New infinite series and integral representations for error function

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"All things were made by him; and without him was not any thing made that was made." - John 1:3.

ABSTRACT. I proved some infinity series power and integral representations for error function and relations for special functions.

1. INTRODUCTION

In this paper, I demonstrated that

$$\frac{\sqrt{\pi}}{2}\mathrm{erf}(u)=\sum_{k=0}^{\infty}\frac{u^{4k+1}}{(2k)!}\left[\frac{1}{4k+1}-\frac{u^2}{2k+1}+\frac{2u^2}{4k+3}\right],$$

$$\frac{\sqrt{\pi}}{2}\mathrm{erf}(u)=\sum_{k=0}^{\infty}\frac{\cos\left(\frac{\pi k}{2}\right)}{(k+1)\Gamma\left(\frac{k}{2}+1\right)}u^{k+1},$$

$$\frac{\sqrt{\pi}}{2}\mathrm{erf}(u)=u\int_0^{\pi/4}\cosh[u^2\tan^2(t)]\sec^2(t)\mathrm{dt}-u^3\int_0^{\pi/4}\cosh[u^2\tan(t)]\sec^2(t)\mathrm{dt}$$

$$+2u^3\int_0^{\pi/4}\cosh[u^2\tan^2(t)]\sin^2(t)\sec^4(t)\mathrm{dt},$$

$$\frac{\sqrt{\pi}\,\mathrm{erf}(u)}{2u}=\int_0^{\pi/4}e^{-u^2\tan^2(t)}\sec^2(t)\,\mathrm{dt},$$

$$\frac{\sqrt{\pi}}{2}\mathrm{erf}(u)=\int_0^{u}[(2t^2+1)\cosh(t^2)-u^2\cosh(ut)]\mathrm{dt}$$

and the relation for special functions

$$2\mathrm{Shi}\,u+2\mathrm{Chi}\,u=2\mathrm{Ei}\,u+\ln\frac{1}{u}+\ln u,$$

for $\mathrm{Re}(u)<0$.

2. INFINITY SERIES AND INTEGRAL REPRESENTATIONS FOR ERROR FUNCTION

Theorem 1. *For* $u\in\mathbb{R}$, *then*

$$(i)\;\frac{\sqrt{\pi}}{2}\mathrm{erf}(u)=\sum_{k=0}^{\infty}\frac{u^{4k+1}}{(2k)!}\left[\frac{1}{4k+1}-\frac{u^2}{2k+1}+\frac{2u^2}{4k+3}\right]$$

and

$$(ii)\ \frac{\sqrt{\pi}}{2}\mathrm{erf}(u) = \sum_{k=0}^{\infty} \frac{\cos\left(\frac{\pi k}{2}\right)}{(k+1)\Gamma\left(\frac{k}{2}+1\right)} u^{k+1},$$

where erf(u) *denotes the error function and* $k!$ *denotes the factorial function and* $\Gamma(k)$ *denotes the gamma function.*

Proof. In the previous paper [1], I demonstrated that

$$e^{-z} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}\left[1 - \frac{z}{2k+1}\right] \tag{1}$$

and, I calculate that

$$e^{-z} = \sum_{k=0}^{\infty} \frac{\cos\left(\frac{\pi k}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)} z^{k/2}, \tag{2}$$

for $z \in \mathbb{R}$.

In [2, page 333, formula 3.326], I have

$$\int_0^u \exp(-t^2)\mathrm{dt} = \frac{\sqrt{\pi}}{2}\mathrm{erf}(u), \tag{3}$$

for $\Re(u) > 0$.

Let $z \to t^2$ in (1) and (2) and integrate from 0 at u with respect to t:

$$\begin{aligned} \frac{\sqrt{\pi}}{2}\mathrm{erf}(u) &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left[\int_0^u t^{4k}\mathrm{dt} - \frac{1}{2k+1}\int_0^u t^{4k+2}\mathrm{dt}\right] \\ &= \sum_{k=0}^{\infty} \frac{u^{4k+1}}{(2k)!}\left[\frac{1}{4k+1} - \frac{u^2}{(2k+1)(4k+3)}\right] \\ &= \sum_{k=0}^{\infty} \frac{u^{4k+1}}{(2k)!}\left[\frac{1}{4k+1} - \frac{u^2}{2k+1} + \frac{2u^2}{4k+3}\right], \end{aligned} \tag{4}$$

and

$$\begin{aligned} \frac{\sqrt{\pi}}{2}\mathrm{erf}(u) &= \sum_{k=0}^{\infty} \frac{\cos\left(\frac{\pi k}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)} \int_0^u t^k \mathrm{dt} \\ &= \sum_{k=0}^{\infty} \frac{\cos\left(\frac{\pi k}{2}\right)}{(k+1)\Gamma\left(\frac{k}{2}+1\right)} u^{k+1}, \end{aligned}$$

which proves the results desired. □

Theorem 2. *For* $u \in \mathbb{R}$, *then*

$$\begin{aligned} \frac{\sqrt{\pi}}{2}\mathrm{erf}(u) = u\int_0^{\pi/4} \cosh[u^2\tan^2(t)]\sec^2(t)\mathrm{dt} - u^3\int_0^{\pi/4} \cosh[u^2\tan(t)]\sec^2(t)\mathrm{dt} \\ +2u^3\int_0^{\pi/4} \cosh[u^2\tan^2(t)]\sin^2(t)\sec^4(t)\mathrm{dt} \end{aligned}$$

and

$$\frac{\sqrt{\pi}\mathrm{erf}(u)}{2u} = \int_0^{\pi/4} e^{-u^2\tan^2(t)}\sec^2(t)\,\mathrm{dt},$$

where $\operatorname{erf}(u)$ *denotes the error function,* e^u *denotes the exponential function,* $\cosh(t)$ *denotes the hyperbolic cosine function,* $\tan(t)$ *denotes the tangent function and* $\sec(t)$ *denotes the secant function.*

Proof. In [2, page 389, formula 3.624], I encounter

$$\frac{1}{k+1}=\int_0^{\pi/4}\frac{\sin^k(t)}{\cos^{k+2}(t)}\mathrm{dt} \tag{5}$$

I substitute (8) in Theorem 1, and obtain

$$\frac{\sqrt{\pi}}{2}\operatorname{erf}(u)=\sum_{k=0}^{\infty}\frac{u^{4k+1}}{(2k)!}\left[\frac{1}{4k+1}-\frac{u^2}{2k+1}+\frac{2u^2}{4k+3}\right]$$

$$=u\int_0^{\pi/4}\sum_{k=0}^{\infty}\frac{u^{4k}}{(2k)!}\frac{\sin^{4k}(t)}{\cos^{4k+2}(t)}\mathrm{dt}-u^3\int_0^{\pi/4}\sum_{k=0}^{\infty}\frac{u^{4k}}{(2k)!}\frac{\sin^{2k}(t)}{\cos^{2k+2}(t)}\mathrm{dt}$$

$$+2u^3\int_0^{\pi/4}\sum_{k=0}^{\infty}\frac{u^{4k}}{(2k)!}\frac{\sin^{4k+2}(t)}{\cos^{4k+4}(t)}\mathrm{dt}$$

$$=u\int_0^{\pi/4}\frac{1}{\cos^2(t)}\sum_{k=0}^{\infty}\frac{[u^2\tan^2(t)]^{2k}}{(2k)!}\mathrm{dt}-u^3\int_0^{\pi/4}\frac{1}{\cos^2(t)}\sum_{k=0}^{\infty}\frac{[u^2\tan(t)]^{2k}}{(2k)!}\mathrm{dt}$$

$$+2u^3\int_0^{\pi/4}\frac{\sin^2(t)}{\cos^4(t)}\sum_{k=0}^{\infty}\frac{[u^2\tan^2(t)]^{2k}}{(2k)!}\mathrm{dt}$$

$$=u\int_0^{\pi/4}\frac{\cosh[u^2\tan^2(t)]}{\cos^2(t)}\mathrm{dt}-u^3\int_0^{\pi/4}\frac{\cosh[u^2\tan(t)]}{\cos^2(t)}\mathrm{dt}$$

$$+2u^3\int_0^{\pi/4}\frac{\sin^2(t)}{\cos^4(t)}\cosh[u^2\tan^2(t)]\mathrm{dt}$$

$$=u\int_0^{\pi/4}\cosh[u^2\tan^2(t)]\sec^2(t)\mathrm{dt}-u^3\int_0^{\pi/4}\cosh[u^2\tan(t)]\sec^2(t)\mathrm{dt}$$

$$+2u^3\int_0^{\pi/4}\cosh[u^2\tan^2(t)]\sin^2(t)\sec^4(t)\mathrm{dt}$$

and

$$\frac{\sqrt{\pi}}{2}\operatorname{erf}(u)=\sum_{k=0}^{\infty}\frac{\cos\left(\frac{\pi k}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)}\int_0^{\pi/4}\frac{\sin^k(t)}{\cos^{k+2}(t)}\mathrm{dt}\,u^{k+1}$$

$$=u\int_0^{\pi/4}\frac{1}{\cos^2 t}\sum_{k=0}^{\infty}\frac{\cos\left(\frac{\pi k}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)}\frac{\sin^k(t)}{\cos^k(t)}u^k\mathrm{dt}$$

$$=u\int_0^{\pi/4}\frac{1}{\cos^2 t}\sum_{k=0}^{\infty}\frac{\cos\left(\frac{\pi k}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)}[u\tan(t)]^k\mathrm{dt}$$

$$=u\int_0^{\pi/4}e^{-u^2\tan^2(t)}\sec^2(t)\,\mathrm{dt},$$

which proves the results desired. □

Question 3. *These questions below are left as an exercise for the reader:*

$$\int_0^{\pi/4} \cosh[u\tan(t)]\sec^2(t)\mathrm{dt} = \frac{\sinh(u)}{u},$$

$$\int_0^{\pi/4} \sinh[u\tan(t)]\sec^2(t)\mathrm{dt} = \frac{\cosh(u)-1}{u},$$

$$\int_0^{\pi/4} \cosh[u^2\tan^2(t)]\sec^2(t)\mathrm{dt} = \frac{\sqrt{\pi}}{4u}(\mathrm{erfi}(u)+\mathrm{erf}(u)),$$

$$\int_0^{\pi/4} \sinh[u^2\tan^2(t)]\sec^2(t)\mathrm{dt} = \frac{\sqrt{\pi}}{4u}(\mathrm{erfi}(u)-\mathrm{erf}(u)),$$

$$\int_0^{\pi/4} e^{-u\tan(t)}\sec^2(t)\mathrm{dt} = \frac{1-e^{-u}}{u},$$

$$\int_0^{\pi/4} e^{u\tan(t)}\sec^2(t)\mathrm{dt} = \frac{e^u-1}{u},$$

$$\int_0^{\pi/4} \tanh[u\tan(t)]\sec^2(t)\mathrm{dt} = \frac{\ln(\cosh u)}{u},$$

$$\int_0^{\pi/4} \mathrm{sech}[2u\tan(t)]\sec^2(t)\mathrm{dt} = \frac{\tan^{-1}(\tanh u)}{u},$$

$$\int_0^{\pi/4} \cos[u\tan(t)]\sec^2(t)\mathrm{dt} = \frac{\sin(u)}{u},$$

$$\int_0^{\pi/4} \cos\big[u\sqrt{\tan(t)}\,\big]\sec^2(t)\mathrm{dt} = \frac{2[u\sin(u)+\cos(u)-1]}{u^2},$$

$$\int_0^{\pi/4} \cos[2u\tan(t)]\sec^2(t)\mathrm{dt} = \frac{\sin(u)\cos(u)}{u},$$

$$\int_0^{\pi/4} \sin[u\tan(t)]\sec^2(t)\mathrm{dt} = \frac{1-\cos(u)}{u},$$

$$\int_0^{\pi/4} \sin\big[u\sqrt{\tan(t)}\,\big]\sec^2(t)\mathrm{dt} = \frac{2[\sin(u)-u\cos(u)]}{u^2},$$

$$\int_0^{\pi/4} \sin[2u\tan(t)]\sec^2(t)\mathrm{dt} = \frac{\sin^2(u)}{u},$$

$$\int_0^{\pi/4} \tan[u\tan(t)]\sec^2(t)\mathrm{dt} = -\frac{\ln(\cos u)}{u},$$

$$\int_0^{\pi/4} \cos[u\tan^2(t)]\sec^2(t)\mathrm{dt} = \sqrt{\frac{\pi}{2u}}C\left(\sqrt{\frac{2u}{\pi}}\right),$$

$$\int_0^{\pi/4} \sin[u\tan^2(t)]\sec^2(t)\mathrm{dt} = \sqrt{\frac{\pi}{2u}}S\left(\sqrt{\frac{2u}{\pi}}\right),$$

Theorem 4. *For* $u\in\mathbb{R}$, *then*

$$\frac{\sqrt{\pi}}{2}\mathrm{erf}(u) = \int_0^u [(2t^2+1)\cosh(t^2) - u^2\cosh(ut)]\mathrm{dt},$$

where $\mathrm{erf}(u)$ *denotes the error function and* $\cosh(t)$ *denotes the hyperbolic cosine function.*

Proof. By Theorem 1 (*ii*), I encounter

$$\begin{aligned}\frac{\sqrt{\pi}}{2}\mathrm{erf}(u)&=\sum_{k=0}^{\infty}\frac{1}{(2k)!}\left[\int_0^u t^{4k}\mathrm{dt}-u^{2k+2}\int_0^u t^{2k}\mathrm{dt}+2\int_0^u t^{4k+2}\mathrm{dt}\right]\\&=\int_0^u\sum_{k=0}^{\infty}\frac{t^{4k}}{(2k)!}\mathrm{dt}-u^2\int_0^u\sum_{k=0}^{\infty}\frac{(ut)^{2k}}{(2k)!}\mathrm{dt}+2\int_0^u\sum_{k=0}^{\infty}\frac{t^{4k+2}}{(2k)!}\mathrm{dt}\\&=\int_0^u\cosh(t^2)\mathrm{dt}-u^2\int_0^u\cosh(ut)\mathrm{dt}+2\int_0^u t^2\cosh(t^2)\mathrm{dt}\\&=\int_0^u[\cosh(t^2)-u^2\cosh(ut)+2t^2\cosh(t^2)]\mathrm{dt}\\&=\int_0^u[(2t^2+1)\cosh(t^2)-u^2\cosh(ut)]\mathrm{dt},\end{aligned}$$

which proves the result desired. □

3. Relations for Special Functions

Theorem 5. *If* $\mathrm{Re}(u)>0$, *then*

$$\Gamma(0,u)=\mathrm{Shi}\,u-\mathrm{Chi}\,u,$$

where $\Gamma(x,u)$ *denotes the incomplete gamma function,* $\mathrm{Shi}(u)$ *denotes the hyperbolic sine integral function and* $\mathrm{Chi}(u)$ *denotes the hyperbolic cosine integral function.*

Proof. In Question 3, I have

$$\int_0^{\pi/4}e^{-x\tan(v)}\sec^2(v)\mathrm{dv}=\frac{1-e^{-x}}{x}. \tag{6}$$

Integrate (6) from 0 at u with respect to x, thus

$$\begin{aligned}&\int_0^{\pi/4}\left[\int_0^u e^{-x\tan(v)}\mathrm{dx}\right]\sec^2(v)\mathrm{dv}=\int_0^u\frac{1-e^{-x}}{x}\mathrm{dx}\\\Rightarrow&\int_0^{\pi/4}\left(1-e^{-u\tan(v)}\right)\cot(v)\sec^2(v)\mathrm{dv}=\int_0^u\frac{1-e^{-x}}{x}\mathrm{dx}.\end{aligned} \tag{7}$$

I evaluate

$$\int_0^{\pi/4}\left(1-e^{-u\tan(v)}\right)\cot(v)\sec^2(v)\mathrm{dv}=\gamma+\ln u+\mathrm{Shi}\,u-\mathrm{Chi}\,u \tag{8}$$

and

$$\int_0^u\frac{1-e^{-x}}{x}\mathrm{dx}=\gamma+\ln u+\Gamma(0,u), \tag{9}$$

for $\mathrm{Re}(u)>0$.

From (8) and (9), I obtain the equation

$$\gamma+\ln u+\mathrm{Shi}\,u-\mathrm{Chi}\,u=\gamma+\ln u+\Gamma(0,u)\Rightarrow\mathrm{Shi}\,u-\mathrm{Chi}\,u=\Gamma(0,u),$$

and this completes the proof. □

Theorem 6. *If* $\mathrm{Re}(u)<0$, *then*

$$2\mathrm{Shi}\,u+2\mathrm{Chi}\,u=2\mathrm{Ei}\,u+\ln\frac{1}{u}+\ln u,$$

where $\mathrm{Shi}(u)$ *denotes the hyperbolic sine integral function,* $\mathrm{Chi}(u)$ *denotes the hyperbolic cosine integral function,* $\mathrm{Ei}(u)$ *denotes the exponential integral function and* $\ln u$ *denotes the natural logarithm function.*

Proof. In Question 3, I have

$$\int_0^{\pi/4}e^{x\tan(v)}\sec^2(v)\mathrm{dv}=\frac{e^x-1}{x}. \tag{10}$$

Integrate (10) from 0 at u with respect to x, thus

$$\begin{aligned}&\int_0^{\pi/4}\left[\int_0^u e^{x\tan(v)}\mathrm{dx}\right]\sec^2(v)\mathrm{dv}=\int_0^u\frac{e^x-1}{x}\mathrm{dx}\\ \Rightarrow&\int_0^{\pi/4}\left(e^{u\tan(v)}-1\right)\cot(v)\sec^2(v)\mathrm{dv}=\int_0^u\frac{e^x-1}{x}\mathrm{dx}.\end{aligned}\tag{11}$$

I evaluate

$$\int_0^{\pi/4}\left(e^{u\tan(v)}-1\right)\cot(v)\sec^2(v)\mathrm{dv}=\mathrm{Ei}\,u+\frac{1}{2}\left(\ln\frac{1}{u}-\ln u-2\gamma\right)\tag{12}$$

and

$$\int_0^u\frac{e^x-1}{x}\mathrm{dx}=\mathrm{Chi}\,u+\mathrm{Shi}\,u-\ln u-\gamma,\tag{13}$$

for $\mathrm{Re}(u)<0$.

From (12) and (13), I obtain the equation

$$\begin{aligned}&\mathrm{Ei}\,u+\frac{1}{2}\left(\ln\frac{1}{u}-\ln u-2\gamma\right)=\mathrm{Chi}\,u+\mathrm{Shi}\,u-\ln u-\gamma\\ \Rightarrow&\mathrm{Ei}\,u+\frac{1}{2}\ln\frac{1}{u}-\frac{1}{2}\ln u-\gamma=\mathrm{Chi}\,u+\mathrm{Shi}\,u-\ln u-\gamma\\ \Rightarrow&\mathrm{Shi}\,u+\mathrm{Chi}\,u=\mathrm{Ei}\,u+\frac{1}{2}\ln\frac{1}{u}+\frac{1}{2}\ln u\\ \Rightarrow&2\mathrm{Shi}\,u+2\mathrm{Chi}\,u=2\mathrm{Ei}\,u+\ln\frac{1}{u}+\ln u,\end{aligned}$$

this completes the proof. □

Theorem 7. *If* $-1<\mathrm{Re}(\alpha)<0$ *and* $u\in\mathbb{R}$*, then*

$$\Gamma(\alpha+1,u)-\alpha\Gamma(\alpha,u)=\Gamma(\alpha+1)-\alpha\Gamma(\alpha)+u^\alpha e^{-u},$$

where $\mathrm{Shi}\,u$ *denotes the hyperbolic sine integral function,* $\mathrm{Chi}\,u$ *denotes the hyperbolic cosine integral function,* $\mathrm{Ei}\,u$ *denotes the exponential integral function and* $\ln u$ *denotes the natural logarithm function.*

Proof. In Question 3, I have

$$\int_0^{\pi/4}e^{-t\tan(v)}\sec^2(v)\mathrm{dv}=\frac{1-e^{-t}}{t}.\tag{14}$$

Multiply (14) by t^α and integrate from 0 at u with respect to t, thus

$$\begin{aligned}&\int_0^{\pi/4}\left[\int_0^u e^{-t\tan(v)}t^\alpha\,\mathrm{dt}\right]\sec^2(v)\mathrm{dv}=\int_0^u\frac{(1-e^{-t})t^\alpha}{t}\mathrm{dt}\\ \Rightarrow&\int_0^{\pi/4}[\Gamma(\alpha+1)-\Gamma(\alpha+1,u\tan(v))]\frac{\cot(v)}{\tan^\alpha(v)}\sec^2(v)\mathrm{dv}=\int_0^u\frac{(1-e^{-t})t^\alpha}{t}\mathrm{dt}.\end{aligned}\tag{15}$$

I evaluate

$$\begin{aligned}&\int_0^{\pi/4}[\Gamma(\alpha+1)-\Gamma(\alpha+1,u\tan(v))]\frac{\cot(v)}{\tan^\alpha(v)}\sec^2(v)\mathrm{dv}\\ =&\Gamma(\alpha+1)\int_0^{\pi/4}\frac{\cot(v)}{\tan^\alpha(v)}\sec^2(v)\mathrm{dv}-\int_0^{\pi/4}\Gamma(\alpha+1,u\tan(v))\frac{\cot(v)}{\tan^\alpha(v)}\sec^2(v)\mathrm{dv}\\ =&-\frac{\Gamma(\alpha+1)}{\alpha}+\frac{u^\alpha(\sinh u-\cosh u+1)+\Gamma(\alpha+1,u)}{\alpha},\end{aligned}\tag{16}$$

for $\mathrm{Re}(\alpha)<0$.

and

$$\int_0^u \frac{(1-e^{-t})t^{\alpha}}{t}\mathrm{dt} = \frac{u^{\alpha}}{\alpha} - \Gamma(\alpha) + \Gamma(\alpha, u), \tag{17}$$

for $\mathrm{Re}(\alpha) > -1$.

From (16) and (17), I obtain the equation

$$\begin{aligned}
-\frac{\Gamma(\alpha+1)}{\alpha} + \frac{u^{\alpha}(\sinh u - \cosh u + 1) + \Gamma(\alpha+1, u)}{\alpha} &= \frac{u^{\alpha}}{\alpha} - \Gamma(\alpha) + \Gamma(\alpha, u) \\
\Rightarrow -\Gamma(\alpha+1) + u^{\alpha}(\sinh u - \cosh u) + \Gamma(\alpha+1, u) &= -\alpha\Gamma(\alpha) + \alpha\Gamma(\alpha, u) \\
\Rightarrow \Gamma(\alpha+1, u) - \alpha\Gamma(\alpha, u) &= \Gamma(\alpha+1) - \alpha\Gamma(\alpha) - u^{\alpha}(\sinh u - \cosh u) \\
\Rightarrow \Gamma(\alpha+1, u) - \alpha\Gamma(\alpha, u) &= \Gamma(\alpha+1) - \alpha\Gamma(\alpha) + u^{\alpha}(\cosh u - \sinh u) \\
\Rightarrow \Gamma(\alpha+1, u) - \alpha\Gamma(\alpha, u) &= \Gamma(\alpha+1) - \alpha\Gamma(\alpha) + u^{\alpha}e^{-u}.
\end{aligned}$$

and this completes the proof. □

4. More Integral Representations

Theorem 8. *For* $u \in \mathbb{R}$, *then*

$$\frac{\mathrm{Shi}(u)}{2} = \int_0^{\pi/4} \frac{\sinh(u\tan(v))}{\sin(2v)}\mathrm{dv},$$

where $\mathrm{Shi}(u)$ *denotes the hyperbolic sine integral function and* $\sinh(v)$ *denotes the hyperbolic sine function, and* $\sin(v)$ *denotes the sine function.*

Proof. In Question 3, I have

$$\int_0^{\pi/4} \cosh[t\tan(v)]\sec^2(v)\mathrm{dv} = \frac{\sinh(t)}{t}. \tag{18}$$

Integrate (18) from 0 at u with respect to t, thus

$$\begin{aligned}
\int_0^{\pi/4}\left[\int_0^u \cosh[t\tan(v)]\,\mathrm{dt}\right]\sec^2(v)\mathrm{dv} &= \int_0^u \frac{\sinh(t)}{t}\mathrm{dt} \\
\Rightarrow \int_0^{\pi/4} \sinh(u\tan(v))\cot(v)\sec^2(v)\mathrm{dv} &= \int_0^u \frac{\sinh(t)}{t}\mathrm{dt}.
\end{aligned} \tag{19}$$

I know that

$$\int_0^u \frac{\sinh(t)}{t}\mathrm{dt} = \mathrm{Shi}(u). \tag{20}$$

From (19) and (20), I obtain the integral representation

$$\begin{aligned}
\int_0^{\pi/4} \sinh(u\tan(v))\cot(v)\sec^2(v)\mathrm{dv} &= \mathrm{Shi}(u) \\
\Rightarrow \int_0^{\pi/4} \sinh(u\tan(v))\csc(v)\sec(v)\mathrm{dv} &= \mathrm{Shi}(u) \\
\Rightarrow 2\int_0^{\pi/4} \frac{\sinh(u\tan(v))}{\sin(2v)}\mathrm{dv} &= \mathrm{Shi}(u),
\end{aligned}$$

and this completes the proof. □

Theorem 9. *For* $u \in \mathbb{R}$, *then*

$$\frac{\mathrm{Chi}(u) - \ln(u) - \gamma}{2} = \int_0^{\pi/4} \frac{\cosh[u\tan(v)] - 1}{\sin(2v)}\mathrm{dv},$$

where $\mathrm{Chi}(u)$ *denotes the hyperbolic cosine integral function,* $\ln(u)$ *denotes the logarithm natural function,* γ *denotes the Euler's constant,* $\cosh(v)$ *denotes the hyperbolic cosine function,* $\tan(v)$ *denotes the tangent function and* $\sin(v)$ *denotes the sine function.*

Proof. In Question 3, I have

$$\int_0^{\pi/4} \sinh[t\tan(v)]\sec^2(v)\mathrm{dv} = \frac{\cosh(t)-1}{t}. \tag{21}$$

Multiply (14) by t^{α} and integrate from 0 at u with respect to t, thus

$$\begin{aligned}\int_0^{\pi/4}\left[\int_0^u \sinh[t\tan(v)]\,\mathrm{dt}\right]\sec^2(v)\mathrm{dv} &= \int_0^u \frac{\cosh(t)-1}{t}\mathrm{dt}\\ \Rightarrow \int_0^{\pi/4}\{\cosh[u\tan(v)]-1\}\cot(v)\sec^2(v)\mathrm{dv} &= \int_0^u \frac{\cosh(t)-1}{t}\mathrm{dt}.\end{aligned} \tag{22}$$

I know that

$$\int_0^u \frac{\cosh(t)-1}{t}\mathrm{dt} = \mathrm{Chi}(u) - \ln(u) - \gamma. \tag{23}$$

From (22) and (23), I obtain the integral representation

$$\begin{aligned}\int_0^{\pi/4}\{\cosh[u\tan(v)]-1\}\cot(v)\sec^2(v)\mathrm{dv} &= \mathrm{Chi}(u)-\ln(u)-\gamma\\ \Rightarrow \int_0^{\pi/4}\{\cosh[u\tan(v)]-1\}\csc(v)\sec(v)\mathrm{dv} &= \mathrm{Chi}(u)-\ln(u)-\gamma\\ \Rightarrow 2\int_0^{\pi/4}\frac{\cosh[u\tan(v)]-1}{\sin(2v)}\mathrm{dv} &= \mathrm{Chi}(u)-\ln(u)-\gamma,\end{aligned}$$

and this completes the proof. □

Theorem 10. *For* $\mathrm{Re}(x) < -1$ *or* $\mathrm{Re}(x) > 1$ *then*

$$\frac{\tanh^{-1}\left(\frac{1}{x}\right)}{x} = \int_0^{\pi/4} \frac{\csc(v)\sec(v)}{x^2\cot(v)-\tan(v)}\mathrm{dv},$$

where $\tanh^{-1}(x)$ *denotes the inverse hyperbolic tangent function,* $\csc(v)$ *denotes the cosecant function,* $\sec(v)$ *denotes the secant function,* $\cot(v)$ *denotes the cotangent function and* $\tan(v)$ *denotes the tangent function.*

Proof. In Theorem 8, I encounter

$$\int_0^{\pi/4} \sinh(u\tan(v))\cot(v)\sec^2(v)\mathrm{dv} = \mathrm{Shi}(u). \tag{24}$$

Multiply (24) by e^{-ux} and integrate from 0 at infinity with respect to u, thus

$$\begin{aligned}\int_0^{\pi/4}\left[\int_0^{\infty} \sinh(u\tan(v))e^{-ux}\mathrm{du}\right]\cot(v)\sec^2(v)\mathrm{dv} &= \int_0^{\infty}\mathrm{Shi}(u)e^{-xu}\mathrm{du}\\ \Rightarrow \int_0^{\pi/4}\frac{\cot(v)\sec^2(v)}{x^2\cot(v)-\tan(v)}\mathrm{dv} &= \int_0^{\infty}\mathrm{Shi}(u)e^{-xu}\mathrm{du}.\end{aligned} \tag{25}$$

I evaluate

$$\int_0^{\infty}\mathrm{Shi}(u)e^{-xu}\mathrm{du} = \frac{1}{x}\tanh^{-1}\left(\frac{1}{x}\right), \tag{26}$$

for $\mathrm{Re}(x) > 1$ and $\mathrm{Im}(x) = 0$.

From (25) and (26), I have

$$\begin{aligned}\int_0^{\pi/4}\frac{\cot(v)\sec^2(v)}{x^2\cot(v)-\tan(v)}\mathrm{dv} &= \frac{\tanh^{-1}\left(\frac{1}{x}\right)}{x}\\ \Rightarrow \int_0^{\pi/4}\frac{\csc(v)\sec(v)}{x^2\cot(v)-\tan(v)}\mathrm{dv} &= \frac{\tanh^{-1}\left(\frac{1}{x}\right)}{x},\end{aligned}$$

and this completes the proof. □

Question 11. *These questions below are left as an exercise for the reader:*

$$\frac{\mathrm{Ci}(u)-\ln(u)-\gamma}{2}=\int_0^{\pi/4}\frac{\cos[u\tan(v)]-1}{\sin(2v)}\mathrm{dv},$$

$$\frac{\mathrm{Si}(u)}{2}=\int_0^{\pi/4}\frac{\sin[u\tan(v)]}{\sin(2v)}\mathrm{dv},$$

References

[1] Guedes, Edigles, *New Infinity Series Representation for the Bessel Function of the first kind*, 2014, to appear.

[2] Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals, Series and Products*, Academic Press, 2000.