On filter (α)-convergence and exhaustiveness of function nets in lattice groups and applications

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Abstract: We consider (strong uniform) continuity of the limit of a pointwise convergent net of lattice group-valued functions, (strong weak) exhaustiveness and (strong) (α)-convergence with respect to a pair of filters, which in the setting of nets are more natural than the corresponding notions formulated with respect to a single filter. Some comparison results are given between such concepts, in connection with suitable properties of filters. Moreover, some modes of filter (strong uniform) continuity for lattice group-valued functions are investigated, giving some characterization. As an application, we get some Ascoli-type theorem in an abstract setting.

A *bornology* on a topological space X is a family $\mathcal{B}$ of nonempty subsets of X which covers X, stable with respect to finite unions and with $B' \in \mathcal{B}$ for each nonempty subset B' of any element $B \in \mathcal{B}$.

If X is a topological space and $x \in X$, then we say that a function $f: X \to R$ is *continuous at* x iff there is an (O)-sequence $(\sigma_p)_p$ (depending on X) with the property that for each $p \in \mathbb{N}$ and $x \in X$ there exists a neighborhood U_x of x with $|f(x) - f(z)| \le \sigma_p$ whenever $z \in U_x$. We say that $f \in R^X$ is *globally continuous on* X iff it is continuous at every point $x \in X$ with respect to a single (O)-sequence, which can be taken independently of x.

Let $X = (X, \mathcal{D})$ be a uniform space. The elements of $\mathcal{D}$ are often called *entourages*. If $\emptyset \neq B \subset X$, then a function $f: X \to R$ is *strongly uniformly continuous on* B iff there exists an (O)-sequence $(\sigma_p)_p$ such that for every $p \in \mathbb{N}$ there is an entourage $D \in \mathcal{D}$ with $|f(\beta) - f(x)| \le \sigma_p$ whenever $x \in X$, $\beta \in B$ and $(x, \beta) \in D$. If $\mathcal{B}$ is a bornology on X, then we say that $f: X \to R$ is *strongly uniformly continuous on* $\mathcal{B}$ iff it is strongly uniformly continuous on B for every $B \in \mathcal{B}$, with respect to an (O)-sequence independent of B.

We denote by $\mathcal{F}_{\text{cofin}}$ the filter of all subsets of $\mathbb{N}$ whose complement is finite, by $\mathcal{I}_{\text{fin}}$ its dual ideal, namely the family of all finite subsets of $\mathbb{N}$, by $\mathcal{F}_{\text{st}}$ the filter of all subsets of $\mathbb{N}$ having asymptotic density 1, and by $\mathcal{I}_{\text{st}}$ its dual ideal, that is the family of all subsets of $\mathbb{N}$ with asymptotic density 0, by $\mathcal{F}_\Lambda$ the class $\{A \subset \Lambda$ and $A \supset M_\lambda :\ \lambda \in \Lambda\}$ and by $\mathcal{I}_\Lambda$ the dual ideal of $\mathcal{F}_\Lambda$. Observe that $\mathcal{F}_{\mathbb{N}} = \mathcal{F}_{\text{cofin}}$ and $\mathcal{I}_{\mathbb{N}} = \mathcal{I}_{\text{fin}}$.

We will sometimes consider free filters $\mathcal{F}$ of $\mathbb{N}$, with the property that there is a partition of the type $\mathbb{N} = \bigcup_{k=1}^{\infty} \Delta_k$, such that

$$\mathcal{I} = \{A \subset \mathbb{N} : A \text{ intersects at most a finite number of } \Delta_k\text{'s}\}, \qquad (1)$$

where $\mathcal{I}$ denotes the dual ideal of $\mathcal{F}$ (see also [6]).

Remarks 1 (a) Observe that the ideal $\mathcal{I}_{\text{fin}}$ satisfies condition (1): indeed it is enough to take $\Delta_k = \{k\}$ for each $k \in \mathbb{N}$.

(b) If $\mathcal{I}$ is as in (1), and $(A_j)_j$ is any sequence of subsets of $\mathbb{N}$, with $A_j \notin \mathcal{I}$ for all $j \in \mathbb{N}$, then there exists a disjoint sequence $(B_j)_j$ in $\mathcal{I}$, with $B_j \subset A_j$ for every $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \notin \mathcal{I}$.

(c) The ideal $\mathcal{I}_{\rm st}$ does not fulfil condition (1). To this aim, it is enough to show that for every partition $(\Delta_k)_k$ of $\mathbb{N}$ there is a set belonging to $\mathcal{I}_{\rm st}$ which intersects infinitely many Δ_k's. Set $q(1)=1$: there is $k_1 \in \mathbb{N}$ with $1 \in \Delta_{k_1}$. At the second step, take a natural number $q(2)$ greater than $q(1)+3$ and belonging to Δ_{k_2}, where k_2 is a suitable integer strictly greater than k_1. At the $n+1$-th step, if we have chosen $q(n)$, let $q(n+1)$ be an integer greater than $q(n)+2n+1$ and belonging to $\Delta_{k_{n+1}}$, where $k_1 < k_2 < \ldots < k_n < k_{n+1}$. It is not difficult to check that the set $A = \{q(n) : n \in \mathbb{N}\}$ has asymptotic density smaller or equal than that of the set of squares, that is 0, and thus $A \in \mathcal{I}_{\rm st}$. Moreover, by construction, A intersects infinitely many Δ_k's.

Let X be any Hausdorff topological space, $x \in X$ and $\mathcal{F}$ be a (Λ)-free filter of Λ. A net $(x_\lambda)_{\lambda \in \Lambda}$ be *$\mathcal{F}$-converges to x* (shortly, $(\mathcal{F})\lim_\lambda x_\lambda = x$) iff $\{\lambda \in \Lambda : x_\lambda \in U\} \in \mathcal{F}$ for each neighborhood U of x.

We say that a net $(x_\lambda)_{\lambda \in \Lambda}$ in R *$(O\mathcal{F})$-converges to* $x \in R$ (briefly, $(O\mathcal{F})\lim_\lambda x_\lambda = x$) iff there exists an (O)-sequence $(\sigma_p)_p$ with $\{\lambda \in \Lambda : | x_\lambda - x | \le \sigma_p\} \in \mathcal{F}$ for each $p \in \mathbb{N}$.

Let Ξ be any nonempty set. A family $\{(x_{\lambda,\xi})_\lambda : \xi \in \Xi\}$ *$(RO\mathcal{F})$-converges* to $x_\xi \in R$ iff there is an (O)-sequence $(\sigma_p)_p$ in R such that for each $p \in \mathbb{N}$ and $\xi \in \Xi$ we get $\{\lambda \in \Lambda : | x_{\lambda,\xi} - x_\xi | \le \sigma_p\} \in \mathcal{F}$. We will denote by *$(RO)$-convergence* the $(RO\mathcal{F}_\Lambda)$-convergence. Observe that (RO)-convergence coincides with the usual pointwise (O)-convergence of a family with respect to a single (O)-sequence and, when $R = \mathbb{R}$, $(RO\mathcal{F})$-convergence coincides with filter convergence in the ordinary sense.

Let $(X, \mathcal{D})$ be a uniform space, $\varnothing \neq B \subset X$, $\Xi = (\Xi, \ge)$ be a directed set and $\mathcal{S}$ be a (Ξ)-free filter of Ξ. We say that the pair of nets $(z_\xi)_{\xi \in \Xi}$, $(x_\xi)_{\xi \in \Xi}$, *satisfies condition H1) with respect to* $\mathcal{S}$ iff x_ξ, $z_\xi \in B$ for each $\xi \in \Xi$, and for every $D \in \mathcal{D}$ there is a set $F \in \mathcal{S}$ with $(x_\xi, z_\xi) \in D$ whenever $\xi \in F$.

Let $\mathcal{S}$ and $\mathcal{F}$ be any two fixed (Ξ)-free filters of Ξ. A function $f : X \to R$ is said to be *strongly $(\mathcal{S},\mathcal{F})$-uniformly continuous on B* iff there is an (O)-sequence $(\sigma_p)_p$ in R such that for every pair of nets $(z_\xi)_{\xi \in \Xi}$, $(x_\xi)_{\xi \in \Xi}$, satisfying condition H1) with respect to $\mathcal{S}$, we have:

for each $p \in \mathbb{N}$ there is $F^* \in \mathcal{F}$ with $| f(x_\xi) - f(z_\xi) | \le \sigma_p$ for each $\xi \in F^*$.

If $\mathcal{B}$ is a bornology on X, we say that $f : X \to R$ is *strongly $(\mathcal{S},\mathcal{F})$-uniformly continuous on $\mathcal{B}$* iff it is strongly $(\mathcal{S},\mathcal{F})$-uniformly continuous on every $B \in \mathcal{B}$ with respect to a single (O)-sequence, independent of B.

Let X be any Hausdorff topological space. We say that $f : X \to R$ is *$(\mathcal{S},\mathcal{F})$-continuous at* $x \in X$ iff there is an (O)-sequence $(\sigma_p)_p$ in R such that, for every net $(x_\xi)_{\xi \in \Xi}$ $\mathcal{S}$-convergent to x, the net $(f(x_\xi))_{\xi \in \Xi}$ $(O\mathcal{F})$-converges to $f(x)$ with respect to $(\sigma_p)_p$. We say that $f : X \to R$ is *$(\mathcal{S},\mathcal{F})$-continuous on X* iff f is $(\mathcal{S},\mathcal{F})$-continuous at every $x \in X$ with respect to a single (O)-sequence, independent of $x \in X$ and of $(x_\xi)_\xi$.

Let $(X, \mathcal{D})$ be a uniform space and $\varnothing \neq B \subset X$. A net of functions $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to be *strongly* $\mathcal{F}$*-exhaustive on* B iff there is an (O)-sequence $(\sigma_p)_p$ such that for any $p \in \mathbb{N}$ there exist an entourage $D \in \mathcal{D}$ and a set $A \in \mathcal{F}$ such that for each $\lambda \in A$ and $x \in X$, $\beta \in B$ with $(x, \beta) \in D$ we have $| f_\lambda(x) - f_\lambda(\beta) | \le \sigma_p$.

We say that a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *strongly weakly* $\mathcal{F}$*-exhaustive on* B iff there is an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there is an entourage $D \in \mathcal{D}$ such that, for every $x \in X$ and $\beta \in B$ with $(x, \beta) \in D$, there is $A \in \mathcal{F}$ (depending on x and β) with $| f_\lambda(x) - f_\lambda(\beta) | \le \sigma_p$ whenever $\lambda \in A$.

Given a bornology $\mathcal{B}$ on X, we say that $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to be *strongly (weakly)* $\mathcal{F}$*-exhaustive on* $\mathcal{B}$ iff it is strongly (weakly) $\mathcal{F}$-exhaustive on every $B \in \mathcal{B}$ with respect to a single (O)-sequence, independent of B.

Let $x \in X$. We say that a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is $\mathcal{F}$*-exhaustive at* x iff there is an (O)-sequence $(\sigma_p)_p$ with the property that for any $p \in \mathbb{N}$ there exist a neighborhood U of x and a set $A \in \mathcal{F}$ such that for each $\lambda \in A$ and $z \in U$ we have $| f_\lambda(z) - f_\lambda(x) | \le \sigma_p$.

A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *weakly* $\mathcal{F}$*-exhaustive at* x iff there is an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there is a neighborhood U of x with the property that for any $z \in U$ there is $A_z \in \mathcal{F}$ with $| f_\lambda(z) - f_\lambda(x) | \le \sigma_p$ whenever $\lambda \in A_z$.

We say that $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *(weakly)* $\mathcal{F}$*-exhaustive on* X iff it is (weakly) $\mathcal{F}$-exhaustive at every $x \in X$ with respect to a single (O)-sequence, independent of $x \in X$.

Let $\mathcal{S}$ and $\mathcal{F}$ be as above, $(X, \mathcal{D})$ be a uniform space and $\varnothing \neq B \subset X$. A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to be *strongly* $(\mathcal{S}, \mathcal{F})$*-exhaustive* on B iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that, for every pair of nets $(x_\xi)_{\xi \in \Xi}$, $(z_\xi)_{\xi \in \Xi}$, satisfying H1) with respect to $\mathcal{S}$ and for any $p \in \mathbb{N}$ there are $S \in \mathcal{S}$, $F \in \mathcal{F}$ with $| f_\lambda(x_\xi) - f_\lambda(z_\xi) | \le \sigma_p$ for every $\xi \in S$ and $\lambda \in F$.

The net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *strongly weakly* $(\mathcal{S}, \mathcal{F})$*-exhaustive* on B iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that, for each pair of nets $(x_\xi)_{\xi \in \Xi}$, $(z_\xi)_{\xi \in \Xi}$, satisfying H1) with respect to $\mathcal{S}$ and for every $p \in \mathbb{N}$ there is a set $S \in \mathcal{S}$ such that for each $\xi \in S$ there exists $F_\xi \in \mathcal{F}$ with $| f_\lambda(x_\xi) - f_\lambda(z_\xi) | \le \sigma_p$ whenever $\lambda \in F_\xi$.

Given a bornology $\mathcal{B}$ on X, we say that the net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *strongly (weakly)* $(\mathcal{S}, \mathcal{F})$*-exhaustive* on $\mathcal{B}$ iff it is strongly (weakly) $(\mathcal{S}, \mathcal{F})$-exhaustive on every $B \in \mathcal{B}$, with respect to a single (O)-sequence, independent of B.

Let X be any Hausdorff topological space and $x \in X$. A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to be $(\mathcal{S}, \mathcal{F})$*-exhaustive* at $x \in X$ iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that, for every net $(x_\xi)_{\xi \in \Xi}$ $\mathcal{S}$-convergent to x and for any $p \in \mathbb{N}$ there are $S \in \mathcal{S}$, $F \in \mathcal{F}$ with $| f_\lambda(x_\xi) - f_\lambda(x) | \le \sigma_p$ for every $\xi \in S$ and $\lambda \in F$.

The net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *weakly* $(\mathcal{S}, \mathcal{F})$*-exhaustive* at $x \in X$ iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that, for each net $(x_\xi)_{\xi \in \Xi}$ $\mathcal{S}$-convergent to x and for every $p \in \mathbb{N}$ there is a set $S \in \mathcal{S}$ such that for each $\xi \in S$ there exists $F_\xi \in \mathcal{F}$ with $| f_\lambda(x_\xi) - f_\lambda(x) | \le \sigma_p$ whenever $\lambda \in F_\xi$.

The net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *(weakly)* $(\mathcal{S},\mathcal{F})$*-exhaustive* on X iff it is (weakly) $(\mathcal{S},\mathcal{F})$-exhaustive at every $x \in X$ with respect to a single (O)-sequence, independent of x.

Note that the analogous concepts of (strong weak) filter exhaustiveness can be formulated analogously for *sequences* of functions, by taking $\Lambda = \mathbb{N}$ with the usual order.

Let $\mathcal{B}$ be a bornology on X. We say that a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, $(\mathcal{FB})$*-converges* to $f : X \to R$ iff there exists an (O)-sequence $(\sigma_p)_p$ such that $(f_\lambda)_\lambda$ is $(RO\mathcal{F})$-convergent to f with respect to $(\sigma_p)_p$, and for every $B \in \mathcal{B}$ and $p \in \mathbb{N}$ there is $F \in \mathcal{F}$ with $| f_\lambda(x) - f(x) | \le \sigma_p$ for each $x \in B$ and $\lambda \in F$.

We now consider (α)-convergence of nets $(f_\lambda)_{\lambda\in\Lambda}$ of (ℓ)-group-valued functions, defined on a Hausdorff topological space X. When we deal with nets, in general it is not always advisable to follow an approach similar as that used for sequences, since the cardinality of Λ can be larger than the one of X, and we want to consider possibly nets in X of the type x_ξ, $\xi \in \Xi$, whose points are all distinct. We say that $f_n : X \to R$, $n \in \mathbb{N}$, c*-converges (continuously converges) to* $f : X \to R$ *at* $x \in X$ iff there is an (O)-sequence $(\sigma_p)_p$ such that for each sequence $(x_n)_n$ in X with $\lim_n x_n = x$ we get $(O)\lim_n f_n(x_n) = f(x)$ (with respect to the (O)-sequence $(\sigma_p)_p$).

The sequence $(f_n)_n$ c*-converges to* $f : X \to R$ *on* X iff it c-converges to f at every $x \in X$ with respect to a single (O)-sequence, independent of $x \in X$.

Let $\mathcal{F}$ be a fixed free filter of $\mathbb{N}$. We say that a sequence $(f_n)_n$ in R^X $(\mathcal{F}c)$*-converges (filter continuously converges) to* $f \in R^X$ *at* $x \in X$ iff there exists an (O)-sequence $(\sigma_p)_p$ such that for each sequence $(x_n)_n$ in X with $(\mathcal{F})\lim_n x_n = x$ we get $(O\mathcal{F})\lim_n f_n(x_n) = f(x)$ with respect to $(\sigma_p)_p$.

The sequence $(f_n)_n$ is $(\mathcal{F}c)$*-convergent to* $f : X \to R$ *on* X iff it $(\mathcal{F}c)$-converges to f at every $x \in X$ with respect to a single (O)-sequence $(\sigma_p)_p$, independent of the choice of x.

Note that $(f_n)_n$ is c-convergent to f if and only if $(f_n)_n$ is $(\mathcal{F}_{\text{cofin}}c)$-convergent to f.

Let now Λ and Ξ be two directed sets, $\mathcal{S}$ and $\mathcal{F}$ be a (Ξ)-free filter of Ξ and a (Λ)-free filter of Λ respectively. We say that a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, $(\mathcal{S},\mathcal{F}\alpha)$*-converges to* $f : X \to R$ at $x \in X$ iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that, for every net $(x_\xi)_{\xi\in\Xi}$ $\mathcal{S}$-convergent to x and for each $p \in \mathbb{N}$ there are $S \in \mathcal{S}$, $F \in \mathcal{F}$ with $| f_\lambda(x_\xi) - f(x) | \le \sigma_p$ whenever $\xi \in S$ and $\lambda \in F$.

A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, $(\mathcal{S},\mathcal{F}\alpha)$*-converges to* $f : X \to R$ on $x \in X$ iff it $(\mathcal{S},\mathcal{F}\alpha)$-converges to $f : X \to R$ at every $x \in X$ with respect to a single (O)-sequence, independent of the choice of x.

Let $(X,\mathcal{D})$ be a uniform space and $\varnothing \ne B \subset X$. We say that a sequence $(f_n)_n$ in R^X *strongly* $(\mathcal{F}c)$*-converges to* $f \in R^X$ *on* B iff there exists an (O)-sequence $(\sigma_p)_p$ such that for each pair of sequences $(x_n)_n$, $(z_n)_n$ in X satisfying condition H1) with respect to $\mathcal{F}$ we get $(O\mathcal{F})\lim_n f_n(x_n) = f(x)$ with respect to $(\sigma_p)_p$.

Given a bornology $\mathcal{B}$ on X, we say that a sequence $(f_n)_n$ in R^X is *strongly* $(\mathcal{F}c)$*-convergent to* $f : X \to R$ *on* $\mathcal{B}$ iff it strongly $(\mathcal{F}c)$-converges to f on every $B \in \mathcal{B}$ with respect to an (O)-

sequence $(\sigma_p)_p$, independent of B.

A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to be *strongly* $(\mathcal{S}, \mathcal{F}\alpha)$*-convergent to* $f : X \to R$ on B iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that, for every pair of nets $(x_\xi)_{\xi\in\Xi}$, $(z_\xi)_{\xi\in\Xi}$ satisfying condition H1) with respect to $\mathcal{S}$ and for each $p \in \mathbb{N}$ there are $S \in \mathcal{S}$, $F \in \mathcal{F}$ with $| f_\lambda(z_\xi) - f(x_\xi) | \le \sigma_p$ whenever $\xi \in S$ and $\lambda \in F$.

A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, *strongly* $(\mathcal{S}, \mathcal{F}\alpha)$*-converges to* $f : X \to R$ on $\mathcal{B}$ iff it strongly $(\mathcal{S}, \mathcal{F}\alpha)$-converges to $f : X \to R$ on every $B \in \mathcal{B}$ with respect to a single (O)-sequence, independently of B.

A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to *(strongly)* $(\mathcal{F}\alpha)$*-converge to* $f : X \to R$ at $x \in X$ and on X (resp. on B and on $\mathcal{B}$) iff it is (strongly) $(\mathcal{S}, \mathcal{F}\alpha)$-convergent to $f : X \to R$ at $x \in X$ and on X (resp. on B and on $\mathcal{B}$) for every directed set $\Xi = (\Xi, \ge)$ and for each (Ξ)-free filter $\mathcal{S}$ of Ξ.

We will prove that the limit of any (strongly) $(\mathcal{F}\alpha)$-convergent net is a (strongly uniformly) continuous function.

Remarks 2 (a) Note that, even when $R = \mathbb{R}$ and $\mathcal{F} = \mathcal{F}_\Lambda$, to use arbitrary nets of the type $(x_\xi)_{\xi\in\Xi}$ instead of arbitrary sequences $(x_n)_n$ is essential. Indeed there are nets $(f_\lambda)_\lambda$ and functions f in $\mathbb{R}^X$ such that $(f_\lambda)_\lambda$ does not $(\mathcal{F}_\Lambda\alpha)$-converge to f, but for every $\varepsilon > 0$, $x \in X$ and for each sequence $(x_n)_n$ convergent to x in the usual sense there are $n_0 \in \mathbb{N}$ and $\lambda_0 \in \Lambda$, with $| f_\lambda(x_n) - f(x) | \le \varepsilon$ whenever $n \ge n_0$ and $\lambda \ge \lambda_0$.

(b) Observe that, in general, $(\mathcal{S}, \mathcal{F}\alpha)$-convergence is strictly weaker than strong $(\mathcal{S}, \mathcal{F}\alpha)$-convergence. Indeed, let $\Lambda = \mathbb{N}$ be with the usual order, $X = [0,1]$ with the usual metric, $R = \mathbb{R}$, $f_n(x) = x^n$ for each $n \in \mathbb{N}$ and $x \in [0,1]$, $f(1) = 1$ and $f(x) = 0$ for every $x \in [0,1)$. We prove that for every (Ξ)-free filter $\mathcal{S}$ of Ξ and for each free filter $\mathcal{F}$ of $\mathbb{N}$ the sequence $(f_n)_n$ $(\mathcal{S}, \mathcal{F}\alpha)$-converges to f, but does not strongly $(\mathcal{S}, \mathcal{F}\alpha)$-converge to f on $B := [0,1)$. Indeed, if $(x_\xi)_{\xi\in\Xi}$ is any net in B, $\mathcal{S}$-convergent to $x_0 \in B$, then $f_n(x_\xi) = x_\xi^n$ and $f(x_0) = 0$, and there exist $S \in \mathcal{S}$ and $x_0 < y < 1$ with $x_\xi < y$ whenever $\xi \in S$. Choose arbitrarily $\varepsilon > 0$. Since $0 < y < 1$, there exists $n' \in \mathbb{N}$ with $y^n < \varepsilon$ for each $n \ge n'$. Hence for such n's and $\xi \in S$ we get $0 < x_\xi^n < y^n < \varepsilon$. Thus $(f_n)_n$ $(\mathcal{S}, \mathcal{F}\alpha)$-converges to f on B. On the other hand, pick any pair of nets $(x_\xi)_\xi$, $(z_\xi)_\xi \in B$, with $(\mathcal{S})\lim_\xi x_\xi = (\mathcal{S})\lim_\xi z_\xi = 1$. Then we get $f(x_\xi) = 0$ for each $\xi \in \Xi$. Now we claim that

$$\text{for every } S \in \mathcal{S} \text{ and } F \in \mathcal{F} \text{ there are } \xi' \in S, n' \in F \text{ with } z_{\xi'}^{n'} > \frac{1}{3}. \tag{2}$$

Choose arbitrarily $S \in \mathcal{S}$ and $F \in \mathcal{F}$. As $\lim_n \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$, there is $n_0 \in \mathbb{N}$ with

$$\left(1 - \frac{1}{n}\right)^n > \frac{1}{3} \quad \text{for every } n \ge n_0. \tag{3}$$

Let n' be any integer greater than n_0 and belonging to F. Since $(\mathcal{S})\lim_\xi z_\xi = 1$, in correspondence with n' there is $S_{n'} \in \mathcal{S}$ with $z_\xi > 1 - \frac{1}{n'}$ whenever $\xi \in S_{n'}$. Let $\xi' \in S \cap S_{n'}$. Since $z_{\xi'} > 1 - \frac{1}{n'}$, taking into account (3) we get

$$z_{\xi'}^{n'} > \left(1 - \frac{1}{n'}\right)^{n'} > \frac{1}{3},$$

that is (2). Thus, the sequence $(f_n)_n$ does not strongly $(\mathcal{S},\mathcal{F}\alpha)$-converge to f on B.

Theorem 3 *Let X be a Hausdorff topological space, x be a fixed element of X, $f_n : X \to R$, $n \in \mathbb{N}$, be a sequence, $(RO\mathcal{F})$-convergent to $f : X \to R$. If $(f_n)_n$ is $\mathcal{F}$-exhaustive at x, then $(f_n)_n$ $(\mathcal{F}c)$-converges to f at x.*

Conversely, if $\mathcal{F}$ satisfies condition (1) and $(f_n)_n$ is $(\mathcal{F}c)$-convergent to f at x, then $(f_n)_n$ is $\mathcal{F}$-exhaustive at x.

Theorem 4 *Let $(X,\mathcal{D})$ be any uniform space, $\varnothing \neq B \subset X$, $\mathcal{S}$ and $\mathcal{F}$ be two free filters of $\mathbb{N}$ with $\mathcal{S} \supset \mathcal{F}$, and $f_n : X \to R$, $n \in \mathbb{N}$, be a function sequence, strongly $(\mathcal{S},\mathcal{F}\alpha)$-convergent to $f \in R^X$ on B. Then $(f_n)_n$ is strongly $(\mathcal{S}c)$-convergent to f on B.*

Theorem 5 *Let X be a Hausdorff topological space, $x \in X$, $\mathcal{S}$ and $\mathcal{F}$ be as in Theorem 3.2 and $f_n : X \to R$, $n \in \mathbb{N}$, be a sequence, $(\mathcal{S},\mathcal{F}\alpha)$-convergent to $f \in R^X$ at x. Then $(f_n)_n$ is $(\mathcal{S}c)$-convergent to f at x.*

Theorem 6 *Let $(X,\mathcal{D})$ be a uniform space with a decreasing base $(U_k)_k$ of entourages, $\varnothing \neq B \subset X$, $\mathcal{S}$ be a (Ξ)-free filter of Ξ, $\mathcal{F}$ be a free filter of $\mathbb{N}$, satisfying condition (1), and $f_n : X \to R$, $n \in \mathbb{N}$, be a sequence, strongly $(\mathcal{F}c)$-convergent to $f \in R^X$ on B. Then $(f_n)_n$ is strongly $(\mathcal{S},\mathcal{F}\alpha)$-convergent to f on B.*

Theorem 7 *Let X be a Hausdorff topological space, $x \in X$, $(U_k)_k$ be a decreasing base of neighborhoods of x, $\mathcal{S}$ be a (Ξ)-free filter of Ξ, $\mathcal{F}$ be a free filter of $\mathbb{N}$, satisfying condition (1), and $f_n : X \to R$, $n \in \mathbb{N}$, be a sequence, $(\mathcal{F}c)$-convergent to $f \in R^X$ at x. Then $(f_n)_n$ is $(\mathcal{S},\mathcal{F}\alpha)$-convergent to f at x.*

Theorem 8 *Let $(X,\mathcal{D})$ be any uniform space, $\varnothing \neq X \subset B$, Λ and Ξ be two directed sets, $\mathcal{S}$ and $\mathcal{F}$ be a (Ξ)-free filter of Ξ and a (Λ)-free filter of Λ respectively, and $f_\lambda : X \to R$, $\lambda \in \Lambda$, be a function net, strongly $\mathcal{F}$-exhaustive on B. Then $(f_\lambda)_\lambda$ is strongly $(\mathcal{S},\mathcal{F})$-exhaustive on B.*

Conversely, if $(D_\xi)_{\xi\in\Xi}$ is a decreasing net in $\mathcal{D}$, such that

for each $U \in \mathcal{D}$ there exists $\xi \in \Xi$ with $D_\xi \subset U$ (4)

and $(f_\lambda)_\lambda$ is strongly $(\mathcal{S},\mathcal{F})$-exhaustive on B, then $(f_\lambda)_\lambda$ is strongly $\mathcal{F}$-exhaustive on B.

Remark 9 It is easy to check that the set $\Xi = \mathcal{D}$, endowed with the order $D_1 \geq D_2$ if and only if $D_1 \subset D_2$, is a directed set, and that (4) is satisfied.

Theorem 10 *Let X be a Hausdorff topological space, $x \in X$ be fixed, $(\mathcal{T}_x, \subset)$ be the set of all neighborhoods of x; Λ, Ξ, $\mathcal{S}$, $\mathcal{F}$ be as in Theorem 8, and $f_\lambda : X \to R$, $\lambda \in \Lambda$, be a net, $\mathcal{F}$-exhaustive at x. Then $(f_\lambda)_\lambda$ is $(\mathcal{S},\mathcal{F})$-exhaustive at x.*

Conversely, if $(D_\xi)_{\xi\in\Xi}$ is a decreasing net in $\mathcal{T}_x$ satisfying (5), and $(f_\lambda)_\lambda$ is $(\mathcal{S},\mathcal{F})$-exhaustive at x, then $(f_\lambda)_\lambda$ is $\mathcal{F}$-exhaustive at x.

Theorem 11 *Let $(X,\mathcal{D})$ be any uniform space, $\mathcal{B}$ be a bornology on X, Ξ and Λ be as* *above, $\mathcal{S}$ and $\mathcal{F}$ be any two (Ξ)-free and (Λ)-free filters of Ξ and Λ respectively, and $f_\lambda : X \to R$, $\lambda \in \Lambda$, be a net of functions, $(\mathcal{FB})$-convergent to $f : X \to R$. Let $B \in \mathcal{B}$ be fixed. Then $(f_\lambda)_\lambda$ is strongly $(\mathcal{S},\mathcal{F}\alpha)$-convergent to f on B if and only if $(f_\lambda)_\lambda$ is strongly $(\mathcal{S},\mathcal{F})$-exhaustive on B.*

Theorem 12 *Let X be a Hausdorff topological space, $x \in X$, $\mathcal{S}$ and $\mathcal{F}$ be as in Theorem 11, and $f_\lambda : X \to R$, $\lambda \in \Lambda$, be a net of functions, $(RO\mathcal{F})$-convergent to $f : X \to R$. Then $(f_\lambda)_\lambda$ is $(\mathcal{S},\mathcal{F}\alpha)$-convergent to f at x if and only if $(f_\lambda)_\lambda$ is $(\mathcal{S},\mathcal{F})$-exhaustive at x.*

Theorem 13 *Let X, R, Λ, Ξ, $\mathcal{F}$, $\mathcal{S}$, $\mathcal{B}$ be as in Theorem 11, $B \in \mathcal{B}$ be fixed, and $f_\lambda : X \to R$, $\lambda \in \Lambda$, be a function net, $(RO\mathcal{F})$-convergent to $f \in R^X$.*

Then $(f_\lambda)_\lambda$ is strongly weakly $(\mathcal{S},\mathcal{F})$-exhaustive on B if and only if f is strongly $(\mathcal{S},\mathcal{S})$-uniformly continuous on B.

Theorem 14 *Let X, R, Λ, Ξ, $\mathcal{F}$, $\mathcal{S}$, be as in Theorem 12, $x \in X$ be fixed, and $f_\lambda : X \to R$, $\lambda \in \Lambda$, be a net, $(RO\mathcal{F})$-convergent to $f \in R^X$.*

Then $(f_\lambda)_\lambda$ is weakly $(\mathcal{S},\mathcal{F})$-exhaustive at x if and only if f is $(\mathcal{S},\mathcal{S})$-continuous at x.

Theorem 15 *Let $(X,\mathcal{D})$ be a uniform space, $(\Xi,\geq)$ be a directed set, $f : X \to R$, $\varnothing \neq B \subset X$, $\mathcal{S}_1$ and $\mathcal{S}_2$ be any two fixed (Ξ)-free filters of Ξ. Suppose that for every point $x \in X$ there is a net $(y_\xi)_{\xi\in\Xi}$ in X, with*

$$(\mathcal{S}_1)\lim_{\xi} y_\xi = x. \tag{5}$$

Then the following results hold.

(a) If $\mathcal{S}_1 \setminus \mathcal{S}_2 \neq \varnothing$, then f is strongly $(\mathcal{S}_1,\mathcal{S}_2)$-uniformly continuous on B if and only if f is constant.

(b) If $(D_\xi)_{\xi\in\Xi}$ is a decreasing net in $\mathcal{D}$, satisfying (5), and $\mathcal{S}_1 \subset \mathcal{S}_2$, then f is strongly $(\mathcal{S}_1,\mathcal{S}_2)$-uniformly continuous on B if and only if f is strongly uniformly continuous on B.

Theorem 16 *Let X be a Hausdorff topological space, $f : X \to R$, $(\Xi,\geq)$ be a directed set, $\mathcal{S}_1$ and $\mathcal{S}_2$ be two fixed (Ξ)-free filters of Ξ, $x \in X$ be such that there is a net $(y_\xi)_{\xi\in\Xi}$ in X, fulfilling (5). Then the following results hold.*

(a) If $\mathcal{S}_1 \setminus \mathcal{S}_2 \neq \varnothing$, then f is $(\mathcal{S}_1,\mathcal{S}_2)$-continuous at x if and only if f is constant.

(b) If $(\mathcal{T}_x,\subset)$ is the set of all neighborhoods of x, $(D_\xi)_{\xi\in\Xi}$ is a decreasing net in $\mathcal{T}_x$ fulfilling (5) and $\mathcal{S}_1 \subset \mathcal{S}_2$, then f is $(\mathcal{S}_1,\mathcal{S}_2)$-continuous at x if and only if it is continuous at x.

A consequence of the previous theorems is that, if X is a Hausdorff topological space (resp. a uniform space) and a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is (strongly) $(\mathcal{F}\alpha)$-convergent to $f : X \to R$ on X (resp. on $\mathcal{B}$), then it $(RO\mathcal{F})$-converges to f, and f is (strongly uniform) continuous on X (resp. on $\mathcal{B}$).

As an application, we give an Ascoli-type theorem. Given a topological space X, a nonempty set $\Phi \subset R^X$ and a convergence (σ) on Φ, we say that Φ is (σ)*-compact* iff every net $(f_\lambda)_{\lambda\in\Lambda}$ in Φ admits a subnet $(f_{\lambda_\kappa})_{\kappa\in\Lambda}$, (σ)-convergent to an element $f \in \Phi$, and that Φ is (σ)*-closed* iff $f \in \Phi$ whenever $(f_\lambda)_{\lambda\in\Lambda}$ is a net in Φ, (σ)-convergent to $f \in R^X$. The (σ)*-closure* of Φ is the set of the functions $f \in R^X$, having a net $(f_\lambda)_{\lambda\in\Lambda}$ in Φ (σ)-convergent to f. A set Φ is (σ)-closed if and only if it coincides with its (σ)-closure.

Theorem 17 *Let X be a Hausdorff topological space, $\mathcal{S}$ and $\mathcal{F}$ be as above.*

If $\Phi \subset \Psi \subset R^X$, where Φ is $(\mathcal{S},\mathcal{F}\alpha)$-closed and Ψ is $(RO\mathcal{F})$-compact, and

every net $(f_\lambda)_{\lambda\in\Lambda}$, $(RO\mathcal{F})$-convergent in Φ, has a subnet $(f_{\lambda_\kappa})_{\kappa\in\Lambda}$, $(RO\mathcal{F})$-convergent (in R^X) and $(\mathcal{S},\mathcal{F})$-exhaustive,

then Φ is $(\mathcal{S},\mathcal{F}\alpha)$-compact.

Moreover, if Φ is $(\mathcal{S},\mathcal{F}\alpha)$-compact, then Φ satisfies condition H').

In our setting, a related question, which arises naturally, is to find some necessary and/or sufficient conditions under which the limit function of a suitably pointwise convergent net is constant, or $(\mathcal{S},\mathcal{F})$-continuous. In this framework, it is advisable to consider the following extensions of the concepts of $(\mathcal{S},\mathcal{F})$-exhaustiveness and corresponding (α)-convergence.

Let X be any Hausdorff topological space, $x\in X$, R be any Dedekind complete lattice group, $(\Xi,\geq)$ and $(\Lambda,\geq)$ be two directed sets, $\mathcal{F}_1$ and $\mathcal{F}_2$ be two (Ξ)-free filters of Ξ, $\mathcal{F}_3$ be a (Λ)-free filter of Λ, and $f_\lambda : X\to R$, $\lambda\in\Lambda$, be a function net. We say that $(f_\lambda)_\lambda$ is $(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)$*-exhaustive* at $x\in X$ iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that, for every net $(x_\xi)_{\xi\in\Xi}$ $\mathcal{F}_1$-convergent to x and for any $p\in\mathbb{N}$ there are $F_2\in\mathcal{F}_2$ and $F_3\in\mathcal{F}_3$ with $|f_\lambda(x_\xi)-f_\lambda(x)|\leq\sigma_p$ for every $\xi\in F_2$ and $\lambda\in F_3$. The net $f_\lambda : X\to R$, $\lambda\in\Lambda$, is *weakly* $(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)$*-exhaustive* at $x\in X$ iff there is an (O)-sequence $(\sigma_p)_p$ in R such that, for each net $(x_\xi)_{\xi\in\Xi}$ $\mathcal{F}_1$-convergent to x and for every $p\in\mathbb{N}$ there is a set $F_2\in\mathcal{F}_2$ such that for each $\xi\in F_2$ there exists $F_\xi\in\mathcal{F}_3$ with $|f_\lambda(x_\xi)-f_\lambda(x)|\leq\sigma_p$ whenever $\lambda\in F_\xi$. The net $f_\lambda : X\to R$, $\lambda\in\Lambda$, $(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\alpha)$*-converges to* $f : X\to R$ at $x\in X$ iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that, for every net $(x_\xi)_{\xi\in\Xi}$ $\mathcal{F}_1$-convergent to x and for each $p\in\mathbb{N}$ there are $F_2\in\mathcal{S}$, $F_3\in\mathcal{F}$ with $|f_\lambda(x_\xi)-f(x)|\leq\sigma_p$ whenever $\xi\in F_2$ and $\lambda\in F_3$.

Theorem 18 *Let $f_\lambda : X\to R$, $\lambda\in\Lambda$, be a net of functions, $(RO\mathcal{F}_3)$-convergent to $f : X\to R$, and $x\in X$. Then $(f_\lambda)_\lambda$ is $(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\alpha)$-convergent to f at x if and only if $(f_\lambda)_\lambda$ is $(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)$-exhaustive at x. Moreover, $(f_\lambda)_\lambda$ is weakly $(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)$-exhaustive at x if and only if f is $(\mathcal{F}_1,\mathcal{F}_2)$-continuous at x.*