SUBSETHOOD MEASURE FOR SINGLE VALUED NEUTROSOPHIC SETS

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Abstract. The main aim of this paper is to introduce a neurosophic subsethood measure for single valued neutrosophic sets. For this purpose, we first introduce a system of axioms for subsethood measure of single valued neutrosophic sets. Then we give a simple subsethood measure based to distance measure. Finally, to show effectiveness of intended subsethood measure, an application is presented in multicriteria decision making problem and results obtained are discussed. Though having a simple measure for calculation, the subsethood measure presents a new approach to deal with neutrosophic information.

1. Introduction

Fuzzy entropy, distance measure and similarity measure are three basic concept forming framework of fuzzy set theory. There is directly a relationship between subsethood measure and these concepts. Usually subsethood measures are constructed using implication operators, t-norms or t-conorns, entropy measures or cardinalities. In classical theory, it is said that a set A is a subset of B and is denoted by $A \subset B$ if every element of A is an element of B, whenever X is a universal set and A, B are two sets in X. Therefore, subsethood measure should be two valued for crisp sets. That is, either A is precisely subset of B or vice versa. But since an element x in universal set X can belong to a fuzzy set A to varying degrees, it is notable to consider situations describing as being "more and less" a subset of another set and to measure the degree of this subsethood. Fuzzy subsethood allows a given fuzzy set to contain another to some degree between 0 and 1. According to Zadeh's fuzzy set containment, a fuzzy set B contains a fuzzy set A if $m_A(x) \leq m_B(x)$, for all x in X, in which m_A and m_B are the membership functions of A and B, respectively. Various researcher have proposed different subsethood measures [3, 5, 8, 9, 12, 13].

Smarandache [10] introduced the concept of neutrosophic set which has three components such that the truth-membership, the indeterminacy-membership, and the falsity-membership. Neutrosophic set is a powerful general formal framework which generalizes the concepts of the classic set, fuzzy set [16], interval-valued fuzzy set [11], intuitionistic fuzzy set [1] and interval-valued intuitionistic fuzzy set [2]. Wang et al. [4] proposed single valued neutrosophic set (SVNS), which is an

1991 *Mathematics Subject Classification.* 2010 Primary 62C86; Secondary 90B50.

Key words and phrases. Subsethood measure, distance measure, fuzzy subsethood, neutrosophic subsethood, multicriteria decision-making.

instance of neutrosophic set and is more suitable for the scientific and engineering applications in the real word. In the literature, there is no more work on entropy, distance measure and similarity measure of neutrosophic sets. Ye [14] has presented the correlation and correlation coefficient of single-valued neutrosophic sets based on the extension of the correlation of intuitionistic fuzzy sets and has established a new multicriteria decision-making method under single-valued neutrosophic environment by means of the weighted correlation coefficient or the weighted cosine similarity measure. Majumdar et al. [7] have investigated the notions of distance and similarity between two single valued neutrosophic sets as well as entropy of a single valued neutrosophic set. As the case of fuzzy sets, we define a subsethood measure which measures the degree of being a subset of another set in neutrosophic information. Our subsethood measure is based on distance measure. To demonstrate the effectiveness of the proposed subsethood measure, we consider the multicriteria decision-making problem adapted from Ye [14].

In this paper, we firstly review the systems of axioms of Young's fuzzy subsethood measure and give a new systems of axioms for subsethood measure of single valued neutrosophic sets. We then propose a simple subsethood measure by the normalized Hamming distance. Examples given show necessity and remarkability of the subsethood measure. The remaining part of this paper is organized as follows. Section 2 contains basic definitions and notations that are used in the remaining parts of the paper. We investigate subsethood measure for single valued neutrosophic sets in Section 3. In section 4, we demonstrate its effectiveness by a multicriteria decision making problem. Finally, conclusion is made in section 5.

2. Preliminaries

In this section we recall some definitions, operations and properties regarding subsethood measure and neutrosophic (single valued) sets, which will be used in the rest of the paper. We denote the set of all fuzzy sets on X by $FS(X)$. For any fuzzy set A, its complement A^c is defined by $m_{A^c}(x) = 1 - m_A(x)\,, \forall\, x \in X$.

2.1. **Subsethood measure.** It is well known that subsethood measures can be generated from distance measures. In fuzzy set theory, fuzzy subsethood is an important concept. Zadeh's subsethood definition is given by for fuzzy sets A and B

$$A \subseteq B \Longleftrightarrow m_A(x) \leq m_B(x)\,, \forall x \in X.$$

Since a element x in universal set X can belong to a fuzzy set A to varying degrees, it is more natural to consider an indicator of degree to which A is subset of B. In general, such an indicator is a mapping $I : FS(X) \times FS(X) \to [0, 1]$ satisfying special properties, called an inclusion indicator or subsethood measure. A systems of axioms of fuzzy subsethood characterized by Young is given in the following:

Definition 2.1. [15] *A mapping $\alpha : FS(X)\times FS(X) \to [0,1]$ is called a fuzzy subsethood measure, if α satisfies the following properties (for all $A,B,C \in FS(X)$):*

- (α_1) $\alpha(A,B) = 1$ *if and only if* $A \subseteq B$.
- (α_2) *Let* $\left[\frac{1}{2}\right] \subseteq A$, *where* $\left[\frac{1}{2}\right]$ *is the fuzzy set of* X *defined by* $m_{\left[\frac{1}{2}\right]}(x) = \frac{1}{2}$ *for each* $x \in X$. *Then* $\alpha(A,A^c) = 0$ *if and only if* $A = X$.
- (α_3) *If* $A \subseteq B \subseteq C$, *then* $\alpha(C,A) \leq \alpha(B,A)$; *and if* $A \subseteq B$, *then* $\alpha(C,A) \leq \alpha(C,B)$.

2.2. **Neutrosophic sets.**

Definition 2.2. [10] *Let X be a space of points (objects). A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]0^-,1^+[$. That is $T_A(x) : X \longrightarrow\ ]0^-,1^+[$, $I_A(x) : X \longrightarrow\ ]0^-,1^+[$ and $F_A(x) : X \longrightarrow\ ]0^-,1^+[$.*

There is no restriction on the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$. So $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]0^-,1^+[$. In real life application in scientific and engineering problems, since it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]0^-,1^+[$, it is considered the neutrosophic set (single valued neutrosophic set) which takes the value from the subset of $[0,1]$. Here, we will use the notions u_A, w_A and v_A instead of notions T_A, I_A and F_A, respectively.

Neutrosophic set is a powerful general set theory that has been recently proposed. However, neutrosophic set needs to be specified from a technical point of view. To this effect, Wang et al. [4] define the concept of single valued neutrosophic set (SVNS) which is an instance of neutrosophic set.

2.3. **Single valued neutrosophic sets.** A single valued neutrosophic set has been defined in [4] as follows:

Definition 2.3. *Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form:*

$$A = \{\langle x, u_A(x), w_A(x), v_A(x)\rangle : x \in X\},$$

where $u_A : X \longrightarrow [0,1]$, $w_A : X \longrightarrow [0,1]$ *and* $v_A : X \longrightarrow [0,1]$ *with the condition*

$$0 \leq u_A(x) + w_A(x) + v_A(x) \leq 3, \forall x \in X.$$

The numbers $u_A(x), w_A(x)$ and $v_A(x)$ denote the degree of truth-membership, indeterminacy-membership and falsity-membership of x to X, respectively.

Definition 2.4. [4] *Let A and B be two single valued neutrosophic sets,*

$$A = \{\langle x, u_A(x), w_A(x), v_A(x)\rangle : x \in X\} \text{ and } B = \{\langle x, u_B(x), w_B(x), v_B\rangle : x \in X\}.$$

Then some operations can be defined as follows:

(1) $A \cup B = \{\langle x : \{\max u_A(x), u_B(x)\}, \max\{w_A(x), w_B(x)\}, \min\{v_A(x), v_B(x)\}\rangle\}$;
(2) $A \cap B = \{\langle x : \{\min u_A(x), u_B(x)\}, \min\{w_A(x), w_B(x)\}, \max\{v_A(x), v_B(x)\}\rangle\}$;
(3) $A \subseteq B$ *if and only if* $u_A(x) \leq u_B(x)$, $w_A(x) \geq w_B(x)$ *and* $v_A(x) \geq v_B(x)$, $\forall x \in X$;
(4) $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$;
(5) $A^c = \{\langle x, v_A(x), 1 - w_A(x), u_A(x)\rangle\}$.

In the single valued neutrosophic information, since each membership values are independently of one another, there exist definitons of different neutrosophic empty set and thus consequently absolute neutrosophic set. For our purposes here we adopt the following definitions.

Definition 2.5. [10] *Let A be a single valued neutrosophic set on X.*

(1) *A single valued neutrosophic set A is empty, denoted by $\underline{0} = \langle 0, 1, 1\rangle$ if $u_A(x) = 0$, $w_A(x) = 1$ and $v_A(x) = 1$ for each $x \in X$.*
(2) *A single valued neutrosophic set A is absolute, denoted by $\underline{1} = \langle 1, 0, 0\rangle$ if $u_A(x) = 1$, $w_A(x) = 0$ and $v_A(x) = 0$ for each $x \in X$.*

3. Subsethood measures for single valued neutrosophic sets

Grzegorzewski et al. [6] have introduced a simple but very useful subsethoot measure for intuitionistic fuzzy sets. In this section, we first give a formal definition of subsethood measure for single valued neutrosophic sets. By generalizing the Grzegorzewski's subsethoot measure to single valued neutrosophic sets , we propose a subsethood measure based on distance between single valued neutrosophic sets.

Definition 3.1. *A mapping $S : SVNS(X) \times SVNS(X) \to [0, 1]$ is called a neutrosophic subsethood measure, if S satisfies the following properties (for all $A, B, C \in SVNS(X)$):*

(S1) $S(A, B) = 1$ *if* $A \subseteq B$ *(in Definition 2.4 (3)).*
(S2) $S(A, A^c) = 1 \Leftrightarrow \forall x \in X$, $u_A(x) \leq v_A(x)$ *and* $w_A(x) \leq 0.5$.
(S3) $S(\underline{1}, \underline{0}) = 0$, *where $\underline{1}$ is the single valued absolute neutrosophic set and $\underline{0}$ is the single valued empty neutrosophic set.*
(S4) $A \subseteq B \subseteq C \Rightarrow S(C, A) \leq S(B, A)$ *and* $S(C, A) \leq S(C, B)$.

Suppose that $d : SVNS(X) \times SVNS(X) \longrightarrow \mathbb{R}^{+} \cup \{0\}$ is a distance between single valued neutrosophic sets in X. To establish the inclusion indicator expressing the degree to which A belongs to B, we can use the distance between single valued neutrosophic sets A and $A \cap B$. If it is considered the subsethood measure based on distance measure, we have the formal given by

$$S_d(A,B) = 1 - d\left(A, A \cap B\right)$$

In this paper, we adopt the normalized Hamming distance between single valued neutrosophic sets $A = \{\langle x, u_A(x), w_A(x), v_A(x)\rangle : x \in X\}$ and $B = \{\langle x, u_B(x), w_B(x), v_B(x)\rangle : x \in X\}$,

$$D_{nH} = \frac{1}{n}\sum_{i=1}^{n} \max\left\{\left|u_A(x_i) - u_B(x_i)\right|, \left|w_A(x_i) - w_B(x_i)\right|, \left|v_A(x_i) - v_B(x_i)\right|\right\}$$

In many situations, the weight of the element $x_i \in X$ should be taken into account, for example, in multiple attribute decision making, the considered attributes usually have different importance, and thus need to be assigned with different weights. So we further extend the normalized Hamming distance and define a weighted distance as follows:

$$\bar{D}_{nH} = \frac{1}{n}\sum_{i=1}^{n} \omega_i \left\{\max\left\{\left|u_A(x_i) - u_B(x_i)\right|, \left|w_A(x_i) - w_B(x_i)\right|, \left|v_A(x_i) - v_B(x_i)\right|\right\}\right\}$$

where ω_i is the weight of the criterion $x_i (i = 1, 2, ..., n)$ entered by the decision-maker with $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$.

Theorem 3.1. *Suppose that $S : SVNS(X) \times SVNS(X) \rightarrow [0,1]$ such that*

$$S(A,B) = 1 - d_{nH}\left(A, A \cap B\right)$$

where d_{nH} is a normalized Hamming distance between single valued neutrosophic sets, is a mapping. Then $S(A,B)$ is a subsethood meusure expressing the degree to which A belongs to B.

Proof. $(S1)$ It is clear from definition of subset of single valued neutrosophic sets.

$(S2)$ For $A \in SVNS(X)$,

$S(A, A^c) = 1 \Longleftrightarrow d(A, A \cap A^c) = 0$

$\Longleftrightarrow \frac{1}{n}\sum_{i=1}^{n} \max\{\left|u_A(x) - \min\left\{u_A(x), u_{A^c}(x)\right\}\right|,$

$\left|w_A(x) - \min\left\{w_A(x), w_{A^c}(x)\right\}\right|, \left|v_A(x) - \max\left\{v_A(x), v_{A^c}(x)\right\}\right|\} = 0, \forall x \in X$

$\Longleftrightarrow \left|u_A(x) - \min\left\{u_A(x), u_{A^c}(x)\right\}\right| = 0$, $\left|w_A(x) - \min\left\{w_A(x), w_{A^c}(x)\right\}\right| = 0$

and $\left|v_A(x) - \max\left\{v_A(x), v_{A^c}(x)\right\}\right| = 0$, $\forall x \in X$

$\Longleftrightarrow u_A(x) \leq u_{A^c}(x)$, $w_A(x) \leq w_{A^c}(x)$ and $v_{A^c}(x) \leq v_A(x)$, $\forall x \in X$

$\Longleftrightarrow u_A(x) \le v_A(x),\ w_A(x) \le 1 - w_A(x),\ \forall x \in X$

$\Longleftrightarrow \forall x \in X,\ u_A(x) \le v_A(x)$ and $w_A(x) \le 0.5$.

$(S3)$ It is clear from definitions of single valued absolute (empty) neutrosophic set.

$(S4)$ Let $A = \{\langle x, u_A(x), w_A(x), v_A(x)\rangle : x \in X\}$, $B = \{\langle x, u_B(x), w_B(x), v_B(x)\rangle : x \in X\}$ and $C = \{\langle x, u_C(x), w_C(x), v_C(x)\rangle : x \in X\}$. To prove that $S(C, A) \le S(B, A)$, we show that $d(B, B \cap A) \le d(C, C \cap A)$.

$d(B, B \cap A)$

$= \frac{1}{n}\sum_{i=1}^{n} \max\{|u_B(x_i) - \min\{u_B(x_i), u_A(x_i)\}|, |w_B(x_i) - \min\{w_B(x_i), w_A(x_i)\}|,$

$|v_B(x_i) - \max\{v_B(x_i), v_A(x_i)\}|\}$

$= \frac{1}{n}\sum_{i=1}^{n} \max\{|u_B(x_i) - u_A(x_i)|, |w_B(x_i) - w_B(x_i)|, |v_B(x_i) - v_A(x_i)|\}$

$\le \frac{1}{n}\sum_{i=1}^{n} \max\{|u_C(x_i) - u_A(x_i)|, |w_C(x_i) - w_C(x_i)|, |v_C(x_i) - v_A(x_i)|\}$

$= \frac{1}{n}\sum_{i=1}^{n} \max\{|u_C(x_i) - \min\{u_C(x_i), u_A(x_i)\}|, |w_C(x_i) - \min\{w_C(x_i), w_A(x_i)\}|,$

$|v_C(x_i) - \max\{v_C(x_i), v_A(x_i)\}|\} = d(C, C \cap A)$. Then $S(C, A) \le S(B, A)$.

Similarly, it can be shown that $S(C, A) \le S(C, B)$. □

Example 3.1. *Let* $X = \{x_1, x_2\}$ *.Suppose that*

$$\begin{aligned} A &= \{\langle x_1, 0.3, 0.7, 0.5\rangle, \langle x_2, 0.4, 0.2, 0.6\rangle\} \\ B &= \{\langle x_1, 0.4, 0.6, 0.2\rangle, \langle x_3, 0.5, 0.1, 0.3\rangle\} \end{aligned}$$

Then $S(A, B) = 1$ *and* $S(B, A) = 0$ *in Wang's subset sense. Here we say that* A *is precisely a subset of* B*. However, it may be situations being "more and less" a subset of another set.*

Example 3.2. *Let* $X = \{x_1, x_2\}$ *.Suppose that*

$$\begin{aligned} A &= \{\langle x_1, 1, 0, 0\rangle, \langle x_2, 0.7, 0.3, 0.6\rangle\} \\ B &= \{\langle x_1, 0, 0, 1\rangle, \langle x_2, 0.2, 0.4, 0.4\rangle\} \end{aligned}$$

It is clear that either $A \subset B$ *or* $B \subset A$*. But since* $S(A, B) = 0.25$ *and* $S(B, A) = 0.9$*, we can say that* B *is much more a subset of* A*.*

Nex we present an application of the neutrosophic subsethood measure in a multicriteria decision-making problem.

4. Multicriteria Decision-Making

In the following, we apply the above subsethood measures to multicriteria decision making based on SVNSs. We discuss the multicriteria decision-making problem [14].

Let $A = \{A_1, A_2, ..., A_m\}$ be a set of alternatives and $X = \{x_1, x_2, ...x_n\}$ be a set of criteria. Assume that the weight of the criterion $x_i (i = 1, 2, ..., n)$, entered by the decision-maker, is ω_i, $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. In this case, the characteristic of the alternative $A_j (j = 1, 2, ..., m)$ is represented by the following SVNS:

$$A_j = \left\{ \langle x_i, u_{A_j}(x_i), w_{A_j}(x_i), v_{A_j}(x_i) \rangle : x_i \in X \right\}$$

where $u_{A_j}(x_i), w_{A_j}(x_i), v_{A_j}(x_i) \in [0, 1]$, $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$. Hence it can defined a single-valued neutrosophic decision matrix $D = (\alpha_{ji})_{m \times n}$.

In multicriteria decision-making environments, the concept of ideal point has been used to help identify the best alternative in the decision set. Ye [14] have defined the ideal alternative A^* as the SVNS $\alpha_i^* = \langle a_i^*, b_i^*, c_i^* \rangle = \langle 1, 0, 0 \rangle$, $i = 1, 2, ..., n$.

There is an investment company, which wants to invest a sum of money in the best option. There is a panel with four possible alternatives to invest the money:

(1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company.

The investment company must take a decision according to the following three criteria:

(1) x_1 is the risk analysis; (2) x_2 is the growth analysis; (3) x_3 is the environmental impact analysis.

Then, the weight vector of the criteria is given by $\omega = (0.35, 0.25, 0.40)$. Thus, when the four possible alternatives with respect to the above three criteria are evaluated by an expert, it can obtained the following single-valued neutrosophic decision matrix D:

$$D = \begin{bmatrix} \langle 0.4, 0.2, 0.3 \rangle & \langle 0.4, 0.2, 0.3 \rangle & \langle 0.2, 0.2, 0.5 \rangle \\ \langle 0.6, 0.1, 0.3 \rangle & \langle 0.6, 0.1, 0.2 \rangle & \langle 0.5, 0.2, 0.2 \rangle \\ \langle 0.3, 0.2, 0.3 \rangle & \langle 0.5, 0.2, 0.3 \rangle & \langle 0.5, 0.3, 0.2 \rangle \\ \langle 0.7, 0.0, 0.1 \rangle & \langle 0.6, 0.1, 0.2 \rangle & \langle 0.4, 0.3, 0.2 \rangle \end{bmatrix}$$

Now, we utilize the developed subsethood measure to obtain the most desirable alternative(s). By the weighted distance, we can obtain the subsethood measures between the ideal alternative and all alternative as follows:

$$S(A^*, A_1) = 0.773,\ S(A^*, A_2) = 0.853$$
$$S(A^*, A_3) = 0.810,\ S(A^*, A_4) = 0.851$$

Thus, we have the ranking $A_2 \succ A_4 \succ A_3 \succ A_1$.

From the above numerical results, we say that the alternative A_2 is the ideal alternative. Note that the ranking is the same as that Ye [14]. Above example show that this kind of similarity measure is much simpler than other and is well suitable for neutrosophic information.

5. Conclusion

In this article, we proposed a measure to express the degree to which a single valued neutrosophic A belongs to another B. To show a real application, we used the data set from Ye [14]. We found that our proposed measure contains less computation in the multicriteria decision making problem which is selection the most ideal from all the alternatives. Though having a simple measure for calculation, the subsethood measure presents a new method to deal with neutrosophic information. We hope that the findings in this paper will help the researchers to enhance and promote the further study on subsethood measure to carry out general framework for the applications in practical life.

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