\bar{a}\dot{e}\bar{c}] < [\bar{b}\dot{e}\bar{c}])$,

$(OA.38)$ $(1 \leq \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge (\bar{c} < [1-1]) \Rightarrow ([\bar{b}+\dot{e}\bar{c}] < [\bar{a}+\dot{e}\bar{c}])$,

$(OA.39)$ $(1 \leq \bar{a}) \wedge (1 \leq \bar{b}) \wedge ([1-1] < \bar{c}) \wedge ([\bar{a}\dot{e}\bar{c}] < [\bar{b}\dot{e}\bar{c}]) \Rightarrow (\bar{a} < \bar{b})$,

$(OA.40)$ $(1 \leq \bar{a}) \wedge (1 \leq \bar{b}) \wedge (\bar{c} < [1-1]) \wedge ([\bar{a}+\dot{e}\bar{c}] < [\bar{b}+\dot{e}\bar{c}]) \Rightarrow (\bar{b} < \bar{a})$,

$(OA.41)$ $(1 < \bar{a}) \wedge (\bar{b} < \bar{c}) \Rightarrow ([\bar{a}\dot{e}\bar{b}] < [\bar{a}\dot{e}\bar{c}])$,

$(OA.42)$ $(1 < \bar{a}) \wedge \bar{b} \wedge \bar{c} \wedge ([\bar{a}\dot{e}\bar{b}] < [\bar{a}\dot{e}\bar{c}]) \Rightarrow (\bar{b} < \bar{c})$,

$(OA.43)$ $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a}-\bar{b}] = [\bar{a}/\bar{b}])$,

$(OA.44)$ $\bar{a} \wedge \neg(\bar{b} = [1-1]) \Rightarrow ([\bar{a}--\bar{b}] = [\bar{a}//\bar{b}])$,

$(OA.45)$ $\bar{a} \Rightarrow ([\bar{a}-\bar{a}] = [1-1])$,

$(OA.46)$ $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a}+\bar{b}] = [\bar{b}+\bar{a}])$,

$(OA.47)$ $\bar{a} \wedge \bar{b} \Rightarrow ([[\bar{a}-\bar{b}]+\bar{b}] = \bar{a})$,

$(OA.48)$ $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([[\bar{a}-\bar{b}]+\bar{c}] = [[\bar{a}+\bar{c}]-\bar{b}])$,

$(OA.49)$ $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}+[\bar{b}+\bar{c}]] = [[\bar{a}+\bar{b}]+\bar{c}])$,

$(OA.50)$ $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a}+[\bar{b}-\bar{c}]] = [[\bar{a}+\bar{b}]-\bar{c}])$,

$$(OA.51) \quad \bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} + \bar{c}]] = [[\bar{a} - \bar{b}] - \bar{c}]),$$

$$(OA.52) \quad \bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} - \bar{c}]] = [[\bar{a} - \bar{b}] + \bar{c}]),$$

$$(OA.53) \quad \bar{a} \Rightarrow ([\bar{a} + + 1] = \bar{a}) \wedge ([\bar{a} - - 1] = \bar{a}),$$

$$(OA.54) \quad \neg(\bar{a} = [1 - 1]) \Rightarrow ([\bar{a} - - \bar{a}] = 1),$$

$$(OA.55) \quad \bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + + \bar{b}] = [\bar{b} + + \bar{a}]),$$

$$(OA.56) \quad \bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + + [\bar{b} + + \bar{c}]] = [[\bar{a} + + \bar{b}] + + \bar{c}]),$$

$$(OA.57) \quad \bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + + [\bar{b} + \bar{c}]] = [[\bar{a} + + \bar{b}] + [\bar{a} + + \bar{c}]]),$$

$$(OA.58) \quad \bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + + [\bar{b} - \bar{c}]] = [[\bar{a} + + \bar{b}] - [\bar{a} + + \bar{c}]]),$$

$$(OA.59) \quad \bar{a} \wedge \neg(\bar{b} = [1 - 1]) \Rightarrow ([[\bar{a} - - \bar{b}] + + \bar{b}] = \bar{a}),$$

$$(OA.60) \quad \bar{a} \wedge \neg(\bar{b} = [1 - 1]) \wedge \bar{c} \Rightarrow ((([[\bar{a} - - \bar{b}] + + \bar{c}] = [[\bar{a} + + \bar{c}] - - \bar{b}]) \wedge ([[\bar{a} + \bar{c}] - - \bar{b}] = [[\bar{a} - - \bar{b}] + [\bar{c} - - \bar{b}]]) \wedge ([[\bar{a} - \bar{c}] - - \bar{b}] = [[\bar{a} - - \bar{b}] - [\bar{c} - - \bar{b}]])),$$

$$(OA.61) \quad \bar{a} \wedge \neg(\bar{b} = [1 - 1]) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow (([\bar{a} + + [\bar{b} - - \bar{c}]] = [[\bar{a} + + \bar{b}] - - \bar{c}]) \wedge ([\bar{a} - - [\bar{b} + + \bar{c}]] = [[\bar{a} - - \bar{b}] - - \bar{c}]) \wedge ([\bar{a} - - [\bar{b} - - \bar{c}]] = [[\bar{a} - - \bar{b}] + + \bar{c}])),$$

$$(OA.62) \quad \bar{a} \Rightarrow ([\bar{a} + + + 1] = \bar{a}) \wedge ([\bar{a} - - - 1] = \bar{a}),$$

$$(OA.63) \quad \bar{a} \Rightarrow ([1 + + + \bar{a}] = 1),$$

$$(OA.64) \quad \neg(\bar{a} = [1 - 1]) \Rightarrow ([\bar{a} + + + [1 - 1]] = 1),$$

$$(OA.65) \quad ([1 - 1] < \bar{a}) \Rightarrow ([[1 - 1] + + + \bar{a}] = [1 - 1]),$$

$$(OA.66) \quad ([1 - 1] < \bar{a}) \wedge \neg(\bar{b} = [1 - 1]) \wedge \bar{c} \Rightarrow ((([[\bar{a} - - - \bar{b}] + + + \bar{b}] = \bar{a}) \wedge ([[\bar{a} - - - \bar{b}] + + + \bar{c}] = [[\bar{a} + + + \bar{c}] - - - \bar{b}]) \wedge ([\bar{a} + + + [\bar{c} - - \bar{b}]] = [[\bar{a} + + + \bar{c}] - - - \bar{b}])),$$

$$(OA.67) \quad ([1 - 1] < \bar{a}) \wedge ([1 - 1] < \bar{b}) \wedge \bar{c} \Rightarrow ((([\bar{a} + + + [\bar{b} /// \bar{a}]] = \bar{b}) \wedge ([[\bar{a} + + + \bar{c}] /// \bar{b}] = [\bar{c} + + [\bar{a} /// \bar{b}]]) \wedge ([[\bar{a} - - \bar{b}] + + + \bar{c}] = [[\bar{a} + + + \bar{c}] - - [\bar{b} + + + \bar{c}]])),$$

$$(OA.68) \quad ([1 - 1] < \bar{a}) \wedge ([1 - 1] < \bar{b}) \wedge ([1 - 1] < \bar{c}) \Rightarrow ((([[\bar{a} /// \bar{c}] - - [\bar{b} /// \bar{c}]] = [\bar{a} /// \bar{b}]) \wedge ([[\bar{a} + + \bar{b}] /// \bar{c}] = [[\bar{a} /// \bar{c}] + [\bar{b} /// \bar{c}]]) \wedge ([[\bar{a} - - \bar{b}] /// \bar{c}] = [[\bar{a} /// \bar{c}] - [\bar{b} /// \bar{c}]])),$$

$$(OA.69) \quad \neg(\bar{a} = [1 - 1]) \wedge \neg(\bar{b} = [1 - 1]) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow ((([[\bar{a} + + \bar{b}] + + + \bar{c}] = [[\bar{a} + + + \bar{c}] + + [\bar{b} + + + \bar{c}]]) \wedge ([\bar{a} + + + [\bar{b} + + \bar{c}]] = [[\bar{a} + + + \bar{b}] + + + \bar{c}]) \wedge ([\bar{a} + + + [\bar{b} + \bar{c}]] = [[\bar{a} + + + \bar{b}] + + [\bar{a} + + + \bar{c}]]) \wedge ([\bar{a} + + + [\bar{b} - \bar{c}]] = [[\bar{a} + + + \bar{b}] - - [\bar{a} + + + \bar{c}]])),$$

$$(OA.70) \quad ([1 - 1] < \bar{a}) \wedge \neg(\bar{b} = [1 - 1]) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow (([\bar{a} - - - [\bar{b} + + \bar{c}]] = [[\bar{a} - - - \bar{b}] - - - \bar{c}]) \wedge ([\bar{a} - - - [\bar{b} - - \bar{c}]] = [[\bar{a} - - - \bar{b}] + + + \bar{c}])),$$

$$(OA.71) \quad ([1 - 1] < \bar{a}) \wedge ([1 - 1] < \bar{b}) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow ((([[\bar{a} + + \bar{b}] - - - \bar{c}] = [[\bar{a} - - - \bar{c}] + + [\bar{b} - - - \bar{c}]]) \wedge ([[\bar{a} - - \bar{b}] - - - \bar{c}] = [[\bar{a} - - - \bar{c}] - - [\bar{b} - - - \bar{c}]])),$$

$$(OA.72) \quad (1 \leq \bar{a}) \Rightarrow ([\bar{a} + \dot{e} 1] = \bar{a}),$$

$(OA.73)\quad (1 \leq \bar{a}) \Rightarrow ([\bar{a} + +\dot{e}[1-1]] = 1),$

$(OA.74)\quad (1 \leq \bar{a}) \Rightarrow ([\bar{a} - \dot{f}1] = \bar{a}),$

$(OA.75)\quad \bar{a} \Rightarrow ([1 + +\dot{e}\bar{a}] = 1),$

$(OA.76)\quad \neg(\bar{a} = [1-1]) \Rightarrow ([1 - -\dot{f}\bar{a}] = 1),$

$(OA.77)\quad (1 < \bar{a}) \Rightarrow ([1//\dot{g}\bar{a}] = [1-1]),$

$(OA.78)\quad (1 < \bar{a}) \Rightarrow ([\bar{a}/\dot{g}\bar{a}] = 1),$

$(OA.79)\quad (1 \leq \bar{a}) \wedge ([1-1] < \bar{b}) \Rightarrow ((([[\bar{a}\dot{i}\bar{b}]\dot{h}\bar{b}] = \bar{a}) \wedge ([[\bar{a}\dot{h}\bar{b}]\dot{i}\bar{b}] = \bar{a})),$

$(OA.80)\quad ([1-1] < \bar{a}) \wedge (\bar{a} \leq 1) \wedge (\bar{b} < [1-1]) \Rightarrow ((([[\bar{a}\dot{i}\bar{b}]\dot{h}\bar{b}] = \bar{a}) \wedge ([[\bar{a}\dot{h}\bar{b}]\dot{i}\bar{b}] = \bar{a})),$

$(OA.81)\quad ([1-1] < \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ((([\bar{b}\dot{h}[\bar{a}\dot{j}\bar{b}]] = \bar{a}) \wedge ([[\bar{b}\dot{h}\bar{a}]\dot{j}\bar{b}] = \bar{a})),$

$(OA.82)\quad (1 \leq \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([\bar{a} + + + \dot{e}\bar{b}] = [\bar{a} + +\dot{e}[\bar{a} + + + \dot{e}[\bar{b} - 1]]]),$

$(OA.83)\quad (1 \leq \bar{a}) \wedge ([1-1] \leq \bar{b}) \wedge (\bar{b} \leq 1) \wedge (\bar{c} = [[\bar{a} - 1] + +[1 + [1+1]]]) \wedge$
$(\bar{d} = [[\bar{a} - 1] + +[1+1]]) \wedge (\bar{e} = [\bar{b} + + + [1+1]]) \wedge$
$(\bar{f} = [\bar{b} + + + [1 + [1+1]]]) \Rightarrow$
$([\bar{a} + + + \dot{h}\bar{b}] = [[1 + [\bar{c} + +[\bar{e} + + + \dot{k}]]] - [\bar{d} + +[\bar{f} + + + \dot{k}]]]),$

$(OA.84)\quad (1 \leq \bar{a}) \wedge (\bar{b} < [1-1]) \Rightarrow ([\bar{a} + + + \dot{e}\bar{b}] = [1 - -[\bar{a} + + + \dot{e}[[1-1] - \bar{b}]]]),$

$(OA.85)\quad (1 \leq \bar{a}) \wedge \bar{b} \Rightarrow ([1-1] < [\bar{a} + + + \dot{e}\bar{b}])$

$\},$

$\Psi\{$

$(OA.86)\quad (\bar{a} \subseteq \bar{b}) \Leftrightarrow (\bar{b} = \bar{c}\bar{a}\bar{d}),$

$(OA.87)\quad (\bar{a} \to \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \to \bar{e}) \Rightarrow (\bar{a} \to \bar{b}\bar{e}\bar{d}),$

$(OA.88)\quad (\bar{\bar{a}} \to \bar{\bar{b}}|\bar{\bar{c}}) \Rightarrow ((\bar{\bar{a}} \to \bar{\bar{b}}) \wedge (\bar{\bar{a}} \to \bar{\bar{c}})),$

$(OA.89)\quad (\bar{\bar{a}}|\bar{\bar{b}} \to \bar{\bar{c}}) \Rightarrow ((\bar{\bar{a}} \to \bar{\bar{c}}) \wedge (\bar{\bar{b}} \to \bar{\bar{c}})),$

$(OA.90)\quad (\bar{a} < \bar{b}) \Rightarrow \neg(\bar{b} < \bar{a}),$

$(OA.91)\quad (\bar{a} < \bar{b}) \Rightarrow \neg(\bar{a} = \bar{b}),$

$(OA.92)\quad (\bar{a} < \bar{b}) \wedge (\bar{b} < \bar{c}) \Rightarrow (\bar{a} < \bar{c}),$

$(OA.93)\quad (\bar{a} < \bar{b}) \wedge (\bar{a} \in (C \circ C)) \wedge (\bar{b} \in (C \circ C)) \Rightarrow (\bar{a} \wedge \bar{b}),$

$(OA.94)\quad (\bar{a} < \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} = \bar{e}) \Rightarrow (\bar{a} < \bar{b}\bar{e}\bar{d}),$

$(OA.95)\quad (\bar{a}\bar{b}\bar{c} < \bar{d}) \wedge (\bar{b} = \bar{e}) \Rightarrow (\bar{a}\bar{e}\bar{c} < \bar{d}),$

$(OA.96)\quad (\bar{a} < \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \to \bar{e}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}\}) \Rightarrow (\bar{a} < \bar{b}\bar{e}\bar{d}),$

$(OA.97)\quad (\bar{a} < \bar{b}\bar{c}\bar{d}\bar{c}\bar{e}) \wedge (\bar{c} \to \bar{f}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a} < \bar{b}\bar{f}\bar{d}\bar{f}\bar{e}),$

$(OA.98)\quad (\bar{a}\bar{b}\bar{c} < \bar{d}) \wedge (\bar{b} \to \bar{e}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}\}) \Rightarrow (\bar{a}\bar{e}\bar{c} < \bar{d}),$

$(OA.99)\quad (\bar{a}\bar{b}\bar{c} < \bar{d}\bar{b}\bar{e}) \wedge (\bar{b} \to \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c} < \bar{d}\bar{f}\bar{e}),$

$(OA.100)\quad (\bar{a}\bar{b}\bar{c} < \bar{d}\bar{b}\bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \to \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c} < \bar{d}\bar{g}\bar{e}\bar{g}\bar{f}),$

$(OA.101)\quad (\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}) \wedge (\bar{b} \to \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c}\bar{f}\bar{d} < \bar{e}),$

$(OA.102)\quad (\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \to \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c}\bar{g}\bar{d} < \bar{e}\bar{g}\bar{f}),$

$(OA.103)\quad (\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}\bar{b}\bar{f}\bar{b}\bar{g}) \wedge (\bar{b} \to \bar{h}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Rightarrow (\bar{a}\bar{h}\bar{c}\bar{h}\bar{d} < \bar{e}\bar{h}\bar{f}\bar{h}\bar{g}),$

$$(OA.104)\quad \bar a=\bar a,$$
$$(OA.105)\quad (\bar a=\bar b)\Rightarrow(\bar b=\bar a),$$
$$(OA.106)\quad (\bar a=\bar b)\Rightarrow\neg(\bar a<\bar b),$$
$$(OA.107)\quad (\bar a=\bar b\bar c\bar d)\wedge(\bar c=\bar e)\Rightarrow(\bar a=\bar b\bar e\bar d),$$
$$(OA.108)\quad (\bar a\bar b\bar c)\wedge(\bar b=\bar d)\Rightarrow(\bar a\bar b\bar c=\bar a\bar d\bar c),$$
$$(OA.109)\quad (\bar a=\bar b\bar c\bar d)\wedge(\bar c\to\bar e)\wedge\neg(\bar c\subseteq\{\bar a,\bar b,\bar d\})\Rightarrow(\bar a=\bar b\bar e\bar d),$$
$$(OA.110)\quad (\bar a=\bar b\bar c\bar d\bar c\bar e)\wedge(\bar c\to\bar f)\wedge\neg(\bar c\subseteq\{\bar a,\bar b,\bar d,\bar e\})\Rightarrow(\bar a=\bar b\bar f\bar d\bar f\bar e),$$
$$(OA.111)\quad (\bar a\bar b\bar c=\bar d\bar b\bar e)\wedge(\bar b\to\bar f)\wedge\neg(\bar b\subseteq\{\bar a,\bar c,\bar d,\bar e\})\Rightarrow(\bar a\bar f\bar c=\bar d\bar f\bar e),$$
$$(OA.112)\quad (\bar a\bar b\bar c=\bar d\bar b\bar e\bar b\bar f)\wedge(\bar b\to\bar g)\wedge\neg(\bar b\subseteq\{\bar a,\bar c,\bar d,\bar e,\bar f\})\Rightarrow(\bar a\bar g\bar c=\bar d\bar g\bar e\bar g\bar f),$$
$$(OA.113)\quad (\bar a\bar b\bar c\bar b\bar d=\bar e\bar b\bar f\bar b\bar g)\wedge(\bar b\to\bar h)\wedge\neg(\bar b\subseteq\{\bar a,\bar c,\bar d,\bar e,\bar f,\bar g\})\Rightarrow(\bar a\bar h\bar c\bar h\bar d=\bar e\bar h\bar f\bar h\bar g),$$
$$(OA.114)\quad (\bar a\le\bar b)\Leftrightarrow((\bar a<\bar b)\vee(\bar a=\bar b)),$$
$$(OA.115)\quad (\bar a\le\bar b)\wedge(\bar b\le\bar c)\Rightarrow(\bar a\le\bar c)$$
$$\}.$$

In the following, we will deduce some numbers and equalities as examples.

$$(A1)\quad (\dot a\to 1|[\dot a\dot b\dot a])\Rightarrow(\dot a\to 1)\qquad by(OA.16),(OA.88)$$
$$(A2)\quad \Rightarrow(\dot a\to[\dot a\dot b\dot a])\qquad by(OA.16),(OA.88)$$
$$(A3)\quad \Rightarrow(\dot a\to[1\dot b\dot a])\qquad by(A2),(A1),(OA.87)$$
$$(A4)\quad \Rightarrow(\dot a\to[1\dot b1])\qquad by(A3),(A1),(OA.87)$$
$$(A5)\quad (\dot b\to +|-)\Rightarrow(\dot b\to -)\qquad by(OA.17),(OA.88)$$
$$(A6)\quad \Rightarrow(\dot a\to[1-1])\qquad by(A4),(A5),(OA.87)$$
$$(A7)\quad (\dot a<[1+\dot a])\Rightarrow(1<[1+1])\qquad by(OA.30),(A1),(OA.99)$$
$$(A8)\quad \Rightarrow 1\qquad by(OA.93)$$
$$(A9)\quad \Rightarrow[1+1]\qquad by(A7),(OA.93)$$
$$(A10)\quad (\dot a<[1+\dot a])\Rightarrow([1-1]<[1+[1-1]])\qquad by(OA.30),(A6),(OA.99)$$
$$(A11)\quad \Rightarrow([1-1]<[[1-1]+1])\qquad by(A10),(OA.46),(OA.94)$$
$$(A12)\quad \Rightarrow([1-1]<1)\qquad by(A11),(OA.47),(OA.94)$$
$$(A13)\quad \Rightarrow[1-1]\qquad by(OA.93)$$
$$(A14)\quad (\dot b\to +|-)\Rightarrow(\dot b\to +)\qquad by(OA.17),(OA.88)$$
$$(A15)\quad \Rightarrow(\dot a\to[1+1])\qquad by(A4),(A14),(OA.87)$$
$$(A16)\quad (\dot a<[1+\dot a])\Rightarrow([1+1]<[1+[1+1]])\qquad by(OA.30),(A15),(OA.99)$$
$$(A17)\quad \Rightarrow([1-1]<[1+1])\qquad by(A12),(A7),(OA.92)$$
$$(A18)\quad \Rightarrow\neg([1+1]=[1-1])\qquad by(A17),(OA.91)$$
$$(A19)\quad \Rightarrow([[1-1]--[1+1]]<[1--[1+1]])\qquad by(A12),(A17),(OA.32)$$
$$(A20)\quad \Rightarrow[1--[1+1]]\qquad by(OA.93)$$
$$(A21)\quad \Rightarrow([[1-1]--[1+1]]=[[1--[1+1]]-[1--[1+1]]])\qquad by(A8),(A18),(OA.60)$$
$$(A22)\quad \Rightarrow([[1--[1+1]]-[1--[1+1]]]=[1-1])\qquad by(A20),(OA.45)$$

$(A23)$	$\Rightarrow ([[1-1]--[1+1]]=[1-1])$	$by(A21),(A22),(OA.107)$
$(A24)$	$\Rightarrow ([1-1]<[1--[1+1]])$	$by(A19),(A23),(OA.95)$
$(A25)$	$\Rightarrow ([1--[1+1]]<[[1+1]--[1+1]])$	$by(A12),(A7),$ $(A17),(OA.32)$
$(A26)$	$\Rightarrow ([[1+1]--[1+1]]=1)$	$by(A18),(OA.54)$
$(A27)$	$\Rightarrow ([1--[1+1]]<1)$	$by(A25),(A26),(OA.94)$
$(A28)$	$([[1-1]+1]<[[1--[1+1]]+1])$	$by(A24),(A8),$ $(OA.31)$
$(A29)$	$(1<[[1--[1+1]]+1])$	$by(A28),(OA.47),$ $(OA.95)$
$\vdots$	$\vdots$	$\vdots$

Then according to Definition 2.1, we can deduce from $R\{\Phi,\Psi\}$ the numbers as follows:

$$\{1,[1+1],[1-1],[1+[1+1]],[1-[1+1]],[1--[1+[1+1]]],$$
$$[[1+[1+1]]----[1+1]],[[1+[1+1]]--[1+1]]\cdots\}$$

$(B1)$	$(\bar{a}=[1+[1+1]])\wedge(\bar{b}=[1--[1+1]])\wedge$ $(\bar{c}=[[\bar{a}-1]++[1+[1+1]]])\wedge(\bar{d}=[[\bar{a}-1]++[1+1]])\wedge$ $(\bar{e}=[\bar{b}+++[1+1]])\wedge(\bar{f}=[\bar{b}+++[1+[1+1]]])$	by (Premise)
$(B2)$	$\Rightarrow ([[1+[1+1]]++++[1--[1+1]]]=[1+1])$	$by(A7),(A24),$ $(A27),(OA.83)$
$(B3)$	$([[1-1]+1]<[[1--[1+1]]+1])$	$by(A24),(A8),$ $(OA.31)$
$(B4)$	$[[1-1]+1]=1$	$by(A13),(A8),$ $(OA.47)$
$(B5)$	$\Rightarrow (1<[[1--[1+1]]+1])$	$by(B3),(B4),$ $(OA.95)$
$(B6)$	$([[1+[1+1]]+++\dot{e}[[1--[1+1]]+1]]=$ $[[1+[1+1]]++\dot{e}[[1+[1+1]]+++\dot{e}$ $[[[1--[1+1]]+1]-1]]])$	$by(A7),(B5),$ $(OA.82)$
$(B7)$	$\Rightarrow ([[1+[1+1]]++++[[1--[1+1]]+1]]=$ $[[1+[1+1]]+++[[1+[1+1]]++++$ $[[[1--[1+1]]+1]-1]]])$	$by(B6),(OA.19),$ $(OA.112)$
$(B8)$	$([[[1--[1+1]]+1]-1]=[[[1--[1+1]]-1]+1])$	$by(A20),(A8),$ $(OA.48),(OA.105)$
$(B9)$	$\Rightarrow ([[[1--[1+1]]-1]+1]=[1--[1+1]])$	$by(A20),(A8),$

$$(OA.47)$$

$$(B10) \quad \Rightarrow ([[[1--[1+1]]+1]-1]=[1--[1+1]]) \quad by(B8),(B9),$$
$$(OA.107)$$

$$(B11) \quad \Rightarrow ([[1+[1+1]]++++[[1--[1+1]]+1]] =$$
$$[[1+[1+1]]+++[[1+[1+1]]++++[1--[1+1]]]]]) \quad by(B7),(B10),$$
$$(OA.107)$$

$$(B12) \quad \Rightarrow ([[1+[1+1]]++++[[1--[1+1]]+1]] =$$
$$[[1+[1+1]]+++[1+1]]) \quad by(B11),(B2),$$
$$(OA.107)$$

$$\vdots \qquad \vdots \qquad \vdots$$

Then we can deduce from $R\{\Phi, \Psi\}$ the equalities as follows:

$$\begin{aligned} [[1+1]++[[1+1]---[1+1]]] &= [[[1+1]---[1+1]]++[1+1]], \\ [[1+[1+1]]++++[[1--[1+1]]+1]] &= [[1+[1+1]]+++[1+1]], \\ \vdots \quad &\vdots \quad \ddots \end{aligned}$$

The deduced numbers correspond to real numbers as follows:

$$\begin{aligned} \vdots \quad &\vdots \quad \ddots, \\ [[[1+[1+1]]----[1+1]]++[1+1]] &\equiv \\ \vdots \quad &\vdots \quad \ddots, \\ [1+[1+1]] &\equiv 3, \\ \vdots \quad &\vdots \quad \ddots, \\ [[1+1]+[1--[1+1]]] &\equiv \frac{5}{2}, \\ \vdots \quad &\vdots \quad \ddots, \\ [1+1] &\equiv 2, \\ \vdots \quad &\vdots \quad \ddots, \\ [[1+[1+1]]----[1+1]] &\equiv \\ \vdots \quad &\vdots \quad \ddots, \\ [[1+[1+1]]///[1+1]] &\equiv \log_2 3, \\ \vdots \quad &\vdots \quad \ddots, \\ [1+[1--[1+1]]] &\equiv \frac{3}{2}, \\ \vdots \quad &\vdots \quad \ddots, \\ [[1+1]---[1+1]] &\equiv \sqrt[2]{2}, \\ \vdots \quad &\vdots \quad \ddots, \end{aligned}$$

$$\begin{array}{rcl}
[[1+1]---[1+[1+1]]] & \equiv & \sqrt[3]{2}, \\
\vdots & \vdots & \vdots, \\
1 & \equiv & 1, \\
\vdots & \vdots & \vdots, \\
[[1+1]///[1+[1+1]]] & \equiv & \log_3 2, \\
\vdots & \vdots & \vdots, \\
[1--[1+1]] & \equiv & \frac{1}{2}, \\
\vdots & \vdots & \vdots, \\
[1-1] & \equiv & 0, \\
\vdots & \vdots & \vdots, \\
[[1-1]-[1--[1+1]]] & \equiv & -\frac{1}{2}, \\
\vdots & \vdots & \vdots, \\
[[1-1]-1] & \equiv & -1, \\
\vdots & \vdots & \vdots, \\
[[1-1]-[[1+1]---[1+[1+1]]]] & \equiv & -\sqrt[3]{2}, \\
\vdots & \vdots & \vdots, \\
[[1-1]-[[1+1]---[1+1]]] & \equiv & -\sqrt[2]{2}, \\
\vdots & \vdots & \vdots, \\
[[1-1]-[[1+[1+1]]----[1+1]]] & \equiv & \\
\vdots & \vdots & \vdots.
\end{array}$$

The equalities on deduced numbers correspond to addition, subtraction, multiplication, division, exponentiation operation, root-extraction operation, logarithm operation and more other operations in real number system. These deduced numbers hold the consist order relation and equivalence relation, so the logical calculus $R\{\Phi,\Psi\}$ is a consist axiom.

Since $Q\{\Phi,\Psi\}$ is dense in $R\{\Phi,\Psi\}$(with its standard topology), every deduced number has rational numbers arbitrary close to it. So every deduced number holds only one position in number line.

Note that in the correspondence above, some deduced numbers such as $[[[1+[1+1]]----[1+1]]++[1+1]]$, $[[1+[1+1]]----[1+1]]$ and $[[1-1]-[[1+[1+1]]----[1+1]]]$ do not correspond to any known real numbers. So the logical calculus $R\{\Phi,\Psi\}$ can deduce more real numbers than before.

In fact, the logical calculus $R\{\Phi,\Psi\}$ not only deduces more real numbers than before, but also makes its deduced numbers join in algebraical operations. So the logical calculus $R\{\Phi,\Psi\}$ intuitively and logically denote real number system.

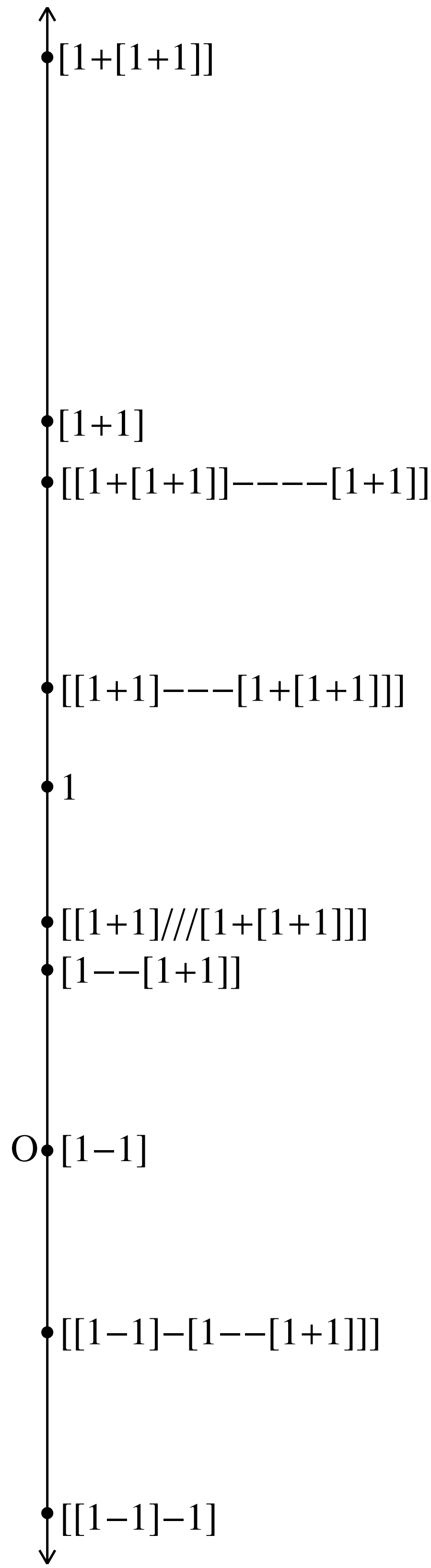

FIGURE 1. The semantics of the Operator axioms

2.2. **Semantics Of The Operator Axioms.** The syntax of the Operator axioms may map to many semantics, but only one semantics is shown here to explain the syntax of the Operator axioms. As shown in Figure 1, one semantics of the Operator axioms is a line. We will explain the semantics according to Figure 1 as follows.

The symbol 'O' in Figure 1 stand for "the origin". The number '1' maps to "a point move up unit length from the origin". The symbol '$>$' maps to "upon". The symbol '$<$' maps to "under". The symbol '$=$' maps to "overlap". The numbers such as "$[1+1]$", "$[[1+[1+1]]----[1+1]]$", "$[[1+1]---[1+[1+1]]]$", "$[1-1]$" map to the points arranged in the line in Figure 1. The number "$[1-1]$" overlap "the origin O".

Supposing that 'p', 'q', "$[p+q]$" are numbers, then "$[p+q]$" maps to "a point move up q units from the point p". The symbol "++" maps to "iteratively +". The symbol "$+++$" maps to "iteratively ++". And so on, the symbols "$++++$", "$+++++$", "$++++++$", $\cdots$ can map to similar semantics.

Supposing that 'p', 'q', "$[p-q]$" are numbers, then "$[p-q]$" maps to "a point can move to the point p by the operation $[[p-q]+q]$". Supposing that 'p', 'q', "$[p--q]$" are numbers, then "$[p--q]$" maps to "a point can move to the point p by the operation $[[p--q]++q]$". Supposing that 'p', 'q', "$[p---q]$" are numbers, then "$[p---q]$" maps to "a point can move to the point p by the operation $[[p---q]+++q]$". And so on, the symbols "$----$", "$-----$", "$------$", $\cdots$ can map to similar semantics.

Supposing that 'p', 'q', "$[p/q]$" are numbers, then "$[p/q]$" maps to "a point can move to the point p by the operation $[q+[p/q]]$". Supposing that 'p', 'q', "$[p//q]$" are numbers, then "$[p//q]$" maps to "a point can move to the point p by the operation $[q++[p//q]]$". Supposing that 'p', 'q', "$[p///q]$" are numbers, then "$[p///q]$" maps to "a point can move to the point p by the operation $[q+++[p///q]]$". And so on, the symbols "////", "/////", "//////", $\cdots$ can map to similar semantics.

3. Fundamental Operator Functions

On the basis of the Operator axioms, we define two fundamental operator functions in this section to study the analytic properties of the Operator axioms. It is supposed that the constant $d \in R$ with $[1-1] < d$. It is supposed that the constant $e \in R$ with $1 < e$.

Definition 3.1. *VE Function* is the function $f : \big[1, +\infty\big) \to R$ defined by $f(x) = [x+++\dot{e}d]$.

Definition 3.2. *EV Function* is the function $f : R \to R$ defined by $f(x) = [e+++\dot{e}x]$.

Definition 3.3. *Fundamental operator functions* are VE Function and EV Function.

Theorem 3.4. *The VE Function $f(x) = [x+++\dot{e}d]$ is continuous, unbounded and strictly increasing.*

Proof. According to (OA.19), the symbol '$\dot{e}$' represents some successive '+'—"$+\cdots+$". According to (OA.20), the symbol '$\dot{f}$' represents some successive '$-$'—"$-\cdots-$". According to (OA.21), the symbol '$\dot{g}$' represents some successive '/'—"$/\cdots/$".

$$
\begin{array}{lll}
(A1) & \text{Supposing that } x_1, x_2 \in [1, +\infty) \text{ with } x_1 < x_2. & \\
(A2) & \Rightarrow (1 \leq x_1) \wedge (x_1 < x_2) & \\
(A3) & [1-1] < d & \text{by (Premise)} \\
(A4) & \Rightarrow [x_1 +++ \dot{e}d] < [x_2 +++ \dot{e}d] & \text{by (A2),(A3),(OA.37)} \\
(A5) & f(x_1) = [x_1 +++ \dot{e}d] & \text{by (Premise)} \\
(A6) & \Rightarrow f(x_1) < [x_2 +++ \dot{e}d] & \text{by (A4),(A5),(OA.95)} \\
(A7) & f(x_2) = [x_2 +++ \dot{e}d] & \text{by (Premise)} \\
(A8) & \Rightarrow f(x_1) < f(x_2) & \text{by (A6),(A7),(OA.94)}
\end{array}
$$

(A1)$\sim$(A8) derive that $f(x)$ is strictly increasing.

For any number $x_0 \in \big[1, +\infty\big)$ and any number $[1-1] < \varepsilon$, we can always construct δ_0 as follows:

$$
\begin{array}{lll}
(B1) & [1 +++ \dot{e}d] \leq [x_0 +++ \dot{e}d] & \text{by (OA.37),(OA.104)} \\
(B2) & \Rightarrow 1 \leq [x_0 +++ \dot{e}d] & \text{by (OA.75),(OA.95)} \\
(B3) & \Rightarrow [1-1] \leq [[x_0 +++ \dot{e}d] - 1] & \text{by (OA.31)} \\
(B4) & [1-1] < \varepsilon & \text{by (Premise)} \\
(B5) & \Rightarrow [[1-1] - -[1+1]] < [\varepsilon - -[1+1]] & \text{by (OA.32)} \\
(B6) & \Rightarrow [[1 - -[1+1]] - [1 - -[1+1]]] < [\varepsilon - -[1+1]] & \text{by (OA.60), (OA.95)} \\
(B7) & \Rightarrow [1-1] < [\varepsilon - -[1+1]] & \text{by (OA.45), (OA.95)} \\
(B8) & \Rightarrow [[1-1] + [\varepsilon - -[1+1]]] < [[\varepsilon - -[1+1]] + [\varepsilon - -[1+1]]] & \text{by (OA.31)} \\
(B9) & \Rightarrow [[1-1] + [\varepsilon - -[1+1]]] < [[\varepsilon + \varepsilon] - -[1+1]] & \text{by (OA.60), (OA.94)} \\
(B10) & \Rightarrow [[\varepsilon - -[1+1]] + [1-1]] < [[\varepsilon + \varepsilon] - -[1+1]] & \text{by (OA.46), (OA.95)} \\
(B11) & \Rightarrow [[[\varepsilon - -[1+1]] + 1] - 1] < [[\varepsilon + \varepsilon] - -[1+1]] & \text{by (OA.50), (OA.95)} \\
(B12) & \Rightarrow [[[\varepsilon - -[1+1]] - 1] + 1] < [[\varepsilon + \varepsilon] - -[1+1]] & \text{by (OA.48), (OA.95)} \\
(B13) & \Rightarrow [\varepsilon - -[1+1]] < [[\varepsilon + \varepsilon] - -[1+1]] & \text{by (OA.47), (OA.95)} \\
(B14) & \Rightarrow [\varepsilon - -[1+1]] < [[\varepsilon + +[1+1]] - -[1+1]] & \text{by (OA.57),(OA.53),} \\
 & & \text{(OA.94)} \\
(B15) & \Rightarrow [\varepsilon - -[1+1]] < [\varepsilon + +[[1+1] - -[1+1]]] & \text{by (OA.61),(OA.94)} \\
(B16) & \Rightarrow [\varepsilon - -[1+1]] < [\varepsilon + +1] & \text{by (OA.54),(OA.94)} \\
(B17) & \Rightarrow [\varepsilon - -[1+1]] < \varepsilon & \text{by (OA.53),(OA.94)} \\
(B18) & \delta_0 = [\varepsilon - -[1+1]] &
\end{array}
$$

We construct δ according to $[[x_0 +++ \dot{e}d] - \delta_0]$.

(1) $1 \leq [[x_0 +++ \dot{e}d] - \delta_0]$.

We construct δ as follows:

$$(C1)\quad \delta_1 = [x_0 - [[[x_0 + + + \dot{h}d] - \delta_0] - - - \dot{i}d]]$$

$$(C2)\quad \delta_2 = [[[[x_0 + + + \dot{h}d] + \delta_0] - - - \dot{i}d] - x_0]$$

$$(C3)\quad \delta = \min\{\delta_1, \delta_2\}$$

$(D1)$	$\delta_0 = [\varepsilon - -[1+1]]$	by (B18)
$(D2)$	$\Rightarrow [1-1] < \delta_0$	by (B7),(D1), (OA.94)
$(D3)$	$[\varepsilon - -[1+1]] < \varepsilon$	by (B17)
$(D4)$	$\Rightarrow \delta_0 < \varepsilon$	by (D3),(D1), (OA.95)
$(D5)$	$\delta = \min\{\delta_1, \delta_2\}$	by (C3)
$(D6)$	$\Rightarrow \delta \le \delta_1$	
$(D7)$	$\Rightarrow \delta \le \delta_2$	
$(D8)$	$\delta_1 = [x_0 - [[[x_0 + + + \dot{h}d] - \delta_0] - - - \dot{i}d]]$	by (C1)
$(D9)$	$\Rightarrow [x_0 - \delta_1] = [x_0 - [x_0 - [[[x_0 + + + \dot{h}d] - \delta_0] - - - \dot{i}d]]]$	by (OA.108)
$(D10)$	$\Rightarrow [x_0 - \delta_1] = [[x_0 - x_0] + [[[x_0 + + + \dot{h}d] - \delta_0] - - - \dot{i}d]]$	by (OA.52),(OA.107)
$(D11)$	$\Rightarrow [x_0 - \delta_1] = [[[[x_0 + + + \dot{h}d] - \delta_0] - - - \dot{i}d] + [x_0 - x_0]]$	by (OA.46),(OA.107)
$(D12)$	$\Rightarrow [x_0 - \delta_1] = [[[[[x_0 + + + \dot{h}d] - \delta_0] - - - \dot{i}d] + x_0] - x_0]$	by (OA.50),(OA.107)
$(D13)$	$\Rightarrow [x_0 - \delta_1] = [[[[[x_0 + + + \dot{h}d] - \delta_0] - - - \dot{i}d] - x_0] + x_0]$	by (OA.48),(OA.107)
$(D14)$	$\Rightarrow [x_0 - \delta_1] = [[[x_0 + + + \dot{h}d] - \delta_0] - - - \dot{i}d]$	by (OA.47),(OA.107)
$(D15)$	$1 \le [[[x_0 + + + \dot{h}d] - \delta_0] - - - \dot{i}d]$	by (Premise),(OA.34), (OA.76)
$(D16)$	$\Rightarrow 1 \le [x_0 - \delta_1]$	by (D14),(D15), (OA.94)
$(D17)$	$\Rightarrow [x_0 - \delta_1] \le [x_0 - \delta]$	by (D6),(OA.31)
$(D18)$	$\Rightarrow [[x_0 - \delta_1] + + + \dot{h}d] \le [[x_0 - \delta] + + + \dot{h}d]$	by (D17),(OA.37), (OA.108)
$(D19)$	$\Rightarrow [[[[x_0 + + + \dot{h}d] - \delta_0] - - - \dot{i}d] + + + \dot{h}d] \le [[x_0 - \delta] + + + \dot{h}d]$	by (D18),(D14), (OA.95)
$(D20)$	$\Rightarrow [[x_0 + + + \dot{h}d] - \delta_0] \le [[x_0 - \delta] + + + \dot{h}d]$	by (OA.79),(OA.95)
$(D21)$	$\Rightarrow [[x_0 + + + \dot{h}d] - \varepsilon] < [[x_0 + + + \dot{h}d] - \delta_0]$	by (D4),(OA.31)
$(D22)$	$\Rightarrow [[x_0 + + + \dot{e}d] - \varepsilon] < [[x_0 - \delta] + + + \dot{e}d]$	by (D21),(D20), (OA.92)
$(D23)$	$\Rightarrow [f(x_0) - \varepsilon] < f([x_0 - \delta])$	by (D22),(Premise)

$(D24)$	$\delta_2 = [[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] - x_0]$	by (C2)
$(D25)$	$\Rightarrow [x_0 + \delta_2] = [x_0 + [[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] - x_0]]$	by (OA.108)
$(D26)$	$\Rightarrow [x_0 + \delta_2] = [[[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] - x_0] + x_0]$	by (OA.46),(OA.107)
$(D27)$	$\Rightarrow [x_0 + \delta_2] = [[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d]$	by (OA.47),(OA.107)
$(D28)$	$[1 - 1] < \delta_0$	by (D2)
$(D29)$	$\Rightarrow [[1 - 1] + 1] < [\delta_0 + 1]$	by (D28),(OA.31)
$(D30)$	$\Rightarrow 1 < [\delta_0 + 1]$	by (OA.47),(OA.95)
$(D31)$	$\Rightarrow 1 < [1 + \delta_0]$	by (OA.46),(OA.94)
$(D32)$	$1 \leq [x_0 +++ \dot{h}d]$	by (B2)
$(D33)$	$\Rightarrow [1 + \delta_0] \leq [[x_0 +++ \dot{h}d] + \delta_0]$	by (OA.31)
$(D34)$	$\Rightarrow 1 < [[x_0 +++ \dot{h}d] + \delta_0]$	by (D31),(D33), (OA.92)
$(D35)$	$\Rightarrow 1 < [[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d]$	by (OA.34)
$(D36)$	$\Rightarrow [[x_0 + \delta_2] +++ \dot{h}d] =$ $[[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] +++ \dot{h}d]$	by (D27),(D35), (OA.108)
$(D37)$	$\Rightarrow [[x_0 + \delta_2] +++ \dot{h}d] = [[x_0 +++ \dot{h}d] + \delta_0]$	by (OA.79),(OA.107)
$(D38)$	$\delta_0 < \varepsilon$	by (D4)
$(D39)$	$\Rightarrow [\delta_0 + [x_0 +++ \dot{h}d]] < [\varepsilon + [x_0 +++ \dot{h}d]]$	by (OA.31)
$(D40)$	$\Rightarrow [[x_0 +++ \dot{h}d] + \delta_0] < [\varepsilon + [x_0 +++ \dot{h}d]]$	by (OA.46),(OA.95)
$(D41)$	$\Rightarrow [[x_0 +++ \dot{h}d] + \delta_0] < [[x_0 +++ \dot{h}d] + \varepsilon]$	by (OA.46),(OA.94)
$(D42)$	$\Rightarrow [[x_0 + \delta_2] +++ \dot{h}d] < [[x_0 +++ \dot{h}d] + \varepsilon]$	by (D41),(D37), (OA.95)
$(D43)$	$\Rightarrow [[x_0 +++ \dot{h}d] - \delta_0] < [[x_0 +++ \dot{h}d] - [1 - 1]]$	by (D2),(OA.31)
$(D44)$	$\Rightarrow [[x_0 +++ \dot{h}d] - \delta_0] < [[[x_0 +++ \dot{h}d] - 1] + 1]$	by (OA.52),(OA.94)
$(D45)$	$\Rightarrow [[x_0 +++ \dot{h}d] - \delta_0] < [x_0 +++ \dot{h}d]$	by (OA.47),(OA.94)
$(D46)$	$\Rightarrow [[x_0 +++ \dot{h}d] - \delta_0] =$ $[[[[x_0 +++ \dot{h}d] - \delta_0] --- \dot{i}d] +++ \dot{h}d]$	by (Premise),(OA.79)
$(D47)$	$\Rightarrow [x_0 +++ \dot{h}d] =$ $[[[x_0 +++ \dot{h}d] --- \dot{i}d] +++ \dot{h}d]$	by (B2),(OA.79)
$(D48)$	$\Rightarrow [[[[x_0 +++ \dot{h}d] - \delta_0] --- \dot{i}d] +++ \dot{h}d] <$ $[[[x_0 +++ \dot{h}d] --- \dot{i}d] +++ \dot{h}d]$	by (D45),(D46), (D47),(OA.94), (OA.95)
$(D49)$	$\Rightarrow [[[x_0 +++ \dot{h}d] - \delta_0] --- \dot{i}d] < [[x_0 +++ \dot{h}d] --- \dot{i}d]$	by (D15),(D48),

(OA.34),(OA.39)

($D50$) $\Rightarrow [[[x_0 +++ \dot{h}d] - \delta_0] --- \dot{i}d] < x_0$ by (Premise),(OA.79), (OA.94)

($D51$) $\Rightarrow [x_0 - x_0] < [x_0 - [[[x_0 +++ \dot{h}d] - \delta_0] --- \dot{i}d]]$ by (OA.31)

($D52$) $\Rightarrow [x_0 - x_0] < \delta_1$ by (C1),(OA.94)

($D53$) $\Rightarrow [1 - 1] < \delta_1$ by (OA.45),(OA.95)

($D54$) $\Rightarrow [[1 - 1] + [x_0 +++ \dot{h}d]] < [\delta_0 + [x_0 +++ \dot{h}d]]$ by (D2),(OA.31)

($D55$) $\Rightarrow [[1 - 1] + [x_0 +++ \dot{h}d]] < [[x_0 +++ \dot{h}d] + \delta_0]$ by (OA.46),(OA.94)

($D56$) $\Rightarrow [[x_0 +++ \dot{h}d] + [1 - 1]] < [[x_0 +++ \dot{h}d] + \delta_0]$ by (OA.46),(OA.95)

($D57$) $\Rightarrow [[[x_0 +++ \dot{h}d] + 1] - 1] < [[x_0 +++ \dot{h}d] + \delta_0]$ by (OA.50),(OA.95)

($D58$) $\Rightarrow [[[x_0 +++ \dot{h}d] - 1] + 1] < [[x_0 +++ \dot{h}d] + \delta_0]$ by (OA.48),(OA.95)

($D59$) $\Rightarrow [x_0 +++ \dot{h}d] < [[x_0 +++ \dot{h}d] + \delta_0]$ by (OA.47),(OA.95)

($D60$) $\Rightarrow [x_0 +++ \dot{h}d] = [[[x_0 +++ \dot{h}d] --- \dot{i}d] +++ \dot{h}d]$ by (B2),(OA.79)

($D61$) $\Rightarrow [[x_0 +++ \dot{h}d] + \delta_0] = [[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] +++ \dot{h}d]$ by (D34),(OA.79)

($D62$) $\Rightarrow [[[x_0 +++ \dot{h}d] --- \dot{i}d] +++ \dot{h}d] < [[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] +++ \dot{h}d]$ by (D59),(D60), (D61),(OA.94), (OA.95)

($D63$) $\Rightarrow [[x_0 +++ \dot{h}d] --- \dot{i}d] < [[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d]$ by (D35),(D62), (OA.39)

($D64$) $\Rightarrow x_0 < [[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d]$ by (Premise),(OA.79), (OA.95)

($D65$) $\Rightarrow [x_0 - x_0] < [[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] - x_0]$ by (OA.31)

($D66$) $\Rightarrow [1 - 1] < [[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] - x_0]$ by (OA.45),(OA.95)

($D67$) $\Rightarrow [1 - 1] < \delta_2$ by (C2),(OA.94)

($D68$) $[1 - 1] < \delta$ by (D5),(D53), (D67)

($D69$) $\Rightarrow [[1 - 1] + 1] < [\delta + 1]$ by (OA.31)

($D70$) $\Rightarrow 1 < [\delta + 1]$ by (OA.47),(OA.95)

($D71$) $\Rightarrow 1 < [1 + \delta]$ by (OA.46),(OA.94)

($D72$) $1 \leq x_0$ by (Premise)

($D73$) $\Rightarrow [1 + \delta] \leq [x_0 + \delta]$ by (OA.31)

($D74$) $\Rightarrow 1 < [x_0 + \delta]$ by (D71),(D73),

		(OA.92)
$(D75)$	$\Rightarrow [\delta + x_0] \leq [\delta_2 + x_0]$	by (D7),(OA.31)
$(D76)$	$\Rightarrow [x_0 + \delta] \leq [\delta_2 + x_0]$	by (OA.46),(OA.95)
$(D77)$	$\Rightarrow [x_0 + \delta] \leq [x_0 + \delta_2]$	by (OA.46),(OA.94)
$(D78)$	$\Rightarrow [[x_0 + \delta] +++ \dot{h}d] \leq [[x_0 + \delta_2] +++ \dot{h}d]$	by (D74),(D77), (OA.37)
$(D79)$	$\Rightarrow [[x_0 + \delta] +++ \dot{e}d] < [[x_0 +++ \dot{e}d] + \varepsilon]$	by (D78),(D42), (OA.92)
$(D80)$	$\Rightarrow f([x_0 + \delta]) < [f(x_0) + \varepsilon]$	by (D79),(Premise)

Since $f(x)$ is strictly increasing, $f([x_0 - \delta]) < f(x)$ and $f(x) < f([x_0 + \delta])$ hold for all $x \in \big([x_0 - \delta], [x_0 + \delta]\big)$. (D23) and (D80) derive that $[f(x_0) - \varepsilon] < f(x)$ and $f(x) < [f(x_0) + \varepsilon]$ hold for all $x \in \big([x_0 - \delta], [x_0 + \delta]\big)$.

(2) $[[x_0 +++ ed] - \delta_0] < 1$.

We construct δ as follows:

$$(E1) \quad \delta = [[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] - x_0]$$

$(F1)$	$\dot{a} \to 1\vert[\dot{a}\dot{b}\dot{a}]$	by (OA.16)
$(F2)$	$\Rightarrow (\dot{a} \to 1)$	by (OA.88)
$(F3)$	$\Rightarrow (\dot{a} \to [\dot{a}\dot{b}\dot{a}])$	by (F1),(OA.88)
$(F4)$	$\Rightarrow (\dot{a} \to [1\dot{b}\dot{a}])$	by (F3),(F2),(OA.87)
$(F5)$	$\Rightarrow (\dot{a} \to [1\dot{b}1])$	by (F3),(F2),(OA.87)
$(F6)$	$\dot{b} \to +\vert-$	by (OA.17)
$(F7)$	$\Rightarrow (\dot{b} \to -)$	by (OA.88)
$(F8)$	$\Rightarrow (\dot{a} \to [1-1])$	by (F5),(F7),(OA.87)
$(F9)$	$\dot{a} < [1 + \dot{a}]$	by (OA.30)
$(F10)$	$\Rightarrow 1 < [1 + 1]$	by (F9),(F2),(OA.99)
$(F11)$	$\Rightarrow 1$	by (OA.93)
$(F12)$	$\Rightarrow [1-1] < [1 + [1-1]]$	by (F9),(F8),(OA.99)
$(F13)$	$\Rightarrow [1-1] < [[1-1] + 1]$	by (OA.46),(OA.94)
$(F14)$	$\Rightarrow [1-1] < 1$	by (F11),(OA.47), (F13),(OA.94)
$(F15)$	$\Rightarrow [1-1] < [x_0 +++ \dot{h}d]$	by (F14),(B2), (OA.94),(OA.92)
$(F16)$	$\Rightarrow [1-1]$	by (OA.93)
$(F17)$	$\Rightarrow [x_0 +++ \dot{h}d]$	by (F15),(OA.93)
$(F18)$	$\delta_0 = [\varepsilon --[1+1]]$	by (B18)
$(F19)$	$\Rightarrow [1-1] < \delta_0$	by (B7),(F18),

(OA.94)

$(F20)$ $\Rightarrow [[1-1]+[x_0+++\dot{h}d]] < [\delta_0+[x_0+++\dot{h}d]]$ by (F19),(F17),(OA.31)

$(F21)$ $\Rightarrow [[x_0+++\dot{h}d]+[1-1]] < [\delta_0+[x_0+++\dot{h}d]]$ by (F16),(F17),(OA.46),(OA.95)

$(F22)$ $\Rightarrow [[[x_0+++\dot{h}d]+1]-1] < [\delta_0+[x_0+++\dot{h}d]]$ by (OA.50),(OA.95)

$(F23)$ $\Rightarrow [[[x_0+++\dot{h}d]-1]+1] < [\delta_0+[x_0+++\dot{h}d]]$ by (OA.48),(OA.95)

$(F24)$ $\Rightarrow [x_0+++\dot{h}d] < [\delta_0+[x_0+++\dot{h}d]]$ by (OA.47),(OA.95)

$(F25)$ $\Rightarrow [x_0+++\dot{h}d] < [[x_0+++\dot{h}d]+\delta_0]$ by (OA.46),(OA.94)

$(F26)$ $\Rightarrow 1 < [[x_0+++\dot{h}d]+\delta_0]$ by (B2),(F25),(OA.95),(OA.92)

$(F27)$ $\Rightarrow [x_0+++\dot{h}d] = [[[x_0+++\dot{h}d]---\dot{i}d]+++\dot{h}d]$ by (B2),(OA.79),(OA.94)

$(F28)$ $\Rightarrow [[x_0+++\dot{h}d]+\delta_0] = [[[[x_0+++\dot{h}d]+\delta_0]---\dot{i}d]+++\dot{h}d]$ by (F26),(OA.79),(OA.94)

$(F29)$ $\Rightarrow [[[x_0+++\dot{h}d]---\dot{i}d]+++\dot{h}d] < [[x_0+++\dot{h}d]+\delta_0]$ by (F25),(F27),(OA.95)

$(F30)$ $\Rightarrow [[[x_0+++\dot{h}d]---\dot{i}d]+++\dot{h}d] < [[[[x_0+++\dot{h}d]+\delta_0]---\dot{i}d]+++\dot{h}d]$ by (F29),(F28),(OA.94)

$(F31)$ $\Rightarrow 1 \leq [[x_0+++\dot{h}d]---\dot{i}d]$ by (B2),(OA.34),(OA.76)

$(F32)$ $\Rightarrow 1 < [[[x_0+++\dot{h}d]+\delta_0]---\dot{i}d]$ by (F26),(OA.34)

$(F33)$ $\Rightarrow [[x_0+++\dot{h}d]---\dot{i}d] < [[[x_0+++\dot{h}d]+\delta_0]---\dot{i}d]$ by (F31),(F32),(F30),(OA.39)

$(F34)$ $\Rightarrow x_0 < [[[x_0+++\dot{h}d]+\delta_0]---\dot{i}d]$ by (OA.79),(OA.95)

$(F35)$ $\Rightarrow [x_0-x_0] < [[[[x_0+++\dot{h}d]+\delta_0]---\dot{i}d]-x_0]$ by (OA.31)

$(F36)$ $\Rightarrow [1-1] < [[[[x_0+++\dot{h}d]+\delta_0]---\dot{i}d]-x_0]$ by (OA.45),(F35),(OA.95)

$(F37)$ $\Rightarrow [1-1] < \delta$ by (F36),(E1),(OA.94)

$(F38)$ $[[x_0+++\dot{h}d]-\delta_0] < 1$ by (Premise)

$(F39)$ $\Rightarrow [[x_0+++\dot{h}d]-[\varepsilon--[1+1]]] < 1$ by (B18),(OA.95)

$(F40)$ $[1-1] < \varepsilon$ by (Premise)

$(F41)$ $\Rightarrow [\varepsilon--[1+1]] < [\varepsilon--1]$ by (F40),(F10),(F14),(OA.33)

$(F42)$ $[\varepsilon--1] = \varepsilon$ by (OA.53)

$(F43)$ $\Rightarrow [\varepsilon--[1+1]] < \varepsilon$ by (F41),(F42),(OA.94)

$(F44)$ $\Rightarrow [[x_0+++\dot{h}d]-\varepsilon] < [[x_0+++\dot{h}d]-[\varepsilon--[1+1]]]$ by (OA.31)

$(F45)$	$\Rightarrow [[x_0 +++ \dot{e}d] - \varepsilon] < 1$	by (F44),(F39),(OA.92)
$(F46)$	$\Rightarrow [f(x_0) - \varepsilon] < 1$	
$(F47)$	$f(1) = [1 +++ \dot{e}d]$	
$(F48)$	$\Rightarrow f(1) = 1$	by (OA.75)
$(F49)$	$\Rightarrow 1 \leq f(x)$	by (A1)∼(A8),(OA.95)
$(F50)$	$\Rightarrow [f(x_0) - \varepsilon] < f(x)$	by (F46),(F49),(OA.92)
$(F51)$	$\delta_0 = [\varepsilon - -[1 + 1]]$	by (B18)
$(F52)$	$\Rightarrow [1 - 1] < \delta_0$	by (B7),(F51), (OA.94)
$(F53)$	$[\varepsilon - -[1 + 1]] < \varepsilon$	by (B17)
$(F54)$	$\Rightarrow \delta_0 < \varepsilon$	by (F53),(F51), (OA.95)
$(F55)$	$\delta = [[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] - x_0]$	by (E1)
$(F56)$	$\Rightarrow [x_0 + \delta] = [x_0 + [[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] - x_0]]$	by (OA.108)
$(F57)$	$\Rightarrow [x_0 + \delta] = [[[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] - x_0] + x_0]$	by (OA.46),(OA.107)
$(F58)$	$\Rightarrow [x_0 + \delta] = [[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d]$	by (OA.47),(OA.107)
$(F59)$	$[1 - 1] < \delta_0$	by (F52)
$(F60)$	$\Rightarrow [[1 - 1] + 1] < [\delta_0 + 1]$	by (F59),(OA.31)
$(F61)$	$\Rightarrow 1 < [\delta_0 + 1]$	by (OA.47),(OA.95)
$(F62)$	$\Rightarrow 1 < [1 + \delta_0]$	by (OA.46),(OA.94)
$(F63)$	$1 \leq [x_0 +++ \dot{h}d]$	by (B2)
$(F64)$	$\Rightarrow [1 + \delta_0] \leq [[x_0 +++ \dot{h}d] + \delta_0]$	by (OA.31)
$(F65)$	$\Rightarrow 1 < [[x_0 +++ \dot{h}d] + \delta_0]$	by (F62),(F64), (OA.92),(OA.94)
$(F66)$	$\Rightarrow 1 < [[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d]$	by (OA.34)
$(F67)$	$\Rightarrow [[x_0 + \delta] +++ \dot{h}d] =$ $[[[[x_0 +++ \dot{h}d] + \delta_0] --- \dot{i}d] +++ \dot{h}d]$	by (F58),(F66), (OA.108)
$(F68)$	$\Rightarrow [[x_0 + \delta] +++ \dot{h}d] = [[x_0 +++ \dot{h}d] + \delta_0]$	by (OA.79),(OA.107)
$(F69)$	$\delta_0 < \varepsilon$	by (F54)
$(F70)$	$\Rightarrow [\delta_0 + [x_0 +++ \dot{h}d]] < [\varepsilon + [x_0 +++ \dot{h}d]]$	by (OA.31)
$(F71)$	$\Rightarrow [[x_0 +++ \dot{h}d] + \delta_0] < [\varepsilon + [x_0 +++ \dot{h}d]]$	by (OA.46),(OA.95)
$(F72)$	$\Rightarrow [[x_0 +++ \dot{h}d] + \delta_0] < [[x_0 +++ \dot{h}d] + \varepsilon]$	by (OA.46),(OA.94)
$(F73)$	$\Rightarrow [[x_0 + \delta] +++ \dot{e}d] < [[x_0 +++ \dot{e}d] + \varepsilon]$	by (F72),(F68), (OA.95)
$(F74)$	$\Rightarrow f([x_0 + \delta]) < [f(x_0) + \varepsilon]$	by (F73),(Premise)

Since $f(x)$ is strictly increasing, $f(1) \leq f(x)$ and $f(x) < f([x_0+\delta])$ hold for all $x \in \big([x_0-\delta],[x_0+\delta]\big)$. (F50) and (F74) derive that $[f(x_0)-\varepsilon] < f(x)$ and $f(x) < [f(x_0)+\varepsilon]$ hold for all $x \in \big([x_0-\delta],[x_0+\delta]\big)$.

Items 1∼2 derive that $f(x)$ is continuous.

(F49) derives that $1 \leq f(x)$ holds on the domain $\big[1,+\infty\big)$. For any number $1 \leq \varepsilon$, (OA.34) and (OA.76) always derive that $[[\varepsilon - - - \dot{i}d]+1] \in \big[1,+\infty\big)$. So there always exists $x_0 = [[\varepsilon - - - \dot{i}d]+1]$ on the domain $\big[1,+\infty\big)$.

$(G1)$	$[1-1] < 1$	by (F14)
$(G2)$	$\Rightarrow [[1-1]+[\varepsilon - - - \dot{i}d]] < [1+[\varepsilon - - - \dot{i}d]]$	by (OA.31)
$(G3)$	$\Rightarrow [[\varepsilon - - - \dot{i}d]+[1-1]] < [1+[\varepsilon - - - \dot{i}d]]$	by (OA.46),(OA.95)
$(G4)$	$\Rightarrow [[[\varepsilon - - - \dot{i}d]+1]-1] < [1+[\varepsilon - - - \dot{i}d]]$	by (OA.50),(OA.95)
$(G5)$	$\Rightarrow [[[\varepsilon - - - \dot{i}d]-1]+1] < [1+[\varepsilon - - - \dot{i}d]]$	by (OA.48),(OA.95)
$(G6)$	$\Rightarrow [\varepsilon - - - \dot{i}d] < [1+[\varepsilon - - - \dot{i}d]]$	by (OA.47),(OA.95)
$(G7)$	$\Rightarrow [\varepsilon - - - \dot{i}d] < [[\varepsilon - - - \dot{i}d]+1]$	by (OA.46),(OA.94)
$(G8)$	$x_0 = [[\varepsilon - - - \dot{i}d]+1]$	by (Premise)
$(G9)$	$\Rightarrow [\varepsilon - - - \dot{i}d] < x_0$	by (G7),(G8),(OA.94)
$(G10)$	$\Rightarrow [[\varepsilon - - - \dot{i}d] + + + \dot{h}d] < [x_0 + + + \dot{h}d]$	by (G9),(OA.34), (Premise),(OA.37)
$(G11)$	$\Rightarrow \varepsilon < [x_0 + + + \dot{e}d]$	by (Premise),(OA.79), (OA.95)
$(G12)$	$\Rightarrow \varepsilon < f(x_0)$	by (G11),(Premise)

(G1)∼(G12) derive that $f(x)$ is unbounded. □

Theorem 3.5. *The EV Function* $f(x) = [e + + + \dot{e}x]$ *is continuous, unbounded and strictly increasing.*

Proof. According to (OA.19), the symbol '$\dot{e}$' represents some successive '+'—"$+\cdots+$". According to (OA.20), the symbol '$\dot{f}$' represents some successive '−'—"$-\cdots-$". According to (OA.21), the symbol '$\dot{g}$' represents some successive '/'—"$/\cdots/$".

$(A1)$	Supposing that $x_1, x_2 \in R$ with $x_1 < x_2$.	
$(A2)$	$\Rightarrow (x_1 < x_2)$	
$(A3)$	$1 < e$	by (Premise)
$(A4)$	$\Rightarrow [e + + + \dot{e}x_1] < [e + + + \dot{e}x_2]$	by (A2),(A3),(OA.41)
$(A5)$	$f(x_1) = [e + + + \dot{e}x_1]$	by (Premise)
$(A6)$	$\Rightarrow f(x_1) < [e + + + \dot{e}x_2]$	by (A4),(A5),(OA.95)
$(A7)$	$f(x_2) = [e + + + \dot{e}x_2]$	by (Premise)
$(A8)$	$\Rightarrow f(x_1) < f(x_2)$	by (A6),(A7),(OA.94)

(A1)∼(A8) derive that $f(x)$ is strictly increasing.

For any number $x_0 \in R$ and any number $[1-1] < \varepsilon$, we can always construct δ_0 as follows:

$(B1)$	$(1 < e) \wedge x_0$	by (Premise)
$(B2)$	$\Rightarrow [1-1] < [e+++\dot{e}x_0]$	by (OA.85)
$(B3)$	$\Rightarrow [[1-1]+1] < [[e+++\dot{e}x_0]+1]$	by (OA.31)
$(B4)$	$[1-1] < \varepsilon$	by (Premise)
$(B5)$	$\Rightarrow [[1-1]--[1+1]] < [\varepsilon--[1+1]]$	by (OA.32)
$(B6)$	$\Rightarrow [[1--[1+1]]-[1--[1+1]]] < [\varepsilon--[1+1]]$	by (OA.60), (OA.95)
$(B7)$	$\Rightarrow [1-1] < [\varepsilon--[1+1]]$	by (OA.45), (OA.95)
$(B8)$	$\Rightarrow [[1-1]+[\varepsilon--[1+1]]] < [[\varepsilon--[1+1]]+[\varepsilon--[1+1]]]$	by (OA.31)
$(B9)$	$\Rightarrow [[1-1]+[\varepsilon--[1+1]]] < [[\varepsilon+\varepsilon]--[1+1]]$	by (OA.60), (OA.94)
$(B10)$	$\Rightarrow [[\varepsilon--[1+1]]+[1-1]] < [[\varepsilon+\varepsilon]--[1+1]]$	by (OA.46), (OA.95)
$(B11)$	$\Rightarrow [[[\varepsilon--[1+1]]+1]-1] < [[\varepsilon+\varepsilon]--[1+1]]$	by (OA.50), (OA.95)
$(B12)$	$\Rightarrow [[[\varepsilon--[1+1]]-1]+1] < [[\varepsilon+\varepsilon]--[1+1]]$	by (OA.48), (OA.95)
$(B13)$	$\Rightarrow [\varepsilon--[1+1]] < [[\varepsilon+\varepsilon]--[1+1]]$	by (OA.47), (OA.95)
$(B14)$	$\Rightarrow [\varepsilon--[1+1]] < [[\varepsilon++[1+1]]--[1+1]]$	by (OA.57),(OA.53), (OA.94)
$(B15)$	$\Rightarrow [\varepsilon--[1+1]] < [\varepsilon++[[1+1]--[1+1]]]$	by (OA.61),(OA.94)
$(B16)$	$\Rightarrow [\varepsilon--[1+1]] < [\varepsilon++1]$	by (OA.54),(OA.94)
$(B17)$	$\Rightarrow [\varepsilon--[1+1]] < \varepsilon$	by (OA.53),(OA.94)
$(B18)$	$\delta_0 = [\varepsilon--[1+1]]$	

We construct δ according to $[[e+++\dot{e}x_0]-\delta_0]$.

(1) $[1-1] < [[e+++\dot{e}x_0]-\delta_0]$.

We construct δ as follows:

$$(C1)\quad \delta_1 = [x_0 - [[[e+++\dot{h}x_0]-\delta_0]///\dot{j}e]]$$
$$(C2)\quad \delta_2 = [[[[e+++\dot{h}x_0]+\delta_0]///\dot{j}e]-x_0]$$
$$(C3)\quad \delta = \min\{\delta_1, \delta_2\}$$

$(D1)$	$\delta_0 = [\varepsilon--[1+1]]$	by (B18)
$(D2)$	$\Rightarrow [1-1] < \delta_0$	by (B7),(D1), (OA.94)
$(D3)$	$[\varepsilon--[1+1]] < \varepsilon$	by (B17)
$(D4)$	$\Rightarrow \delta_0 < \varepsilon$	by (D3),(D1), (OA.95)
$(D5)$	$\delta = \min\{\delta_1, \delta_2\}$	by (C3)
$(D6)$	$\Rightarrow \delta \le \delta_1$	
$(D7)$	$\Rightarrow \delta \le \delta_2$	

$(D8)$	$\delta_1 = [x_0 - [[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e]]$	by (C1)
$(D9)$	$\Rightarrow [x_0 - \delta_1] = [x_0 - [x_0 - [[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e]]]$	by (OA.108)
$(D10)$	$\Rightarrow [x_0 - \delta_1] = [[x_0 - x_0] + [[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e]]$	by (OA.52),(OA.107)
$(D11)$	$\Rightarrow [x_0 - \delta_1] = [[[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e] + [x_0 - x_0]]$	by (OA.46),(OA.107)
$(D12)$	$\Rightarrow [x_0 - \delta_1] = [[[[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e] + x_0] - x_0]$	by (OA.50),(OA.107)
$(D13)$	$\Rightarrow [x_0 - \delta_1] = [[[[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e] - x_0] + x_0]$	by (OA.48),(OA.107)
$(D14)$	$\Rightarrow [x_0 - \delta_1] = [[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e]$	by (OA.47),(OA.107)
$(D15)$	$\Rightarrow [x_0 - \delta_1] \leq [x_0 - \delta]$	by (D6),(OA.31)
$(D16)$	$\Rightarrow [e+++\dot{h}[x_0 - \delta_1]] \leq [e+++\dot{h}[x_0 - \delta]]$	by (D15),(OA.41), (OA.108)
$(D17)$	$\Rightarrow [e+++\dot{h}[[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e]] \leq$ $[e+++\dot{h}[x_0 - \delta]]$	by (D16),(D14), (OA.95),(OA.107)
$(D18)$	$\Rightarrow [[e+++\dot{h}x_0] - \delta_0] \leq [e+++\dot{h}[x_0 - \delta]]$	by (OA.81),(OA.95)
$(D19)$	$\Rightarrow [[e+++\dot{h}x_0] - \varepsilon] < [[e+++\dot{h}x_0] - \delta_0]$	by (D4),(OA.31)
$(D20)$	$\Rightarrow [[e+++\dot{e}x_0] - \varepsilon] < [e+++\dot{e}[x_0 - \delta]]$	by (D19),(D18), (OA.92),(OA.94)
$(D21)$	$\Rightarrow [f(x_0) - \varepsilon] < f([x_0 - \delta])$	by (D20),(Premise)
$(D22)$	$\delta_2 = [[[[e+++\dot{h}x_0] + \delta_0]///\dot{j}e] - x_0]$	by (C2)
$(D23)$	$\Rightarrow [x_0 + \delta_2] = [x_0 + [[[[e+++\dot{h}x_0] + \delta_0]///\dot{j}e] - x_0]]$	by (OA.108)
$(D24)$	$\Rightarrow [x_0 + \delta_2] = [[[[[e+++\dot{h}x_0] + \delta_0]///\dot{j}e] - x_0] + x_0]$	by (OA.46),(OA.107)
$(D25)$	$\Rightarrow [x_0 + \delta_2] = [[[e+++\dot{h}x_0] + \delta_0]///\dot{j}e]$	by (OA.47),(OA.107)
$(D26)$	$[1-1] < \delta_0$	by (D2)
$(D27)$	$\Rightarrow [[1-1] + [1-1]] < [[1-1] + \delta_0]$	by (D26),(OA.31),(OA.46)
$(D28)$	$\Rightarrow [[[1-1] + 1] - 1] < [[1-1] + \delta_0]$	by (OA.50),(OA.95)
$(D29)$	$\Rightarrow [1-1] < [[1-1] + \delta_0]$	by (OA.47),(OA.95)
$(D30)$	$[1-1] < [e+++\dot{h}x_0]$	by (B2)
$(D31)$	$\Rightarrow [[1-1] + \delta_0] < [[e+++\dot{h}x_0] + \delta_0]$	by (OA.31)
$(D32)$	$\Rightarrow [1-1] < [[e+++\dot{h}x_0] + \delta_0]$	by (D29),(D31), (OA.92)
$(D33)$	$\Rightarrow [[e+++\dot{h}x_0] + \delta_0]$	by (OA.93)
$(D34)$	$\Rightarrow [e+++\dot{h}[x_0 + \delta_2]] =$ $[e+++\dot{h}[[[e+++\dot{h}x_0] + \delta_0]///\dot{j}e]]$	by (D25),(OA.108)
$(D35)$	$\Rightarrow [e+++\dot{h}[x_0 + \delta_2]] = [[e+++\dot{h}x_0] + \delta_0]$	by (D34),(D32), (OA.81),(OA.107)

$(D36)$	$\delta_0 < \varepsilon$	by (D4)
$(D37)$	$\Rightarrow [\delta_0 + [e+++\dot{h}x_0]] < [\varepsilon + [e+++\dot{h}x_0]]$	by (OA.31)
$(D38)$	$\Rightarrow [[e+++\dot{h}x_0] + \delta_0] < [\varepsilon + [e+++\dot{h}x_0]]$	by (OA.46),(OA.95)
$(D39)$	$\Rightarrow [[e+++\dot{h}x_0] + \delta_0] < [[e+++\dot{h}x_0] + \varepsilon]$	by (OA.46),(OA.94)
$(D40)$	$\Rightarrow [e+++\dot{h}[x_0 + \delta_2]] < [[e+++\dot{h}x_0] + \varepsilon]$	by (D39),(D35), (OA.95)
$(D41)$	$\Rightarrow [[e+++\dot{h}x_0] - \delta_0] < [[e+++\dot{h}x_0] - [1-1]]$	by (D2),(OA.31)
$(D42)$	$\Rightarrow [[e+++\dot{h}x_0] - \delta_0] < [[[e+++\dot{h}x_0] - 1] + 1]$	by (OA.52),(OA.94)
$(D43)$	$\Rightarrow [[e+++\dot{h}x_0] - \delta_0] < [e+++\dot{h}x_0]$	by (OA.47),(OA.94)
$(D44)$	$\Rightarrow [[e+++\dot{h}x_0] - \delta_0] =$ $[e+++\dot{h}[[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e]]$	by (Premise),(OA.81)
$(D45)$	$\Rightarrow [e+++\dot{h}x_0] =$ $[e+++\dot{h}[[e+++\dot{h}x_0]///\dot{j}e]]$	by (B2),(OA.81)
$(D46)$	$\Rightarrow [e+++\dot{h}[[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e]] <$ $[e+++\dot{h}[[e+++\dot{h}x_0]///\dot{j}e]]$	by (D43),(D44), (D45),(OA.94), (OA.95)
$(D47)$	$\Rightarrow [[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e] < [[e+++\dot{h}x_0]///\dot{j}e]$	by (D46),(B2), (OA.35),(OA.42)
$(D48)$	$\Rightarrow [[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e] < x_0$	by (B2),(OA.81),(OA.94)
$(D49)$	$\Rightarrow [x_0 - x_0] < [x_0 - [[[e+++\dot{h}x_0] - \delta_0]///\dot{j}e]]$	by (OA.31)
$(D50)$	$\Rightarrow [x_0 - x_0] < \delta_1$	by (C1),(OA.94)
$(D51)$	$\Rightarrow [1-1] < \delta_1$	by (OA.45),(OA.95)
$(D52)$	$\Rightarrow [[1-1] + [e+++\dot{h}x_0]] < [\delta_0 + [e+++\dot{h}x_0]]$	by (D2),(OA.31)
$(D53)$	$\Rightarrow [[1-1] + [e+++\dot{h}x_0]] < [[e+++\dot{h}x_0] + \delta_0]$	by (OA.46),(OA.94)
$(D54)$	$\Rightarrow [[e+++\dot{h}x_0] + [1-1]] < [[e+++\dot{h}x_0] + \delta_0]$	by (OA.46),(OA.95)
$(D55)$	$\Rightarrow [[[e+++\dot{h}x_0] + 1] - 1] < [[e+++\dot{h}x_0] + \delta_0]$	by (OA.50),(OA.95)
$(D56)$	$\Rightarrow [[[e+++\dot{h}x_0] - 1] + 1] < [[e+++\dot{h}x_0] + \delta_0]$	by (OA.48),(OA.95)
$(D57)$	$\Rightarrow [e+++\dot{h}x_0] < [[e+++\dot{h}x_0] + \delta_0]$	by (OA.47),(OA.95)
$(D58)$	$\Rightarrow [e+++\dot{h}x_0] =$ $[e+++\dot{h}[[e+++\dot{h}x_0]///\dot{j}e]]$	by (B2),(OA.81)
$(D59)$	$\Rightarrow [[e+++\dot{h}x_0] + \delta_0] =$ $[e+++\dot{h}[[[e+++\dot{h}x_0] + \delta_0]///\dot{j}e]]$	by (D32),(OA.81)
$(D60)$	$\Rightarrow [e+++\dot{h}[[e+++\dot{h}x_0]///\dot{j}e]] <$ $[e+++\dot{h}[[[e+++\dot{h}x_0] + \delta_0]///\dot{j}e]]$	by (D57),(D58),

		(D59),(OA.94), (OA.95)
$(D61)$	$\Rightarrow [[e+++\dot{h}x_0]///\dot{j}e] < [[[e+++\dot{h}x_0]+\delta_0]///\dot{j}e]$	by (D60),(B2), (OA.35),(OA.42)
$(D62)$	$\Rightarrow x_0 < [[[e+++\dot{h}x_0]+\delta_0]///\dot{j}e]$	by (B2),(OA.81),(OA.95)
$(D63)$	$\Rightarrow [x_0-x_0] < [[[[e+++\dot{h}x_0]+\delta_0]///\dot{j}e]-x_0]$	by (OA.31)
$(D64)$	$\Rightarrow [1-1] < [[[[e+++\dot{h}x_0]+\delta_0]///\dot{j}e]-x_0]$	by (OA.45),(OA.95)
$(D65)$	$\Rightarrow [1-1] < \delta_2$	by (C2),(OA.94)
$(D66)$	$[1-1] < \delta$	by (D5),(D51), (D65)
$(D67)$	$\Rightarrow [\delta+x_0] \leq [\delta_2+x_0]$	by (D7),(OA.31)
$(D68)$	$\Rightarrow [x_0+\delta] \leq [\delta_2+x_0]$	by (OA.46),(OA.95)
$(D69)$	$\Rightarrow [x_0+\delta] \leq [x_0+\delta_2]$	by (OA.46),(OA.94)
$(D70)$	$\Rightarrow [e+++\dot{h}[x_0+\delta]] \leq [e+++\dot{h}[x_0+\delta_2]]$	by (D69),(Premise), (OA.41)
$(D71)$	$\Rightarrow [e+++\dot{e}[x_0+\delta]] < [[e+++\dot{e}x_0]+\varepsilon]$	by (D70),(D40), (OA.92),(OA.95)
$(D72)$	$\Rightarrow f([x_0+\delta]) < [f(x_0)+\varepsilon]$	by (D71),(Premise)

Since $f(x)$ is strictly increasing, $f([x_0-\delta]) < f(x)$ and $f(x) < f([x_0+\delta])$ hold for all $x \in \big([x_0-\delta],[x_0+\delta]\big)$. (D21) and (D72) derive that $[f(x_0)-\varepsilon] < f(x)$ and $f(x) < [f(x_0)+\varepsilon]$ hold for all $x \in \big([x_0-\delta],[x_0+\delta]\big)$.

(2) $[[e+++\dot{e}x_0]-\delta_0] \leq [1-1]$.

We construct δ as follows:

$$(E1)\quad \delta = [[[[e+++\dot{h}x_0]+\delta_0]///\dot{j}e]-x_0]$$

$(F1)$	$\dot{a} \to 1\|[\dot{a}\dot{b}\dot{a}]$	by (OA.16)
$(F2)$	$\Rightarrow (\dot{a} \to 1)$	by (OA.88)
$(F3)$	$\Rightarrow (\dot{a} \to [\dot{a}\dot{b}\dot{a}])$	by (F1),(OA.88)
$(F4)$	$\Rightarrow (\dot{a} \to [1\dot{b}\dot{a}])$	by (F3),(F2),(OA.87)
$(F5)$	$\Rightarrow (\dot{a} \to [1\dot{b}1])$	by (F3),(F2),(OA.87)
$(F6)$	$\dot{b} \to +\|-$	by (OA.17)
$(F7)$	$\Rightarrow (\dot{b} \to -)$	by (OA.88)
$(F8)$	$\Rightarrow (\dot{a} \to [1-1])$	by (F5),(F7),(OA.87)
$(F9)$	$\dot{a} < [1+\dot{a}]$	by (OA.30)
$(F10)$	$\Rightarrow 1 < [1+1]$	by (F9),(F2),(OA.99)
$(F11)$	$\Rightarrow 1$	by (OA.93)
$(F12)$	$\Rightarrow [1-1] < [1+[1-1]]$	by (F9),(F8),(OA.99)

$(F13)$	$\Rightarrow [1-1] < [[1-1]+1]$	by (OA.46),(OA.94)
$(F14)$	$\Rightarrow [1-1] < 1$	by (F11),(OA.47), (F13),(OA.94)
$(F15)$	$\Rightarrow [1-1] < [e+++\dot{h}x_0]$	by (B2)
$(F16)$	$\Rightarrow [1-1]$	by (OA.93)
$(F17)$	$\Rightarrow [e+++\dot{h}x_0]$	by (F15),(OA.93)
$(F18)$	$\delta_0 = [\varepsilon--[1+1]]$	by (B18)
$(F19)$	$\Rightarrow [1-1] < \delta_0$	by (B7),(F18), (OA.94)
$(F20)$	$[[e+++\dot{e}x_0]-\delta_0] \leq [1-1]$	by (Premise)
$(F21)$	$\Rightarrow [[[e+++\dot{e}x_0]-\delta_0]+\delta_0] \leq [[1-1]+\delta_0]$	by (OA.31)
$(F22)$	$\Rightarrow [e+++\dot{e}x_0] \leq [[1-1]+\delta_0]$	by (OA.47),(OA.95),(OA.107)
$(F23)$	$\Rightarrow [e+++\dot{e}x_0] \leq [\delta_0+[1-1]]$	by (OA.46),(OA.94),(OA.107)
$(F24)$	$\Rightarrow [e+++\dot{e}x_0] \leq [[\delta_0+1]-1]$	by (OA.50),(OA.94),(OA.107)
$(F25)$	$\Rightarrow [e+++\dot{e}x_0] \leq [[\delta_0-1]+1]$	by (OA.48),(OA.94),(OA.107)
$(F26)$	$\Rightarrow [e+++\dot{e}x_0] \leq \delta_0$	by (OA.47),(OA.94),(OA.107)
$(F27)$	$\Rightarrow [e+++\dot{e}x_0] \leq [\varepsilon--[1+1]]$	by (F26),(B18), (OA.94),(OA.107)
$(F28)$	$\Rightarrow [e+++\dot{e}x_0] < \varepsilon$	by (F27),(B17), (OA.92),(OA.95)
$(F29)$	$\Rightarrow [[e+++\dot{e}x_0]-\varepsilon] < [\varepsilon-\varepsilon]$	by (OA.31)
$(F30)$	$\Rightarrow [[e+++\dot{e}x_0]-\varepsilon] < [1-1]$	by (F29),(OA.45),(OA.94)
$(F31)$	$\Rightarrow [f(x_0)-\varepsilon] < [1-1]$	by (F30),(Premise),(OA.95)
$(F32)$	$[1-1] < [e+++\dot{e}[x_0-\delta]]$	by (Premise),(OA.85)
$(F33)$	$f([x_0-\delta]) = [e+++\dot{e}[x_0-\delta]]$	by (Premise)
$(F34)$	$\Rightarrow [1-1] < f([x_0-\delta])$	by (F32),(F33),(OA.94)
$(F35)$	$\Rightarrow [f(x_0)-\varepsilon] < f([x_0-\delta])$	by (F31),(F34),(OA.92)
$(F36)$	$\delta_0 = [\varepsilon--[1+1]]$	by (B18)
$(F37)$	$\Rightarrow [1-1] < \delta_0$	by (B7),(F36), (OA.94)
$(F38)$	$[\varepsilon--[1+1]] < \varepsilon$	by (B17)
$(F39)$	$\Rightarrow \delta_0 < \varepsilon$	by (F38),(B18), (OA.95)
$(F40)$	$\delta = [[[[e+++\dot{h}x_0]+\delta_0]///\dot{j}e]-x_0]$	by (E1)
$(F41)$	$\Rightarrow [x_0+\delta] = [x_0+[[[[e+++\dot{h}x_0]+\delta_0]///\dot{j}e]-x_0]]$	by (OA.108)
$(F42)$	$\Rightarrow [x_0+\delta] = [[[[[e+++\dot{h}x_0]+\delta_0]///\dot{j}e]-x_0]+x_0]$	by (OA.46),(OA.107)

$(F43)$ $\Rightarrow [x_0 + \delta] = [[[e + ++ \dot{h}x_0] + \delta_0]///\dot{j}e]$ by (OA.47),(OA.107)

$(F44)$ $[1 - 1] < \delta_0$ by (F37)

$(F45)$ $\Rightarrow [[1 - 1] + [1 - 1]] < [[1 - 1] + \delta_0]$ by (F44),(OA.31),(OA.46)

$(F46)$ $\Rightarrow [[[1 - 1] + 1] - 1] < [[1 - 1] + \delta_0]$ by (OA.50),(OA.95)

$(F47)$ $\Rightarrow [[[1 - 1] - 1] + 1] < [[1 - 1] + \delta_0]$ by (OA.48),(OA.95)

$(F48)$ $\Rightarrow [1 - 1] < [[1 - 1] + \delta_0]$ by (OA.47),(OA.95)

$(F49)$ $[1 - 1] < [e + ++ \dot{h}x_0]$ by (B2)

$(F50)$ $\Rightarrow [[1 - 1] + \delta_0] < [[e + ++ \dot{h}x_0] + \delta_0]$ by (OA.31)

$(F51)$ $\Rightarrow 1 < [[e + ++ \dot{h}x_0] + \delta_0]$ by (F48),(F50), (OA.92)

$(F52)$ $\Rightarrow [1 - 1] < [[[e + ++ \dot{h}x_0] + \delta_0]///\dot{j}e]$ by (F51),(Premise),(OA.35)

$(F53)$ $\Rightarrow [e + ++ \dot{h}[x_0 + \delta]] =$
$[e + ++ \dot{h}[[[e + ++ \dot{h}x_0] + \delta_0]///\dot{j}e]]$ by (F43),(F52), (OA.108)

$(F54)$ $\Rightarrow [e + ++ \dot{h}[x_0 + \delta]] = [[e + ++ \dot{h}x_0] + \delta_0]$ by (OA.81),(OA.107)

$(F55)$ $\delta_0 < \varepsilon$ by (F39)

$(F56)$ $\Rightarrow [\delta_0 + [e + ++ \dot{h}x_0]] < [\varepsilon + [e + ++ \dot{h}x_0]]$ by (OA.31)

$(F57)$ $\Rightarrow [[e + ++ \dot{h}x_0] + \delta_0] < [\varepsilon + [e + ++ \dot{h}x_0]]$ by (OA.46),(OA.95)

$(F58)$ $\Rightarrow [[e + ++ \dot{h}x_0] + \delta_0] < [[e + ++ \dot{h}x_0] + \varepsilon]$ by (OA.46),(OA.94)

$(F59)$ $\Rightarrow [e + ++ \dot{e}[x_0 + \delta]] < [[e + ++ \dot{e}x_0] + \varepsilon]$ by (F58),(F54), (OA.95)

$(F60)$ $\Rightarrow f([x_0 + \delta]) < [f(x_0) + \varepsilon]$ by (F59),(Premise)

Since $f(x)$ is strictly increasing, $f([x_0 - \delta] < f(x)$ and $f(x) < f([x_0 + \delta])$ hold for all $x \in \big([x_0 - \delta], [x_0 + \delta]\big)$. (F35) and (F60) derive that $[f(x_0) - \varepsilon] < f(x)$ and $f(x) < [f(x_0) + \varepsilon]$ hold for all $x \in \big([x_0 - \delta], [x_0 + \delta]\big)$.

Items (1)$\sim$(2) derive that $f(x)$ is continuous.

(OA.85) derives that $[1 - 1] < f(x)$ holds on the domain $\big(-\infty, +\infty\big)$. For any number $1 \leq \varepsilon$, (OA.35) always derives that $[\varepsilon///\dot{j}e] \in R$. So there always exists $x_0 = [[\varepsilon///\dot{j}e]+1]$ such that $x_0 \in R$.

$(G1)$ $[1 - 1] < 1$ by (F14)

$(G2)$ $\Rightarrow [[1 - 1] + [\varepsilon///\dot{j}e]] < [1 + [\varepsilon///\dot{j}e]]$ by (OA.31)

$(G3)$ $\Rightarrow [[\varepsilon///\dot{j}e] + [1 - 1]] < [1 + [\varepsilon///\dot{j}e]]$ by (OA.46),(OA.95)

$(G4)$ $\Rightarrow [[[\varepsilon///\dot{j}e] + 1] - 1] < [1 + [\varepsilon///\dot{j}e]]$ by (OA.50),(OA.95)

$(G5)$ $\Rightarrow [[[\varepsilon///\dot{j}e] - 1] + 1] < [1 + [\varepsilon///\dot{j}e]]$ by (OA.48),(OA.95)

$(G6)$ $\Rightarrow [\varepsilon///\dot{j}e] < [1 + [\varepsilon///\dot{j}e]]$ by (OA.47),(OA.95)

$(G7)$ $\Rightarrow [\varepsilon///\dot{j}e] < [[\varepsilon///\dot{j}e] + 1]$ by (OA.46),(OA.94)

$$
\begin{array}{lll}
(G8) & x_0 = [[\varepsilon///\dot{j}e] + 1] & \text{by (Premise)} \\
(G9) & \Rightarrow [\varepsilon///\dot{j}e] < x_0 & \text{by (G7),(G8),(OA.94)} \\
(G10) & \Rightarrow [e+++\dot{h}[\varepsilon///\dot{j}e]] < [e+++\dot{h}x_0] & \text{by (G9),(OA.35),} \\
 & & \text{(Premise),(OA.41)} \\
(G11) & \Rightarrow \varepsilon < [e+++\dot{e}x_0] & \text{by (Premise),(OA.81),(OA.95)} \\
(G12) & \Rightarrow \varepsilon < f(x_0) & \text{by (G11),(Premise)}
\end{array}
$$

(G1)∼(G12) derive that $f(x)$ is unbounded. □

After further studies with Theorem 3.4 and Theorem 3.5, we pose the conjectures as follows.

Conjecture 3.6. *For a function* $f:\big(1,+\infty\big)\to R$ *defined by* $f(x) = [x+++\dot{e}d]$*, it is smooth.*

Conjecture 3.7. *For a function* $f : R \to R$ *defined by* $f(x) = [e+++\dot{e}x]$*, it is smooth.*

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