Bounds for variables with few occurrences in conjunctive normal forms

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August 5, 2014

Abstract

We investigate connections between SAT (the propositional satisfiability problem) and combinatorics, around the minimum degree of variables in various forms of redundancy-free boolean conjunctive normal forms (clause-sets).

Let $\mu\mathrm{vd}(F) \in \mathbb{N}$ for a clause-set F denote the *minimum* variable-degree, the minimum of the number of occurrences of a variable. A central result is the upper bound $\sigma(F) + 1 \leq \mu\mathrm{vd}(F) \leq \mathrm{nM}(\sigma(F)) \leq \sigma(F) + 1 + \log_2(\sigma(F))$ for *lean clause-sets* $F \in \mathcal{LEAN}$ in dependency on the *surplus* $\sigma(F) \in \mathbb{N}$. Lean clause-sets, defined as having no non-trivial autarkies (partial assignments satisfying some clauses and not touching the other clauses), generalise minimally unsatisfiable clause-sets, i.e., $\mathcal{LEAN} \supset \mathcal{MU}$. For the surplus we have $\sigma(F) \leq \delta(F) = c(F) - n(F)$, using the deficiency $\delta(F)$ of clause-sets, the difference between the number $c(F)$ of clauses and the number $n(F)$ of variables. $\mathrm{nM}(k) \in \mathbb{N}$ is the k-th "non-Mersenne" number, skipping in the sequence of natural numbers all numbers of the form $2^n - 1$. As an application of the upper bound we obtain, that clause-sets F violating $\mu\mathrm{vd}(F) \leq \mathrm{nM}(\sigma(F))$ must have a non-trivial autarky, so clauses can be removed satisfiability-equivalently. We obtain a polynomial time autarky reduction, but where it is open whether such an autarky itself can be found in polynomial time.

We show that the upper bound is sharp, i.e., $\mu\mathrm{vd}(\mathcal{LEAN}_{\delta=k}) = \mathrm{nM}(k)$ for all deficiencies $k \in \mathbb{N}$, where $\mu\mathrm{vd}(\mathcal{LEAN}_{\delta=k})$ is the maximum of $\mu\mathrm{vd}(F)$ over $F \in \mathcal{LEAN}_{\delta=k}$. The determination of $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) =: \mu\mathrm{nM}(k)$ seems to be a much more involved question. We show that for $k \leq 5$ we have $\mu\mathrm{nM}(k) = \mathrm{nM}(k)$, but for $k = 6$ we have $\mu\mathrm{nM}(k) = \mathrm{nM}(k) - 1$. Moreover this correction by -1 causes further corrections by -1 for infinitely many other deficiencies, resulting in the upper-bound function $\mathrm{nM}_1 : \mathbb{N} \to \mathbb{N}$, an instance of a *generalised non-Mersenne function* found by a novel recursion scheme.

Extensive introductions, overviews, conclusions, examples and open problems are provided.

Keywords conjunctive normal form , deficiency , minimally unsatisfiable , variable minimal unsatisfiable , saturated minimal unsatisfiable , marginal minimal unsatisfiable , hitting clause-sets , disjoint tautologies , orthogonal tautologies , lean clause-set , autarky , surplus , matching-lean , non-Mersenne numbers , occurrences of variable , minimum variable degree , polynomial time , fixed-parameter tractable , SAT decision , DP-reduction , variable elimination , singular DP-reduction , singular variables , subsumption resolution , L-matrices , minimally sign-central matrices

*This research was partially supported by NSFC Grand 61272059, NSSFC Grant 13&ZD186 and MOE Grant 11JJD7200020.

Contents

1 Introduction

In this work we aim at bringing together some aspects of combinatorics with the developing theory of SAT. We concentrate on degree considerations in "clause-sets" (conjunctive normal forms as set-systems), which can be considered as generalised hypergraphs, namely hypergraphs with "polarities". The general goal is to develop an *understanding* of propositional (un)satisfiability, which corresponds for hypergraphs to an understanding of (non-)2-colourability.

SAT, the prototypical NP-complete problem ([12]), took a strong development in the past two decades also regarding (industrial) applications (see the handbook [7] for a recent overview). It is often mainly considered as belonging to complexity theory, algorithms and heuristics (with [15, 14] the basic papers here), and finally implementations and experimentation ("SAT solvers"). "Understanding" SAT in a precise sense is considered to be impossible, and only various investigations on random and approximation structures (including "islands of tractability") in general are deemed fruitful. We want to challenge this view, starting to build a new bridge, towards an *understanding of unsatisfiability*. We note here that understanding unsatisfiability seems easier than to understand satisfiability, since unsatisfiability means a form of completion, *all* assignments have been excluded as potential satisfying assignments ("models"), while satisfiability means a lack of such completion. More precisely, we aim at understanding *minimal unsatisfiability*, the building blocks of unsatisfiability — similar to critical colourability, here removal of any clause renders the clause-set satisfiable.

A fundamental question, the subject of this study, is the existence of "simple" variables in clause-sets. "Simple" here means a variable occurring not very often. A major use of the existence of such variables is in inductive proofs of properties of minimally unsatisfiable clause-sets, using splitting on a variable to reduce n, the number of variables, to $n-1$: here it is vital that we have control over the changes imposed by the substitution, and so we want to split on a variable occurring as few times as possible. "Splitting" of a clause-set F on variable v means the consideration of the clause-sets $\langle v \to 0\rangle * F$, $\langle v \to 1\rangle * F$, that is, instantiating variable v by both truth values $0, 1$. A feature of clause-sets is the closure under splitting, and splitting is a major tool for investigations into minimal unsatisfiability.

1.1 Deficiency as the main structural parameter

The definition of the class $\mathcal{CLS}$ of "clause-sets", and of the class $\mathcal{MU} \subset \mathcal{CLS}$ of "minimally unsatisfiable clause-sets", can be quickly (and precisely) given as follows, using (just) natural numbers as "variables":

A "literal" x is an element of $\mathbb{Z} \setminus \{0\}$. A "clause" C is a finite set of literals, such that there is no $x \in C$ with $-x \in C$. Using $-L := \{-x : x \in L\}$ for sets L of literals, the "clash-freeness" condition for C becomes $C \cap -C = \emptyset$. A "clause-set" F is a finite set of clauses, the set of all clause-sets is denoted by $\mathcal{CLS}$. The most basic measurements for $F \in \mathcal{CLS}$ are:

- the number $c(F) := |F| \in \mathbb{N}_0$ of clauses of F;
- the number $n(F) := |\mathrm{var}(F)| \in \mathbb{N}_0$ of variables of F, where $\mathrm{var}(F)$ is the set of $v \in \mathbb{N}$ (variables as positive integers) with $\{v, -v\} \cap \bigcup F \neq \emptyset$;
- The "deficiency" $\delta(F) := c(F) - n(F) \in \mathbb{Z}$. This parameter is only informative when certain (weak) assumptions are made for F, and for general F the "maximal deficiency" $\delta^*(F) := \max_{F' \subseteq F} \delta(F') \in \mathbb{N}_0$ is to be used.

A clause-set F is "satisfiable" if there exists a partial assignment φ, which here in this introduction is just a clause, such that $\varphi \cap D \neq \emptyset$ for all $D \in F$.[1)] The set of all satisfiable clause-sets is $\mathcal{SAT} \subset \mathcal{CLS}$, the set of all unsatisfiable clause-sets is $\mathcal{USAT} := \mathcal{CLS} \setminus \mathcal{SAT}$. Finally $\mathcal{MU} \subset \mathcal{USAT}$ is the set of $F \in \mathcal{USAT}$ such that for all $C \in F$ we have $F \setminus \{C\} \in \mathcal{SAT}$.

The background for the investigations of this report is the enterprise of classifying $F \in \mathcal{MU}$ in dependency on $\delta(F)$. The basic facts are $\delta^*(F) = \delta(F)$ (as will be discussed in Subsection 1.5), and the well-known $\delta(F) \geq 1$, as first shown in [3]. For $\delta(F) = 1$ the structure is completely known ([3, 17, 49]; see Example 3.2), for $\delta(F) = 2$ the structure after reduction of singular variables (occurring in one sign only once) is known ([42]; see Example 3.3), while for $\delta(F) \in \{3, 4\}$ only basic cases have been classified ([99]).

The starting point of our investigation is Lemma C.2 in [49], where it is shown that $F \in \mathcal{MU}$ with $n(F) > 0$ must have a variable $v \in \mathrm{var}(F)$ with at most $\delta(F)$ positive and at most $\delta(F)$ negative occurrences; we write this as $\mathrm{ld}_F(v) \leq \delta(F)$ and $\mathrm{ld}_F(-v) \leq \delta(F)$, using the notion of *literal degrees* (the number of occurrences of the literal), where for a literal x its degree is

$$\mathrm{ld}_F(x) := |\{C \in F : x \in C\}| \in \mathbb{N}_0.$$

Thus we have $\mathrm{vd}_F(v) \leq 2\delta(F)$, using the *variable degree*

$$\mathrm{vd}_F(v) := \mathrm{ld}_F(v) + \mathrm{ld}_F(-v) \in \mathbb{N}_0.$$

Using the *minimum variable degree* (min-var-degree)

$$\mu\mathrm{vd}(F) := \min_{v \in \mathrm{var}(F)} \mathrm{vd}_F(v) \in \mathbb{N}$$

1) The clause φ is the set of satisfied literals of the corresponding "partial assignment". This definition of "satisfying assignments", via clauses intersecting every clause of F, generalises transversals of hypergraphs, by taking complementation into account (φ does not contain clashes).

of F with $n(F) > 0$, the upper bounds becomes $\mu\mathrm{vd}(F) \leq 2\delta(F)$. A main theme of this report is the consideration of $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \in \mathbb{N}$ for $k \in \mathbb{N}$, the maximum of $\mu\mathrm{vd}(F)$ for $F \in \mathcal{MU}$ with $\delta(F) = k$. The upper bound now becomes $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \leq 2k$.

We show a sharper bound on $\mu\mathrm{vd}(F)$, namely we show that the worst-cases $\mathrm{ld}_F(v), \mathrm{ld}_F(-v) \leq \delta(F)$ can not occur at the same time (for a suitable variable), but actually $\mathrm{ld}_F(v) + \mathrm{ld}_F(\overline{v}) - \delta(F)$ only grows logarithmically in $\delta(F)$. The really interesting aspect here is the precise determination of $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k})$, and we investigate the (elementary) number-theoretic function $\mathrm{nM}(k)$, which yields the upper bound $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \leq \mathrm{nM}(k)$ for all $k \in \mathbb{N}$, where the function $\mathrm{nM} : \mathbb{N} \to \mathbb{N}$ fulfils $k + \lfloor \log_2(k+1) \rfloor \leq \mathrm{nM}(k) \leq k + 1 + \lfloor \log_2(k) \rfloor$ for $k \in \mathbb{N}$.

1.2 Refining deficiency by surplus

After having settled this basic min-var-degree upper bound for $\mathcal{MU}_{\delta=k}$, we show a sharper bound on $\mu\mathrm{vd}(F)$ for a larger class of clause-sets F:

- The larger class of clause-sets considered is the class $\mathcal{LEAN}$ of *lean clause-sets* (introduced in [50]), which are clause-sets having no non-trivial autarky. For an overview on the theory of minimally unsatisfiable clause-sets and on the theory of autarkies see [43]. $\mathcal{LEAN} \subset \mathcal{CLS}$ is the set of $F \in \mathcal{CLS}$ such that there is no partial assignment φ (a "non-trivial autarky") with the properties

  - for every clause $D \in F$ with $-\varphi \cap D \neq \emptyset$ we have $\varphi \cap D \neq \emptyset$ (note that this generalises the satisfaction criterion);
  - there exists $v \in \mathrm{var}(F)$ with $\{v, -v\} \cap \varphi \neq \emptyset$.

  Note $\mathcal{LEAN} \cap \mathcal{SAT} = \top$, where $\top := \emptyset \in \mathcal{CLS}$ is the empty clause-set (the standard satisfiable clause-set).

- The deficiency $\delta(F) \in \mathbb{Z}$ is strengthened by the *surplus* $\sigma(F) \in \mathbb{Z}$, defined in case of $n(F) > 0$ as follows.

  Consider the bipartite clause-variable graph of F (generalising the incidence graph of a hypergraph), with the clauses $C \in F$ on one side of the bipartition, and the variables $v \in \mathrm{var}(F)$ on the other side, and an edge between v and C if $\{v, -v\} \cap C \neq \emptyset$. The "expansion" of a set $\emptyset \neq V \subseteq \mathrm{var}(F)$ of variables is $|\Gamma(V)| - |V|$, where $\Gamma(V)$ is the set of neighbours of V (incident clauses), and the surplus then is the minimum expansion, i.e., $\sigma(F) = \min_{\emptyset \neq V \subseteq \mathrm{var}(F)} |\Gamma(V)| - |V|$.

  In the terminology of [73, Section 1.3], $\delta^*(F)$ is the deficiency of the bipartite clause-variable graph (with bipartition $(F, \mathrm{var}(F))$), while $\sigma(F)$ is the surplus of the bipartite variable-clause graph (with bipartition $(\mathrm{var}(F), F)$).

  Note that by considering $V = \mathrm{var}(F)$ we have $\sigma(F) \leq \delta(F)$, and by considering $V = \{v\}$ for $v \in \mathrm{var}(F)$ we get $\sigma(F) \leq \mu\mathrm{vd}(F) - 1$.

  We have $\sigma(F) \geq 1$ for $F \in \mathcal{LEAN}$ with $n(F) > 0$ ([51, Lemma 7.7]), generalising the basic fact $\delta(F) \geq 1$ for $F \in \mathcal{MU}$.

Now a central result of this report (Theorem 9.8) is

$$\mu\mathrm{vd}(F) \leq \mathrm{nM}(\sigma(F))$$

for $F \in \mathcal{LEAN}$ with $n(F) > 0$. As an application we obtain (Theorem 10.2), that via removing satisfiability-equivalently some clauses (via some autarky), we can reduce every (multi-)clause-set F in polynomial time to a (multi-)clause-set F'

containing a variable occurring with degree at most $\sigma(F') + 1 + \log_2(\sigma(F'))$. It is an open problem whether such an autarky can be found in polynomial time (for arbitrary clause-sets F); we conjecture (Conjecture 10.3) that this is possible.

We also show sharpness of the upper bound, i.e., $\mu\mathrm{vd}(\mathcal{LEAN}_{\delta=k}) = \mathrm{nM}(k)$ for all $k \in \mathbb{N}$, in Corollary 9.13 (proving Conjecture 23 from the conference version [62]), which indeed holds for every class of clause-sets between $\mathcal{VMU}$, i.e., "variable-minimally unsatisfiable clause-sets" as introduced in [11], and $\mathcal{LEAN}$; the definition of $\mathcal{VMU}$ is as the set of $F \in \mathcal{USAT}$ such that for all $F' \subseteq F$ with $\mathrm{var}(F') \subset \mathrm{var}(F)$ we have $F' \in \mathcal{SAT}$.

We then come back to the special case of minimal unsatisfiability. Here things turn out to be much more complicated, and the numbers $\mu\mathrm{nM}(k) := \mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \in \mathbb{N}$ for $k \in \mathbb{N}$, the guaranteed minimum variable degrees for minimally unsatisfiable clause-sets of deficiency k, seem to be very complicated (and very interesting) quantities. We proof the sharpened bound $\mu\mathrm{nM}(k) \le \mathrm{nM}_1(k)$, which improves on $\mathrm{nM}(k)$ for infinitely many k.

According to the goal of bringing different communities together, we try to provide and explain much of the relevant background, so that this report is mostly self-contained, and the results cited from the literature can be treated as black-boxes.

1.3 Some basic intuitions about the upper bound nM

As already mentioned, the function $\mathrm{nM} : \mathbb{N} \to \mathbb{N}$ is strictly increasing with range

$$\mathrm{nM}(\mathbb{N}) = \mathbb{N} \setminus \{2^n - 1 : n \in \mathbb{N}\} = \{\underline{2}, \quad 4, 5, \underline{6}, \quad 8, \dots, \underline{14}, \quad 16, 17, \dots\}.$$

We show $\mu\mathrm{vd}(\mathcal{LEAN}_{\delta=k}) = \mathrm{nM}(k)$ for deficiencies $k \in \mathbb{N}$, that is, every lean clause-set F with $n(F) > 0$ contains a variable $v \in \mathrm{var}(F)$ with $\mathrm{vd}_F(v) \le \mathrm{nM}(\delta(F))$, and for every deficiency $k \ge 1$ there are lean clause-sets F with $\mu\mathrm{vd}(F) = \mathrm{nM}(\delta(F))$.

The underlined values $2, 6, 14, \dots$, which have the form $2^n - 2$ for $n \in \mathbb{N}$, are the function values at the "jump positions" $1, 4, 11, \dots$, which are of the form $2^n - n - 1$ for $n \ge 2$ (where the function values changes by $+2$, while otherwise it changes by $+1$ for an increment of the argument). This basic structure of nM can be motivated by the following constructions of $F \in \mathcal{MU}$ with "high" min-var-degree; indeed these considerations only concern the lower bounds, given by appropriate constructions, while the arithmetic nature of $\mathrm{nM}(k)$ rests on different considerations, but for the deficiencies considered here, lower and upper bounds are equal, and the lower bounds are easier to understand here.

The basic clause-sets are the A_n for $n \in \mathbb{N}_0$, which consist of all 2^n sets (clauses) of numbers $\pm 1, \dots, \pm n$, using the natural numbers $1, \dots, n$ as variables. So $A_0 = \{\emptyset\}$, $A_1 = \{\{-1\}, \{1\}\}$, $A_2 = \{\{1, 2\}, \{-1, 2\}, \{1, -2\}, \{-1, -2\}\}$ and so on. It is easy to see that we have $A_n \in \mathcal{MU}$ with $n(A_n) = n$, $c(A_n) = 2^n = \mu\mathrm{vd}(A_n)$, and $\delta(A_n) = 2^n - n$. We will see that the A_n have the largest possible min-var-degree 2^n for given deficiency $2^n - n$, and we also have $\mathrm{nM}(2^n - n) = 2^n$ for $n \in \mathbb{N}$. These deficiencies $k = 2^n - n$ (numerical values are $1, 2, 5, 12, \dots$) are the positions directly after the jump positions (excluding deficiency $k = 1$ as a special case).

How can we obtain from that more clause-sets in $\mathcal{MU}$ with high min-var-degree? Consider A_3: we have e.g. $\{1, 2, 3\}, \{1, 2, -3\} \in A_3$; now logically these two clauses are equivalent to $\{1, 2\}$ (i.e., we have the same satisfying assignments; technically, a "strict full subsumption resolution" is performed), and we obtain $A'_3 := (A_3 \setminus \{\{1, 2, 3\}, \{1, 2, -3\}\}) \cup \{\{1, 2\}\} \in \mathcal{MU}$. Performing this process in general, using $\{1, \dots, n\}, \{1, \dots, n-1, -n\} \in A_n$, yields $A'_n \in \mathcal{MU}$ for $n \ge 2$, with $n(A'_n) = n$, $c(A'_n) = 2^n - 1$, $\delta(A'_n) = 2^n - n - 1$, and $\mu\mathrm{vd}(A'_n) = 2^n - 2$ (the (single) variable with minimum occurrences is n). These deficiencies are precisely the jump positions $2^n - n - 1$, and accordingly we have $\mathrm{nM}(2^n - n - 1) = 2^n - 2$.

Performing the same trick again to A_3', we might replace $\{-1, 2, 3\}, \{-1, -2, 3\} \in \mathcal{A}_n'$ by $\{-1, 3\}$, obtaining $A_3'' \in \mathcal{MU}$. Again for general $n \geq 3$ we get $A_n'' \in \mathcal{MU}$, $n(A_n'') = n$, $c(A_n'') = 2^n - 2$, $\delta(A_n'') = 2^n - n - 2$, and $\mu\mathrm{vd}(A_n'') = 2^n - 3$; note here the crucial difference, that the min-var-degree has only been changed by -1. The reason is that there are two variables now with minimum occurrences, namely $n-1, n$, where the degree of variable n changed first by -2, then by -1, while for variable $n-1$ the degree first changed by -1, and then by -2 (and for the other variables $1, \dots, n-2$ we had degree changes by -1, -1).

Now one might imagine this process of strict full subsumption resolution continuing until deficiency $2^{n-1} - (n-1) + 1$, always with change of the min-var-degree by -1, just before the deficiency of the previous A_{n-1} — this would yield the function nM. However the combinatorial reality is more complicated, and as we prove in this report (Section 14), at least we can not get until $2^{n-1} - (n-1) + 1$ for $n \geq 4$ (in effect), that is, at these deficiencies $k = 6, 13, 28, \dots$ we have $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \leq \mathrm{nM}_1(k) = \mathrm{nM}(k) - 1$.[2)]

1.4 Related work on $\mathcal{MU}$

A general overview on minimally unsatisfiable clause-sets (also "minimal unsatisfiable clause-sets/formulas", or "MU") is [43]; later developments are in [57, 58] (generalisations to non-boolean clause-sets) and in [64, 65] (studying "singular DP-reduction", the elimination of variables which occur in one sign only once).

Two early papers on the complexity aspects are [81, 80], who introduced the complexity class D^P and showed that the decision "$F \in \mathcal{MU}$?" with input $F \in \mathcal{CLS}$ is complete for this class. Another important early paper is [3], which showed $\delta(F) \geq 1$ for $F \in \mathcal{MU}$, where the notion of "deficiency" was introduced by [25]. Furthermore [3] showed polytime-decision of the sub-class $\mathcal{SMU}_{\delta=1} \subset \mathcal{MU}_{\delta=1}$ (called "strongly minimal unsatisfiable" there), where $\mathcal{SMU} \subset \mathcal{MU}$ is the set of $F \in \mathcal{USAT}$ such that for all $C \in F$ and all $x \in \mathbb{Z} \setminus \{0\}$ with $\{x, -x\} \cap C = \emptyset$ holds $(F \setminus \{C\}) \cup \{C \cup \{x\}\} \in \mathcal{SAT}$, that is, adding any literal to any clause renders the clause-set satisfiable. We use the terminology "saturated minimally unsatisfiable" introduced in [24], where the important connection to splitting was introduced, and a simpler proof of $\delta(F) \geq 1$ for $F \in \mathcal{MU}$ was given. Just for this introduction we handle "partial assignments" via clauses φ (containing the satisfied literals; thus $-\varphi$ is the set of falsified literals), so for a literal x the partial assignment $\langle x \to 0 \rangle$ is given by $\{-x\}$, while $\langle x \to 1 \rangle$ is given by $\{x\}$. The application of φ to $F \in \mathcal{CLS}$ is defined as

$$\varphi * F := \{D \setminus -C : D \in F \wedge C \cap D = \emptyset\} \in \mathcal{CLS},$$

that is, removing first the satisfied clauses from F, and then the falsified literals from the remaining clauses. Now for $F \in \mathcal{CLS}$ holds $F \in \mathcal{SMU}$ iff for all $x \in \mathbb{Z} \setminus \{0\}$ holds $\langle x \to 1 \rangle * F \in \mathcal{MU}$ (the "only if"-direction was shown in [24], the "if"-direction in [49]). Due to this property plus the property, that every $F \in \mathcal{MU}$ can be "saturated" by adding literals to clauses, the class $\mathcal{SMU}$ is an important helping class for investigations into $\mathcal{MU}$ via the splitting method, splitting up $F \in \mathcal{MU}$ into $\langle v \to 0 \rangle * F$ and $\langle v \to 1 \rangle * F$ for selected variables v.

We have already mentioned the literature concerned with characterising the classes $\mathcal{MU}_{\delta=k}$ (and subclasses) for small deficiencies $k \leq 4$. Less ambitious is the goal of polytime decision of these classes: the problem was raised in [41], and has been solved via two independent approaches in [49] and [23] (indeed establishing polytime SAT decision for inputs $F \in \mathcal{CLS}$ and fixed $\delta^*(F)$), later strengthened in

2) We do not apply the above method for gaining lower bounds as far as we can, but only as needed in this report; see the end of Subsection 12.1 for some further remarks.

[93] (showing that SAT decision is even fixed-parameter tractable in $\delta^*(F)$; see also [57] for generalisations and simplifications).

1.4.1 MUS

As we have already mentioned, we consider $\mathcal{MU}$ as the "primal" building block for understanding unsatisfiability. In general an unsatisfiable clause-set can contain many minimally unsatisfiable sub-clause-sets, called "MUSs". The task of enumerating all of them or at least some "good" ones is also of practical importance, to extract more information on the "causes" of unsatisfiability. A recent overview is [74], while a clean approach to enumerate all MUSs, via hypergraph transversals, is in [71] (the earliest appearance of the underlying observation seems [10, Theorem 2]; compare also [58, Subsection 4.3] for generalisations of the fundamental approach). See also [61] for a reflection on various types of such sub-clause-sets, and on the connection to autarky theory (compare Subsection 1.6.3).

1.4.2 Tovey's problem (uniform clause-sets)

This report appears to be the first systematic study of the problem of *minimum* variable occurrences / degrees in minimally unsatisfiable clause-sets and generalisations, in dependency on the deficiency — asking for the existence of a variable occurring "infrequently" in general, or for extremal examples where all variables occur not infrequently. The "dual" problem is to consider *maximum* variable occurrences / degrees — asking for the existence of a variable occurring frequently in general, or for extremal examples where all variables occur not frequently. More precisely, the *maximum variable degree* is

$$\nu\mathrm{vd}(F) := \max_{v \in \mathrm{var}(F)} \mathrm{vd}_F(v) \in \mathbb{N},$$

for $n(F) > 0$, while for a class $\mathcal{C} \subseteq \mathcal{CLS}$ of clause-sets, the quantity $\nu\mathrm{vd}(\mathcal{C})$ (to be studied) is the minimum of $\nu\mathrm{vd}(F)$ for $F \in \mathcal{C}$.

This problem has been well-studied for *p-uniform* minimally unsatisfiable clause-sets, starting with [94, 18, 47].[3)] We denote by $p\text{–}\mathcal{CLS} \subset \mathcal{CLS}$ for $p \in \mathbb{N}_0$ the set of all $F \in \mathcal{CLS}$ with $\forall\, C \in \mathcal{CLS} : |C| \le p$, while by $\mathcal{UCLS} \subset \mathcal{CLS}$ we denote the set of all uniform clause-sets, i.e., those $F \in \mathcal{CLS}$ such that for $C, D \in F$, $C \ne D$, holds $|C| = |D|$. Finally $p\text{-}\mathcal{UCLS} := p\text{–}\mathcal{CLS} \cap \mathcal{UCLS}$ and $p\text{-}\mathcal{UMU} := p\text{-}\mathcal{UCLS} \cap \mathcal{MU}$. Now the basic fact is

$$\nu\mathrm{vd}(p\text{-}\mathcal{UMU}) \ge p + 1$$

for $p \in \mathbb{N}$ ([94], generalised in [58, Corollary 7.3]). Trivially $\nu\mathrm{vd}(1\text{-}\mathcal{UMU}) = 2$, and easily one sees $\nu\mathrm{vd}(2\text{-}\mathcal{UMU}) = 3$, while by [94] holds $\nu\mathrm{vd}(3\text{-}\mathcal{UMU}) = 4$. As reported in [38], we have $\nu\mathrm{vd}(4\text{-}\mathcal{UMU}) = 5$, and these are all known precise values of $\nu\mathrm{vd}(p\text{-}\mathcal{UMU})$ (where the notation $f(p) := \nu\mathrm{vd}(p\text{-}\mathcal{UMU}) - 1$ was introduced in [47]). In [38] it was observed that extremal examples might be found in $\mathcal{MU}_{\delta=1}$, and this work was recently extended in [28], establishing the asymptotically tight bound $\lim_{p\to\infty} \frac{2}{e}\frac{2^p}{p} / \nu\mathrm{vd}(p\text{-}\mathcal{UMU}) = 1$ (where indeed $p\text{-}\mathcal{UMU} \cap \mathcal{MU}_{\delta=1}$ is considered).

In our setting, studying the classes $\mathcal{MU}_{\delta=k}$, the max-var-degree is not very relevant, since we have $\nu\mathrm{vd}(\mathcal{MU}_{\delta=1}) = 2$, while $\nu\mathrm{vd}(\mathcal{MU}_{\delta=k}) = 3$ for $k \ge 2$. This can be seen as follows: As already noticed in [94], there is a poly-time transformation from $\mathcal{CLS}$ to the class $\mathcal{CLS}(1,2) \subset \mathcal{CLS}$, consisting of those $F \in \mathcal{CLS}$ where for every variable $v \in \mathrm{var}(F)$ we have $\mathrm{ld}_F(v) = 1$ and $\mathrm{ld}_F(-v) \le 2$. Namely if there

[3)] We remark that typically in the literature the connections to minimally unsatisfiable clause-sets are not emphasised, but it is clear that when considering (uniform) unsatisfiable clause-sets with a maximum variable degree as small as possible, then one can restrict attention to (uniform) minimally unsatisfiable clause-sets (as worst-cases).

is a literal x and two clauses $C, D \in F$ with $x \in C \cap D$, then we can introduce a new variable v, replace x in C, D by v, and add the new clause $\{-v, x\}$, obtaining F'. We study such extensions under the name of "singular DP-extension", but it is also easy to see directly that F' is satisfiable iff F is, that F' is minimally unsatisfiable iff F is, and that $\delta(F') = \delta(F)$. By repeating this transformation, we obtain $t^{1,2} : \mathcal{CLS} \to \mathcal{CLS}(1, 2)$. So for $F \in \mathcal{MU}$ we get $t^{1,2}(F) \in \mathcal{MU} \cap \mathcal{CLS}(1, 2)$ with $\delta(t(F)) = \delta(F)$. Whence for all $k \in \mathbb{N}$ we have $\nu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \leq 3$. Now trivially $\nu\mathrm{vd}(\mathcal{MU}_{\delta=1}) = 2$ due to $\{\{1\}, \{-1\}\} \in \mathcal{MU}_{\delta=1}$. On the other hand, if for $F \in \mathcal{MU}$ holds $\nu\mathrm{vd}(F) \leq 2$ (thus $\nu\mathrm{vd}(F) = 2$), then via so-called singular DP-reduction this clause-set can be reduced to $\{\bot\}$, whence $F \in \mathcal{MU}_{\delta=1}$ (this is well-known; compare Example 3.2 later).

So for the study of the max-var-degree, the uniformity restriction seems essential. This is similar to many investigations into (colour-)critical hypergraphs (discussed in Subsection 1.6.1 below), where uniformity is a crucial assumption, and the clause-length p is the main parameter. For investigations into the case of uniform (general) clause-sets, where clauses share at most one variable, see [83, 88]. The number of clauses in $F \in p\text{-}\mathcal{UMU}$ has been studied in [69], showing that for $p = 2$ holds $c(F) \leq 4n - 2$, while for $p \geq 3$ there are F with $c(F) = \Omega(n(F)^p)$. Finally, the number of conflicts (clashes) in $F \in p\text{-}\mathcal{UMU}$ is considered in [89], and for a review of the use of the Lovász Local Lemma in this context see [27].

In contrast, for the study of the minimum variable degree as in this report, in dependency on the deficiency, the restriction to uniformity seems not interesting, and is also not needed, but unrestricted clause-sets are considered. We remark that for every $p \in \mathbb{N}$, $p \geq 3$, there is a polytime translation $t_p : \mathcal{CLS} \to p\text{-}\mathcal{UCLS}$, such that $t_p(F)$ is satisfiable iff F is, $t_p(F)$ is minimally unsatisfiable iff F is, and $\delta(t_p(F)) = \delta(F)$. This works by replacing clauses C with $|C| < p$ by clauses $C \cup \{v\}, C \cup \{-v\}$ for some new variable v (in the MU-case we will call this a "non-strict full subsumption extension"), and by replacing clauses C with $|C| > C$ by clauses $C' \cup \{v\}$, $C'' \cup \{-v\}$ for some new variable v and choosing clauses C', C'' with $C = C' \cup C''$ and $|C'| = p - 1$, $|C''| \geq p - 1$ (in the MU-case again we have a singular DP-extension). But the transformation t_p appear to be useless, since it completely garbles the structure of F.

We conclude these remarks on p-uniform clause-sets by the observation, that for $p \geq 4$ the instances involved above become quickly very big, and only methods from random analysis are available (which by nature are very rough). It seems that these considerations do not have practical relevance. In contrast, we consider all minimal unsatisfiable clause-sets (and more), that is, the deficiency does not filter out clause-sets, but only organises them in layers. And for a wide range of deficiency values, say, $k = 1, \ldots, 10000$, there are interesting and relevant examples.

1.5 Autarkies

An important tool, used in this report to go beyond $\mathcal{MU}$, is the theory of autarkies, which also provides a strong link to various areas of combinatorics; the relations to hypergraph colouring will be discussed in Subsection 1.6.3. Recall that a partial assignment φ is an autarky for $F \in \mathcal{CLS}$ iff every clause $C \in F$ touched by φ (i.e., $\varphi \cap (C \cup -C) \neq \emptyset$) satisfies C (i.e., $\varphi \cap C \neq \emptyset$), which is equivalent to $\forall F' \subseteq F : \varphi * F' \subseteq F'$. Autarkies were introduced in [78] for improved worst-case upper bounds for SAT decision, applying that for an autarky φ obviously $\varphi * F$ is sat-equivalent to F. For a recent overview see [43].

Autarky reduction. *Autarky reduction*, the reduction of $F \in \mathcal{CLS}$ to $\varphi * F \in \mathcal{CLS}$ for a non-trivial autarky φ, is an essential concept, algorithmically as well as for theoretical understanding; see [43, Subsection 11.10] for an overview on finding autarkies. If we reduce all autarkies, then we obtain the (unique) *lean kernel* of F. If there are no non-trivial autarkies, then we have a lean clause-sets, i.e., $F \in \mathcal{LEAN}$, as already mentioned in Subsection 1.2; the concept was introduced in [50], and [43, Subsection 11.8.3] contains more information. The lean kernel of F is the largest lean sub-clause-set of a clause-set, that is, $\bigcup\{F' \subseteq F : F' \in \mathcal{LEAN}\}$; for a recent paper on the computation of the lean kernel see [75].

The decision of leanness is coNP-complete, and so consideration of special autarkies is of interest; actually, these considerations are not just "algorithmic hacks", but in a sense represent various areas of combinatorics (for example matching theory) via "autarky systems".

Autarky systems. The notion of an "autarky system", as a selection of special autarkies with similar good properties as general autarkies, was introduced in [51], partially further expanded in [56], and overviewed in [43, Subsection 11.11].

The starting point for an autarky system is to single out a restricted notion of autarky. This restricted autarky notion implies a restricted satisfiability notion, namely clause-sets satisfiable via autarky reduction, using only these special autarkies. This is indeed equivalent for "normal autarky systems" to the clause-set being satisfiable by a single special autarky.[4)] Often the general autarkies of the system can be derived from the extreme case of satisfiability through such autarkies. For arbitrary autarky systems also the notions "minimal unsatisfiability" and "lean" are defined, and are central properties.

Balanced autarkies are an example of a rather general autarky system, the basis for autarkies for hypergraph colouring; here for an autarky, touched clauses need not only have some satisfied literal, but also some falsified literal. The corresponding satisfiability notion is "NAE-satisfiability", and will be further discussed in Subsection 1.6.3.

Matching autarkies. The autarky system especially of importance in this report, besides the full system, is that of *matching autarkies*; for a short introduction see [43, Subsection 11.11.2]. They yield the set $\mathcal{MLEAN} \supset \mathcal{LEAN}$ of *matching-lean clause-sets*, and the set $\mathcal{MSAT} \subset \mathcal{SAT}$ of *matching-satisfiable clause-sets* (called "matched clause-sets" in [25]):

- A matching autarky for $F \in \mathcal{CLS}$ is an autarky φ for F such that for all $C \in F$ touched by φ one can select $x_C \in C$ with $x \in \varphi$ such that the underlying variables $\mathrm{var}(x_C)$ are pairwise different.
- We have $F \in \mathcal{MSAT} \Leftrightarrow \forall F' \subseteq F : \delta(F) \leq 0$, i.e., $\delta^*(F) = 0$.
- And $F \in \mathcal{MLEAN} \Leftrightarrow \forall F' \subset F : \delta(F') < \delta(F)$.
- Thus for $F \in \mathcal{MLEAN}$ holds $\delta^*(F) = \delta(F)$, and for $F \neq \top$ holds $\delta(F) \geq 1$ (note $\delta(\top) = 0$), a vast generalisation of this fact for $\mathcal{MU}$.
- More strongly, we have for $F \neq \top$ that $F \in \mathcal{MLEAN} \Leftrightarrow \sigma(F) \geq 1$.
- Every $F \in \mathcal{CLS}$ has a largest matching-lean sub-clause-set, the *matching-lean kernel*, namely $\bigcup\{F' \subseteq F : F' \in \mathcal{MLEAN}\}$, computable in polynomial time (for example via reduction by matching autarkies).

4) "Normal autarky systems" were called "strong autarky systems" in [51, Section 8].

Linear autarkies. A stronger autarky system than matching autarkies is given by "linear autarkies"; we will not use them for the results of this report, but they are an important link to combinatorics, and so we discuss them here; see [43, Subsection 11.11.3] for a more elaborated introduction. "Simple linear autarkies" for $F \in \mathcal{CLS}$ have been introduced in [50], based on linear programming. For $F \in \mathcal{CLS}$ we consider the clause-variable matrix $M(F)$, which is a $c(F) \times n(F)$ matrix over $\mathbb{R}$ (or over $\mathbb{Q}$ for computational purposes), which encodes in the rows the clauses and in the columns the variables, by using 0 for absence of the variable, and ± 1 for positive resp. negative sign. Now the *simple linear autarkies* φ are obtained from solutions $\vec{x} \in \mathbb{R}^{n(F)}$ of $M(F) \cdot \vec{x} \geq 0$, by translating the values $\vec{x}_i$, where the indices i correspond to the variables of F, into "unassigned" for $\vec{x}_i = 0$, "true" (i.e., 1) for $\vec{x}_i > 0$, and "false" (i.e., 0) for $\vec{x}_i < 0$. It is an easy exercise to see that this yields indeed autarkies. We have a non-trivial simple linear autarky iff $M(F) \cdot \vec{x} \geq 0$ has a non-trivial solution. We obtain the classes

- $\mathcal{LLEAN}$ of "linearly lean clause-sets" (not having a non-trivial simple linear autarky), with $\mathcal{LEAN} \subset \mathcal{LLEAN} \subset \mathcal{MLEAN}$;
- $\mathcal{LSAT}$ of "linearly satisfiable clause-sets" (satisfiable by a sequence of simple linear autarkies), with $\mathcal{MSAT} \subset \mathcal{LSAT} \subset \mathcal{SAT}$.

Linear autarkies, as introduced in [51], are obtained from simple linear autarkies by composition, corresponding to iterated reduction by simple linear autarkies; simple linear autarkies yield an autarky system, while linear autarkies yield a normal autarky system. The main point here is, that the reduction to the linearly-lean kernel can be done by a single linear autarky, and linearly satisfiable clause-sets are satisfiable by a single linear autarky. In Subsection 1.6.3 we discuss the special case of "balanced linear autarkies".

1.6 Connections to combinatorics

We now discuss the connections between SAT and combinatorics in a wider context than the degree considerations of this report, concentrating on aspects related to minimal unsatisfiability and autarkies (if one is only interested in the results of this report, then these discussions may be ignored). A general source on SAT is the handbook [7]; a classical connection to combinatorics, random satisfiability, is discussed in Chapter 8 ([2]) there, and of further general interest to combinatorics is Chapter 10 ([86]) on symmetry (group theory), Chapter 13 ([87]) on fixed-parameter tractable problems (for example treewidth and related notions), and Chapter 17 ([100]) on the handling of various combinatorial designs to SAT solving, for example from Ramsey theory. Ramsey theory has strong connections to hypergraph colouring, which we discuss next; we mention, that applying SAT solving to solve hypergraph colouring problems is a powerful tool, and a recent overview can be found in [4] (where especially van-der-Waerden numbers are discussed).

1.6.1 Hypergraph colouring

Hypergraph-colouring, especially 2-colouring, and SAT are closely connected; see [19, Section 5] for a general introduction and overview on hypergraph colouring (from the combinatorial point of view), while a monograph is given with [40]. An overview especially on the question of the minimum number of hyperedges for a given number of vertices in non-k-colourable hypergraphs is given in [46].

Hypergraphs. For this introduction, a hypergraph G is a finite set of finite subsets of $\mathbb{Z}$; so G itself is the set of hyperedges, i.e., $E(G) := G$, while $\bigcup G$ is the set of vertices, i.e., $V(G) := \bigcup G$. The set of all hypergraphs is denoted by $\mathcal{HYP}$. Let the deficiency be $\delta_{\mathrm{H}}(G) := |E(G)| - |V(G)|$. Note that clause-sets are special hypergraphs ($\mathcal{CLS} \subset \mathcal{HYP}$), but their deficiency is defined differently. Hypergraphs G with $\delta_{\mathrm{H}}(G) = 0$ are called *square hypergraphs*. Special hypergraphs are the positive clause-sets, and the set of all positive clause-sets is denoted by $\mathcal{PCLS} := \{F \in \mathcal{CLS} : \bigcup F \subset \mathbb{N}\} = \{G \in \mathcal{HYP} : V(G) \subset \mathbb{N}\}$. For $F \in \mathcal{PCLS}$ we have $\delta(F) = \delta_{\mathrm{H}}(F)$; obviously every hypergraph can be renamed to a positive clause-set. From general clause-sets $F \in \mathcal{CLS}$ we obtain two hypergraphs:

- F itself is a hypergraph (breaking the link between positive and negative literals, which are now just unrelated vertices).

  We note that we could have allows $\mathcal{CLS} = \mathcal{HYP}$, by allowing tautological clauses (i.e., clauses containing clashing literals) and self-complementary literals ($-0 = 0$). In certain contexts allowing such degenerations has advantages, but in our context is seems best to ban them (for example so we have a direct correspondence between clauses and partial assignments).

- The "variable-hypergraph" of F is $\{\mathrm{var}(C) : C \in F\} \in \mathcal{PCLS}$. This formation for example is important to apply methods from matching theory.

For positive clause-sets both formations collapse to the identity, and we treat positive clause-sets as representing (general) hypergraphs by (special) clause-sets.

Colouring. A k-colouring for $k \in \mathbb{N}_0$ of G is a map $f : V(G) \to \{0, \ldots, k-1\}$ such that for all $H \in G$ there are $x, y \in H$ with $f(x) \neq f(y)$; G is called k-colourable if there exists a k-colouring of G. Note that if there are $H \in G$ with $|H| \leq 1$, then G is not k-colourable for any k. A hypergraph G is *critically k-colourable* if G is k-colourable, not $k-1$-colourable, but for all $H \in G$ the hypergraph $G \setminus \{H\}$ is $k-1$-colourable. In the SAT-context there is no need to discard hyperedges containing at most one vertex, and then *minimally non-k-colourability* is more appropriate, that is G is not k-colourable (possibly not colourable at all), while after removal of any hyperedge G becomes k-colourable. The set of all minimally non-k-colourable hypergraphs is denoted by $\mathcal{MNC}^k \subset \mathcal{HYP}$ for $k \in \mathbb{N}_0$. We have $\{\emptyset\}, \{\{x\}\} \in \mathcal{MNC}^k$ for all $k \in \mathbb{N}_0$ and $x \in \mathbb{Z}$.

We are especially interested in $\mathcal{MNC}^2$. For $G \in \mathcal{MNC}^2$ holds $\delta_{\mathrm{H}}(G) \geq 0$, as shown in [90], and so we can consider the sets $\mathcal{MNC}^2_{\delta_{\mathrm{H}}=k}$ for deficiencies $k \in \mathbb{N}_0$ (all minimally non-2-colourable hypergraphs of deficiency (exactly) k).[5)] The famous problem of deciding in polynomial time, whether a directed graph contains an even cycle, is equivalent to the problem of deciding "$F \in \mathcal{MNC}^2_{\delta_{\mathrm{H}}=0}$?" for $F \in \mathcal{HYP}$ (via simple transformations), and this problem was finally solved in [84, 76]. It was conjectured in [56], that for all $k \in \mathbb{N}_0$ the classes $\mathcal{MNC}^2_{\delta_{\mathrm{H}}=k}$ are decidable in polynomial time (see also [43, Conjecture 11.12.1]). More on this in Subsection 1.6.4. In [1] one finds more information on vertex degrees in uniform elements of $\mathcal{MNC}^2_{\delta_{\mathrm{H}}=0}$ (i.e., where all hyperedges have the same length).

Translating hypergraphs into clause-sets. For a positive hypergraph $G \in \mathcal{PCLS}$ we obtain the translation of 2-colouring to satisfiability via

$$F_2(G) := G \cup \{-H : H \in G\} \in \mathcal{CLS}.$$

For a general discussion of such translations, also considering more colours, see [58, Subsection 1.2]. A hypergraph $G \in \mathcal{PCLS}$ is 2-colourable iff $F_2(G)$ is satisfiable, and

[5)] Indeed in [55, Corollary 8.2] it is shown $\delta_{\mathrm{H}}(G) \geq 0$ for all $G \in \mathcal{MNC}^k$ for $k \geq 2$, as a simple application of the autarky method; note that for $G := \{\{1, \ldots, n\}\} \in \mathcal{MNC}^k$ for $k \leq 1$ and $n \geq 2$ holds $\delta_{\mathrm{H}}(G) = 1 - n < 0$.

G is minimally non-2-colourable iff $F_2(G)$ is minimally unsatisfiable, i.e., $F_2(G) \in \mathcal{MU} \Leftrightarrow G \in \mathcal{MNC}^2$ (this is easy to prove, and a special case of [55, Lemma 8.1]). Regarding the deficiency we have $\delta(F_2(G)) = \delta_{\mathrm{H}}(G) + |E(G)|$ for $\emptyset \notin G$, and thus e.g. $F_2(\mathcal{MNC}^2_{\delta_{\mathrm{H}}=0} \cap \mathcal{PCLS})$ is not contained in any $\mathcal{MU}_{\delta=k}$ for some $k \in \mathbb{N}$.

A slight generalisation of the image $F_2(\mathcal{PCLS})$ under this translation is the class of complementation-invariant clause-sets $F \in \mathcal{CLS}$, characterised by $C \in F \Leftrightarrow -C \in F$ for clauses C, as introduced in [56] (see also [43, Subsection 11.4.5]), while the image $F_2(\mathcal{PCLS})$ is the set of complementation-invariant PN-clause-sets, that is, clause-sets F where every clause $C \in F$ is positive (i.e., $C \subset \mathbb{N}$) or negative ($-C \subset \mathbb{N}$). See Subsection 1.6.4 for how "autarkies", as considered on $F_2(G)$, can help understanding G.

Translating clause-sets into hypergraphs. In the other direction a translation was provided in [72]. For $F \in \mathcal{CLS}$ let

$$e(F) := \{C \cup \{0\} : C \in F\} \cup \{\{v, -v\} : v \in \mathrm{var}(F)\} \in \mathcal{HYP}.$$

The hypergraph $e(F)$ is 2-colourable iff F is satisfiable, and F is minimally unsatisfiable iff $e(F)$ is minimally non-2-colourable, i.e., $e(F) \in \mathcal{MNC}^2 \Leftrightarrow F \in \mathcal{MU}$ (the direction "$\Leftarrow$" of the latter statement is stated in the proof of Theorem 3 in [3], the other direction is (also) very easy). Furthermore $\delta_{\mathrm{H}}(e(F)) = \delta(F) - 1$. Thus e embeds the classes $\mathcal{MU}_{\delta=k}$ into $\mathcal{MNC}^2_{\delta_{\mathrm{H}}=k-1}$, which motivates the conjecture, that all $\mathcal{MNC}^2_{\delta_{\mathrm{H}}=k}$ for $k \in \mathbb{N}_0$ are polytime decidable, as a strengthening of the polytime decision of the $\mathcal{MU}_{\delta=k}$ for $k \in \mathbb{N}$ (recall Subsection 1.4).

We remark that via this embedding e we obtain a proof of $\delta(F) \geq 1$ for $F \in \mathcal{MU}$ from $\delta_{\mathrm{H}}(G) \geq 0$ for $G \in \mathcal{MNC}$ (this is one of the proofs given in [3]). In [3] also an alternative proof of $\delta_{\mathrm{H}}(G) \geq 0$ is given, based on matching theory, plus one further proof of $\delta(F) \geq 1$, using linear algebra, as in [90]. In Subsection 1.6.3 we will further comment on these proofs, as they are unfolded in the theory of "autarkies".

We also remark, that the hypergraph class $e(\mathcal{MU}_{\delta=1}) \subset \mathcal{MNC}^2_{\delta_{\mathrm{H}}=0}$ has the property, that every hypergraph in it different from $\{\{0\}\}$ has a vertex of degree 2 (since every $F \in \mathcal{MU}_{\delta=1}$ different from $\{\emptyset\}$ has a variable of degree 2). More generally, for all $k \in \mathbb{N}$ every hypergraph in $e(\mathcal{MU}_{\delta=k}) \setminus \{\{\{0\}\}\} \subset \mathcal{MNC}^2_{\delta_{\mathrm{H}}=k-1}$ has a vertex of degree at most $k+1$. We do not know whether the minimum vertex-degrees of general $G \in \mathcal{MNC}^2_{\delta_{\mathrm{H}}=k}$ for any (fixed) $k \in \mathbb{N}_0$ are bounded.

1.6.2 Hypergraph transversals

For $G \in \mathcal{HYP}$ let $\mathrm{Tr}(G) \in \mathcal{HYP}$, the *transversal hypergraph* of G, be defined as the set of all minimal $T \subseteq V(G)$ such that $T \cap H \neq \emptyset$ for all $H \in G$. The *Transversal Hypergraph Problem* is the computational problem, given $G, G' \in \mathcal{HYP}$, to decide whether $\mathrm{Tr}(G) = G'$ holds. Equivalently, the input is $G \in \mathcal{HYP}$, and it is to be decided whether $G = \mathrm{Tr}(G)$ holds (obviously this is a special case of the Transversal Hypergraph Problem, and by a polynomial-time translation the general case can be reduced to it). For an overview on this important problem and its many guises see [22]. It is known that the problem is solvable in quasi-polynomial time, and the long outstanding problem is whether it can be solved in polynomial time.

An *intersecting hypergraph* is a hypergraph $G \in \mathcal{HYP}$, such that for $H, H' \in G$ with $H \neq H'$ holds $H \cap H' \neq \emptyset$, the class of all intersecting hypergraphs is denoted by $\mathcal{IHYP} \subset \mathcal{HYP}$. By definition we have $G \subseteq \mathrm{Tr}(G)$ for $G \in \mathcal{IHYP}$, and it is not hard to see that for $G \in \mathcal{IHYP}$ holds $G \in \mathcal{MNC}^2$ iff $\mathrm{Tr}(G) = G$. Thus the Transversal Hypergraph Problem is equivalent to the problem, deciding whether an intersecting hypergraph is minimally non-2-colourable. The natural question arises for the decision of the classes $(\mathcal{MNC} \cap \mathcal{IHYP})_{\delta_{\mathrm{H}}=k}$ for $k \in \mathbb{N}_0$. The case

$k = 0$ has been handled in [90], indeed not just deciding the class in polynomial time, but efficiently classifying the elements. The cases $k \geq 1$ appear to be open, and whether decision is possible in polynomial time for fixed k, or is even fixed-parameter tractable (fpt) in k, is an interesting test case for the general Hypergraph Transversal Problem, as well as it is relevant for the understanding of minimally non-2-colourable hypergraphs.

The translation of intersecting hypergraphs $G \in \mathcal{IHYP}$ into clause-sets $F_2(G) \in \mathcal{CLS}$ yields also a natural and interesting class of clause-sets. *Bihitting clause-sets*, introduced in [26, Subsection 4.2], are those $F \in \mathcal{CLS}$ where $F', F'' \subseteq F$ with $F' \cup F'' = F$, $F' \cap F'' = \emptyset$ exist, such that for all $C' \in F', C'' \in F''$ holds $C' \cap -C'' \neq \emptyset$, while F', F'' itself are clash-free (i.e., $(\bigcup F') \cap -(\bigcup F') = \emptyset$, and $(\bigcup F'') \cap -(\bigcup F'') = \emptyset$). Obviously, the images under F_2 of intersecting hypergraphs are precisely the bihitting complementation-invariant PN-clause-sets (i.e., the set of bihitting clause-sets in the image of F_2), and deciding their minimal unsatisfiability is thus another manifestation of the Hypergraph Transversal Problem (directly related to the decision "$G = \mathrm{Tr}(G))$?"). And another one is to decide SAT for general bihitting clause-sets (as can be easily seen, and is discussed in [26, Subsection 4.3]; directly related to the decision "$\mathrm{Tr}(G) = G'$?").

In [55, Theorem 8.14] (the first 6 sections are covered by [57, 58]) the characterisation of [90] (the intersecting $G \in \mathcal{MNC}^2_{\delta_{\mathrm{H}}=0}$) is translated into this language.

1.6.3 Autarkies for hypergraphs

We discuss here now two autarky systems (recall Subsection 1.5 for a general introduction), which are especially relevant for hypergraph colouring.

Balanced autarkies. *Balanced autarkies* for $F \in \mathcal{CLS}$ (introduced in [56]; [43, Subsection 11.11.4] provides an introduction) are partial assignments φ, which in every clause of F they touch satisfy as well as falsify at least one literal (that is, for $C \in F$ with $C \cap (\varphi \cup -\varphi) \neq \emptyset$ holds $C \cap \varphi \neq \emptyset$ as well as $C \cap -\varphi \neq \emptyset$). This is a normal autarky system, and thus we basically have all the good property general autarkies have. Balanced autarkies are closely related to hypergraph colouring. The balanced autarkies for F are precisely the autarkies of $F \cup \{-C : C \in F\}$, and every autarky for a complementation-invariant clause-set is automatically balanced. A clause-set is balanced-satisfiable, i.e., can be satisfied by a balanced autarky, iff it is NAE-satisfiable ("not-all-equal"; see [82] for basic results).

Balanced autarkies provide the general autarky form for $\mathcal{PCLS}$ (whose elements are all trivially satisfiable, and thus unrestricted autarkies are not of interest here), which represents hypergraphs for the 2-colouring problem: an $F \in \mathcal{PCLS}$ is 2-colourable iff it is balanced-satisfiable, and F is minimally non-2-colourable iff it is minimally balanced-unsatisfiable. Finally we have *balanced lean* clause-sets (i.e., having no non-trivial balanced autarkies), and this is the appropriate notion of "leanness" for hypergraphs, as represented by the class $\mathcal{PCLS}$; more precisely, a hypergraph G is lean iff for an isomorphic $F \in \mathcal{PCLS}$ (isomorphic as hypergraph) we have that F is balanced lean. For lean hypergraphs G we have $\delta_{\mathrm{H}}(G) \geq 0$, and this is properly treated by "balanced linear autarkies".

Balanced linear autarkies. The special case of "balanced linear autarkies" was introduced in [51, Section 6]; these are the simple linear autarkies for $F \cup \{-C : C \in F\}$ (recall Subsection 1.5).[6)] Equivalently, the balanced linear autarkies φ for

[6)] More precisely one should speak of "balanced simple linear autarkies", but for convenience "simple" is dropped. We note that "balanced linear autarkies" are balanced and linear autarkies, but in general a balanced and linear autarky need not be a balanced linear autarky, and one should speak of "balanced-linear autarkies"; again this is an abuse of language, motivated by the fact

$F \in \mathcal{CLS}$ are obtained from solutions $\vec{x} \in \mathbb{R}^{n(F)}$ of $M(F) \cdot \vec{x} = 0$, by translating the values $\vec{x}_i$ as discussed before (it is an easy exercise to see that this yields indeed balanced autarkies). We have a non-trivial balanced linear autarky iff $M(F) \cdot \vec{x} = 0$ has a non-trivial solution, and so, in other words, F is balanced linearly lean iff the columns of $M(F)$ are linearly independent (iff $\operatorname{rank}(M(F)) = n(F)$). Thus if $F \in \mathcal{CLS}$ is balanced linearly-lean, then $\delta(F) \geq 0$ holds; furthermore, as shown in [55, Lemma 7.2], there is then a matching in the clause-variable graph covering all variable nodes, and thus even $\delta^*(F) = \delta(F)$ holds. By noting that $F \in \mathcal{CLS}$ is balanced linearly lean iff $F \cup \{-C : C \in F\}$ is linearly lean, and considering $\mathcal{PCLS}$, we obtain that for lean hypergraphs G (especially, minimally non-2-colourable) we have $\delta_{\mathrm{H}}(G) \geq 0$. To say the argument again explicitly: Consider a hypergraph $G \in \mathcal{PCLS}$; then G (as a clause-set) is balanced linearly lean iff the variable-clause matrix has linearly independent rows, iff $F_2(G)$ is linearly lean (again, as a clause-set), which is implied by $F_2(G)$ being minimally unsatisfiable (or weaker, being lean), which in turn is equivalent to G (as a hypergraph) being minimally-non-2-colourable. This conclusion "The rows of the incidence matrix [our variable-clause matrix] of a minimally-non-2-colourable hypergraph are linearly independent over $\mathbb{R}$." is shown in [90]; see [32, Lemma 4.7] for this and related results, while the conclusion "$\delta_{\mathrm{H}}(G) \geq 0$" is discussed as Principle 2.1 in [32]. For properties of minimally balanced linearly unsatisfiable clause-sets see [56, Section 4].

Fundamental inequalities. We have yet seen two fundamental inequalities, namely $\delta(F) \geq 1$ for $F \in \mathcal{MLEAN}$, as first shown in [3] for minimally unsatisfiable clause-sets, and $\delta(F) \geq 0$ for balanced linearly lean clause-sets, first shown in [90] (as $\delta_{\mathrm{H}}(G) \geq 0$ for minimally non-2-colourable hypergraphs).[7] Autarky theory shows the general structure of the arguments: We find "obstructions", which prevent these bounds from holding, where such obstructions are given by a subset $F' \subseteq F$ where there is a partial assignment φ with $\varphi * F' = \top$, while $\operatorname{var}(\varphi) \cap \operatorname{var}(F' \setminus F) = \emptyset$. Now minimally unsatisfiable F do not have such F', and thus the envisaged bound holds for them, and this is the argumentation in e.g. [90, 3].

But one can go beyond this, exploiting autarky *reduction*. Note that φ is precisely an autarky, and furthermore possibly one of a special structure. If we just look at general autarkies, then we obtain the first generalisation, to lean clause-sets or balanced lean clause-sets (covering the hypergraph cases). However often, due to the special structure, these special autarkies can be found in polynomial time, and their application yields some $F' \subseteq F$, such that the bound holds for F' (while for $F \in \mathcal{MU}$ we just have $F' = F$). If we have even an "autarky system", then F' is uniquely determined, that is, does not rely on the choice of the autarkies in the reduction process. The case of main importance for this report is $\delta(F) \geq 1$, where the autarkies are matching autarkies, and the reduced F' is the matching-lean kernel of F, while those F with $F' = F$ are precisely the $F \in \mathcal{MLEAN}$. On the other hand, for hypergraph colouring the fundamental fact is $\delta(F) \geq 0$ for balanced linearly-lean clause-sets, where the autarkies are balanced linear autarkies, and the reduced F' is the balanced-linearly-lean kernel of F. In fact, via autarky reduction we obtain a general method to study *decompositions*, which we will discuss in the context of "QMA".

1.6.4 Qualitative matrix analysis (QMA)

QMA can be understood as the analysis of matrices M over the real numbers in abstraction of the absolute value of the entries, but only their signs count, that is,

that linear autarkies which are also balanced are apparently too general a concept to be useful.

[7] An application yielding Fisher's inequality (design theory) is discussed in Subsection 7.4 of [55] (while Seymour's inequality is discussed there in Subsection 8.2).

one considers the *qualitative class* $\mathfrak{Q}(M)$, which consists of all matrices with the same dimensions as M, which have entry-wise the same signs as M (positive, zero, negative), and investigates when a property of M holds for all $M' \in \mathfrak{Q}(M)$. For example, a matrix M, such that all $M' \in \mathfrak{Q}(M)$ have linearly independent rows, is called an *L-matrix*. The monography [9] is an excellent source on QMA until the 1990's, while a more recent overview is given in [36].

Starting from [16], which exploits Farkas' lemma to understand (un)satisfiability, the connections to QMA have been first explored in Sections 3 and 5 in [51]; see [43, Subsection 11.12.1] for a more substantial introduction. It is shown in [51, Remark 5 in Section 5], that L-matrices correspond (nearly) precisely (up to transposition and handling of zero-rows/columns and repeated rows/columns) to balanced lean clause-sets, while lean clause-sets correspond (nearly) precisely to so-called *L^+-matrices* (as investigated in [70]). The square L-matrices are called *SNS-matrices*; SNS-matrices are at the heart of the poly-time decision for $\mathcal{MNC}^2_{\delta_{\mathrm{H}}=0}$ (recall Subsection 1.6.1), and the connections to autarky theory are explored in [56]; see [43, Subsection 11.12.2] for an overview.

Further in the translation of terms, now regarding unsatisfiability: unsatisfiable clause-sets correspond to *sign-central matrices*, minimally unsatisfiable clause-sets correspond to *minimally sign-central matrices*. So [9, Theorem 5.4.3] is yet another proof of $\delta(F) \geq 1$ for $F \in \mathcal{MU}$. The variable-degree, as studied in the current report, corresponds to the number of non-zero entries in the rows of the matrices (while the deficiency is the difference of the number of columns and the number of rows). The elements of $\mathcal{MU}_{\delta=1}$ correspond to *S-matrices*, the elements of $\mathcal{SMU}_{\delta=1}$ correspond to *maximal S-matrices*.

As mentioned, autarky systems $\mathcal{A}$ (like balanced autarkies, matching autarkies, etc.) also yield a framework for decomposition theorems. The basic decomposition is into $\mathcal{A}$-lean and $\mathcal{A}$-satisfiable sub-clause-sets, as given in [51, Theorem 8.5] (for normal autarky systems), which corresponds to a certain unique decomposition of the clause-variable matrix into a triangular shape with two blocks on the diagonal, and generalises various matrix decompositions in QMA, as discussed in [51, Footnote 7, Page 246]. $\mathcal{A}$-lean clause-sets itself can be further decomposed, and the main result is [56, Lemma 6] (reviewed in [43, Subsection 11.11.5]), generalising [9, Theorem 2.2.5], stating that a clause-set $F \in \mathcal{CLS}$ is minimally $\mathcal{A}$-unsatisfiable iff F is barely $\mathcal{A}$-lean (it is lean, but removal of any single clause destroys this) and $\mathcal{A}$-indecomposable (no triangular decomposition into $\mathcal{A}$-lean blocks is possible for the clause-variable matrix).

1.6.5 Biclique partitions of (multi-)graphs, and algebraic graph theory

We finish this overview on related themes in combinatorics by a field of graph theory, which, like QMA, can be understood as a study of clause-sets from a special angle, focusing on the conflict-structure of clauses.

Certain aspects of algebraic graph theory. The starting point is [33], where the problem of "addressing a graph" was introduced. One considers a symmetric matrix D of dimension $m \in \mathbb{N}$ over $\mathbb{N}_0$, with a zero-diagonal, where the entries are interpreted as "distances" (in [33] the $D_{i,j}$ are the distances between the nodes of some graph), and asks for the smallest $N \in \mathbb{N}_0$, such that there are m codewords $c_1, \ldots, c_m \in \{0, 1, *\}^N$ with the property, that the modified Hamming distance between c_i and c_j, which simply ignores positions with $*$, is $D_{i,j}$.

From our point of view, such a codeword is nothing else than a clause over the variables $1, \ldots, N$, while the modified Hamming distance is the number of clashes (conflicts). So the question is about the existence of clauses C_i for $i \in \{1, \ldots, m\}$

over variables $1, \dots, N$, such that $D_{i,j} = |C_i \cap -C_j|$ for $i, j \in \{1, \dots, m\}$. However yet "clause-sets" are not perceived/known as combinatorial objects, and their perspective is missing from the literature. See Chapter 9 of [95, Chapter 9] for an introduction. A basic result of [33] is that if D has all entries outside the diagonal equal to 1, then $N = m - 1$ (see also [32, Lemma 6.6] for a discussion in the context of eigenvalue methods; for a direct combinatorial proof see [96]). This follows from the general result $N \geq \max(n_+(D), n_-(D))$ of [33] (the "Lemma of Witsenhausen"), where $n_+(D)$ resp. $n_-(D)$ is the number of positive/negative eigenvalues of D. For the general case in [98] it is shown, that if the distances $D_{i,j}$ are indeed the distances between the nodes of some graph, then we have $N \leq m - 1$.

Actually, the Lemma of Witsenhausen works for arbitrary matrices D over $\mathbb{N}_0$ with zero diagonal. Taking up the clause-set perspective again, the minimal number N of variables in a clause-set representing D (it is an easy exercise to see that N is finite, i.e., a representation is always possible) is equal to the minimal number of $0,1$-matrices of rank 1 which add up to D: A variable contributes precisely a "rectangle", i.e., a matrix which is 1 at the entries $I \times J$ for some $\emptyset \neq I, J \subseteq \{1, \dots, m\}$, and otherwise 0, and these are precisely the $0,1$-matrices of rank 1. Considering D as the adjacency matrix $A = D$ of some multigraph (where parallel edges are allowed), we see that N is also equal to the minimum number of bicliques into which the edge-set of that multigraph can be partitioned, and N is therefore denoted by $\mathrm{bcp}(A) \in \mathbb{N}_0$ (the "biclique partition number" of A resp. the corresponding multigraph).[8)]

The notion "hermitian rank" has been introduced and studied in [35] for arbitrary hermitian matrices A (square matrices with complex numbers as entries, such that transposing the matrix and taking the complex conjugate of each entry yields back the original matrix), denoted by $h(A) := \max(n_+(A), n_-(A)) \in \mathbb{N}_0$. So the Lemma of Witsenhausen takes the form, that for symmetric matrices A over $\mathbb{N}_0$ with zero diagonal holds $\mathrm{bcp}(A) \geq h(A)$.

Conflict analysis. The essential observation is now that we can go back and forth between biclique partitions of multigraphs and clause-sets. In one direction we can understand clause-sets F as representations of biclique partitions of multigraphs, where for each vertex we get a clause, and from each biclique we obtain a variable, where the two sides of the biclique are the positive and negative occurrences of the variable. So we can understand a multigraph together with a biclique partition as a clause-set, and we can use tools from clause-set-logic to analyse the pair multigraph with biclique-partition. The deficiency then becomes the difference between the number of nodes and the number of bicliques. Satisfiability means that it is possible to select from each biclique one side such that all vertices are covered.

In the other direction we can understand biclique partitions of multigraphs (or, equivalently, representing a matrix A as above as a sum of rank-1 matrices over $\{0, 1\}$) as representations of clause-sets F, namely the nodes of the *conflict multigraph* $\mathrm{cmg}(F)$ are given by the clauses, while the edges are the conflicts (clashing literal occurrences $x, -x$), and the bicliques are given by the variables (their positive and negative occurrences). In this way we can analyse the influence of the "conflict structure" on properties of clause-sets; the basic notions, as introduced in [54] with underlying report [53], are as follows.

For $F \in \mathcal{CLS}$ let $\mathrm{CM}(F)$ (the *conflict matrix*) be the square matrix of dimension $c(F)$ over $\mathbb{N}_0$, with entries $|C \cap -D|$ for $C, D \in F$ (thus with zero diagonal), i.e., $\mathrm{CM}(F)$ is the adjacency matrix of $\mathrm{cmg}(F)$. So we can use the hermitian rank as a measure $h : \mathcal{CLS} \to \mathbb{N}_0$ (as first done in [54, Subsection 3.2]), namely

$$h(F) := h(\mathrm{CM}(F));$$

8) In [54] we used "$n_{\mathrm{s}}(A)$ instead, the "symmetric conflict number".

see Points 1, 3 in [26, Section 2] for various equivalent characterisations.[9)] By definition we have $\mathrm{bcp}(F) := \mathrm{bcp}(\mathrm{CM}(F)) \le n(F)$, and thus $h(F) \le n(F)$. Since for a principal submatrix A' of a hermitian matrix A holds $h(A') \le h(A)$ (this follows by "interlacing"; see [31, Theorem 9.1.1]), we get $h(\varphi * F) \le h(F)$ for all partial assignments φ, and also $h(F') \le h(F)$ for all $F' \subseteq F$, which gives motivation to consider $h(F)$ as a *complexity measure* for $F \in \mathcal{CLS}$.

In [54] also the *hermitian defect* $\delta_{\mathrm{h}} : \mathcal{CLS} \to \mathbb{N}_0$ has been introduced as

$$\delta_{\mathrm{h}}(F) := c(F) - h(F),$$

and thus $\delta(F) \le \delta_{\mathrm{h}}(F)$; see Point 2 in [26, Section 2] for a geometric characterisation (as the "Witt index" of the quadratic form associated with $\mathrm{CM}(F)$). Actually $\delta^*(F) \le \delta_{\mathrm{h}}(F)$ holds and even stronger properties (see [54, Subsection 3.3]). An important property is (again) $\delta_{\mathrm{h}}(\varphi * F) \le \delta_{\mathrm{h}}(F)$ for all $F \in \mathcal{CLS}$ and partial assignments φ together with $\delta_{\mathrm{h}}(F') \le \delta_{\mathrm{h}}(F)$ for $F' \subseteq F$, by [54, Corollary 9], and so we might consider the hermitian defect as a stabilised version of the maximal defect (both are also complexity measures; recall that we have fixed-parameter tractable SAT decision for input $F \in \mathcal{CLS}$ in the parameter $\delta^*(F)$). Note that in general we can have $\delta^*(\varphi * F) > \delta^*(F)$, for example $F := \{\{1\}\}$ has $\delta^*(F) = \delta(F) = 0$, while for $F' := \langle 1 \to 0 \rangle$ we get $F' = \{\bot\}$, and thus $\delta^*(F') = \delta(F') = 1$. See [49, Subsection 3.3] and [57, Subsection 11.2] for more information on $\delta^*(\varphi * F)$; splitting on a single(!) variable is very important for this report, with the basic fact $\delta^*(\langle x \to 1 \rangle * F) \le \delta(F)$ for $F \in \mathcal{MU}$ and any literal x.

The first direct applications applied $\delta(F) \le \delta_{\mathrm{h}}(F)$ for $F \in \mathcal{CLS}$, namely that for a hitting clause-set $F \in \mathcal{HIT}$ (equivalently, all entries of $\mathrm{CM}(F)$ outside the diagonal are non-zero) with a regular conflict multigraph (i.e., all entries of $\mathrm{CM}(F)$ outside the diagonal are equal) we have $\delta_{\mathrm{h}}(F) \le 1$, and thus $\delta(F) \le 1$ ([54, Theorem 33]).[10)] We get that $\mathcal{SMU}_{\delta=1} = \mathcal{UHIT}_{\delta=1}$ is (precisely) the class of unsatisfiable hitting clause-sets with regular conflict multigraph ([54, Corollary 34]; a combinatorial proof of this was independently found in [91, Lemma 11]), and is also (precisely) the class of unsatisfiable clause-sets F with $\delta_{\mathrm{h}}(F) \le 1$ ([54, Theorem 26]).

A clause-set $F \in \mathcal{CLS}$ is called *exact* ([54, Subsection 3.4]) if $\mathrm{bcp}(F) = n(F)$, that is, F is optimal in realising $\mathrm{cmg}(F)$ with respect to the number of variables. Deciding exactness is coNP-complete, while the special class of *eigensharp clause-sets*, defined by $h(F) = n(F)$, or, equivalently, $\delta_{\mathrm{h}}(F) = \delta(F)$, is decidable in polynomial time. With [54, Theorem 14] every eigensharp clause-set is matching lean. This leads to [54, Conjecture 16], "Every exact clause-set, whose conflict-matrix is the distance matrix of some connected graph, is matching lean.", which generalises the already mentioned main result of [98] (the proof of the "squashed cube conjecture").

As already mentioned, we consider $h(F)$ for $F \in \mathcal{CLS}$ as some form of complexity measure, measuring the complexity of representing the conflicts of F via simple matrices. In [26] polytime SAT decision in case $h(F) \le 1$ was shown, while the cases $h(F) \le k$ for fixed $k \ge 2$ are open; an interesting stepping stone is to show polytime SAT decision for $F \in \mathcal{CLS}$ with $\mathrm{bcp}(F) \le k$ (recall $\mathcal{CLS}_{\mathrm{bcp} \le k} \subseteq \mathcal{CLS}_{h \le k}$). The notion of *blocked clauses*, a special type of clauses which can be removed sat-equivalently, introduced in [48], is important here, and [26, Theorem 3] shows, that from $F \in \mathcal{CLS}_{h \le 1}$ after elimination of all blocked clauses (which yields a unique result) we obtain $F' \subseteq F$ with $\mathrm{bcp}(F') \le 1$. We recall from Subsection 1.6.2, that SAT-decision for F' is now a special case of the Transversal Hypergraph Problem, namely, as shown in [26, Lemma 11], the problem is exactly the *Exact Transversal Hypergraph Problem*, where every transversal must be "exact", that is, must intersect every hyperedge in exactly one vertex; this problem is decidable in

[9)] For Point 1(c) there it must be a "diagonal matrix A'".

[10)] In [54] unfortunately the term "uniform" was (mis)used instead of "regular".

polynomial time by [21], and thus we get SAT-decision for $\mathcal{CLS}_{h\le 1}$ in polynomial time. The characterisation of $F \in \mathcal{MU}$ with $\operatorname{bcp}(F) \le 1$ is an open problem (while we have polytime membership decision for $\mathcal{MU}_{\operatorname{bcp}\le 1}$), and by [26, Conjecture 16] they would have a very simple structure.

We conclude by mentioning that in [58, Section 6] the basic facts are generalised to non-boolean clause-sets, and that by extending the reduction of multi-clique partitions to biclique partitions from [34] a new and interesting translation from non-boolean to boolean clause-sets was obtained.

1.7 Overview on results

Sections 2 to 6 provide foundations for the main results in the later sections. In Section 2 basic notions and concepts regarding clause-sets and autarkies are reviewed. In Section 3 we discuss minimal unsatisfiability, with some auxiliary results on saturation (adding literal occurrences to clauses, to make minimal unsatisfiability robust against splitting) and splitting. Section 4 reviews the concept of "variable-minimal unsatisfiability", as introduced in [11], i.e., the class $\mathcal{MU} \subset \mathcal{VMU} \subset \mathcal{USAT}$. There are mistakes in this paper, and we rectify them here:

- we show that $\mathcal{VMU} \subset \mathcal{LEAN}$ holds (Lemma 4.3);
- we provide a corrected characterisation of $\mathcal{VMU}$ (Lemma 4.5);
- and we give a corrected proof of polytime decision of $\mathcal{VMU}_{\delta=k}$ for fixed k in Theorem 4.7, where we also obtain the stronger result, that decision of $\mathcal{VMU}$ (i.e., deciding whether for input $F \in \mathcal{CLS}$ holds $F \in \mathcal{VMU}$ or not) is fixed-parameter tractable in the deficiency $\delta(F)$.

Section 5 is then concerned with singular variables, eliminating them via singular DP-reduction, and creating them via "singular extensions". An important auxiliary result is Lemma 5.4, showing that eliminating singular variables is harmless for bounds on the minimum variable-degree; we also show various auxiliary results on unit-clauses in minimally unsatisfiable clause-sets. This block of preparatory sections is concluded by Section 6 on full subsumption resolution, an ubiquitous reduction (and extension); as an application, in Theorem 6.13 we can determine precisely the possible $n(F)$ and $c(F)$ for $F \in \mathcal{MU}_{\delta=k}$.

The first main results (but still on the preparation side) one finds in Section 7, which introduces the numbers $\operatorname{nM}(k) \in \mathbb{N}$ and proves exact formulas and sharp lower and upper bounds; the point here is that the introduction of $\operatorname{nM}(k)$ happens via a recursion which is tailor-made for our application in Section 8, but which makes it somewhat difficult to determine the numbers in a global way. A main technical result is Theorem 7.15, while Theorem 7.21 proves the general formula.

In Section 8 then we find a basic central result of this report, the upper bound $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \le \operatorname{nM}(k)$ (Theorem 8.3). Section 9 is concerned with generalising this upper bound. An interesting auxiliary class $\mathcal{SED} \subset \mathcal{CLS}$, clause-sets where deficiency and surplus coincide, is introduced in Subsection 9.1; the main lemma here is Lemma 9.5, which shows that unsatisfiable elements of $\mathcal{SED}$ are in fact in $\mathcal{VMU}$. In Subsection 9.2 the upper bound for $\mathcal{MU}$ then is lifted to lean clause-sets in Theorem 9.8, and also sharpened via replacing the deficiency δ by the surplus σ. Theorem 9.12 shows that the upper bound is sharp for any class between $\mathcal{VMU} \cap \mathcal{SED}$ and $\mathcal{LEAN}$.

Section 10 concerns algorithmic applications. A corollary of Theorem 9.8 is, that if the asserted upper bound on the minimum variable degree is not fulfilled, then a non-trivial autarky must exist (Lemma 10.1). Since the variable-set of such a non-trivial autarky is polytime computable, we show in Theorem 10.2 that we can indeed establish the upper bound shown for lean clause-sets also for general clause-sets, after a polytime autarky-reduction. In Subsection 10.2 then the problem of finding such autarky (that is, finding the assignment) is discussed, with Conjecture 10.3 making precise our believe that one can find such autarkies efficiently. Theorem 10.9 pinpoints the "critical" class $\mathcal{MLCR} \subset \mathcal{SAT}$, which is polytime decidable, and where we know that these clause-sets are satisfiable, but we even don't know how to find any non-trivial autarky efficiently. This block on generalisations of the min-var-degree upper bound is concluded by Section 11, where we discuss the possibilities to generalise it to matching-lean clause-sets (where only the absence of special (non-trivial) autarkies is guaranteed).

In Section 12 we then turn to the study of the numbers $\mu\mathrm{nM}(k) := \mu\mathrm{vd}(\mathcal{MU}_{\delta=k})$, looking now for improved upper bounds and matching lower bounds. We present two infinite classes of deficiencies k with $\mu\mathrm{nM}(k) = \mathrm{nM}(k)$, and present a general method of obtaining lower bounds for $\mu\mathrm{nM}(k)$, via counting full clauses (clauses containing all variables — these clause are strong structural anchors). In Section 13 we introduce a general recursive method to obtain upper bounds like $\mathrm{nM}(k)$, via the "non-Mersenne operator" $\mathrm{NM}(f)$, which takes a "valid bounds function" f, that is, some partial information on $\mu\mathrm{nM}(k)$, and improves it (Definition 13.12). Theorem 13.10 shows that this indeed yields a valid method for improving upper bounds on $\mu\mathrm{nM}(k)$, while in Theorem 13.15 we demonstrate how this method recovers $\mathrm{nM}(k)$, by just starting with the information $\mu\mathrm{nM}(1) = 2$. In Section 14 we harvest (first) fruits of these methods. First in Theorem 14.1 we show $\mu\mathrm{nM}(k) = \mathrm{nM}(k)$ for $k \le 5$. Then in Theorem 14.3 we prove $\mu\mathrm{nM}(6) = \mathrm{nM}(k) - 1$ (using a variety of structural results on $\mathcal{MU}$ provided in this report). Plugging this information on $\mu\mathrm{nM}$ into our machinery, we obtain the improved upper bound $\mu\mathrm{nM} \le \mathrm{nM}_1$ in Theorem 14.5, while in Theorem 14.5 we determine $\mathrm{nM}_1(k)$ numerically.

Finally, in Section 15 open problems are stated, thoroughly discussing research perspectives, including nine conjectures. Subsection 15.2 discusses improved upper bounds for $\mu\mathrm{nM}(k)$ from the forthcoming work [68]. Subsection 15.3 is about improved lower bounds, via counting full clauses. In Lemma 15.2 we cite from the work in progress [66] (to be completed soon), which provides improved lower bounds via the "Smarandache primitive function" $S_2(k)$, yielding the first-order asymptotic determination of $\mu\mathrm{nM}(k) \sim k$ (Corollary 15.4), where now the open question is about the asymptotic determination of $\mu\mathrm{nM}(k) - k$. In Subsection 15.4 we discuss generalisations to non-boolean clause-sets.

The central Conjecture 15.6 of the project of "understanding MU", on the finitely many "characteristic patterns" for each $\mathcal{MU}_{\delta=k}$, is discussed in Subsection 15.5. An important special case is Conjecture 15.7 (now a fully precise statement), about the classification of unsatisfiable hitting clause-sets (or "disjoint/orthogonal tautologies" in the terminology of DNFs). In Lemma 15.9 we show how two of the conjectures together yield computability of $\mathrm{nM}(k)$.

This report is a substantial extension of the conference paper [62]: Section 3 there has been extended to Section 7 here, with considerable more details and examples on non-Mersenne numbers. Section 4 there is covered by Sections 8, 9 and 10, with various additional results (for example showing sharpness of the upper bound for $\mathcal{LEAN}$). And the results for Section 5 there are contained in Subsection 11 here. All other sections in this report are new.

2 Preliminaries

We follow the general notations and definitions as outlined in [43], where also further background on autarkies and minimal unsatisfiability can be found. We use $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For the binary logarithm we use $\mathrm{ld}(x) := \log_2(x)$

("logarithm dualis").

We apply standard set-theoretic concepts, like that of a map as a set of pairs, and standard set-theoretic notations, like $f(S) = \{f(x) : x \in S\}$ for maps f and $S \subseteq \mathrm{dom}(f)$, and "$\subset$" for the strict subset-relation. We use also the less-known notation "$A \mathbin{\dot\cup} B$" for union in case A, B are disjoint, that is, $A \mathbin{\dot\cup} B := A \cup B$ is only defined for $A \cap B = \emptyset$. For maps f, g with the same domain X we use $f \le g :\Leftrightarrow \forall\, x \in X : f(x) \le g(x)$ (i.e., pointwise comparison), while $f < g :\Leftrightarrow \forall\, x \in X : f(x) < g(x)$.

2.1 Clause-sets

The basic structure is a set $\mathcal{LIT}$, the elements called "literals", together with a fixed-point free involution called "complementation", written $x \in \mathcal{LIT} \mapsto \overline{x} \in \mathcal{LIT}$; so the laws are $\overline{x} \neq x$ and $\overline{\overline{x}} = x$ for $x, y \in \mathcal{LIT}$. We assume $\mathbb{Z} \setminus \{0\} \subseteq \mathcal{LIT}$, with $\overline{x} = -x$ for $x \in \mathbb{Z} \setminus \{0\}$. For a set L of literals we define $\overline{L} := \{\overline{x} : x \in L\}$. Furthermore a set $\mathbb{N} \subseteq \mathcal{VA} \subset \mathcal{LIT}$, the elements called "variables", is given, with $\mathcal{LIT} = \mathcal{VA} \mathbin{\dot\cup} \overline{\mathcal{VA}}$. Variables are also called "positive literals", while literals $\overline{v}$ for $v \in \mathcal{VA}$ are called "negative literals". The "underlying variable" of a literal is given by the operation $\mathrm{var} : \mathcal{LIT} \to \mathcal{VA}$ ("forgetting complementation"), with $\mathrm{var}(v) := v$ and $\mathrm{var}(\overline{v}) := v$ for $v \in \mathcal{VA}$.

Example 2.1 *We can thus write e.g.* $1, 6$ *for two (different) variables, and* $1, 5, -1$ *for three (different) literals. In examples we will also use* v, w, a, b, c *and such letters for variables (as it is customary), and accordingly* $\overline{v}$ *etc. for literals, and in this context (only) it is then understood that these variables are pairwise different. So* $\{v, w, x, \overline{x}\}$*, when given in an example (without further specification), denotes a set of literals with* $|\{v, w, x, \overline{x}\}| = 4$ *and* $|\{v, w, x, \overline{x}\} \cap \mathcal{VA}| = 3$.

Without restriction we could assume $\mathcal{LIT} = \mathbb{Z} \setminus \{0\}$ *(as we did in the Introduction), but it is often convenient to use arbitrary mathematical objects as variables. All our objects build from literals are finite, and thus, because of the infinite supply of variables, there will always be "new variables" (that's the mathematical point of having natural numbers as variables — we won't use the arithmetical structure).*

A **clause** C is a finite and clash-free set of literals (i.e., $C \cap \overline{C} = \emptyset$), the set of all clauses is $\mathcal{CL}$. A **clause-set** is a finite set of clauses, the set of all clause-sets is $\mathcal{CLS}$. The simplest clause is the *empty clause* $\bot := \emptyset \in \mathcal{CL}$, the simplest clause-set is the *empty clause-set* $\top := \emptyset \in \mathcal{CLS}$. The set of all **hitting clause-sets** is denoted by $\mathcal{HIT} \subset \mathcal{CLS}$, those $F \in \mathcal{CLS}$ such that two different clauses $C, D \in F$, $C \neq D$, have at least one clash, i.e., $C \cap \overline{D} \neq \emptyset$. In the language of DNF, hitting clause-sets are known as "orthogonal" or "disjoint" DNF's; see [13, Chapter 7].

Example 2.2 *We have e.g.* $\{1, 2, -3\} \in \mathcal{CL}$*, while* $\{-1, 1\} \notin \mathcal{CL}$*. The only clause-set in* $\mathcal{HIT}$ *containing the empty clause is* $\{\bot\} \in \mathcal{HIT}$*. An example of a non-hitting clause-set is* $\{\{1, 2\}, \{-1, 2\}, \{3\}\} \in \mathcal{CLS} \setminus \mathcal{HIT}$*, where we obtain an element of* $\mathcal{HIT}$ *if we add literal* -2 *to the third clause.*

We use $\mathrm{var}(F) := \bigcup_{C \in F} \mathrm{var}(C)$ for the set of variables of $F \in \mathcal{CLS}$, where $\mathrm{var}(C) := \{\mathrm{var}(x) : x \in C\}$ is the set of variables of clause $C \in \mathcal{CL}$. The possible literals for a clause-set F are given by $\mathrm{lit}(F) := \mathrm{var}(F) \mathbin{\dot\cup} \overline{\mathrm{var}(F)}$, while the actually occurring literals are just given by $\bigcup F$ (the union of the clauses of F). A literal x is **pure** for F if $\overline{x} \notin \bigcup F$. For a clause-set F we use the following measurements:

- $n(F) := |\mathrm{var}(F)| \in \mathbb{N}_0$ is the number of variables,
- $c(F) := |F| \in \mathbb{N}_0$ is the number of clauses,

- $\delta(F) := c(F) - n(F) \in \mathbb{Z}$ is the **deficiency** (the difference of the number of clauses and the number of variables),
- $\ell(F) := \sum_{C \in F} |C| \in \mathbb{N}_0$ is the number of literal occurrences.

We call a clause C **full** for a clause-set F if $C \in F$ and $\mathrm{var}(C) = \mathrm{var}(F)$, while a clause-set F is called full if every clause is full. For a finite set V of variables let

$$\boldsymbol{A(V)} := \{C \in \mathcal{CL} : \mathrm{var}(C) = V\} \in \mathcal{CLS}.$$

Obviously $A(V) \in \mathcal{HIT}$ is the set of all $2^{|V|}$ full clauses over V, and $F \in \mathcal{CLS}$ is full iff $F \subseteq A(\mathrm{var}(F))$. We use $\boldsymbol{A_n} := A(\{1, \dots, n\})$ for $n \in \mathbb{N}_0$. Dually, a variable $v \in \mathcal{VA}$ is called **full** for a clause-set F if for all $C \in F$ holds $v \in \mathrm{var}(C)$. A clause-set is full iff every $v \in \mathrm{var}(F)$ is full.

Example 2.3 *For $F := \{\bot, \{1\}, \{-1, 2\}\}$ we have:*

1. $\mathrm{var}(F) = \{1, 2\}$, $\mathrm{lit}(F) = \{-1, 1, -2, 2\}$, $\bigcup F = \{-1, 1, 2\}$.
2. *Literal 2 is pure for F (the other literals in $\mathrm{lit}(F)$ are not pure).*
3. $n(F) = 2$, $c(F) = 3$, $\delta(F) = 1$, $\ell(F) = 3$.
4. $\{-1, 2\}$ *is a full clause of F, while the two other clauses are not full.*
5. *F has no full variable, while $F \setminus \{\bot\}$ has the (single) full variable 1.*

The standard "complete" full clause-sets are $A_0 = \{\bot\}$, $A_1 = \{\{-1\}, \{1\}\}$,

$$A_2 = \{\{-1, -2\}, \{-1, 2\}, \{1, -2\}, \{1, 2\}\},$$

and so on.

We often define a class of clause-sets via some measure μ as follows:

Definition 2.4 *Consider a class $\mathcal{C} \subseteq \mathcal{CLS}$ and a measure $\mu : \mathcal{CLS} \to \mathbb{R}$. For $a \in \mathbb{R}$ we use $\boldsymbol{\mathcal{C}_{\mu=a}} := \{F \in \mathcal{C} : \mu(F) = a\}$, and similarly we use $\boldsymbol{\mathcal{C}_{\mu<a}}$ and analogous notations.*

When we use the form "$\mathcal{C}_{\mu \Box a}$", then μ stands for a measure (e.g., $\mu = \delta$ or $\mu = n$).

Example 2.5 $\mathcal{CLS}_{n=0} = \mathcal{CLS}_{\ell=0} = \{\top, \{\bot\}\}$, $\mathcal{CLS}_{c=0} = \{\top\}$, *and* $\mathcal{CLS}_{n<0} = \emptyset$.

2.2 Semantics

A **partial assignment** is a map $\varphi : V \to \{0, 1\}$ for some finite (possibly empty) set $V \subset \mathcal{VA}$ of variables, where $\mathrm{var}(\varphi) := V$ and $\mathrm{lit}(\varphi) := \mathrm{var}(\varphi) \mathbin{\dot\cup} \overline{\mathrm{var}(\varphi)}$. The set of all partial assignments is denoted by $\mathcal{PASS}$. For a literal $x \in \mathrm{lit}(\varphi)$ we also define $\varphi(x) \in \{0, 1\}$, via $\varphi(\overline{v}) := 1 - \varphi(v)$ for $v \in \mathrm{var}(\varphi)$. Via a small abuse of language we define $\varphi^{-1}(\varepsilon) := \{x \in \mathrm{lit}(\varphi) : \varphi(x) = \varepsilon\} \in \mathcal{CL}$ for $\varepsilon \in \{0, 1\}$. Special partial assignments are the *empty partial assignment* $\langle\rangle := \emptyset$, and for literals $x \in \mathcal{LIT}$ and $\varepsilon \in \{0, 1\}$ the partial assignment $\boldsymbol{\langle x \to \varepsilon \rangle} \in \mathcal{PASS}$, with $\mathrm{var}(\langle x \to \varepsilon \rangle) = \{\mathrm{var}(x)\}$ and $\langle x \to \varepsilon \rangle(x) = \varepsilon$.

The application of a partial assignment $\varphi \in \mathcal{PASS}$ to a clause-set F is denoted by $\boldsymbol{\varphi * F}$, which yields the clause-set obtained from F by removing all satisfied clauses (which have at least one literal set to 1), and removing all falsified literals from the remaining clauses:

$$\varphi * F := \{C \setminus \varphi^{-1}(0) : C \in F \wedge C \cap \varphi^{-1}(1) = \emptyset\} \in \mathcal{CLS}.$$

This definition is motivated by the default interpretation of a clause-set as a "conjunctive normal form" (CNF), where a clause is understood as a disjunction of literals (thus is satisfied iff at least one literal in it is satisfied), while a clause-set is understood as a conjunction of its clauses (thus is satisfied iff all clauses are satisfied). A clause-set F is *satisfiable* iff there is a partial assignment φ with $\varphi * F = \top$, otherwise F is *unsatisfiable*. The set of satisfiable clause-sets is denoted by $\boldsymbol{\mathcal{SAT}} \subset \mathcal{CLS}$, while $\boldsymbol{\mathcal{USAT}} := \mathcal{CLS} \setminus \mathcal{SAT}$ denotes the set of all unsatisfiable clause-sets.

Example 2.6 *If $F \in \mathcal{USAT}$ and for $F' \in \mathcal{CLS}$ holds $F \subseteq F'$, then also $F' \in \mathcal{USAT}$ (satisfying a clause-sets gets harder the more clauses there are).*

By definition we have $\varphi * F = \top$ iff $\forall\, D \in F : \varphi^{-1}(1) \cap D \neq \emptyset$; thus $F \in \mathcal{SAT}$ iff there is a clause $C \in \mathcal{CL}$ with $C \cap D \neq \emptyset$ for all $D \in F$. (We could write "$C \cap D \neq \bot$" here, but it appears somewhat more natural to use "$\emptyset$" here.)

The unsatisfiable hitting clause-sets are denoted by $\boldsymbol{\mathcal{UHIT}} := \mathcal{USAT} \cap \mathcal{HIT}$.

Example 2.7 *$\top \in \mathcal{SAT} \cap \mathcal{HIT}$ and $\{\bot\} \in \mathcal{UHIT}$. In general a full clause-set F is unsatisfiable iff $F = A(\mathrm{var}(F)))$, and thus $A(V) \in \mathcal{UHIT}$ for all finite $V \subset \mathcal{VA}$.*

The fundamental property for $F \in \mathcal{HIT}$: Consider $\varphi, \psi \in \mathcal{PASS}$, such that there are $C, D \in F$, $C \neq D$, with $\varphi * \{C\} = \psi * \{D\} = \{\bot\}$ (that is, $\bot \in \varphi * F \cap \psi * F$, where there are different falsified clauses for these two partial assignments). Then φ, ψ are incompatible, i.e., there is $v \in \mathrm{var}(\varphi) \cap \mathrm{var}(\psi)$ with $\varphi(v) \neq \psi(v)$.

It follows easily that for $F \in \mathcal{HIT}$ holds $F \in \mathcal{USAT} \Leftrightarrow \sum_{C \in F} 2^{-|C|} = 1$.

A nice exercise is to show $\mathcal{UHIT}_{\delta \leq 0} = \emptyset$ (in Section 2.6 a more general result is stated).

Finally, the *semantical implication* $F \models C$ for $F \in \mathcal{CLS}$ and clauses $C \in \mathcal{CL}$ holds iff $\forall\, \varphi \in \mathcal{PASS} : \varphi * F = \top \Rightarrow \varphi * \{C\} = \top$. We have $F \in \mathcal{USAT} \Leftrightarrow F \models \bot$.

2.3 Resolution

Two clauses $C, D \in \mathcal{CL}$ are **resolvable** if $|C \cap \overline{D}| = 1$, i.e., they clash in exactly one variable (called the resolution variable $\mathrm{var}(x)$, while x is called the resolution literal). For two resolvable clauses C and D, the **resolvent** $C \diamond D := (C \cup D) \setminus \{x, \overline{x}\} \in \mathcal{CL}$ for $C \cap \overline{D} = \{x\}$ is the union of the two clauses minus the resolution literal and its complement. As it is well-known (the earliest source is [8, 77]), a clause-set F is unsatisfiable iff via resolution (i.e., closing F under addition of resolvents) we can derive $\bot$, and, more generally, we have $F \models C$ iff from F via resolution a clause $C' \subseteq C$ is derivable.

An important reduction for clause-sets $F \in \mathcal{CLS}$ and variables $v \in \mathcal{VA}$, resulting in a clause-set satisfiability-equivalent to F (satisfiable iff F is; sometimes called "equi-satisfiable") and with variable v eliminated, is **DP-reduction**

$$\mathbf{DP}_{\boldsymbol{v}}\boldsymbol{(F)} := \{C \in F : v \notin \mathrm{var}(C)\} \cup \{C \diamond D : C, D \in F \wedge C \cap \overline{D} = \{v\}\} \in \mathcal{CLS}$$

(also called "variable elimination"), obtained from F by removing all clauses containing variable v from F, and replacing them by their resolvents on v. See [65] for a fundamental study of DP-reduction. The satisfying assignments φ of $\mathrm{DP}_v(F)$ (i.e., $\varphi * \mathrm{DP}_v(F) = \top$) with $\mathrm{var}(\varphi) = \mathrm{var}(F) \setminus \{v\}$ are precisely the satisfying assignments φ of F with $\mathrm{var}(\varphi) = \mathrm{var}(F)$, when restricting φ to $\mathrm{var}(F) \setminus \{v\}$. Logically, $\mathrm{DP}_v(F)$ is equivalent to $\exists\, v : F$, the existential quantification of v for F (but we do not use quantifiers in this report, so this remark might be ignored here).

2.4 Multi-clause-sets and restrictions

These notions are generalised to **multi-clause-sets**, which are maps $F : \mathcal{CL} \to \mathbb{N}_0$, such that the underlying set of clauses $\{C \in \mathcal{CL} : F(C) \neq 0\}$ is finite, and so we speak of the **underlying clause-set**; the set of all multi-clause-sets is denoted by $\underline{\mathcal{CLS}} := \{F \in \mathbb{N}_0^{\mathcal{CL}} : \mathcal{CL} \setminus F^{-1}(0) \text{ is finite}\}$.[11] Clause-sets are implicitly promoted to multi-clause-sets, if needed, by using their characteristic functions, and multi-clause-sets are implicitly cast down, if needed, to clause-sets by considering the underlying clause-set; "if needed" refers to operations which either require multi-clause-sets or clause-sets. If however we want to make explicit these operations, we use $\mathrm{cls} : \underline{\mathcal{CLS}} \to \mathcal{CLS}$ (with $\mathrm{cls}(F) := \mathcal{CL} \setminus F^{-1}(0)$) and $\underline{\mathrm{cls}} : \mathcal{CLS} \to \underline{\mathcal{CLS}}$ (with $\underline{\mathrm{cls}}(F)(C) := 1$ if $C \in F$, and $\underline{\mathrm{cls}}(F)(C) := 0$ otherwise). For $F \in \underline{\mathcal{CLS}}$ we extend the basic operations in the obvious way:

- $\mathrm{var}(F) := \mathrm{var}(\mathrm{cls}(F))$, $\mathrm{lit}(F) := \mathrm{lit}(\mathrm{cls}(F))$, $\bigcup F := \bigcup \mathrm{cls}(F)$.
- $n(F) := n(\mathrm{cls}(F)) \in \mathbb{N}_0$, $c(F) := \sum_{C \in \mathcal{CL}} F(C) \in \mathbb{N}_0$, $\delta(F) := c(F) - n(F) \in \mathbb{Z}$, $\ell(F) := \sum_{C \in F} F(C) \cdot |C| \in \mathbb{N}_0$.

The application of partial assignments $\varphi \in \mathcal{PASS}$ to a multi-clause-set $F \in \underline{\mathcal{CLS}}$ yields a *multi*-clause-set $\varphi * F \in \underline{\mathcal{CLS}}$, where the multiplicity of a clause $C \in \mathcal{CL}$ in $\varphi * F$ is the sum of all multiplicities of clauses $D \in F$ (i.e., $D \in \mathrm{cls}(F)$) which are shortened to C by φ:

$$(\varphi * F)(C) := \sum_{D \in F,\, D \cap \varphi^{-1}(1) = \emptyset,\, D \setminus \varphi^{-1}(0) = C} F(D).$$

Example 2.8 *If φ is a total assignment for F (assigns all variables of F, that is, $\mathrm{var}(\varphi) = \mathrm{var}(F)$), then $\varphi * F$ is $\{m * \bot\}$, denoting the multiplicity of a clause by a (formal) factor, with $m = \sum_{C \in F,\, C \cap \varphi^{-1}(1) = \emptyset} F(C) \in \mathbb{N}_0$ (so $m = 0 \Leftrightarrow \varphi * F = \top$).*

For us the clause-sets are the objects of interests, while multi-clause-sets are only auxiliary devices, created by the operation of "restriction" defined next (Definition 2.9). However we have to take care of the details, and thus together with introducing a class $\mathcal{C} \subseteq \mathcal{CLS}$ we also introduce the corresponding class $\underline{\mathcal{C}} \subseteq \underline{\mathcal{CLS}}$ of multi-clause-sets (using the generalised definition of $\mathcal{C}$), where we must discuss the relation. To start with, the classes $\underline{\mathcal{SAT}}$ and $\underline{\mathcal{USAT}}$ are **invariant under multiplicities**, that is, a multi-clause-set is in it iff the underlying clause-set is in the underlying class of clause-sets ($\mathcal{SAT}$ resp. $\mathcal{USAT}$). The other extreme we have with the class $\underline{\mathcal{HIT}}$ of multi-hitting-clause-sets, which **disallows multiplicities**, that is, all multiplicities must be 1 (since clauses can not clash with themselves, by definition of clauses), and thus up to the canonical identification the classes $\mathcal{HIT}$ and $\underline{\mathcal{HIT}}$ are identical.

An important operation with multi-clause-set is the "restriction" to a set of variables (see Subsection 3.5 in [57] for more information):

Definition 2.9 *For a set $V \subseteq \mathcal{VA}$ of variables and a multi-clause-set $F \in \underline{\mathcal{CLS}}$ by $\boldsymbol{F[V]} \in \underline{\mathcal{CLS}}$ the **restriction** of F to V is denoted, which is the multi-clause-set obtained by removing clauses from F which have no variables in common with V, and removing from the remaining clauses all literals where the underlying variable is not in V:*

$$F[V](C) := \sum_{D \in F,\, \mathrm{var}(D) \cap V \neq \emptyset,\, D \cap (V \mathbin{\dot\cup} \overline{V}) = C} F(D).$$

Here it is essential that $F[V]$ is a multi-clause-set, and if previously unequal clauses become equal, then accordingly the multiplicity is increased.

[11] In earlier papers we used "$\mathcal{MCLS}$" instead of "$\underline{\mathcal{CLS}}$".

Example 2.10 $\{\{a\}, \{a, b\}, \{b\}, \{\overline{a}, \overline{b}\}\}[\{a\}] = \{2 * \{a\}, \{\overline{a}\}\}$.

Simple properties of this operation are (for multi-clause-sets F and $V, V' \subseteq \mathcal{VA}$):

1. $F[\emptyset] = \top$, $F[V] = F \setminus \{\bot\}$ for $\mathrm{var}(F) \subseteq V$ (where $F \setminus F'$ for a clause-set F' means that all occurrences of clauses from F' are removed from F).
2. $(F[V])[V'] = F[V \cap V']$.

Clause-sets F, G are called **isomorphic**, if the variables of F can be renamed and potentially flipped so that F is turned into G. More precisely, an isomorphism α from F to G is a bijection $\alpha : \mathrm{lit}(F) \to \mathrm{lit}(G)$ which preservers complementation ($\alpha(\overline{x}) = \overline{\alpha(x)}$), and which maps the clauses of F precisely to the clauses of G; when considering multi-clause-sets, then the isomorphism must preserve the multiplicity of clauses (that is, $G(\alpha(C)) = F(C)$ for all $C \in \mathcal{CL}$).

2.5 Degrees

For the number of occurrences of a literal $x \in \mathcal{LIT}$ in a (multi-)clause-set $F \in \underline{\mathcal{CLS}}$ we write

$$\boldsymbol{\mathrm{ld}_F(x)} := \sum_{C \in F,\, x \in C} F(C) \in \mathbb{N}_0,$$

called the **literal-degree**, while the **variable-degree** of a variable v is defined as $\boldsymbol{\mathrm{vd}_F(v)} := \mathrm{ld}_F(v) + \mathrm{ld}_F(\overline{v}) \in \mathbb{N}_0$. A (multi-)clause-set F is called **variable-regular** if all variables $v \in \mathrm{var}(F)$ have the same degree, or, stronger, **literal-regular**, if all literals $x \in \mathrm{lit}(F)$ have the same degree. A **singular variable** in a (multi-)clause-set F is a variable occurring in one sign only once (i.e., $1 \in \{\mathrm{ld}_F(v), \mathrm{ld}_F(\overline{v})\}$). A (multi-)clause-set is called **non-singular** if it does not have singular variables. The central concept for this report is the degree of a variable with minimal occurrences:

Definition 2.11 *We define the* ***minimum variable degree*** $\boldsymbol{\mu\mathrm{vd}} : \mathcal{CLS} \to \mathbb{N} \cup \{+\infty\}$ *("min-var-degree" for short) as follows (which also works for multi-clause-sets F):*

- *For $F \in \mathcal{CLS}$ with $n(F) \neq 0$ we let $\mu\mathrm{vd}(F) := \min_{v \in \mathrm{var}(F)} \mathrm{vd}_F(v) \in \mathbb{N}$.*
- *While for $n(F) = 0$ we set $\mu\mathrm{vd}(F) := +\infty$.*

For a class $\mathcal{C} \subseteq \mathcal{CLS}$ of clause-sets let $\boldsymbol{\mu\mathrm{vd}(\mathcal{C})} \in \mathbb{N}_0 \cup \{+\infty\}$ be the supremum of $\mu\mathrm{vd}(F)$ for $F \in \mathcal{C}$ with $n(F) > 0$, where we set $\mu\mathrm{vd}(\mathcal{C}) := 0$ if there is no such F (while otherwise we have $\mu\mathrm{vd}(\mathcal{C}) \geq 1$).

By definition we have $\mu\mathrm{vd}(\mathcal{C}) \leq \mu\mathrm{vd}(\mathcal{C}')$ for $\mathcal{C} \subseteq \mathcal{C}' \subseteq \mathcal{CLS}$.

Example 2.12 *For $F := \{2 * \{a, b\}, \{\overline{a}, b\}, \{\overline{b}, c\}\} \in \underline{\mathcal{CLS}}$ we have:*

- $\mathrm{ld}_F(a) = 2$, $\mathrm{ld}_F(\overline{a}) = 1$, $\mathrm{ld}_F(b) = 3$, $\mathrm{ld}_F(\overline{b}) = 1$, $\mathrm{ld}_F(c) = 1$, $\mathrm{ld}_F(\overline{c}) = 0$.
- $\mathrm{vd}_F(a) = 3$, $\mathrm{vd}_F(b) = 4$, $\mathrm{vd}_F(c) = 1$.
- $\mu\mathrm{vd}(F) = 1$.

Examples for $\mu\mathrm{vd}(\mathcal{C})$ are:

- $\mu\mathrm{vd}(\emptyset) = 0$.
- $\mu\mathrm{vd}(\mathcal{CLS}) = +\infty$.

- $\mu\mathrm{vd}(\{\top, \{\bot\}, \{\{v\}, \{\overline{v}\}\}, F\}) = 2$.

The simplest but relevant class of clause-sets for us is given by the $A(V)$ (the unsatisfiable full clause-sets; these are the simplest unsatisfiable clause-sets):

Lemma 2.13 *For $n \in \mathbb{N}_0$ we have*

1. $n(A_n) = n$, $c(A_n) = 2^n$, $\delta(A_n) = 2^n - n$.
2. *A_n is full and unsatisfiable, and thus $A_n \in \mathcal{UHIT}_{\delta=2^n-n}$.*
3. *A_n is literal-regular (thus variable-regular), with $\mu\mathrm{vd}(A_n) = 2^n$.*

Further properties of unsatisfiable full clause-sets one finds in Example 2.20, Lemma 3.9, Corollary 3.11, Lemmas 6.9, 6.10, Corollaries 6.11, 6.12, and Examples 9.2, 9.10. Properties of satisfiable full clause-sets are found in Example 10.7.

2.6 Autarkies

Besides algorithmic considerations, which were always present since the introduction of the notion of an "autarky" in [78], also a kind of a "combinatorial SAT theory" has been developed around this notion of generalised satisfying assignments. A general overview is given in [43], with recent additions and generalisations in [57]. An **autarky** (see [43, Section 11.8]) for a clause-set $F \in \mathcal{CLS}$ is a partial assignment $\varphi \in \mathcal{PASS}$ which satisfies every clause $C \in F$ it touches, i.e., for all $C \in F$ with $\mathrm{var}(\varphi) \cap \mathrm{var}(C) \neq \emptyset$ holds $\varphi * \{C\} = \top$; equivalently, for all $C \in F$ holds $C \cap \varphi^{-1}(0) \neq \emptyset \Rightarrow C \cap \varphi^{-1}(1) \neq \emptyset$. The simplest examples for autarkies are as follows:

Example 2.14 *The empty partial assignment $\langle\rangle$ is an autarky for every $F \in \mathcal{CLS}$ (no clause is touched), and more generally all $\varphi \in \mathcal{PASS}$ with $\mathrm{var}(\varphi) \cap \mathrm{var}(F) = \emptyset$ are autarkies for F, the **trivial autarkies**. On the other end of the spectrum every satisfying assignment for F (i.e., $\varphi * F = \top$) is an autarky for F (every clause is satisfied). A literal $x \in \mathcal{LIT}$ is a pure literal for F iff $\langle x \to 1\rangle$ is an autarky for F.*

If φ is an autarky for F, then $\varphi * F \subseteq F$ holds, and thus $\varphi * F$ is satisfiability-equivalent to F. Autarkies mark redundancies, and the corresponding notion of clause-sets without such redundancies was introduced in [50], namely a clause-set F is **lean** if there is no non-trivial autarky for F, and the set of all lean clause-sets is denoted by $\mathcal{LEAN} \subset \mathcal{USAT} \dot\cup \{\top\}$. The class $\underline{\mathcal{LEAN}}$ of lean multi-clause-sets is invariant under multiplicities.

Example 2.15 *Some simple examples:*

1. $\top, \{\bot\}, \{\{v\}, \{\overline{v}\}\}, \{\{v\}, \{\overline{v}\}, \{w\}, \{\overline{w}\}\} \in \mathcal{LEAN}$.
2. *If $F, F' \in \mathcal{LEAN}$, then $F \cup F' \in \mathcal{LEAN}$.*
3. *If $F \in \mathcal{LEAN}$ and $F' \in \mathcal{CLS}$ with $\mathrm{var}(F') \subseteq \mathrm{var}(F)$, then $F \cup F' \in \mathcal{LEAN}$.*
4. $\{\{v\}, \{\overline{v}\}, \{w\}\} \notin \mathcal{LEAN}$.

A weakening is the notion of a **matching-lean** clause-set F (introduced in [51, Section 7]; see [43, Section 11.11] for an overview), which has no non-trivial **matching autarky**, which are special autarkies given by a matching condition: for every clause touched, a satisfied literal with unique underlying variable must be selectable; the class of all matching-lean clause-sets is denoted by $\mathcal{MLEAN} \supset \mathcal{LEAN}$.

Example 2.16 *A clause-set $F \in \mathcal{CLS}$ is matching-lean ($F \in \mathcal{MLEAN}$) iff for all $F' \subset F$ holds $\delta(F') < \delta(F)$ ([51, Theorem 7.5]). Thus for every matching-lean multi-clause-set $F \neq \top$ we have $\delta(F) \geq 1$ ([51], generalising [3]).*

$F := \{\{1,3\},\{2,-3\},\{3\},\{-3\}\}$ has the matching autarky $\langle 1 \to 1, 2 \to 1 \rangle$, while for $F' := F \cup \{\{1,2\}\}$ we have $F' \in \mathcal{MLEAN}$. Note $\delta(F) = 1 = \delta(\{\{3\},\{-3\}\})$, while $\delta(F') = 2$.

It is decidable in polynomial time whether $F \in \mathcal{MLEAN}$ holds (which follows for example by the characterisation of $\mathcal{MLEAN}$ via the surplus below). The class $\underline{\mathcal{MLEAN}} \supset \underline{\mathcal{LEAN}}$ of matching-lean multi-clause-sets is not invariant under multiplicities: For a multi-clause-set F holds $\operatorname{cls}(F) \in \mathcal{MLEAN} \Rightarrow F \in \underline{\mathcal{MLEAN}}$, but not the other way around:

Example 2.17 *$\{\{v\}\} \notin \mathcal{MLEAN}$, but $\{2 * \{v\}\} \in \underline{\mathcal{MLEAN}}$, and more generally $\{\{x_1,\dots,x_n\}\} \notin \mathcal{MLEAN}$ for $n \geq 1$, while $\{(n+1) * \{x_1,\dots,x_n\}\} \in \underline{\mathcal{MLEAN}}$. Indeed it is easy to see that for every $F \in \mathcal{CLS}$ there is $F' \in \underline{\mathcal{CLS}}$ with $\operatorname{cls}(F') = F$ and $F' \in \underline{\mathcal{MLEAN}}$.*

The process of applying autarkies as long as possible to a clause-set $F \in \mathcal{CLS}$ is confluent, yielding the **lean kernel** of F (the largest lean sub-clause-set of F, that is, $\bigcup\{F' \subseteq F : F' \in \mathcal{LEAN}\}$; see [50, Section 3]). Computation of the lean kernel is NP-hard, since the lean kernel of satisfiable clause-sets is $\top$. But the **matching-lean kernel** of F (the largest matching-lean sub-clause-set of F, that is, $\bigcup\{F' \subseteq F : F' \in \mathcal{MLEAN}\}$; see [51, Section 3]), now obtained by applying *matching* autarkies as long as possible, which again is a confluent process, is computable in polynomial time. Note that a clause-set F is lean resp. matching lean iff the lean resp. matching-lean kernel is F itself. Due to the polytime computability of the matching-lean kernel, which is a sub-clause-set obtained by removing clauses redundant in a strong sense, "w.l.o.g." for the purpose of SAT-decision one might consider the inputs as matching-lean.

Example 2.18 *For inputs $F \in \mathcal{MLEAN}$ by [93, Theorem 4] we have SAT-decision in time $O(2^{\delta(F)} \cdot n(F)^3)$ (see [57] for generalisations), and thus SAT-decision for inputs $F \in \mathcal{MLEAN}$ is fixed-parameter tractability (fpt) in the parameter $\delta(F)$.*

We note here (though we won't use it in this report), that for inputs $F \in \mathcal{MLEAN}_{\delta=k}$ the computation of the lean kernel can be done in polynomial time for fixed k ([57, Theorem 10.3]; this computational problem appears not to be fpt).

A multi-clause-set $F \neq \top$ is matching lean iff for the **surplus** we have $\sigma(F) \geq 1$ ([51, Lemma 7.7]), which is defined as follows (see Subsection 11.1 in [57] for more information; in [93] a clause-set has "q-expansion" iff $\sigma(F) \geq q$):

Definition 2.19 *For a multi-clause-set F let $\boldsymbol{\sigma(F)} \in \mathbb{Z}$ be defined as the minimum of $\delta(F[V])$ (recall Definition 2.9) over all $\emptyset \neq V \subseteq \operatorname{var}(F)$ if $n(F) > 0$, while $\sigma(F) := 0$ in case of $n(F) = 0$.*

Note that for $\emptyset \neq V \subseteq \operatorname{var}(F)$ we have

$$\delta(F[V]) = c(F[V]) - |V| = \sum_{C \in F,\, \operatorname{var}(C) \cap V \neq \emptyset} F(C) \; - |V|.$$

The surplus is computable in polynomial time. Some basic properties of the surplus are (for multi-clause-sets F):

1. $\sigma(F) \leq \delta(F[\operatorname{var}(F)]) = \delta(F \setminus \{\bot\}) \leq c(F)$.

2. For every $\emptyset \subset V \subseteq \mathrm{var}(F)$ holds $\sigma(F[V]) \geq \sigma(F)$.

3. For $F' \leq F$ with $\mathrm{var}(F') = \mathrm{var}(F)$ we have $\sigma(F') \leq \sigma(F)$.

4. $-n(F) \leq \sigma(F)$, and for $n(F) > 0$ holds $1 - n(F) \leq \sigma(F) \leq c(F) - 1$.

Example 2.20 *$\sigma(A_0) = 0$, while $\sigma(A_n) = 2^n - n = \delta(A_n)$ for $n \in \mathbb{N}$ (since every variable occurs in every clause). If we take $F \in \mathcal{CLS}$ and some $v \in \mathcal{VA} \setminus \mathrm{var}(F)$, then $\sigma(F \mathbin{\dot\cup} \{\{v\}\}) \leq 0$.*

3 Minimally unsatisfiable clause-sets

In this section we review minimally unsatisfiable clause-sets; see [43] for an overview, while [58, 65] contain recent developments. First the basic definitions and examples are given in Subsection 3.1. In Subsection 3.2 we consider in some detail the fundamental process of "saturation", which is about adding "missing literal occurrences" to minimally unsatisfiable clause-sets. Saturation repairs the problem that splitting of $F \in \mathcal{MU}$ into $\langle v \to 0\rangle * F$ and $\langle v \to 1\rangle * F$ may destroy minimal unsatisfiable, i.e., $\langle v \to 0\rangle * F \notin \mathcal{MU}$ or $\langle v \to 1\rangle * F \notin \mathcal{MU}$ might hold, due to some clauses missed to be deleted, and this process is considered in Subsection 3.3.

3.1 $\mathcal{MU}$ and subclasses

An unsatisfiable clause-set F is called **minimally unsatisfiable**, if for every clause $C \in F$ the clause-set $F \setminus \{C\}$ is satisfiable, and the set of minimally unsatisfiable clause-sets is denoted by $\boldsymbol{\mathcal{MU}} \subset \mathcal{USAT}$. A clause-set $F \in \mathcal{MU}$ is called **saturated**, if replacing any $C \in F$ by any super-clause $C' \supset C$ yields a satisfiable clause-set, and the set of saturated minimally unsatisfiable clause-sets is denoted by $\boldsymbol{\mathcal{SMU}} \subset \mathcal{MU}$.

Example 3.1 *The simplest element of $\mathcal{USAT} \setminus \mathcal{MU}$ is $\{\bot, \{1\}\}$, while the simplest element of $\mathcal{MU} \setminus \mathcal{SMU}$ is $\{\{1, 2\}, \{-1\}, \{-2\}\}$ (see Example 3.5 for a "saturation").*

Unsatisfiable hitting clause-sets fulfil $\mathcal{UHIT} \subset \mathcal{SMU}$ (see [65, Lemma 2] for the proof). The subsets of non-singular elements (i.e., there is no literal occurring only once) are denoted by $\boldsymbol{\mathcal{MU}'} \subset \mathcal{MU}$, $\boldsymbol{\mathcal{SMU}'} \subset \mathcal{SMU}$, and $\boldsymbol{\mathcal{UHIT}'} \subset \mathcal{UHIT}$.

Example 3.2 *By [17] holds $\mathcal{MU}'_{\delta=1} = \mathcal{SMU}'_{\delta=1} = \mathcal{UHIT}'_{\delta=1} = \{\{\bot\}\}$, while for the characterisation of $\mathcal{MU}_{\delta=1} \supset \mathcal{SMU}_{\delta=1} = \mathcal{UHIT}_{\delta=1}$ see also [3, 49]. As shown in [17], for $F \in \mathcal{MU}_{\delta=1}$ with $n(F) > 0$ holds $\mu\mathrm{vd}(F) = 2$.*

We consider the "reasons" for unsatisfiability as given by the elements of $\mathcal{MU}_{\delta=1}$ as "noise", only "masking" the pure contradiction of the only element of $\mathcal{MU}'_{\delta=1} = \{\{\bot\}\}$ (in Section 5 the elimination of singular variables will be discussed). "Real reasoning" starts with deficiency 2:

Example 3.3 *By [42], the elements of $\mathcal{MU}'_{\delta=2}$ are up to isomorphism precisely the $\mathcal{F}_n$ for $n \geq 2$, where*

$$\mathcal{F}_n := \{\{1, \ldots, n\}, \{-1, \ldots, -n\}, \{-1, 2\}, \ldots, \{-(n-1), n\}, \{-n, 1\}\}.$$

All $\mathcal{F}_n$ are literal-regular, with $\mu\mathrm{vd}(\mathcal{F}_n) = 4$. It is easy to see that all $\mathcal{F}_n$ are saturated, and thus $\mathcal{MU}'_{\delta=2} = \mathcal{SMU}'_{\delta=2}$. The only hitting clause-sets amongst the

$\mathcal{F}_n$ are for $n = 2, 3$, and thus up to isomorphism the elements of $\mathcal{UHIT}'_{\delta=2}$ are $\mathcal{F}_2, \mathcal{F}_3$, with $\mathcal{F}_2 = A_2$ and

$$\mathcal{F}_3 = \{\{1,2,3\}, \{-1,-2,-3\}, \{-1,2\}, \{-2,3\}, \{-3,1\}\}.$$

We have $\sigma(\mathcal{F}_n) = 2 = \delta(\mathcal{F}_n)$, since any $m \leq n$ variables occur at least in m different binary clauses plus in the two "long clauses". Further properties of the $\mathcal{F}_n$ we have in Examples 3.13, 6.2, 8.2, 9.2. See Section 7 in [65] for more information.

As shown in [65, Theorem 74], for every $F \in \mathcal{MU}_{\delta=2}$ there is a unique $n \geq 2$ such that $\mathcal{F}_n$ "embeds" into F, and this n is called the "non-singularity type" of F. So for $\mathcal{MU}_{\delta=2}$ we have identified the (in a sense) unique reason of unsatisfiability, the (possibly hidden) presence of a cycle $v_1 \to \ldots \to v_n \to v_1$ together with the assertions, that one v_i must be true and one must be false (only the n is unique in general, not the v_i). We will come back to the theme of classifying $\mathcal{MU}_{\delta=k}$ in the Conclusion, Subsection 15.5.

By definition, $\underline{\mathcal{MU}}$ disallows multiplicities (since a duplicated clause is the trivial logical redundancy), and this also holds for the subclasses $\underline{\mathcal{SMU}}$ and $\underline{\mathcal{UHIT}}$ (as well as for all other subclasses of $\mathcal{MU}$ considered here). A fundamental fact is $\delta(F) \geq 1$ for all $F \in \mathcal{MU}$ (note that every minimally unsatisfiable clause-set is lean), which motivates the investigation of the layers $\mathcal{MU}_{\delta=1}, \mathcal{MU}_{\delta=2}, \ldots$. Special elements of $\mathcal{UHIT}$ are the $A(V)$ for finite sets V of variables (recall Lemma 2.13), which are the minimally unsatisfiable clause-sets with maximal deficiency for a given number of variables, as we will see in Corollary 6.11.

3.2 Saturation

We recall the fact ([24, 62]) that every minimally unsatisfiable clause-set $F \in \mathcal{MU}$ can be **saturated**, i.e., by adding literal occurrences to F we obtain $F' \in \mathcal{SMU}$ with $\mathrm{var}(F') = \mathrm{var}(F)$ such that there is a bijection $\alpha : F \to F'$ with $C \subseteq \alpha(C)$ for all $C \in F$. Since we need to consider saturation in many situations, we introduce some special notations for it from [65] (Subsection 2.2). First we introduce the notation $\mathrm{S}(F, C, x)$ for adding a literal x to a clause C in a clause-set F:

Definition 3.4 ([65]) *The operation (adding literal x to clause C in F)*

$$\mathbf{S}(\boldsymbol{F}, \boldsymbol{C}, \boldsymbol{x}) := (F \setminus \{C\}) \cup (C \mathbin{\dot\cup} \{x\}) \in \mathcal{CLS}$$

is defined if $F \in \mathcal{CLS}$, $C \in F$, and x is a literal with $\mathrm{var}(x) \in \mathrm{var}(F) \setminus \mathrm{var}(C)$.

Some technical remarks:

1. $\mathrm{var}(\mathrm{S}(F, C, x)) = \mathrm{var}(F)$.
2. If $C \mathbin{\dot\cup} \{x\} \notin F$, then $c(\mathrm{S}(F, C, x)) = c(F)$, and thus also $\delta(\mathrm{S}(F, C, x)) = \delta(F)$.
3. For $F \in \mathcal{MU}$ we have:
   (a) $\mathrm{S}(F, C, x) \in \mathcal{MU}$ iff $\mathrm{S}(F, C, x)$ is unsatisfiable (since all what happened is that a clause has been weakened, i.e., extended).
   (b) If $\mathrm{S}(F, C, x) \in \mathcal{MU}$, then $c(\mathrm{S}(F, C, x)) = c(F)$ (no subsumption here).
   (c) F is saturated iff there are no C, x such that $\mathrm{S}(F, C, x) \in \mathcal{USAT}$.

Example 3.5 *For $F := \{\{a, b\}, \{\overline{a}\}, \{\overline{b}\}\} \in \mathcal{MU} \setminus \mathcal{SMU}$ we have $\mathrm{S}(F, \{\overline{a}\}, b) = \{\{a, b\}, \{\overline{a}, b\}, \{\overline{b}\}\} \in \mathcal{SMU}$.*

A "saturation" of a minimally unsatisfiable clause-set is obtained by adding literals to clauses as long as possible while maintaining unsatisfiability (which is the same as maintaining minimal unsatisfiability):

Definition 3.6 ([65]) *A **saturation** $F' \in \mathcal{SMU}$ of $F \in \mathcal{MU}$ is obtained by a **saturation sequence** $F = F_0, \dots, F_m = F'$, $m \in \mathbb{N}_0$, such that*

(i) for $0 \le i < m$ there are C_i, x_i with $F_{i+1} = \mathrm{S}(F_i, C_i, x_i)$,

(ii) for all $1 \le i \le m$ we have $F_i \in \mathcal{USAT}$,

(iii) the sequence cannot be extended (without violating conditions (i) or (ii)).

*Note that $n(F') = n(F)$ and $c(F') = c(F)$ holds (and thus $\delta(F') = \delta(F)$). If we drop requirement (iii), then we speak of a **partial saturation sequence**, while $F' \in \mathcal{MU}$ is a **partial saturation** of $F \in \mathcal{MU}$.*

Some technical remarks:

1. $F' \in \mathcal{MU}$ is a partial saturation of $F \in \mathcal{MU}$ iff
   - $\mathrm{var}(F') = \mathrm{var}(F)$
   - there is a bijection $\alpha : F \to F'$ such that for all $C \in F$ we have $C \subseteq \alpha(C)$.
2. For a partial saturation sequence $F_0, \dots, F_m$ we have $\ell(F_m) = \ell(F_0) + m$.
3. F' is a saturation of $F \in \mathcal{MU}$ iff F' is a partial saturation of F with $F' \in \mathcal{SMU}$.

Example 3.7 *A saturation sequence for $F := \{\{a, b, c\}, \{\overline{a}\}, \{\overline{b}\}, \{\overline{c}\}\}$ with $m = 3$ is obtained by adding literals b, c to clause $\{\overline{a}\}$, and adding literal c to clause $\{\overline{b}\}$.*

We can perform a partial saturation $F \rightsquigarrow \mathrm{S}(F, C, x)$ iff F without C implies (logically) $C \mathbin{\dot\cup} \{\overline{x}\}$ (note that $C \mathbin{\dot\cup} \{x\}, C \mathbin{\dot\cup} \{\overline{x}\}$ implies C):

Lemma 3.8 *Consider $F \in \mathcal{MU}$, $C \in F$, and a literal x with $\mathrm{var}(x) \in \mathrm{var}(F) \setminus \mathrm{var}(C)$. Then $\mathrm{S}(F, C, x)$ is a partial saturation of F if and only if $F \setminus \{C\} \models C \mathbin{\dot\cup} \{\overline{x}\}$.*

Proof: First assume that $\mathrm{S}(F, C, x)$ is a partial saturation of F, but $F \setminus \{C\} \not\models C \mathbin{\dot\cup} \{\overline{x}\}$. So there is a partial assignment φ with $\varphi * (F \setminus \{C\}) = \top$ but $\varphi * \{C \mathbin{\dot\cup} \{\overline{x}\}\} = \{\bot\}$ (whence $\varphi(x) = 1$). But then we have $\varphi * \mathrm{S}(F, C, x) = \top$. Reversely assume $F \setminus \{C\} \models C \mathbin{\dot\cup} \{\overline{x}\}$, but that $\mathrm{S}(F, C, x)$ is not a partial saturation of F. So $\mathrm{S}(F, C, x)$ has a satisfying assignment φ; due to $F \in \mathcal{USAT}$ we have $\varphi(x) = 1$ and $\varphi * \{C\} = \{\bot\}$. But this yields $\varphi * (F \setminus \{C\}) = \top$ and $\varphi * \{C \mathbin{\dot\cup} \{\overline{x}\}\} = \{\bot\}$. □

See Lemma 6.5, Part 6, for another characterisation of partial saturations. The dual notion of "saturated" is "marginal": $F \in \mathcal{MU}$ is **marginal** iff replacing any clause by a strict subclause yields a clause-set not in $\mathcal{MU}$. The decision "F marginal minimally unsatisfiable ?" for inputs $F \in \mathcal{CLS}$ is D^P-complete ([45, Theorem 2]). By [44, Theorem 8] however this decision is easy for inputs $F \in \mathcal{SMU}$, namely there is the following characterisation of minimally unsatisfiable clause-sets which are marginal and saturated at the same time:

Lemma 3.9 ([44]) *$F \in \mathcal{MU}$ is marginal and saturated iff $F = A(\mathrm{var}(F))$.*

Thus $F \in \mathcal{SMU}$ is marginal iff $F = A(\mathrm{var}(F))$; so if $F \in \mathcal{SMU}$ is not full, then there is a literal occurrence which can be removed without destroying minimal unsatisfiability, that is, there exists a clause $C \in F$ and $x \in C$ such that $F' := (F \setminus \{C\}) \mathbin{\dot\cup} \{C \setminus \{x\}\} \in \mathcal{MU}$ (note that $F' \in \mathcal{USAT}$ in any case, but minimality in general is not maintained). And for inputs $F \in \mathcal{SMU}$ the existence of such C, x is decidable in linear time (namely they exist iff F is not full). But finding such C, x should be hard in general, that is, the decision problem, whether a concrete literal can be removed, even for inputs $F \in \mathcal{SMU}$ should be NP-complete:

Question 3.10 *Is the promise problem for input $F \in \mathcal{SMU}$, $C \in F$, $x \in C$, whether "$(F \setminus \{C\}) \mathbin{\dot\cup} \{C \setminus \{x\}\} \in \mathcal{MU}$?", NP-complete? (That is, is there a polytime computation $G \in \mathcal{CLS} \rightsquigarrow (F, C, x) \in \mathcal{SMU} \times \mathcal{CL} \times \mathcal{LIT}$, with $x \in C \in F$, such that $G \in \mathcal{SAT} \Leftrightarrow (F \setminus \{C\}) \mathbin{\dot\cup} \{C \setminus \{x\}\} \in \mathcal{MU}$?)*

And is the promise problem for input $F \in \mathcal{MU}$, whether F is marginal, NP-complete? (That is, is there a polytime computation $G \in \mathcal{CLS} \rightsquigarrow F \in \mathcal{MU}$, such that $G \in \mathcal{SAT}$ iff F is marginal?)

Back to saturation: precisely all saturated clause-sets except the $A(V)$ are obtained as non-trivial saturations of some minimally unsatisfiable clause-set:

Corollary 3.11 *Consider $F \in \mathcal{SMU}$.*

1. *F is trivially the saturation of itself.*
2. *If $F = A(\mathrm{var}(F))$, then this is also the only possibility for F being a saturation, that is, if F is the saturation of some $F' \in \mathcal{MU}$, then we have $F' = F$.*
3. *Otherwise F is a saturation of some clause-set other than it itself, that is, if $F \neq A(\mathrm{var}(F))$, then there is some $F' \in \mathcal{MU}$ with $F' \neq F$ such that F is a saturation of F'.*

Proof: Part 1 is trivial. For Part 2 assume that $F = A(\mathrm{var}(F))$, and we have $F = \mathrm{S}(F', C, x)$ for some $F' \in \mathcal{MU}$: But since F is marginal, F' is not minimally unsatisfiable. Finally for Part 3 note, that if $F \neq A(\mathrm{var}(F))$, then by Lemma 3.9 F is not marginal, and thus there is $C \in F$ and $x \in C$ such that for $C' := C \setminus \{x\}$ and $F' := (F \setminus \{C\}) \mathbin{\dot\cup} \{C'\}$ we have $F' \in \mathcal{MU}$. Now $F = \mathrm{S}(F', C', x)$. $\square$

As discussed above, we expect the decision whether for inputs $F \in \mathcal{SMU}$, $C \in F$ and $x \in C$ we have $(F \setminus \{C\}) \mathbin{\dot\cup} \{C'\} \in \mathcal{MU}$ to be NP-complete. But this decision is easy for $F \in \mathcal{UHIT}$, namely iff no "subsumption resolution" with another clause containing $\overline{x}$ can be performed, that is, there is no $D \in F$ with $\overline{x} \in D$ and $C \setminus \{x\} \subseteq D$:

Lemma 3.12 *Consider $F \in \mathcal{UHIT}$, $C \in F$ and $x \in C$. Let $C' := C \setminus \{x\}$, and let $F' := (F \setminus \{C\}) \mathbin{\dot\cup} \{C'\}$. Then we have $F' \in \mathcal{MU}$ iff there is no $D \in F \setminus \{C\}$ with $C' \subset D$.*

Proof: If there is $D \in F \setminus \{C\}$ with $C' \subset D$, then $F' \notin \mathcal{MU}$. So assume there is no such D. Assume $F' \notin \mathcal{MU}$. Thus there is $E \in F'$ with $F' \setminus \{E\}$ unsatisfiable. We must have $E \neq C'$, since otherwise $F \setminus \{C\}$ would be unsatisfiable. Since F is hitting, E clashes with every clause of $F' \setminus \{C'\}$. It follows that $C' \subset E$ must hold (since the falsifying assignments for E are disjoint with those for any clause in $F' \setminus \{C'\}$), contradicting the minimal unsatisfiability of F. $\square$

Some examples on removable literal occurrences illustrate Lemma 3.12:

Example 3.13 *For* $F := \{\{a,b\},\{\overline{a},b\},\{\overline{b}\}\} \in \mathcal{UHIT}_{\delta=1}$ *we can exactly remove one of the two literal-occurrences of* b *and still obtain a clause-set in* $\mathcal{MU}$ *(of course not in* $\mathcal{UHIT}$ *anymore; the resulting clause-sets are in fact marginally minimally unsatisfiable). For* $\mathcal{F}_2 = A_2 = \{\{1,2\},\{-1,-2\},\{-1,2\},\{-2,1\}\} \in \mathcal{UHIT}'_{\delta=2}$ *we can not remove any literal occurrence without leaving* $\mathcal{MU}$ *(i.e.,* $\mathcal{F}_2$ *is marginal).*

3.3 Splitting

An important (equivalent) characterisation of saturation for $F \in \mathcal{CLS}$, as shown in [58], is that splitting on any variable v will yield minimally unsatisfiable clause-sets $\langle v \to 0\rangle * F$, $\langle v \to 1\rangle * F$. This enables induction on the number of variables, which is a central method for this report; see Lemma 8.1 for the basic example. We also have that if for some variable both splitting results are minimally unsatisfiable resp. saturated, then this can be lifted to the original clause-set, provided that no contraction takes place. These basic facts are collected in the following lemma.

Lemma 3.14 *For all clause-sets* $F \in \mathcal{CLS}$ *we have:*

1. $F \in \mathcal{SMU}$ *iff for all* $v \in \operatorname{var}(F)$ *and all* $\varepsilon \in \{0,1\}$ *we have* $\langle v \to \varepsilon\rangle * F \in \mathcal{MU}$.
2. *If for some variable* v *holds* $\langle v \to 0\rangle * F \in \mathcal{MU}$ *and* $\langle v \to 1\rangle * F \in \mathcal{MU}$*, and if for all* $C \in F$ *with* $v \in \operatorname{var}(C)$ *we have* $C \setminus \{v,\overline{v}\} \notin F$*, then* $F \in \mathcal{MU}$.
3. *If for some variable* v *holds* $\langle v \to 0\rangle * F \in \mathcal{SMU}$ *and* $\langle v \to 1\rangle * F \in \mathcal{SMU}$*, and if for all* $C \in F$ *with* $v \in \operatorname{var}(C)$ *we have* $C \setminus \{v,\overline{v}\} \notin F$*, then* $F \in \mathcal{SMU}$.

Proof: Part 1 is Corollary 5.3 in [58]. For Part 2 assume $F \notin \mathcal{MU}$; thus there is $C \in F$ with $F' := F \setminus \{C\} \in \mathcal{USAT}$. We consider three cases:

1. $v \notin \operatorname{var}(C)$: Due to the assumption on subsumption-freeness we have $C \mathbin{\dot\cup} \{v\} \notin F'$. Now $C \in \langle v \to 0\rangle * F$, while $(\langle v \to 0\rangle * F) \setminus \{C\} = \langle v \to 0\rangle * F' \in \mathcal{USAT}$, contradicting $\langle v \to 0\rangle * F \in \mathcal{MU}$.
2. $v \in C$: By assumption holds $C' := C \setminus \{v\} \notin F'$. Now $C' \in \langle v \to 0\rangle * F$, while $(\langle v \to 0\rangle * F) \setminus \{C'\} = \langle v \to 0\rangle * F' \in \mathcal{USAT}$, contradicting $\langle v \to 0\rangle * F \in \mathcal{MU}$.
3. $\overline{v} \in C$: By assumption holds $C' := C \setminus \{\overline{v}\} \notin F'$. Now $C' \in \langle v \to 1\rangle * F$, while $(\langle v \to 1\rangle * F) \setminus \{C'\} = \langle v \to 1\rangle * F' \in \mathcal{USAT}$, contradicting $\langle v \to 1\rangle * F \in \mathcal{MU}$.

Finally consider Part 3. By Part 2 we already know that $F \in \mathcal{MU}$ holds. Assume that $F \notin \mathcal{SMU}$; thus there is $C \in F$ and a literal x with $F' := \mathrm{S}(F,C,x) \in \mathcal{USAT}$. So by Lemma 3.8 we have $F \setminus \{C\} \models C' := C \mathbin{\dot\cup} \{\overline{x}\}$. There exists at least one $\varepsilon \in \{0,1\}$ with $\langle v \to \varepsilon\rangle * \{C'\} \neq \top$, and then $\langle v \to \varepsilon\rangle * (F \setminus \{C\}) \models \langle v \to \varepsilon\rangle * C'$. If $\operatorname{var}(x) = v$, then this contradicts minimal unsatisfiability of $\langle v \to \varepsilon\rangle * F$. And if $\operatorname{var}(x) \neq v$, then $\langle v \to \varepsilon\rangle * F \setminus \langle v \to \varepsilon\rangle * \{C\} \models (\langle v \to \varepsilon\rangle * C) \mathbin{\dot\cup} \{\overline{x}\}$, contradicting saturatedness of $\langle v \to \varepsilon\rangle * F$ by Lemma 3.8. □

The additional assumption $C \setminus \{v,\overline{v}\} \notin F$ for Parts 2, 3 is equivalent to saying that when applying $\langle v \to 0\rangle, \langle v \to 1\rangle$, then no contraction takes place. An alternative way of stating this would be to use multi-clause-sets, since then no contractions would be performed, and the doubled clauses would destroy minimal unsatisfiability. In [62] (Lemma 1) (and in the underlying report [63], Lemma 2.1) that additional assumption for Parts 2, 3 is missing by mistake:

Example 3.15 *An example for* $\langle v \to 0\rangle * F \in \mathcal{UHIT}$ *and* $\langle v \to 1\rangle * F \in \mathcal{UHIT}$*, but* $F \notin \mathcal{MU}$ *is trivially given by* $\{\bot,\{v\},\{\overline{v}\}\}$ *(note that* F *is a set — for a multi-clause-set* F *the contraction would not occur).*

4 Variable-minimal unsatisfiability

In [11] the generalisation of minimal unsatisfiability to "variable-minimal unsatisfiability" has been introduced, and the class of all such clause-sets is denoted by $\mathcal{VMU}$, the set of clause-sets $F \in \mathcal{USAT}$ such that for every $F' \subseteq F$ with $F' \in \mathcal{USAT}$ holds $\mathrm{var}(F') = \mathrm{var}(F)$. The corresponding class $\underline{\mathcal{VMU}}$ of multi-clause-sets is invariant under multiplicities. Thus, as with $\mathcal{LEAN}$ (and different from $\mathcal{MU}$), regarding variable-minimal unsatisfiability w.l.o.g. multi-clause-sets can be cast down to clause-sets. The basic (trivial) characterisation of $\mathcal{VMU}$ is:

Lemma 4.1 *For $F \in \mathcal{CLS}$ holds $F \in \mathcal{VMU}$ if and only if $F \in \mathcal{USAT}$ and for all $v \in \mathrm{var}(F)$ holds $\{C \in F : v \notin \mathrm{var}(C)\} \in \mathcal{SAT}$.*

By definition we have $\mathcal{MU} \subset \mathcal{VMU}$, moreover, as shown in Lemma 6 of [11], for every deficiency $k \geq 2$ we have $\mathcal{MU}_{\delta=k} \subset \mathcal{VMU}_{\delta=k}$ (for example, for every $F \in \mathcal{MU}_{\delta=k}$, $k \in \mathbb{N}$, and every non-full clause $C \in F$, i.e., $\mathrm{var}(C) \subset \mathrm{var}(F)$, we can add to F a full clause subsumed by C, obtaining $F' \in \mathcal{VMU}_{\delta=k+1} \setminus \mathcal{MU}_{\delta=k+1}$).

In [11] there is the false statement "$\mathcal{VMU} \not\subseteq \mathcal{LEAN}$", based on following erroneous example:

Example 4.2 *[11, Page 266] gives the example $F_4 := \{\{a\}, \{\overline{b}\}, \{\overline{a}, b\}, \{a, \overline{b}\}\}$ with the assertion "$F_4 \in \mathcal{VMU} \setminus \mathcal{LEAN}$". Obviously we have $F_4 \in \mathcal{VMU}$, but we also have $F_4 \in \mathcal{LEAN}$. Using the characterisation from [50] (which is the only characterisation used in [11]), that $F \in \mathcal{LEAN}$ holds iff every clause of F can be used in a tree-resolution refutation of F, we see this as follows: the sole subset of F_4 in $\mathcal{MU}$ is $\{\{a\}, \{\overline{b}\}, \{\overline{a}, b\}\}$, while the clause $\{a, \overline{b}\}$ can(!) also be used in a tree-resolution refutation — it is obviously superfluous, but nevertheless there is a tree-resolution refutation using it, namely via $(\{a\} \diamond \{\overline{a}, b\}) \diamond \{a, \overline{b}\} = \{a\}$.*

Based on the characterisation of lean clause-sets via autarkies, it is easy to show that $\mathcal{VMU}$ consists of special lean clause-sets (thus Figure 1 in [11] needs to be corrected, showing instead that $\mathcal{LEAN}$ is indeed a superclass of $\mathcal{VMU}$):

Lemma 4.3 *$\mathcal{VMU} \subset \mathcal{LEAN} \setminus \{\top\}$.*

Proof: While in [11] the characterisation of $\mathcal{LEAN}$ via variables usable in resolution refutation was (only) used, here we need to use the equivalent characterisation via autarkies, shown in Theorem 3.16 in [50], and used as our definition in Subsection 2.6, namely that for $F \in \mathcal{CLS}$ holds $F \in \mathcal{LEAN}$ iff there is no autarky φ for F with $\mathrm{var}(\varphi) \cap \mathrm{var}(F) \neq \emptyset$: if we had such an autarky for $F \in \mathcal{VMU}$, then $\varphi * F \in \mathcal{USAT}$ with $\varphi * F \subset F$ and $\mathrm{var}(\varphi * F) \subseteq \mathrm{var}(F) \setminus \mathrm{var}(F)$, contradicting $F \in \mathcal{VMU}$. That we indeed have a strict subset can for example be seen by Lemma 3.2 in [50], which shows that if we extended a minimally unsatisfiable clause-sets via Extended Resolution, then we always stay in $\mathcal{LEAN}$; another example is clause-set F_3 on Page 266 in [11]. □

Thus it follows $\mathcal{VMU}_{\delta=1} = \mathcal{MU}_{\delta=1}$ (shown in Lemma 6 in [11]), since by Corollary 5.7 in [50] holds $\mathcal{LEAN}_{\delta=1} \cap \mathcal{USAT} = \mathcal{MU}_{\delta=1}$.

For $F \in \mathcal{VMU}$ obviously there is some $F' \subseteq F$ with $\mathrm{var}(F') = \mathrm{var}(F)$ and $F' \in \mathcal{MU}$; Lemma 5 in [11] asserts the converse, but this is false, as the following simple example shows:

Example 4.4 *Consider $F := \{\bot, \{v\}, \{\overline{v}\}\}$ and $F' := \{\{v\}, \{\overline{v}\}\}$; we have $F' \in \mathcal{MU}$ and $\mathrm{var}(F') = \mathrm{var}(F)$, but $F \notin \mathcal{VMU}$, since $\{\bot\} \in \mathcal{USAT}$. If we don't want to use the empty clause, then we can consider any $F' \in \mathcal{MU}$ with $v \in \mathrm{var}(F')$ and*

$\{\{v\}, \{\overline{v}\}\} \cap F' = \emptyset$*, and let* $F := F' \mathbin{\dot\cup} \{\{v\}, \{\overline{v}\}\}$ *— again we have* $F' \in \mathcal{MU}$ *and* $\mathrm{var}(F') = \mathrm{var}(F)$*, but* $F \notin \mathcal{VMU}$*.*

The corrected version of Lemma 5 from [11] is as follows:

Lemma 4.5 *For* $F \in \mathcal{CLS}$ *let* $\mathbb{U}_F$ *be the set of* $F' \subseteq F$ *with* $\mathrm{var}(F') = \mathrm{var}(F)$*,* $\delta(F') \geq 1$*, and* $F' \in \mathcal{USAT}$*. Then* $F \in \mathcal{VMU}$ *if and only if* $F \in \mathbb{U}_F$ *and all minimal elements of* $\mathbb{U}_F$ *w.r.t. the subset-relation are minimally unsatisfiable.*

Proof: The condition is necessary, since if $F \in \mathcal{VMU}$, then on the one hand we have $F \in \mathcal{LEAN} \setminus \{\top\}$, and thus $\delta(F) \geq 1$ by [50] (or use Lemma 3 in [11]); and on the other hand if there would be a minimal element $F' \in \mathbb{U}_F$ which wouldn't be minimally unsatisfiable, then there would be some $F'' \subset F'$ with $F'' \in \mathcal{MU}$, whence by definition of $\mathbb{U}_F$ we get $\mathrm{var}(F'') \subset \mathrm{var}(F)$ contradicting $F \in \mathcal{VMU}$.

For the other direction assume, that we have $\mathbb{U}_F$ as specified, and we have to show $F \in \mathcal{VMU}$. Since $F \in \mathbb{U}_F$, we have $F \in \mathcal{USAT}$. Consider now some $F' \subseteq F$ with $F' \in \mathcal{USAT}$, and assume $\mathrm{var}(F') \subset \mathrm{var}(F)$. Consider some minimal $F'' \in \mathbb{U}_F$ (regarding inclusion) with $F' \subset F'' \subseteq F$. Furthermore consider a minimal element $G \in \mathbb{U}_F$ with $G \subseteq F''$; by assumption $G \in \mathcal{MU}$, and since $F' \subset F''$, we have $G \subset F''$. If for $C \in F''$ we had $\mathrm{var}(F'' \setminus \{C\}) \subset \mathrm{var}(F'')$, then there would be $x \in C$ such that x or $\overline{x}$ is pure in F'', thus also pure in G, whence $C \notin G$ (since $G \in \mathcal{MU}$), contradicting $\mathrm{var}(G) = \mathrm{var}(F)$. Now choose some $C \in F'' \setminus F'$ (we have $\mathrm{var}(F'' \setminus \{C\}) = \mathrm{var}(F'')$); by minimality of F'' we now have $\delta(F'' \setminus \{C\}) \leq 0$ (otherwise all conditions for $\mathbb{U}_F$ are fulfilled for $F'' \setminus \{C\}$), whence $\delta(F'') = 1$. But then due to $\mathrm{var}(F'') = \mathrm{var}(G)$ and $G \subset F''$ it follows $\delta(G) \leq 0$, contradicting $G \in \mathcal{MU}$. □

The following examples show applications of Lemma 4.5:

Example 4.6 *Consider the two (non-)examples from Example 4.4:*

1. *For* $F = \{\bot, \{v\}, \{\overline{v}\}\}$ *we have the minimal element* $\{\bot, \{v\}\}$ *of* $\mathbb{U}_F$ *which is not minimally unsatisfiable.*
2. *For* $F = F' \mathbin{\dot\cup} \{\{v\}, \{\overline{v}\}\}$ *(note that from the assumptions follows* $\mathrm{var}(F') \supset \{v\}$*) consider a minimal* $F'' \subseteq F'$ *with* $\mathrm{var}(G) = \mathrm{var}(F)$ *and* $\delta(G) \geq 1$ *for* $G := F'' \mathbin{\dot\cup} \{\{v\}, \{\overline{v}\}\}$ *(note* $\top \subset F'' \subset F$*): now* G *is a minimal element of* $\mathbb{U}_F$ *which is not minimally unsatisfiable.*

Based on Lemma 5 in [11], also the proof of Theorem 3 in [11] is false (the procedure goes astray on the clause-sets of Example 4.4). Fortunately we can give a simple proof of the assertion, which even shows fixed-parameter tractability (fpt) of the decision problem "$F \in \mathcal{VMU}_{\delta=k}$?" in the parameter k:

Theorem 4.7 *Membership "*$F \in \mathcal{VMU}_{\delta=k}$ *?" for input* $F \in \mathcal{CLS}$ *is fpt in the parameter* $k \in \mathbb{Z}$*.*

Proof: If $F \notin \mathcal{MLEAN}$, then $F \notin \mathcal{VMU}$ (by Lemma 4.3). So we can assume now $F \in \mathcal{MLEAN}$, and thus we have $\delta(F') < \delta(F)$ for all $F' \subset F$. Now the decisions of Lemma 4.1, as discussed in Example 2.18, are fpt in k. □

5 Eliminating and creating singularity

In this section we continue the study of the handling of singular variables in minimal unsatisfiable clause-sets, as initiated in [64, 65]. In Section 5.1 we study the reduction process, eliminating singular variables. A main insight is Lemma 5.4, showing that the elimination is harmless concerning the minimum variable degree. In Subsection 5.2 we introduce the inverse elimination ("extension"); the main point here is the precise statement of the various conditions. Finally in Subsection 5.3 we consider a special case of singularity, namely unit-clauses.

5.1 Singular DP-reduction

In [65] (Section 3) the process of "singular DP-reduction" has been studied for minimally unsatisfiable clause-sets. By it we can reduce the case of arbitrary $F \in \mathcal{MU}$ to (non-singular) $F' \in \mathcal{MU}'$ (that is, for every $v \in \operatorname{var}(F')$ we have $\operatorname{ld}_{F'}(v), \operatorname{ld}_{F'}(\overline{v}) \geq 2$). The definition is as follows (see Definition 8 in [65]):

Definition 5.1 ([65]) *The relation* $\boldsymbol{F} \xrightarrow{sDP} \boldsymbol{F'}$ *(singular DP-reduction) holds for clause-sets* $F, F' \in \mathcal{CLS}$*, if there is a singular variable* v *in* F*, such that* F' *is obtained from* F *by DP-reduction on* v*, that is,* $F' = \operatorname{DP}_v(F)$*. The reflexive-transitive closure of this relation is denoted by* $\boldsymbol{F} \xrightarrow{sDP}_{*} \boldsymbol{F'}$*.*

By $\operatorname{sDP}(F) \subset \mathcal{MU}$ for $F \in \mathcal{MU}$ the set of non-singular $F' \in \mathcal{MU}$ with $F \xrightarrow{\mathrm{sDP}}_{*} F'$ is denoted. For us the main property of $\operatorname{sDP}(F)$ is that it is not empty. In [65] it is shown that for $S \in \mathcal{SMU}$ we have $|\operatorname{sDP}(F)| = 1$, and that for arbitrary $F \in \mathcal{MU}$ and $F', F'' \in \operatorname{sDP}(F)$ we have $n(F') = n(F'')$.

Example 5.2 *In [65] the following is shown for* $F \in \mathcal{MU}$*:*

1. *For* $\delta(F) = 1$ *we have* $\operatorname{sDP}(F) = \{\bot\}$*.*
2. *For* $\delta(F) = 2$ *all elements of* $\operatorname{sDP}(F)$ *are isomorphic.*
3. *For* $\delta(F) \geq 3$ *in general there are non-isomorphic elements in* $\operatorname{sDP}(F)$*.*

By the results of Sections 3.1, 3.2 in [65] we have the following basic preservation properties:

Lemma 5.3 ([65]) *For* $F, F' \in \mathcal{MU}$ *with* $F \xrightarrow{sDP}_{*} F'$ *we have:*

1. $\delta(F') = \delta(F)$*.*
2. $F \in \mathcal{MU} \Rightarrow F' \in \mathcal{MU}$*.*
3. $F \in \mathcal{SMU} \Rightarrow F' \in \mathcal{SMU}$*.*
4. $F \in \mathcal{UHIT} \Rightarrow F' \in \mathcal{UHIT}$*.*

Although singular DP-reduction can reduce the variable-degree of some variables, it can not decrease the *minimum* variable-degree:

Lemma 5.4 *For* $F, F' \in \mathcal{MU}$ *with* $F \xrightarrow{sDP}_{*} F'$ *we have* $\mu\mathrm{vd}(F') \geq \mu\mathrm{vd}(F)$*.*

Proof: It is sufficient to consider the case $F' = \mathrm{DP}_v(F)$ for a singular variable v. Assume $\mu\mathrm{vd}(F') < \mu\mathrm{vd}(F)$; thus $\mathrm{var}(F') \neq \emptyset$ (otherwise we have $\mu\mathrm{vd}(F') = +\infty$), and we consider $w \in \mathrm{var}(F')$ with $\mathrm{vd}_{F'}(w) = \mu\mathrm{vd}(F')$. So we have $\mathrm{vd}_{F'}(w) < \mathrm{vd}_F(w)$, and thus by Lemma 24 in [65], for all clauses $C \in F$ with $v \in \mathrm{var}(C)$ we have $w \in \mathrm{var}(C)$. But then $\mu\mathrm{vd}(F') = \mathrm{vd}_{F'}(w) \geq \mathrm{vd}_F(v) \geq \mu\mathrm{vd}(F) > \mu\mathrm{vd}(F')$, a contradiction. $\square$

Thus in order to determine the minimum variable-degree for minimally unsatisfiable clause-sets in dependency on the deficiency, w.l.o.g. one can restrict attention to saturated and non-singular instances:

Corollary 5.5 *For all $k \in \mathbb{N}$ holds:*

1. $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) = \mu\mathrm{vd}(\mathcal{SMU}'_{\delta=k})$.
2. $\mu\mathrm{vd}(\mathcal{UHIT}_{\delta=k}) = \mu\mathrm{vd}(\mathcal{UHIT}'_{\delta=k})$.

Proof: For Part 1 we note that by Lemma 5.4 for every $F \in \mathcal{MU}_{\delta=k}$ we can find $F' \in \mathcal{SMU}'_{\delta=k}$ with $\mu\mathrm{vd}(F') \geq \mu\mathrm{vd}(F)$, and thus $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \leq \mu\mathrm{vd}(\mathcal{SMU}'_{\delta=k})$, while $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \geq \mu\mathrm{vd}(\mathcal{SMU}'_{\delta=k})$ holds due to $\mathcal{SMU}'_{\delta=k} \subseteq \mathcal{MU}_{\delta=k}$. The same reasoning applies for Part 2. $\square$

See Lemma 15.9 for some conditions under which $k \in \mathbb{N} \mapsto \mu\mathrm{vd}(\mathcal{UHIT}_{\delta=k})$ and $k \in \mathbb{N} \mapsto \mu\mathrm{vd}(\mathcal{MU}_{\delta=k})$ would be computable.

5.2 Singular DP-extensions

We consider now the reverse direction of singular DP-reduction, from $\mathrm{DP}_v(F)$ to F, as a *singular extension*, and also generalise it to arbitrary clause-sets. This process was mentioned in [65, Examples 15,19,54] for minimally unsatisfiable $\mathrm{DP}_v(F)$, called "inverse singular DP-reduction" there:

Definition 5.6 *Consider a clause-set $G \in \mathcal{CLS}$, a variable $v \in \mathcal{VA} \setminus \mathrm{var}(G)$, and $m \in \mathbb{N}$. A* ***singular m-extension of G with*** *v is a clause-set $F \in \mathcal{CLS}$ obtained as follows (employing four choice steps):*

1. *m different clauses $D_1, \dots, D_m \in G$ are chosen.*
2. *A subset $C \subseteq \bigcap_{i=1}^m D_i'$ is chosen.*
3. *A literal x with $\mathrm{var}(x) = v$ is chosen.*
4. *Clauses $D_i' \in \mathcal{CL}$ for $i \in \{1, \dots, m\}$ with $(D_i \setminus C) \subseteq D_i' \subseteq D_i$ are chosen.*
5. *Let $C' := C \mathbin{\dot\cup} \{x\}$, and let $D_i'' := D_i' \mathbin{\dot\cup} \{\overline{x}\}$ for $i \in \{1, \dots, m\}$.*
6. *F is obtained from G by adding C' and replacing the D_i with D_i'':*

$$F := (G \setminus \{D_1, \dots, D_m\}) \mathbin{\dot\cup} \{C', D_1'', \dots, D_m''\}.$$

Example 5.7 *Consider $G := \{\{a,b,c\},\{a,b,\overline{c}\}\}$, $m := 2$, and the choices $C := \{a\}$, $x := v$, and $D_1' := \{b,c\}$, $D_2' := \{a,b,\overline{c}\}$. Then the 2-extension F of G is $F = \{\{v,a\},\{\overline{v},b,c\},\{\overline{v},a,b,\overline{c}\}\}$.*

By definition we have for an m-extension F of $G \in \mathcal{CLS}$ with v the following simple properties:

1. $c(F) = c(G) + 1$, $n(F) = n(G) + 1$, $\delta(F) = \delta(G)$.

2. v is singular for F, $\mathrm{vd}_F(v) = m + 1$.

3. $\mathrm{DP}_v(F) = G$.

We now show that indeed the process of Definition 5.6 is precisely the inversion of singular DP-reduction:

Lemma 5.8 *Consider $m \in \mathbb{N}$, $G, F \in \mathcal{CLS}$ and $v \in \mathcal{VA}$. Then F is an m-extension of G by v iff the following conditions are fulfilled:*

1. *v is singular for F;*
2. *$\mathrm{vd}_F(v) = m + 1$;*
3. *$\mathrm{DP}_v(F) = G$, $c(G) = c(F) - 1$.*

Proof: If F is an m-extension of G by v, then the three properties hold, as we have already mentioned. So assume these three properties hold. Now let the m clauses $D_1, \dots, D_m$ be the result of singular DP-reduction on v for F; they must be pairwise different, and all m resolutions must be possible, otherwise $c(G) < c(F) - 1$. And let C be singular occurrence of v minus the variable v. Now all properties of a singular m-extension are easily checked. $\square$

Singular extensions behave well regarding minimal unsatisfiability:

Lemma 5.9 *Consider $m \in \mathbb{N}$, $G \in \mathcal{CLS}$ and an m-extension F of G by $v \in \mathcal{VA}$. Then $F \in \mathcal{MU} \Leftrightarrow G \in \mathcal{MU}$.*

Proof: This follows by Lemma 5.8 together with Lemma 9, Parts 1, 2 in [65]. $\square$

In the situation of Lemma 5.9, regarding saturatedness we only have the direction $F \in \mathcal{SMU} \Rightarrow G \in \mathcal{SMU}$, while for the other direction the conditions of [65, Lemma 12] need to be observed (this would yield "saturated extensions", which however we do not need here).

5.3 Unit clauses

We conclude this section by considering unit-clauses in minimally unsatisfiable clause-sets. The following (fundamental, simple) lemma is proven in [65] (Lemma 14); there in Subsection 3.3 one finds further information on unit-clauses in minimally unsatisfiable clause-sets.

Lemma 5.10 ([65]) *Consider $F \in \mathcal{MU}$.*

1. *If v is full and singular in F, then we have $\{v\} \in F$ or $\{\overline{v}\} \in F$.*
2. *If $\{x\} \in F$, then $v := \mathrm{var}(x)$ is singular in F (with $\mathrm{ld}_F(x) = 1$). If here F is saturated, then v is also full in F.*

So unit-clauses in minimally unsatisfiable clause-sets are strong cases of singular variables. They can obviously be removed by singular DP-reduction, while singular ≥ 2-extensions can not remove all unit-clauses:

Lemma 5.11 *Consider a clause-set $F \in \mathcal{MU}$ containing at least one unit-clause, and obtain F' from F by a singular m-extension, where $m \geq 2$. Then also F' must contain at least one unit-clause.*

Proof: For a unit-clause $\{x\} \in F$ to be removed in F', it needs to be one of the D_i (using the terminology of Definition 5.6). Then the intersection C must be empty (otherwise any other D_i needed to contain x, and since $m \geq 2$ this would mean a subsumption in F). Thus the extension introduces the new unit-clause C'. $\square$

The following examples shows that the assumptions $F \in \mathcal{MU}$ and $m \geq 2$ in Lemma 5.11 are needed:

Example 5.12 *First consider* $F := \{\{a\}, \{\overline{a}, b\}, \{\overline{a}, \overline{b}\}\} \in \mathcal{MU}$*. Via a 1-singular extension we obtain* $F' := \{\{v, a\}, \{\overline{v}, a\}, \{\overline{a}, b\}, \{\overline{a}, \overline{b}\}\}$*, which has no unit-clauses. A 2-singular extension of* F*, which touches* $\{a\}$*, has* $C = \bot$*, and thus* C' *is a new unit-clause. If on the other hand we consider* $F := \{\{a\}, \{a, b\}\} \in \mathcal{CLS} \setminus \mathcal{MU}$*, then* $F' := \{\{v, a\}, \{\overline{v}, a\}, \{\overline{v}, b\}\}$ *is a 2-extension without unit-clauses.*

For certain $F \in \mathcal{MU}_{\delta=2}$ the existence of a unit-clause is actually necessary for singularity:

Lemma 5.13 *Consider* $F \in \mathcal{MU}_{\delta=2}$ *with* $\mu\mathrm{vd}(F) \geq 4$*. Then* F *is singular if and only if* F *contains a unit-clause.*

Proof: That if F contains a unit-clause, then F must be singular, follows by Lemma 5.10, Part 2. So assume now that F is singular, and we have to show that F contains a unit-clause. Consider a reduction sequence $F = F_0 \xrightarrow{\mathrm{sDP}} F_1 \xrightarrow{\mathrm{sDP}} \dots \xrightarrow{\mathrm{sDP}} F_m$, where F_m is non-singular (note $m \geq 1$). So there exists $n \geq 2$ such that F_m is isomorphic to $\mathcal{F}_n$ (recall Example 3.3), and thus every variable of F_m has degree 4. So by Lemma 5.4 we know $\mu\mathrm{vd}(F_i) = 4$ for $i \in \{0, \dots, m\}$. We show by induction on m that F contains a unit-clause. If $m = 1$, then in order to obtain the min-var-degree of at least 4, at least 3 side-clauses $D_1, \dots, D_3 \in \mathcal{F}_n$ for the singular extension have to be chosen (using Definition 5.6), but every literal occurs precisely twice in $\mathcal{F}_n$ (because of variable-degree 4 and non-singularity), and thus the intersection C has to be empty, and the new clause introduced by the singular extension is a unit-clause, whence F contains a unit-clause. Finally assume $m > 1$. So by induction hypothesis, F_1 contains a unit-clause, and thus by Lemma 5.11 also F_0 contains a unit-clause. $\square$

We will later see (Theorem 8.3) that the condition $\mu\mathrm{vd}(F) \geq 4$ in Lemma 5.13 is equivalent to $\mu\mathrm{vd}(F) = 4$; the following examples show that this condition can not be improved:

Example 5.14 *A* 1*-singular extension of* A_2 *is*

$$F_1 := \{\{1, 2, 3\}, \{1, 2, -3\}, \{-1, 2\}, \{1, -2\}, \{-1, -2\}\} \in \mathcal{MU}_{\delta=2},$$

where F_1 *is singular, has no unit-clause, and* $\mu\mathrm{vd}(F_1) = 2$*. While a* 2*-singular extension of* A_2 *is*

$$F_2 := \{\{1, 3\}, \{1, 2, -3\}, \{1, -2, -3\}, \{-1, 2\}, \{-1, -2\}\} \in \mathcal{MU}_{\delta=2},$$

where F_2 *is singular, has no unit-clause, and* $\mu\mathrm{vd}(F_2) = 3$*.*

We conclude with a simple form of adding a new variable, by adding it in one sign as unit-clause, and adding it in the other sign to all given clauses:

Definition 5.15 *A* ***full singular unit-extension*** *of a clause-set* $F \in \mathcal{CLS}$ *(by unit-clause* $\{x\}$*) is a clause-set* $F' \in \mathcal{CLS}$ *obtained from* F *by adding a unit-clause* $\{x\}$ *with* $\mathrm{var}(x) \notin \mathrm{var}(F)$*, and by adding literal* $\overline{x}$ *to all clauses of* F*, i.e.,* $F' := \{\{x\}\} \mathbin{\dot\cup} \{C \mathbin{\dot\cup} \{\overline{x}\} : C \in F\}$ *for some* $x \in \mathcal{LIT} \setminus \mathrm{lit}(F)$*.*

A full singular unit-extension F' of F by $\{x\}$ is a case of a singular $c(F)$-extension of F with $\mathrm{var}(x)$, and thus $F' \xrightarrow{\mathrm{sDP}} F$.[12)]

Example 5.16 *Starting from $\{\bot\}$, the (up to the choice of the new literal) first full singular unit-extension is $\{\{v\}, \{\overline{v}\}\}$, the second one is $\{\{w\}, \{v, \overline{w}\}, \{\overline{v}, \overline{w}\}\}$. In this way we get special examples of $\mathcal{SMU}_{\delta=1}$ (since we started with $\{\bot\} \in \mathcal{SMU}_{\delta=1}$).*

If we start with $\top$ instead, then first we get $\{\{v\}\}$, and then $\{\{w\}, \{v, \overline{w}\}\}$.

Example 15, Part 1, in [65] contains two example of "inverse unit elimination", where Example (a) there is an example of a full singular unit-extension, while Example (b) there would be a non-full singular unit-extension (where the new variable is not full; not used in this report). There is the dual notion of a "full variable" for a clause-set F, which is some element of $\bigcap_{C \in F} \mathrm{var}(C)$, which explains why we speak of a "full extension" (namely the new variable is full).

The process of full singular unit-extension of a clause-set F maintains many properties of F, and we list here those we use:

Lemma 5.17 *Consider a full singular unit-extension F' of F (by $\{v\}$):*

1. *$n(F') = n(F) + 1$ and $c(F') = c(F) + 1$.*
2. *$\delta(F') = \delta(F)$.*
3. *$\sigma(F') = \sigma(F)$ for $F \neq \{\bot\}$.*
4. *$\mu\mathrm{vd}(F') = \mu\mathrm{vd}(F)$ for $n(F) > 0$.*
5. *F' is satisfiable iff F is satisfiable.*
6. *For $F \neq \top$: F' is lean iff F is lean.*
7. *F' is (saturated) minimally unsatisfiable iff F is (saturated) minimally unsatisfiable.*
8. *F' is hitting iff F is hitting.*

Proof: Parts 1, 2 follow directly by definition. For Part 3 we notice that for $F = \top$ we have $\sigma(F') = \sigma(F) = 0$, while for $n(F) > 0$ consider $\emptyset \subset V \subseteq \mathrm{var}(F')$: if $v \notin V$, then $F'[V] = F[V]$, and thus the minimisation for $\sigma(F)$ is included in $\sigma(F')$, and if $v \in V$, then $\delta(F'[V]) = c(F') - |V| \geq \delta(F') = \delta(F) \geq \sigma(F)$, and thus these V do not contribute to the minimisation.

For Part 4 we just note that the variables of F keep their degrees in F', while the new variable has degree $\mathrm{vd}_{F'}(v) = c(F') > c(F)$, and thus does not contribute to the min-var degree. Part 5 is trivial, and follows also by the satisfiability-equivalence of $\mathrm{DP}_v(F)$ and F. For Part 6 we note, that an autarky for F' involving v must be a satisfying assignment for F', while the autarkies for F' not involving v are the same as the autarkies for F. Part 7 concerning (just) minimal unsatisfiability follows with Lemma 9 in [65], while regarding saturatedness we can use Lemma 12 in [65] (both assertions also follow easily by direct reasoning). Part 8 is trivial. $\square$

So our fundamental classes are respected by full singular unit-extension:

Corollary 5.18 *If $F \in \mathcal{MU}_{\delta=k}$ ($k \in \mathbb{N}$), then every full singular unit-extension is also in $\mathcal{MU}_{\delta=k}$. If furthermore F is saturated resp. hitting, then every full singular unit-extension is also saturated resp. hitting.*

[12)] The case $m = 0$ is excluded in Definition 5.6, since it is not needed, and would only complicate the formulation.

Obviously, full singular unit-extension is unique up to isomorphism:

Lemma 5.19 *Consider a clause-set $F \in \mathcal{CLS}$ and clause-sets $F', F'' \in \mathcal{CLS}$ obtained from F by repeated full singular unit-extensions. Then F', F'' are isomorphic if and only if $n(F') = n(F'')$.*

Proof: The number of repeated full singular unit-extensions leading to F' resp. F'' is the number of variables in these clause-sets with degree strictly greater than $c(F)$, and sorting these variables by increasing degree yields the sequence of extensions. Thus just from knowing the number of variables in F', F'' we can reconstruct them up to isomorphism (using that a full singular unit-extension of F by $\{x\}$ is isomorphic to one by $\{y\}$, for arbitrary literals x, y with new variables). $\square$

6 Full subsumption resolution / extension

In this section we investigate the second reduction concept for this report, "full subsumption resolution". As with singular DP-reduction from Section 5, in general the reduction uncovers hidden structure, while the inverse process, "full subsumption extension", serves as a generator for minimally unsatisfiable clause-sets with various properties. However in this report, unlike with singular DP-reduction, we will not consider full subsumption resolution for arbitrary $F \in \mathcal{MU}$, but only starting from some $A(V)$, while a deeper use will be important in [67]. Subsection 6.1 discusses the basic definitions (there are various technicalities one needs to be aware of), and first applications are given in Subsection 6.2.

The basic idea is, for a clause-set F containing two clauses $R \mathbin{\dot\cup} \{v\}, R \mathbin{\dot\cup} \{\overline{v}\} \in F$, to replace these two clauses by the clause R, i.e., we consider the case where the resolvent R of parent clauses C, D subsumes both parent clauses (thus the name). This is a very old procedure, based on the trivial observation that $(R \vee v) \wedge (R \vee \neg v)$ is logically equivalent to R. If we perform this in the inverse direction, as an "extension", then every clause-set $F \in \mathcal{CLS}$ can be transformed into its "distinguished CNF" $F' \subseteq A(\mathrm{var}(F))$ (just expand every non-full clause), which is uniquely determined. We however have to be more careful about deficiency and membership in $\mathcal{MU}$, and thus will consider only "full subsumption resolution", where the resolvent must not be present already, while for the "strict" form additionally the resolution variable v must occur also in other clauses. Then from $A(V)$ by strict full subsumption resolution we can obtain precisely the $F \in \mathcal{UHIT}$ with $\mathrm{var}(F) = V$ (Lemma 6.9). For the inverse forms, we have to be even more carefully, making sure that neither any of the two parent clauses is already present (this prevents the above expansion of arbitrary $F \in \mathcal{CLS}$ to $A(\mathrm{var}(F))$).

The (more general) well-known "subsumption resolution" is the reduction $F \leadsto (F \setminus \{C\}) \cup \{C \setminus \{x\}\}$ for $F \in \mathcal{CLS}$, that is the removal of a literal $x \in C$ from a clause $C \in F$, in case there exists $D \in F$ with $\overline{x} \in D$ and $D \setminus \{\overline{x}\} \subseteq C$ (note that $C \diamond D = C \setminus \{x\}$ subsumes C). An early use is in [85] (under the name "replacement principle"), while the terminology "subsumption resolution" is used in [29] (for SAT solving). The earliest sources with a systematic treatment appear to be [59, Section 7] and [60, Section 7]. An experimental study of the practical importance of subsumption resolution in connection with DP-reductions $F \leadsto \mathrm{DP}_v(F)$ (under suitable additional conditions to make DP-reduction feasible; see Subsection 1.3 in [65] for an overview on such restrictions) is performed in [20] (under the name of "self-subsuming resolution"), continued in [37]. A theoretic (similar) use one finds in [79, Section 4], where a variable v is called "DP-simplicial" for $F \in \mathcal{CLS}$ iff all resolutions performed by the reduction $F \leadsto \mathrm{DP}_v(F)$ are subsumption resolutions.

Our special form, where *both* parent clauses are subsumed we call *full subsumption resolution*, namely the reduction $F \leadsto (F \setminus \{C, D\}) \mathbin{\dot\cup} \{C \diamond D\}$ in case of $C, D \in F$ such that $C \cap \overline{D} = \{x\}$ and $C \setminus \{x\} = D \setminus \{\overline{x}\}$. A main tool is Lemma 6.5, where especially Part 6 is somewhat subtle, and can be easily overlooked. Via this tool we have a controlled way of transforming $F \in \mathcal{MU}$ resp. $F \in \mathcal{UHIT}$ into $A(\mathrm{var}(F))$, and in Theorem 6.13 this yields the determination of the possible numbers of variables and clauses in minimally unsatisfiable clause-sets of a given deficiency.

6.1 Basic definitions

Before defining "full subsumption reduction" $F \leadsto (F \setminus \{R \mathbin{\dot\cup} \{v\}, R \mathbin{\dot\cup} \{\overline{v}\}\}) \mathbin{\dot\cup} \{R\}$ in Definition 6.7 (so R is new and the two clauses $R \mathbin{\dot\cup} \{v\}, R \mathbin{\dot\cup} \{\overline{v}\}$ vanish), we introduce the "strict" form, which is more important to us, and which has the additional condition that v must still occur (in other clauses of F; the "non-strict" form on the other hand guarantees that v vanishes (see Definition 6.7)):

Definition 6.1 *For clause-sets $F, F' \in \mathcal{CLS}$ by $\boldsymbol{F \xrightarrow{\mathit{sfsR}} F'}$ we denote that F' is obtained from F by one step of **strict full subsumption resolution**, that is,*

- *there is a clause $R \in F'$*
- *and a literal x with $\mathrm{var}(x) \notin R$*
- *such that for the clauses $C := R \mathbin{\dot\cup} \{x\}$ and $D := R \mathbin{\dot\cup} \{\overline{x}\}$*
- *we have $F = (F' \setminus \{R\}) \mathbin{\dot\cup} \{C, D\}$;*
- *we furthermore require $\mathrm{var}(x) \in \mathrm{var}(F')$.*
- *As usual, the literals $x, \overline{x}$ are the resolution literals, $\mathrm{var}(x)$ is the resolution variable, C, D are the parent clauses, and R is the resolvent.*

We write $F \xrightarrow{\mathit{sfsR}}_k F'$ for $k \in \mathbb{N}_0$ if exactly k steps have been performed, while we write $F \xrightarrow{\mathit{sfsR}}_* F'$ for an arbitrary number of steps (including zero).

We require $R \notin F$, that is, the (full subsumption) resolvent is not already present in the original clause-set. This is of course satisfied if $F \in \mathcal{MU}$. We also require that the variable v does not vanish, for the sake of keeping control on the deficiency.

Example 6.2 *Some simple examples are:*

1. *$\mathcal{F}_2 = \{\{1, 2\}, \{-1, -2\}, \{-1, 2\}, \{-2, 1\}\} \xrightarrow{\mathit{sfsR}} \{\{2\}, \{-1, -2\}, \{-2, 1\}\}$, and no further reduction is possible (note that the only possibility is blocked, since variable 1 would vanish).*
2. *$\{\{v\}, \{\overline{v}\}\} \not\xrightarrow{\mathit{sfsR}} \{\bot\}$, as v vanishes, while $\{\{v\}, \{\overline{v}\}, \{v, x\}\} \xrightarrow{\mathit{sfsR}} \{\bot, \{v, x\}\}$.*
3. *$\{\{v, w\}, \{\overline{v}, w\}, \{v, \overline{w}\}\} \not\xrightarrow{\mathit{sfsR}} \{\{v, w\}, \{\{w\}, \{v, \overline{w}\}\}\}$, as one parent clause is kept, while $\{\{v, w\}, \{\overline{v}, w\}, \{v, \overline{w}\}\} \xrightarrow{\mathit{sfsR}} \{\{\{w\}, \{v, \overline{w}\}\}\}$.*
4. *$\{\{v, w\}, \{\overline{v}, w\}, \{v, \overline{w}\}, \{v\}, \{w\}\}$ can not be reduced by strict-full-subsumption resolution, since all possible resolvents are already there.*

The expansion of a clause R to two clauses $R \mathbin{\dot\cup} \{v\}, R \mathbin{\dot\cup} \{\overline{v}\}$ under the above requirements is called "extension":

Definition 6.3 *For clause-sets $F, F' \in \mathcal{CLS}$ we say that F* ***is obtained from*** *F'* ***by strict full subsumption extension*** *if $F \xrightarrow{sfsR} F'$. And for $k \in \mathbb{N}_0$ we say that F* ***is obtained from*** *F'* ***by strict full subsumption extension with*** *k* ***steps*** *if $F \xrightarrow{sfsR}_k F'$.*

So one step of strict full subsumption extension for a clause-set F uses a non-full clause $R \in F$ and a variable $v \in \mathrm{var}(F) \setminus \mathrm{var}(R)$, and replaces R by the two clauses $R \mathbin{\dot\cup} \{v\}, R \mathbin{\dot\cup} \{\overline{v}\}$, where none of them is already present.

Example 6.4 *From $\{\{a\}, \{b\}\}$ by one step of strict full subsumption extension we can obtain $\{\{a, b\}, \{a, \overline{b}\}, \{b\}\}$ or $\{\{a\}, \{a, b\}, \{\overline{a}, b\}\}$; note that no new variable has been introduced, that the original clause ($\{a\}$ resp. $\{b\}$) vanished, and that its replacement clauses were not already present. For $\{\{a, b\}, \{a\}\}$ no strict full subsumption extension is possible. Further examples are obtained by "reading Example 6.2 backwards".*

The basic properties of strict full subsumption resolution are collected in the following lemma.

Lemma 6.5 *For clause-sets $F, F' \in \mathcal{CLS}$ with $F \xrightarrow{sfsR}_k F'$ ($k \in \mathbb{N}_0$, with resolution variable v and resolvent R) we have:*

1. *F' is logically equivalent to F.*
2. *$\mathrm{var}(F') = \mathrm{var}(F)$.*
3. *$c(F') = c(F) - k$, $\delta(F') = \delta(F) - k$.*
4. *$\mu\mathrm{vd}(F) \geq \mu\mathrm{vd}(F')$.*
5. *$F \in \mathcal{MU} \Rightarrow F' \in \mathcal{MU}$.*
6. *If $k = 1$ and $F' \in \mathcal{MU}$, then exactly one of the following three possibilities holds:*
   (a) $\mathrm{S}(F', R, v)$ is a partial saturation of F' (recall Definition 3.6).
   (b) $\mathrm{S}(F', R, \overline{v})$ is a partial saturation of F'.
   (c) $F \in \mathcal{MU}$.
7. *$F \in \mathcal{SMU} \Rightarrow F' \in \mathcal{SMU}$.*
8. *$F \in \mathcal{HIT} \Leftrightarrow F' \in \mathcal{HIT}$.*

Proof: Parts 1, 2, 3, 4 follow directly from the definition. Parts 5, 7 hold since we strengthen two clauses into one, which on the other hand is logically equivalent to its parent clauses. Part 8 follows by trivial combinatorics.

Now consider Part 6. That the two possibilities for partial saturation exclude each other follows by Lemma 3.8 (and $F' \setminus \{R\} \not\models R$). And that each possibility for partial saturation excludes $F \in \mathcal{MU}$ follows by definition. Finally, that the negation of the two partial saturation possibilities implies $F \in \mathcal{MU}$ follows again by Lemma 3.8. □

Part 6 of Lemma 6.5 handles a subtle source for errors: One could easily think that for $F' \in \mathcal{MU}$ a strict full subsumption extension yields another $F \in \mathcal{MU}$, but this is not so, as there are three possible cases to be considered here, illustrated by the following examples:

Example 6.6 *Consider* $F := \{\{v,a\},\{\overline{v},a\},\{\overline{v}\},\{v,\overline{a}\}\}$. *So* $F \xrightarrow{sfsR}_1 F'$ *for* $F' := \{\{a\},\{\overline{v}\},\{v,\overline{a}\}\}$. *We have* $F' \in \mathcal{MU}$, *but* $F \notin \mathcal{MU}$, *and indeed* $\mathrm{S}(F',R,v) = \{\{a,v\},\{\overline{v}\},\{v,\overline{a}\}\}$ *is a partial saturation of* F' *(while* $\mathrm{S}(F',R,\overline{v})$ *isn't one).*

The condition on the resolution variable for strict full subsumption resolution (that it must not vanish) is exactly needed for Parts 2, 3 of Lemma 6.5. If this condition is dropped, then we speak of *full subsumption resolution*:

Definition 6.7 ***full subsumption resolution*** *is defined as strict full subsumption resolution, but now the resolution variable is allowed to vanish. If the resolution variable definitely vanishes, then we speak of if* ***non-strict full subsumption resolution****. In the other direction we speak of* ***full subsumption extension*** *resp.* ***non-strict full subsumption extension****.*

So if F' is obtained from F by one step of non-strict full subsumption extension, then we have $c(F') = c(F) + 1$, $n(F') = n(F) + 1$ and $\delta(F') = \delta(F)$.

Example 6.8 *Considering the non-examples from Example 6.2:*

1. $\{\{v\},\{\overline{v}\}\} \overset{sfsR}{\not\longrightarrow} \{\bot\}$, *but by full subsumption resolution we obtain* $\{\bot\}$.
2. $\{\{v,w\},\{\overline{v},w\},\{v,\overline{w}\}\} \overset{sfsR}{\not\longrightarrow} \{\{v,w\},\{\{w\},\{v,\overline{w}\}\}\}$, *and the transition is also not possible by full subsumption resolution.*
3. $\{\{v,w\},\{\overline{v},w\},\{v,\overline{w}\},\{v\},\{w\}\}$ *is irreducible by full subsumption resolution.*

As follows from the characterisation of $\mathcal{SMU}_{\delta=1} = \mathcal{UHIT}_{\delta=1}$ *in [49], a clause-set* $F \in \mathcal{CLS}$ *can be reduced by a series of non-strict full subsumption resolutions to* $\{\bot\}$ *iff* $F \in \mathcal{SMU}_{\delta=1} = \mathcal{UHIT}_{\delta=1}$.

6.2 Extensions to full clause-sets

If we start with the full clause-sets $A(V)$, then by strict full subsumption resolution we obtain exactly all unsatisfiable hitting clause-sets:

Lemma 6.9 *If for some finite* $V \subset \mathcal{VA}$ *we have* $A(V) \xrightarrow{sfsR}_* F$, *then* $F \in \mathcal{UHIT}$ *holds. And for* $F \in \mathcal{UHIT}$ *we have* $A(\mathrm{var}(F)) \xrightarrow{sfsR}_* F$.

Proof: The first part follows by Lemma 6.5, Part 8 (and $A(V) \in \mathcal{UHIT}$). And for the second part note, that if $F \in \mathcal{UHIT}$ has a non-full clause, then an strict full subsumption extension step can be applied, where the result is still in $\mathcal{UHIT}$ (again by Lemma 6.5, Part 8; if F has only full clauses, then $F = A(\mathrm{var}(F))$). □

Recall that in Example 6.6 we have seen, that strict full subsumption extension does not maintain minimal unsatisfiability in general. But from arbitrary minimally unsatisfiable F we can obtain $A(\mathrm{var}(F))$, when we additionally allow partial saturation:

Lemma 6.10 *For* $F \in \mathcal{MU}$ *we can obtain* $A(\mathrm{var}(F))$ *from* F *by a series of strict full subsumption extensions in combination with partial saturations.*

Proof: If $F \in \mathcal{MU}$ has a non-full clause, and if strict full subsumption extension can not be applied in order to obtain $F' \in \mathcal{MU}$, then by Lemma 6.5, Part 6, a partial saturation is possible. □

We obtain sharp upper bounds on deficiency and number of clauses in terms of the number of variables:

Corollary 6.11 *For $F \in \mathcal{MU}$ holds:*

1. $\delta(F) \le 2^{n(F)} - n(F)$.
2. $c(F) \le 2^{n(F)}$.

In both cases we have equality iff F is full (i.e., $F = A(\mathrm{var}(F))$).

Proof: For Part 1 note that by Lemma 6.10 we can transform F into $A(\mathrm{var}(F))$ by a series of steps not decreasing the deficiency. Thus $\delta(F) \le \delta(A(\mathrm{var}(F))) = 2^{n(F)} - n(F)$. For Part 2 note $c(F) = \delta(F) + n(F) \le 2^{n(F)}$ (by Part 1). For $F = A(\mathrm{var}(F))$ these inequalities are indeed equalities. If we had $\delta(F) = 2^{n(F)} - n(F)$ for some non-full F, then some strict full subsumption extension must be possible, contradicting the upper bound of Part 1. And if we have $c(F) \le 2^{n(F)}$ for some non-full F, then again some strict full subsumption extension must be possible, contradicting the upper bound of Part 2. □

We explicitly state the instructive reformulation, that the A_n are the minimally unsatisfiable clause-sets of maximal deficiency for given number n of variables:

Corollary 6.12 *Consider $m \in \mathbb{N}_0$ and $F \in \mathcal{MU}_{n=m}$ such that $\delta(F)$ is maximal.*[13] *Then $F = A(\mathrm{var}(F))$. Thus the maximal deficiency for $F \in \mathcal{MU}_{n=m}$ is $2^m - m$ (realised by $A_m \in \mathcal{MU}_{n=m} \cap \mathcal{MU}_{\delta=2^m-m}$).*

So for $m = 0, 1, 2, 3, 4, 5, 6$ variables the maximal deficiency of minimally unsatisfiable clause-sets is $1, 1, 2, 5, 12, 27, 58$; in general the deficiencies of the form $2^m - m$ are central for our investigations (note that the function $m \in \mathbb{N}_0 \mapsto 2^m - m \in \mathbb{N}$ is monotonically increasing). We are now able to determine the numbers of variables and numbers of clauses possible for minimally unsatisfiable clause-sets with a given deficiency:

Theorem 6.13 *For $k \in \mathbb{N}$ let $o(k) \in \mathbb{N}_0$ be the smallest $n \in \mathbb{N}_0$ with $2^n - n \ge k$.*

1. $\{n(F) : F \in \mathcal{MU}_{\delta=k}\} = \{n \in \mathbb{N}_0 : n \ge o(k)\}$.
2. $\{c(F) : F \in \mathcal{MU}_{\delta=k}\} = \{n \in \mathbb{N} : n \ge o(k) + k\}$.

Proof: Part 2 follows by Part 1, so it remains to show Part 1. By Corollary 6.11 we see that the left-hand side is a subset of the right-hand side. To show the other direction, we first note that increasing the number of variables by keeping the deficiency constant is achieved by one non-strict full subsumption extension step. It remains to show the existence of $F \in \mathcal{MU}_{\delta=k}$ with $n(F) = o(k)$. For $k = 1$ we have $F = \{\bot\}$, so assume $k > 1$. Let $F_0 := A(o(k) - 1)$ (so $\delta(F_0) = k - 1$; note $o(k) - 1 \ge 1$). Add a variable by one step of non-strict full subsumption extension, obtaining $F_1 \in \mathcal{MU}_{\delta=k-1}$ with one new variable, and then take a clause in F_1 without that new variable and perform one step of strict full subsumption extension (on that new variable), obtaining F_2 with $n(F_2) = n(F_1) = o(k)$ and $\delta(F_2) = \delta(F_1) + 1 = k$. □

$o(k)$ for $k \ge 1$ by definition is the smallest $n \ge 0$ with $\delta(A_n) \ge k$, and by Theorem 6.13 it is the smallest $n \ge 0$ such that there is $F \in \mathcal{MU}_{\delta=k}$ with $n(F) = n$. We have $o(1) = 0$, $o(2) = 2$, $o(3) = \cdots = o(5) = 3$, $o(6) = \cdots = o(12) = 4$ and $o(13) = \cdots = o(27) = 5$. Except for the first term, the sequence $(o(k))_{k \in \mathbb{N}}$ is sequence `http://oeis.org/A103586` in the "On-Line Encyclopedia of Integer Sequences".

[13] That is, $F \in \mathcal{MU}$, $n(F) = m$, and for all $F' \in \mathcal{MU}$ with $n(F') = m$ we have $\delta(F') \le \delta(F)$.

7 Non-Mersenne numbers

In this section we study the function $\mathrm{nM} : \mathbb{N} \to \mathbb{N}$ via a recursive definition (Definition 7.1; see Table 1). The understanding of this recursion is the underlying topic of this section. This recursion is naturally obtained from splitting on variables with minimum occurrence in minimally unsatisfiable clause-sets, and will be used in Theorem 8.3 later. The sequence nM is sequence `http://oeis.org/A062289` in the "On-Line Encyclopedia of Integer Sequences":

- It can be defined as the enumeration of those natural numbers containing "10" in their binary representation; in other words, exactly the numbers whose binary representation contains only 1's are skipped.
- Thus the sequence leaves out exactly the number of the form $2^n - 1$ for $n \in \mathbb{N}$ (that is, $1, 3, 7, 15, 31, \dots$), whence the name.
- The sequence consists of arithmetic progressions of slope 1 and length $2^m - 1$, $m = 1, 2, \dots$, each such progression separated by an additional step of $+1$.

k	1	2	3	4	5	$\cdots$	11	12	$\cdots$	26	27	$\cdots$	57	58
$\mathrm{nM}(k)$	2	4	5	6	8	$\cdots$	14	16	$\cdots$	30	32	$\cdots$	62	64

Table 1: Values for $\mathrm{nM}(k)$, $k \in \{1, \dots, 58\}$

The key deficiencies in Table 1 are the following two classes:

1. The k-values $k = 1, 2, 5, 12, 27, 58, \dots$ are the deficiencies $k = 2^n - n$ of the clause-sets A_n, $n \in \mathbb{N}$, while the corresponding values $\mathrm{nM}(k) = 2^k$ are the minimum variable-degree of the clause-sets A_n (see Lemma 2.13), as explained in Subsection 1.3.
2. The k-values $1, 4, 11, 26, 57, \dots$ are the positions just before these deficiencies, as also discussed in Subsection 1.3; we call them "jump positions", since precisely at these positions the function value increases by 2 for the next argument (compare Definition 7.12).

The recursion in Definition 7.1 is new, and so we can not use these characterisations, but must directly prove the basic properties; the deficiencies $k = 2^n - n$ will be handled in Corollary 7.23, while the jump positions are handled in Lemma 7.20. Later we will obtain two further alternative characterisations of nM:

- Combinatorial characterisations are obtained in Corollary 9.13, where we will see that $\mathrm{nM}(k)$ for $k \in \mathbb{N}$ is the maximal min-var-degree for lean clause-sets or variable-minimal unsatisfiable clause-sets with deficiency k.
- In Subsection 13.2 we will develop a general recursion scheme, which has the function nM "built-in", as shown in Theorem 13.15.

Definition 7.1 *For $k \in \mathbb{N}$ let $\mathrm{nM}(k) := 2$ if $k = 1$, while else*

$$\mathrm{nM}(k) := \max_{i \in \{2,\dots,k\}} \min(2 \cdot i, \mathrm{nM}(k - i + 1) + i).$$

The intuition underlying Definition 7.1 of $\mathrm{nM}(k)$, as later unfolded in Theorem 8.3, is that we want to get an upper bound on the min-var-degree of an $F \in \mathcal{MU}_{\delta=k}$ (recall Definition 2.11), and for that we consider a variable $v \in \mathrm{var}(F)$ of minimum var-degree, set it to $0, 1$, and infer an upper bound on $\mathrm{vd}_F(v)$ from the two splitting results. The index i runs over the possible literal-degrees of v (thus we have to maximise over it), where i actually is the maximum degree over both signs, and thus we can take the minimum with $i+i$ for the var-degree. In the splitting results $\langle v \to \varepsilon\rangle * F$ ($\varepsilon \in \{0,1\}$) the deficiency is reduced by $i-1$, since i occurrences (i.e., clauses) and one variable are lost, and we apply recursively the lower bound $\mathrm{nM}(k-(i-1))$, where then the i cancelled occurrences have to be re-added.

Example 7.2 *Computing* $\mathrm{nM}(k)$ *for* $2 \le k \le 5$*:*

1. $\mathrm{nM}(2) = \min(2\cdot 2, \mathrm{nM}(2-2+1)+2) = \min(4,4) = 4$.
2. $\mathrm{nM}(3) = \max(\min(2\cdot 2, \mathrm{nM}(3-2+1)+2), \min(2\cdot 3, \mathrm{nM}(3-3+1)+3)) = \max(\min(4,6), \min(6,5)) = 5$.
3. $\mathrm{nM}(4) = \max(\min(2\cdot 2, \mathrm{nM}(4-2+1)+2), \min(2\cdot 3, \mathrm{nM}(4-3+1)+3), \min(2\cdot 4, \mathrm{nM}(4-4+1)+4)) = \max(\min(4,7), \min(6,7), \min(8,6)) = 6$.
4. $\mathrm{nM}(5) = \max(\min(2\cdot 2, \mathrm{nM}(5-2+1)+2), \min(2\cdot 3, \mathrm{nM}(5-3+1)+3), \min(2\cdot 4, \mathrm{nM}(5-4+1)+4), \min(2\cdot 5, \mathrm{nM}(5-5+1)+5) = \max(\min(4,8), \min(6,8), \min(8,8), \min(10,7)) = 8$.

7.1 Basic properties

We begin our investigations into $\mathrm{nM}(k)$ by some simple bounds:

Lemma 7.3 *Consider* $k \in \mathbb{N}$*.*

1. $k+1 \le \mathrm{nM}(k) \le 2\cdot k$ *for* $k \in \mathbb{N}$*.*
2. *For* $k \ge 2$ *we have* $\mathrm{nM}(k) \ge 4$*.*

Proof: The upper bound of Part 1 follows directly from the definition (by the min-component $2i$). The lower bounds follows by induction: $\mathrm{nM}(1) = 2 \ge 1+1$, while for $k > 1$ we have $\mathrm{nM}(k) \ge \min(2k, \mathrm{nM}(k-k+1)+k) = \min(2k, 2+k) = k+2$. Part 1 follows by Part 1 and $\mathrm{nM}(2) = 4$. □

A basic tool for investigating sequences is the Delta-operator, which measures the differences in values between to neighbouring arguments:

Definition 7.4 *For a sequence* $a : \mathbb{N} \to \mathbb{R}$ *and* $k \in \mathbb{N}$ *let* $\boldsymbol{\Delta a(k)} := a(k+1) - a(k)$ *be the step in the value of the sequence from* k *to* $k+1$*.*

A few obvious properties of this Delta-operator are as follows:

1. $\Delta : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is linear: $\Delta(\lambda\cdot a + \mu\cdot b) = \lambda\cdot\Delta(a) + \mu\cdot\Delta(b)$.
2. $a \in \mathbb{R}^{\mathbb{N}}$ is constant iff $\Delta a = (0)$.
3. a is increasing iff $\Delta a \ge 0$, while a is strictly increasing iff $\Delta a > 0$. Here for sequences $a, b : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ of real numbers we use $a \le b :\Leftrightarrow \forall\, n \in \mathbb{N} : a_n \le b_n$, and $a < b :\Leftrightarrow \forall\, n \in \mathbb{N} : a_n < b_n$.

The first key insight is, that the next number in the sequence of non-Mersenne numbers is obtained by adding 1 or 2 to the previous number:

Lemma 7.5 *For* $k \in \mathbb{N}$ *holds* $\Delta\,\mathrm{nM}(k) \in \{1,2\}$*.*

Proof: For $k = 1$ we get $\Delta\,\mathrm{nM}(1) = 2$. Now consider $k \geq 2$. We have

$$\mathrm{nM}(k+1) = \max(\min(4, \mathrm{nM}(k)+2), \max_{i \in \{3,\dots,k+1\}} \min(2i, \mathrm{nM}(k-i+2)+i)) =$$
$$\max_{i \in \{3,\dots,k+1\}} \min(2i, \mathrm{nM}(k-i+2)+i) =$$
$$\max_{i \in \{2,\dots,k\}} \min(2(i+1), \mathrm{nM}(k-(i+1)+2)+(i+1)) =$$
$$\max_{i \in \{2,\dots,k\}} \min(2i+2, \mathrm{nM}(k-i+1)+i+1) = 1 + \max_{i \in \{2,\dots,k\}} \min(2i+1, \mathrm{nM}(k-i+1)+i).$$

Thus on the one hand we have $\mathrm{nM}(k+1) \geq 1 + \max_{i \in \{2,\dots,k\}} \min(2i, \mathrm{nM}(k-i+1)+i) = 1 + \mathrm{nM}(k)$, and on the other hand $\mathrm{nM}(k+1) \leq 1 + \max_{i \in \{2,\dots,k\}} \min(2i+1, \mathrm{nM}(k-i+1)+i+1) = 2 + \mathrm{nM}(k)$. □

Thus increasing the deficiency k by one increases $\mathrm{nM}(k)$ at least by one:

Corollary 7.6 $\mathrm{nM} : \mathbb{N} \to \mathbb{N}$ *is strictly increasing.*

And changing $\mathrm{nM}(a+b)$ to $\mathrm{nM}(a)+b$ can not increase the value:

Corollary 7.7 *We have* $\mathrm{nM}(a+b) \geq \mathrm{nM}(a)+b$ *for* $a \in \mathbb{N}$ *and* $b \in \mathbb{N}_0$, *and thus* $\mathrm{nM}(a-b) \leq \mathrm{nM}(a)-b$ *for* $b < a$.

Proof: We have $\mathrm{nM}(a+b) - \mathrm{nM}(a) = \sum_{i=0}^{b-1} \Delta\,\mathrm{nM}(a+i) \geq b \cdot 1$, whence the first inequality. Applying it yields $\mathrm{nM}(a-b)+b \leq \mathrm{nM}(a-b+b) = \mathrm{nM}(a)$. □

Instead of considering the maximum over $k-1$ cases $i \in \{2,\dots,k\}$ to compute $\mathrm{nM}(k)$ (according to Definition 7.1), we can now simplify the recursion to only one case $\mathrm{i}_{\mathrm{nM}}(k) \in \{2,\dots,k\}$, and for that case also consideration of the minimum is dispensable. $\mathrm{i}_{\mathrm{nM}}(k)$ is the first index i in Definition 7.1, where the minimum is attained by the nM-term, that is, where $2i \geq \mathrm{nM}(k-i+1)+i$:

Definition 7.8 *For* $k \in \mathbb{N}$, $k \geq 2$, *let* $\mathbf{i_{nM}(k)} \in \mathbb{N}$ *be the smallest* $i \in \{2,\dots,k\}$ *with* $i \geq \mathrm{nM}(k-i+1)$ *(note that* $k \geq \mathrm{nM}(k-k+1) = 2$, *and thus* $\mathrm{i}_{\mathrm{nM}}(k)$ *is well-defined).*

Example 7.9 *We have*

1. $\mathrm{i}_{\mathrm{nM}}(2) = 2$.
2. $\mathrm{i}_{\mathrm{nM}}(3) = 3$, *since* $\mathrm{nM}(3-2+1) = 4$, $\mathrm{nM}(3-3+1) = 2$.
3. $\mathrm{i}_{\mathrm{nM}}(4) = 4$, *since* $\mathrm{nM}(4-3+1) = 4$, $\mathrm{nM}(4-4+1) = 2$.
4. $\mathrm{i}_{\mathrm{nM}}(5) = 4$, *since* $\mathrm{nM}(5-3+1) = 5$, $\mathrm{nM}(5-4+1) = 4$.
5. $\mathrm{i}_{\mathrm{nM}}(6) = 5$, *since* $\mathrm{nM}(6-4+1) = 5$, $\mathrm{nM}(6-5+1) = 4$.

As promised, from $\mathrm{i}_{\mathrm{nM}}(k)$ we can compute $\mathrm{nM}(k)$ by one recursive call of nM:

Lemma 7.10 *For* $k \in \mathbb{N}$, $k \geq 2$, *we have:*

1. $0 \leq \mathrm{i}_{\mathrm{nM}}(k) - \mathrm{nM}(k - \mathrm{i}_{\mathrm{nM}}(k)+1) \leq 2$.
2. $\Delta\,\mathrm{i}_{\mathrm{nM}}(k) \in \{0,1\}$.
3. $\mathrm{nM}(k) = \mathrm{nM}(k - \mathrm{i}_{\mathrm{nM}}(k)+1) + \mathrm{i}_{\mathrm{nM}}(k)$.

Proof: For Part 1 we consider the sequence $i \mapsto f_k(i) := i - \mathrm{nM}(k - i + 1)$; this sequence starts with $f_k(2) = 2 - \mathrm{nM}(k-1) \le 0$, and finishes with $f_k(k) = k - \mathrm{nM}(1) \ge 2$, and $\mathrm{i}_{\mathrm{nM}}(k)$ is the smallest i with $f_k(i) \ge 0$. By Lemma 7.5 we have $\Delta f_k(i) = \Delta i(i) - \Delta\,\mathrm{nM}(k-i+1)(i) \in \{1+1, 1+2\} = \{2,3\}$. So for $\mathrm{i}_{\mathrm{nM}}(k) - \mathrm{nM}(k - \mathrm{i}_{\mathrm{nM}}(k)+1) = f_k(\mathrm{i}_{\mathrm{nM}}(k))$ by definition we have $f_k(\mathrm{i}_{\mathrm{nM}}(k)) \ge 0$, while $f_k(\mathrm{i}_{\mathrm{nM}}(k)) \le 2$ due to $\Delta f_k(\mathrm{i}_{\mathrm{nM}}(k)) \le 3$ (otherwise $\mathrm{i}_{\mathrm{nM}}(k)$ wouldn't be minimal).

For Part 2 we consider the sequence $k \mapsto g_i(k) := i - \mathrm{nM}(k-i+1)$. Again by Lemma 7.5 we get $\Delta g_i(k) \in \{-1,-2\}$. It follows immediately $\Delta\,\mathrm{i}_{\mathrm{nM}}(k) \ge 0$. Now assume $\Delta\,\mathrm{i}_{\mathrm{nM}}(k) \ge 1$; thus $-2 \le g_{\mathrm{i}_{\mathrm{nM}}(k)}(k+1) < 0$, whence, as shown before, $g_{\mathrm{i}_{\mathrm{nM}}(k)+1}(k+1) \ge -2+2 = 0$, and thus $\Delta\,\mathrm{i}_{\mathrm{nM}}(k) = 1$.

For Part 3 we consider the sequence $i \mapsto h_k(i) := \mathrm{nM}(k-i+1)+i$; by Lemma 7.5 we have $\Delta h_k(i) \in \{-1+1, -2+1\} = \{0,-1\}$. Thus, and by definition of $\mathrm{i}_{\mathrm{nM}}(k)$, we get $\mathrm{nM}(k) = \max(2\cdot 1, \dots, 2\cdot(\mathrm{i}_{\mathrm{nM}}(k)-1), h_k(\mathrm{i}_{\mathrm{nM}}(k))) = \max(2\,\mathrm{i}_{\mathrm{nM}}(k) - 2, h_k(\mathrm{i}_{\mathrm{nM}}(k)))$. Finally $h_k(\mathrm{i}_{\mathrm{nM}}(k)) \ge 2\,\mathrm{i}_{\mathrm{nM}}(k) - 2 \Leftrightarrow \mathrm{nM}(k - \mathrm{i}_{\mathrm{nM}}(k)+1)+2 \ge \mathrm{i}_{\mathrm{nM}}(k)$, which holds by Part 1. □

We obtain an alternative, functional characterisation of $\mathrm{i}_{\mathrm{nM}}(k)$:

Corollary 7.11 *For $k \in \mathbb{N}$, $k \ge 2$ the value $\mathrm{i}_{\mathrm{nM}}(k) \in \{1, \dots, k\}$ is uniquely characterised by the two inequalities*

$$\begin{aligned} \mathrm{i}_{\mathrm{nM}}(k) &\ge \mathrm{nM}(k - \mathrm{i}_{\mathrm{nM}}(k)+1) \\ \mathrm{i}_{\mathrm{nM}}(k) &\le \mathrm{nM}(k - \mathrm{i}_{\mathrm{nM}}(k)+2). \end{aligned}$$

Proof: As shown in the first part of the proof of Lemma 7.10, the sequence $i \mapsto f_k(i) := i - \mathrm{nM}(k-i+1)$ is strictly increasing. □

7.2 Characterising the jumps

After these preparations we are able to characterise the "jump positions", which are defined as those k where the function nM increases by 2:

Definition 7.12 *Let $\boldsymbol{J} := \{k \in \mathbb{N} : \Delta\,\mathrm{nM}(k) = 2\}$ be the set of **jump positions**.*

Thus $\Delta\,\mathrm{nM}(k) = 1$ iff $k \notin J$, and by Table 1 we see $J = \{1, 4, 11, 26, 57, \dots\}$. Note that $\mathrm{nM}(k) = 1 + k + |\{k' \in J : k' < k\}|$. It is useful to define two auxiliary functions:

Definition 7.13 *Let $\boldsymbol{i'(k)} := k - \mathrm{i}_{\mathrm{nM}}(k) + 1 \in \mathbb{N}$ for $k \in \mathbb{N}$, $k \ge 2$. And let $\boldsymbol{h(k)} := \mathrm{nM}(i'(k)) \in \mathbb{N}$ for $k \in \mathbb{N}$, $k \ge 2$.*

Some basic properties:

1. We have $\Delta i'(k) = 1 - \Delta\,\mathrm{i}_{\mathrm{nM}}(k)$.
2. Thus by Lemma 7.10, Part 2, holds $\Delta i'(k) \in \{0,1\}$.
3. By Lemma 7.10, Part 3, we have $\mathrm{nM}(k) = h(k) + \mathrm{i}_{\mathrm{nM}}(k)$.
4. Thus $\Delta h(k) = \Delta\,\mathrm{nM}(k) - \Delta\,\mathrm{i}_{\mathrm{nM}}(k)$.
5. By Lemmas 7.5 and 7.10, Part 2 we get $\Delta h(k) \in \{0,1,2\}$.
6. By Lemma 7.10, Part 1 we have $\mathrm{i}_{\mathrm{nM}}(k) - h(k) \in \{0,1,2\}$.
7. By Corollary 7.11 we have $h(k) = \mathrm{nM}(i'(k)) \le \mathrm{i}_{\mathrm{nM}}(k) \le \mathrm{nM}(i'(k)+1)$.

It is instructive to consider initial values of the auxiliary functions in Table 2; this table is constructed as follows:

- The values for $\mathrm{nM}(k)$ are from Table 1 (it would be possible to completely construct the whole table row by row, but we leave this as an exercise to the reader, once the section is completed).
- The columns i', h duplicate the columns k, nM, but with repetitions.
- Columns i' and $\mathrm{i_{nM}}$ are connected via $\mathrm{i_{nM}} + i' = k + 1$.
- Columns nM, $\mathrm{i_{nM}}$ and h are connected via $\mathrm{nM} = \mathrm{i_{nM}} + h$.
- The values of column $\mathrm{i_{nM}}$ are determined according to Corollary 7.11 by the condition $h \leq \mathrm{i_{nM}} \leq h'$, where $h'(k)$ is the nM-value following $h(k)$.

k	nM	$\Delta\,$nM	$\mathrm{i_{nM}}$	$\Delta\,\mathrm{i_{nM}}$	i'	$\Delta i'$	h	Δh	$\mathrm{i_{nM}} - h$
1	2	2	-	-	-	-	-	-	-
2	4	1	2	1	1	0	2	0	0
3	5	1	3	1	1	0	2	0	1
4	6	2	4	0	1	1	2	2	2
5	8	1	4	1	2	0	4	0	0
6	9	1	5	0	2	1	4	1	1
7	10	1	5	1	3	0	5	0	0
8	11	1	6	0	3	1	5	1	1
9	12	1	6	1	4	0	6	0	0
10	13	1	7	1	4	0	6	0	1
11	14	2	8	0	4	1	6	2	2
12	16	1	8	1	5	0	8	0	0

Table 2: Values of auxiliary functions; underlined the jump positions

First we show some further simple properties of the auxiliary functions:

Lemma 7.14 *Consider $k \geq 2$.*

1. *If $\Delta\,\mathrm{i_{nM}}(k) = 0$, then:*
   (a) $\Delta\,\mathrm{i_{nM}}(k+1) = 1$.
   (b) $\mathrm{i_{nM}}(k) - h(k) \in \{1, 2\}$.
   (c) $\mathrm{i_{nM}}(k+1) = h(k+1)$.
2. $\Delta\,\mathrm{i_{nM}}(k) = 1 \Leftrightarrow \Delta i'(k) = 0 \Leftrightarrow \Delta h(k) = 0$.
3. *If $\Delta\,\mathrm{i_{nM}}(k) = 1$, then:*
   (a) $k \notin J$.
   (b) $\mathrm{i_{nM}}(k) - h(k) \in \{0, 1\}$.

Proof: For Part 1a assume $\Delta\,\mathrm{i_{nM}}(k+1) = 0$ (and thus $\Delta i'(k+1) = 1$ due to $\Delta i' = 1 - \Delta\,\mathrm{i_{nM}}$). Because of $\Delta h = \Delta\,\mathrm{nM} - \Delta\,\mathrm{i_{nM}}$ we obtain $\Delta h(k+1) \geq 1$. Thus $\mathrm{i_{nM}}(k) = \mathrm{i_{nM}}(k+2) \geq h(k+2) \geq h(k+1)+1 = \mathrm{nM}(i'(k+1))+1 = \mathrm{nM}(i'(k)+1)+1$, contradicting $\mathrm{i_{nM}}(k) \leq \mathrm{nM}(i'(k)+1)$. For the remainder of Part 1 note $\Delta h(k) = \Delta\,\mathrm{nM}(k) \geq 1$.

For Part 1b note $\mathrm{i_{nM}}(k) = \mathrm{i_{nM}}(k+1) \geq h(k+1) \geq h(k) + 1$.

For Part 1c assume $\mathrm{i}_{\mathrm{nM}}(k+1) > h(k+1)$. Thus $\mathrm{i}_{\mathrm{nM}}(k) = \mathrm{i}_{\mathrm{nM}}(k+1) \geq h(k+1)+1 \geq h(k)+2$, whence $\mathrm{i}_{\mathrm{nM}}(k) = h(k)+2$. If we would have $\Delta h(k) = 2$, then $\mathrm{i}_{\mathrm{nM}}(k) = \mathrm{i}_{\mathrm{nM}}(k+1) > h(k+1) = h(k)+2$; thus $h(k+1) = h(k)+1$. Now $\mathrm{i}_{\mathrm{nM}}(k) = h(k)+2 = h(k+1)+1 = \mathrm{nM}(i'(k+1))+1 = \mathrm{nM}(i'(k)+1)+1$, a contradiction.

Part 2 is obvious, and Part 3a follows. Finally, Part 3b follows by $\mathrm{i}_{\mathrm{nM}}(k+1) \leq h(k+1)+2$ and $\mathrm{i}_{\mathrm{nM}}(k+1) = \mathrm{i}_{\mathrm{nM}}(k)+1$, while $h(k+1) = h(k)$ due to Part 2, whence $\mathrm{i}_{\mathrm{nM}}(k) \leq h(k)+1$. □

We obtain the main characterisation of the jump positions via the auxiliary functions:

Theorem 7.15 *For $k \geq 2$ the following conditions are equivalent:*

1. $k \in J$
2. $\Delta h(k) = 2$
3. $\mathrm{i}_{\mathrm{nM}}(k) = h(k)+2$
4. $\Delta\,\mathrm{i}_{\mathrm{nM}}(k-1) = 1$ *and* $\mathrm{i}_{\mathrm{nM}}(k-1) = h(k-1)+1$
5. $\Delta\,\mathrm{i}_{\mathrm{nM}}(k-2) = \Delta\,\mathrm{i}_{\mathrm{nM}}(k-1) = 1$ *(yielding various equivalent forms via Lemma 7.14, Part 2).*

Proof: Condition 1 implies Condition 2 due to $\Delta\,\mathrm{i}_{\mathrm{nM}}(k) = 0$ in case of $k \in J$ by Lemma 7.14, Part 3a. Condition 2 implies Condition 3, since $\Delta h(k) = 2$ implies $\Delta\,\mathrm{i}_{\mathrm{nM}}(k) = 0$, and so by Lemma 7.14, Part 1c we have $\mathrm{i}_{\mathrm{nM}}(k) = \mathrm{i}_{\mathrm{nM}}(k+1) = h(k+1)$, while the assumption says $h(k+1) = h(k)+2$. In turn Condition 3 implies Condition 1, since by Lemma 7.14, Part 3b we get $\Delta\,\mathrm{i}_{\mathrm{nM}}(k) = 0$, and thus $\Delta\,\mathrm{nM}(k) = \Delta h(k)$, where in case of $\Delta h(k) \leq 1$ we would have $h(k)+2 = \mathrm{i}_{\mathrm{nM}}(k) \leq \mathrm{nM}(i'(k)+1) = \mathrm{nM}(i'(k+1)) = h(k+1) \leq h(k)+1$. So now we can freely use the equivalence of these three conditions.

Condition 3 implies Condition 4, since we have $\Delta\,\mathrm{i}_{\mathrm{nM}}(k) = 0$, and thus $\Delta\,\mathrm{i}_{\mathrm{nM}}(k-1) = 1$ with Lemma 7.14, Part 1a, from which we furthermore get $\mathrm{i}_{\mathrm{nM}}(k) = \mathrm{i}_{\mathrm{nM}}(k-1)+1$ and $h(k-1) = h(k)$, and so $\mathrm{i}_{\mathrm{nM}}(k-1) = \mathrm{i}_{\mathrm{nM}}(k)-1 = h(k)+1 = h(k-1)+1$. Condition 4 implies Condition 5, since in case of $\Delta\,\mathrm{i}_{\mathrm{nM}}(k-2) = 0$ we had $\mathrm{i}_{\mathrm{nM}}(k-1) = h(k-1)$ with Lemma 7.14, Part 1c. In turn Condition 5 implies Condition 3, since $\mathrm{i}_{\mathrm{nM}}(k) = \mathrm{i}_{\mathrm{nM}}(k-1)+1 = \mathrm{i}_{\mathrm{nM}}(k-2)+2$, while $h(k) = h(k-1) = h(k-2)$, where by definition $\mathrm{i}_{\mathrm{nM}}(k-2) \geq h(k-2)$ holds, whence $\mathrm{i}_{\mathrm{nM}}(k) \geq h(k)+2$, which implies $\mathrm{i}_{\mathrm{nM}}(k) = h(k)+2$. □

We understand now the shape of the four Δ-sequences:

Corollary 7.16 *By definition the sequence $(\Delta\,\mathrm{nM}(k))_{k\in\mathbb{N}}$ is 1 except at the jump positions k, where it is 2. The other three Δ-sequences are shaped as follows:*

1. *The sequence $(\Delta\,\mathrm{i}_{\mathrm{nM}}(k))_{k\in\mathbb{N},k\geq 2}$ consists of alternating $0,1$'s except the two positions $k-2, k-1$ before a jump position $k \in J$, where we have two consecutive 1's (while at the jump position we have 0).*
2. *The sequence $(\Delta i'(k))_{k\in\mathbb{N},k\geq 2}$ consists of alternating $0,1$'s except two positions before a jump position k, where we have two consecutive 0's.*
3. *The sequence $(\Delta h(k))_{k\in\mathbb{N},k\geq 2}$ consists of alternating $0,1$'s except two positions before a jump position k, where we have two consecutive 0's, followed by a 2 at the jump position k, which is followed by 0.*

Proof: Part 1: By Lemma 7.14, 1a we have $\Delta\,\mathrm{i_{nM}}(k) = 0 \Rightarrow \Delta\,\mathrm{i_{nM}}(k+1) = 1$, while by Theorem 7.15, Part 5 we have $\Delta\,\mathrm{i_{nM}}(k) = \Delta\,\mathrm{i_{nM}}(k+1) = 1 \Rightarrow k+2 \in J$, and by Lemma 7.14, Part 3a we have $k \in J \Rightarrow \Delta\,\mathrm{i_{nM}}(k) = 0$.

Part 2 follows from Part 1 by $\Delta i' = 1 - \Delta\,\mathrm{i_{nM}}$.

Part 3: By Lemma 7.14, Part 2 the $0's$ in the sequence Δh are precisely the 1's in the sequence $\Delta\,\mathrm{i_{nM}}$, while a 0 of $\Delta\,\mathrm{i_{nM}}$ translates into a 2 precisely at the jump positions by Theorem 7.15, Part 2. The assertion follows now by Part 1. $\square$

Especially instructive is understanding of the i'-sequence:

Corollary 7.17 *The i'-sequence $(i'(k))_{k\in\mathbb{N},k\geq 2}$ consists of doublets m, m for consecutive $m = 1, 2, \ldots,$, except for $k \in J\setminus\{1\}$, where we have at positions $k-2, k-1, k$ a triplet m, m, m. These triplet-values occur exactly when $m \in J$.*

Proof: The doublet/triplet structure follows by Corollary 7.16, Part 2. Now consider a triplet $i'(k-2) = i'(k-1) = i'(k) = m$ for $k \in J \setminus \{1\}$, $m \in \mathbb{N}$. By definition we have $\Delta\,\mathrm{nM}(m) = \Delta h(k)$ (due to $h(k) = h(i'(k)) = \mathrm{nM}(m)$, $h(k+1) = \mathrm{nM}(i'(k+1)) = \mathrm{nM}(i'(k)+1) = \mathrm{nM}(m+1)$). By Theorem 7.15, Part 2 we have thus have $\Delta\,\mathrm{nM}(m) = 2$, i.e., $m \in J$. The triplets do not leave out some jump-value in J, since for $m \in J$ and for the last position k with $i'(k) = m$ we have $\Delta\,\mathrm{nM}(m) = \Delta h(k)$. $\square$

Example 7.18 *We see now how we can built up the three columns k, nM, i' of Table 2 together with an enumeration of the set J, which is built up as the set I (I is an initial part of J, which in the limit becomes J):*

1. *We start with the first row $k := 1$, initialising the value n of $\mathrm{nM}(1) = n$ to $n := 2$, while $\mathrm{i_{nM}}$ is undefined; $k = 1$ is the first jump position, that is, $I := \{1\}$.*
2. *We go to the second row, $k := k+1$. We update $n := n+2$ and initialise the running value of $i'(k) = m$ to $m := 1$.*
3. *We repeat the following steps ad infinitum:*
   (a) *If $m \in I$, then three rows are created:*
      i. $\mathrm{i_{nM}}(k) = n$, $i'(k) = m$, $k := k+1$, $n := n+1$
      ii. $\mathrm{i_{nM}}(k) = n$, $i'(k) = m$, $k := k+1$, $n := n+1$
      iii. $\mathrm{i_{nM}}(k) = n$, $i'(k) = m$, $I := I \,\dot\cup \{k\}$, $k := k+1$, $n := n+2$, $m := m+1$.
   (b) *If $m \notin I$, then two rows are created:*
      i. $\mathrm{i_{nM}}(k) = n$, $i'(k) = m$, $k := k+1$, $n := n+1$
      ii. $\mathrm{i_{nM}}(k) = n$, $i'(k) = m$, $k := k+1$, $n := n+1$, $m := m+1$.

Next we show that $i'(k)$ for jump positions is the previous jump position:

Lemma 7.19 *For $k \in J$, $k \geq 2$, holds $i'(k) = \max\{k' \in J : k' < k\}$.*

Proof: We prove the assertion by induction on k (regarding the enumeration of J). We have $i'(4) = 1$, and so the induction holds for $k = 4$, the smallest jump position $k \geq 2$. Now assume that the assertion holds for all elements of $J \cap \{1, \ldots, k-1\}$, where $k > 4$, and we have to show the assertion for k. By Corollary 7.17 we know $i'(k) \in J$, where $2 \leq i'(k) < k$. Assume there is $m \in J$ with $i'(k) < m < k$. By induction hypothesis we get $i'(k) \leq i'(m) < m$. However by Lemma 7.14 we get $\Delta i'(m) = 1$, and thus $i'(k) > i'(m)$ (since $k > m$). $\square$

We obtain the promised characterisation of the jump positions:

Lemma 7.20 *We have* $J = \{2^{m+1} - (m+1) - 1 : m \in \mathbb{N}\}$.

Proof: Let k_m for $m \in \mathbb{N}$ be the mth element of J; so the assertion is $k_m = 2^{m+1} - m - 2$. We have $k_1 = 4 - 1 - 2 = 1 = \min J$; in the remainder assume $m \geq 2$. We prove the assertion by induction, in parallel with $\mathrm{i}_{\mathrm{nM}}(k_m) = 2^{m+1} - 2^m$. For $m = 2$ we have $k_2 = 8 - 2 - 2 = 4 = \min J \setminus \{1\}$, while $\mathrm{i}_{\mathrm{nM}}(4)$ is the smallest $i \in \{2,3,4\}$ with $i \geq \mathrm{nM}(5-i)$, which yields $\mathrm{i}_{\mathrm{nM}}(4) = 4 = 2^3 - 2^2$. Now we consider the induction step, from $m-1$ to m. The induction hypothesis yields $k_{m-1} = 2^m - m - 1$ and $\mathrm{i}_{\mathrm{nM}}(k_{m-1}) = 2^m - 2^{m-1}$. Lemma 7.19 yields $i'(k_m) = k_{m-1}$, from which by $i'(k_m) = k_m - \mathrm{i}_{\mathrm{nM}}(k_m) + 1$ follows

$$k_m = 2^m - m - 2 + \mathrm{i}_{\mathrm{nM}}(k_m).$$

Via a telescoping series we get

$$\mathrm{i}_{\mathrm{nM}}(k_m) = \Delta\,\mathrm{i}_{\mathrm{nM}}(k_m - 1) + \cdots + \Delta\,\mathrm{i}_{\mathrm{nM}}(k_{m-1}) + \mathrm{i}_{\mathrm{nM}}(k_{m-1}).$$

By Corollary 7.16, Part 1 the sequence $\Delta\,\mathrm{i}_{\mathrm{nM}}(k_{m-1}), \ldots, \Delta\,\mathrm{i}_{\mathrm{nM}}(k_m - 1)$ has the form $0,1,\ 0,1,\ \ldots,0,1,1$, and thus their sum has the value $\frac{1}{2}(k_m - k_{m-1} - 1) + 1$. So we get

$$\begin{aligned}\mathrm{i}_{\mathrm{nM}}(k_m) = \frac{1}{2}&(k_m - k_{m-1} - 1) + 1 + \mathrm{i}_{\mathrm{nM}}(k_{m-1}) = \\ &\frac{1}{2}(2^m - m - 2 + \mathrm{i}_{\mathrm{nM}}(k_m) - 2^m + m + 1 - 1) + 1 + 2^m - 2^{m-1} = \\ &\qquad\qquad\qquad\qquad \frac{1}{2}\,\mathrm{i}_{\mathrm{nM}}(k_m) - 1 + 1 + 2^m - 2^{m-1},\end{aligned}$$

from which $\mathrm{i}_{\mathrm{nM}}(k_m) = 2^{m+1} - 2^m$ follows. Finally $k_m = 2^m - m - 2 + 2^{m+1} - 2^m = 2^{m+1} - m - 2$. $\square$

7.3 Applications

Now the closed formula for $\mathrm{nM}(k)$ can be proven:

Theorem 7.21 *For* $k \in \mathbb{N}$ *let* $\mathrm{fld}(k) := \lfloor \mathrm{ld}(k) \rfloor$. *Then we have for* $k \in \mathbb{N}$ *the equality* $\mathrm{nM}(k) = k + \mathrm{fld}(k + 1 + \mathrm{fld}(k+1))$.

Proof: Let $g(k) := \mathrm{fld}(k+1+\mathrm{fld}(k+1))$ and $f(k) := k + g(k)$ (so $\mathrm{nM}(k) = f(k)$ is to be shown, for $k \geq 1$). We have $f(1) = 1 + \mathrm{fld}(2 + \mathrm{fld}(2)) = 1 + \mathrm{fld}(3) = 2 = \mathrm{nM}(1)$. We will now prove that the function $g(k)$ changes values exactly at the transitions $k \mapsto k+1$ for $k \in J$, that is, for indices $k = k_m := 2^{m+1} - m - 2$ (using Lemma 7.20) with $m \in \mathbb{N}$ we have $\Delta g(k_m) = 1$, while otherwise we have $\Delta g(k_m) = 0$, from which the assertion follows (by the definition of J).

We have $g(1) = 1$ and $g(2) = 2$. Now consider $m \in \mathbb{N}$ and $k_m + 1 \leq k \leq k_{m+1}$. We show $g(k) = m + 1$, which proves the claim. Note that $g(k)$ is monotonically increasing. Now $g(k) \geq g(k_m + 1) = \lfloor \mathrm{ld}(2^{m+1} - m + \lfloor \mathrm{ld}(2^{m+1} - m) \rfloor) \rfloor = \lfloor \mathrm{ld}(2^{m+1} - m + m) \rfloor = m + 1$ and $g(k) \leq g(k_{m+1}) = \lfloor \mathrm{ld}(2^{m+2} - m - 2 + \lfloor \mathrm{ld}(2^{m+2} - m - 2) \rfloor) \rfloor \leq \lfloor \mathrm{ld}(2^{m+2} - m - 2 + m + 1) \rfloor = \lfloor \mathrm{ld}(2^{m+2} - 1) \rfloor = m + 1$. $\square$

As a result, we obtain very precise bounds for $\mathrm{nM}(k)$:

Corollary 7.22 *$k + \mathrm{fld}(k+1) \le \mathrm{nM}(k) \le k + 1 + \mathrm{fld}(k)$ holds for $k \in \mathbb{N}$.*

Proof: The lower bound follows trivially. The upper bound holds (with equality) for $k \le 2$, so assume $k \ge 3$. We have to show $g(k) = \mathrm{fld}(k+1+\mathrm{fld}(k+1)) \le 1+\mathrm{fld}(k)$, which follows from $\mathrm{ld}(k+1+\mathrm{fld}(k+1)) \le 1 + \mathrm{ld}(k)$. Now $\mathrm{ld}(k+1+\mathrm{fld}(k+1)) \le \mathrm{ld}(k+1+\mathrm{ld}(k+1)) \le \mathrm{ld}(k+k) = 1 + \mathrm{ld}(k)$. □

Note that $(k+1+\mathrm{fld}(k)) - (k+\mathrm{fld}(k+1)) \in \{0,1\}$, where this difference is zero iff $k+1$ is a power of 2. Finally we can prove the already mentioned characterisation, which motivates the terminology of "non-Mersenne numbers", namely that $(\mathrm{nM}(k))_{n\in\mathbb{N}}$ enumerates $\mathbb{N} \setminus \{2^n - 1 : n \in \mathbb{N}\}$.[14] For that we consider the positions directly after the jump positions, which by Lemma 7.20 are the positions $2^n - n$ for $n \ge 2$. From that position on until the next jump position, which is $2^{n+1} - n - 2$, the nM-values increase constantly by 1 per step. So we just need to understand the values of $\mathrm{nM}(2^n - n)$, to understand all of nM, which is achieved as follows (note that $(2^{n+1} - n - 2) - (2^n - n) = 2^n - 2$):

Corollary 7.23 *Consider $n \in \mathbb{N}$, $k := 2^n - n$, and $m \in \mathbb{N}_0$ with $m \le 2^n - 1$.*

1. $\mathrm{nM}(k) = 2^n$.
2. *More generally for $m < 2^n - 1$ holds* $\mathrm{nM}(k+m) = 2^n + m$.
3. *For $m = 2^n - 1$ we have $k+m = 2^{n+1} - (n+1)$, and thus* $\mathrm{nM}(k+m) = 2^{n+1}$.

Proof: By Theorem 7.21, Part 1 follows with $\mathrm{nM}(2^n - n) = 2^n - n + \mathrm{fld}(2^n - n + 1 + \mathrm{fld}(2^n - n + 1)) = 2^n - n + \mathrm{fld}(2^n - n + 1 + (n-1)) = 2^n - n + \mathrm{fld}(2^n) = 2^n$. Part 2 follows by Lemma 7.20, and Part 3 follows by Part 1. □

Besides $\mathrm{nM}(2^n - n) = 2^n$ also the following special value is of importance:

Corollary 7.24 *For $n \in \mathbb{N}$, $n \ge 2$, we have* $\mathrm{nM}(2^n - n - 1) = 2^n - 2$.

It is also useful to have simple formulas for the $\mathrm{i}_{\mathrm{nM}}(k)$-values around the jump positions:

Corollary 7.25 *For $n \in \mathbb{N}$, $n \ge 3$ the values of $\mathrm{i}_{\mathrm{nM}}(2^n - n + m)$ are as follows, using $p := 2^{n-1}$ (where for $m = -4$ we need $n \ge 4$):*

m	-4	-3	-2	-1	0	1	2	3	4
i_{nM}	$p-2$	$p-2$	$p-1$	p	p	$p+1$	$p+1$	$p+2$	$p+2$

Proof: We have $\mathrm{i}_{\mathrm{nM}}(2^n - n) = 2^{n-1}$ by Corollary 7.11:

$$\begin{aligned} 2^{n-1} &\ge \mathrm{nM}(2^n - n - 2^{n-1} + 1) = \mathrm{nM}(2^{n-1} - (n-1)) = 2^{n-1} \\ 2^{n-1} &\le \mathrm{nM}(2^n - n - 2^{n-1} + 2) = \mathrm{nM}(2^{n-1} - (n-1) + 1) = 2^{n-1} + 1. \end{aligned}$$

The remaining values follow by Corollary 7.16, Part 1. □

We conclude with an alternative characterisation of the jump-set J:

Corollary 7.26 *For $k \in \mathbb{N}$ the following conditions are equivalent:*

1. $\mathrm{nM}(k) < 2 \cdot \mathrm{i}_{\mathrm{nM}}(k) - 1$.
2. $\mathrm{nM}(k) = 2 \cdot \mathrm{i}_{\mathrm{nM}}(k) - 2$.

[14] Note that we are not speaking of "non-Mersenne *primes*".

3. $k \in J$*, that is,* $k = 2^{m+1} - m - 2$ *for some* $m \in \mathbb{N}$*.*

Proof: If $k \in J$, then by Theorem 7.15, Part 3, we have $\mathrm{nM}(k) = 2 \cdot \mathrm{i}_{\mathrm{nM}}(k) - 2 < 2 \cdot \mathrm{i}_{\mathrm{nM}}(k) - 1$. And if $k \notin J$, then by the same lemma we have $\mathrm{i}_{\mathrm{nM}}(k) \le h(k) + 1$, and thus $\mathrm{nM}(k) = h(k) + \mathrm{i}_{\mathrm{nM}}(k) \ge 2 \cdot \mathrm{i}_{\mathrm{nM}}(k) - 1$. $\square$

8 The min-var-degree upper bound for $\mathcal{MU}$

In a sense the main auxiliary lemma of this report is the following statement on the deficiencies obtained when splitting a saturated minimally unsatisfiable clause-set, which receives its importance from the fact that every minimally unsatisfiable clause-set can be saturated (recall Subsection 3.2; this method was first applied in this context in [49]).

Lemma 8.1 *Consider* $F \in \mathcal{SMU}_{\delta=k}$ *for* $k \in \mathbb{N}$ *and a variable* $v \in \mathrm{var}(F)$ *realising the minimum var-degree (i.e.,* $\mathrm{vd}_F(v) = \mu\mathrm{vd}(F)$*). Using* $m_0 := \mathrm{ld}_F(\overline{v})$ *and* $m_1 := \mathrm{ld}_F(v)$ *we have* $\langle v \to \varepsilon \rangle * F \in \mathcal{MU}_{\delta=k-m_\varepsilon+1}$ *for* $\varepsilon \in \{0,1\}$*, where* $n(\langle v \to \varepsilon \rangle * F) = n(F) - 1$*. Since minimally unsatisfiable clause-sets have deficiency at least one, we get* $m_\varepsilon \le k$*.*

Proof: We have $n(\langle v \to \varepsilon \rangle * F) = n(F) - 1$ since F contains no pure variable, while v realises the minimum of var-degrees. Thus $\delta(\langle v \to \varepsilon \rangle * F) = \delta(F) - m_\varepsilon + 1$, while $\langle v \to \varepsilon \rangle * F \in \mathcal{MU}$ by Lemma 3.14, Part 1. $\square$

Some explanations on this fundamental lemma:

Example 8.2 *If in the situation of Lemma 8.1 the value of* m_ε *is minimal, i.e.,* $m_\varepsilon = 1$*, then we have* $\delta(\langle v \to \varepsilon \rangle * F) = \delta(F) = k$*, while if* m_ε *is maximal, i.e.,* $m_\varepsilon = k$*, then we have* $\delta(\langle v \to \varepsilon \rangle * F) = 1$*. The deficiency is strictly decreased for both splitting results iff* v *is non-singular. The point of* v *realising the minimum var-degree is, that we have control over the number of eliminated variables (namely no further variable is eliminated). If* $v \in \mathrm{var}(F)$ *is arbitrary, then* $\delta(\langle v \to \varepsilon \rangle * F) = k - m_\varepsilon + 1 + r$*, where* r *is the number of variables in* F *which occur only in the clauses containing* $\overline{v}$ *for* $\varepsilon = 0$ *resp. in the clauses containing* v *for* $\varepsilon = 1$*.*

A class of concrete examples is given by the $\mathcal{F}_n \in \mathcal{SMU}'_{\delta=2}$ *(*$n \ge 2$*; recall Example 3.3), where for every* $v \in \mathrm{var}(\mathcal{F}_n)$ *and* $\varepsilon \in \{0,1\}$ *holds* $\langle v \to \varepsilon \rangle * \mathcal{F}_n \in \mathcal{MU}_{\delta=1}$ *(since every literal of* $\mathcal{F}_n$ *has degree* 2*).*

The definition of $\mathrm{nM}(k)$ (recall Definition 7.1) matches the recursion-structure of Lemma 8.1, and we obtain an upper bound on the min-var-degree for minimally unsatisfiable clause-sets:

Theorem 8.3 *For all* $k \in \mathbb{N}$ *and* $F \in \mathcal{MU}_{\delta \le k}$ *we have* $\mu\mathrm{vd}(F) \le \mathrm{nM}(k)$*. More precisely, for* $n(F) > 0$ *there exists a variable* $v \in \mathrm{var}(F)$ *with* $\mathrm{vd}_F(v) \le \mathrm{nM}(k)$ *and* $\mathrm{ld}_F(v), \mathrm{ld}_F(\overline{v}) \le k$*.*

Proof: The assertion is known for $k = 1$, so assume $k > 1$, and we apply induction on k. Assume $\delta(F) = k$ (due to $k > 1$ we have $n(F) > 1$). Saturate F and obtain F'. Consider a variable $v \in \mathrm{var}(F')$ realising the min-var-degree of F'. If $\mathrm{vd}_{F'}(v) = 2$ then we are done, so assume $\mathrm{vd}_{F'}(v) \ge 3$. Let $i := \max(\mathrm{ld}_{F'}(v), \mathrm{ld}_{F'}(\overline{v}))$; so $\mathrm{vd}_{F'}(v) \le 2i$. W.l.o.g. assume that $i = \mathrm{ld}_{F'}(v)$. By Lemma 8.1 we get $2 \le i \le k$. Applying the induction hypothesis and Lemma 8.1 we obtain a variable $w \in \mathrm{var}(G)$ for $G := \langle v \to 1 \rangle * F$ with $\mathrm{vd}_G(w) \le \mathrm{nM}(k-i+1)$. By definition we have $\mathrm{vd}_{F'}(w) \le \mathrm{vd}_G(w) + \mathrm{ld}_{F'}(v)$. Altogether $\mu\mathrm{vd}(F) \le \min(2i, \mathrm{nM}(k-i+1) + i) \le \mathrm{nM}(k)$. $\square$

The upper bound on the minimum variable degree of Theorem 8.3 is not sharp, and will be further investigated from Section 12 on. However the bound is attained for infinitely many deficiencies, and we demonstrate now that the jump positions (the set J; recall Definition 7.12) are such deficiencies. Moreover, to investigate the remaining deficiencies, we show that they always have at least two variables realising the bound (if the bound is attained at all); this will be used to prove Theorem 14.3. So we consider "extremal" $F \in \mathcal{MU}_{\delta=k}$ with $\mu\mathrm{vd}(F) = \mu\mathrm{nM}(k)$, and we show that such extremal clause-sets have at least two different variables of minimal degree, if $k \notin J$. First, it is useful to have a notation for the set of variables of minimal degree:

Definition 8.4 *For $F \in \mathcal{CLS}$ let* $\mathbf{var}_{\boldsymbol{\mu}\mathbf{vd}}\boldsymbol{(F)} \subseteq \mathrm{var}(F)$ *be the set of variables of minimal degree, that is,* $\mathrm{var}_{\mu\mathrm{vd}}(F) := \{v \in \mathrm{var}(F) : \mathrm{vd}_F(v) = \mu\mathrm{vd}(F)\}$.

Obviously $\mathrm{var}_{\mu\mathrm{vd}}(F) \neq \emptyset$ iff $n(F) > 0$, and $\mathrm{var}_{\mu\mathrm{vd}}(F) = \mathrm{var}(F)$ holds iff F is variable-regular.

Lemma 8.5 *Consider $k \in \mathbb{N}$.*

1. *For $k \notin J$ and $F \in \mathcal{MU}_{\delta=k}$ with $\mu\mathrm{vd}(F) = \mathrm{nM}(k)$ we have $|\mathrm{var}_{\mu\mathrm{vd}}(F)| \geq 2$.*
2. *For $k \in J$ there is $F \in \mathcal{UHIT}_{\delta=k}$ with $\mu\mathrm{vd}(F) = \mathrm{nM}(k)$ and $|\mathrm{var}_{\mu\mathrm{vd}}(F)| = 1$.*

Proof: First assume $k \notin J$; we have to show the existence of different $v, w \in \mathrm{var}_{\mu\mathrm{vd}}(F)$. W.l.o.g. F is saturated. Consider $v \in \mu\mathrm{vd}(F)$. By Corollary 7.26 we have $\mathrm{nM}(k) \geq 2 \cdot \mathrm{i}_{\mathrm{nM}}(k) - 1$. Because of $\mathrm{ld}_F(v) + \mathrm{ld}_F(\overline{v}) = \mathrm{nM}(k)$ thus w.l.o.g. $e_1 := \mathrm{ld}_F(v) \geq \mathrm{i}_{\mathrm{nM}}(k)$. Let $F' := \langle v \to 1 \rangle * F$. So $\delta(F') = k - e_1 + 1$. Recall $\mathrm{nM}(k) = \mathrm{nM}(k - \mathrm{i}_{\mathrm{nM}}(k) + 1) + \mathrm{i}_{\mathrm{nM}}(k)$ (Lemma 7.10, Part 3), and thus $\mathrm{nM}(k) \geq \mathrm{nM}(k - e_1 + 1) + e_1$. Since $n(F) \geq 2$, we can consider $w \in \mathrm{var}_{\mu\mathrm{vd}}(F')$. We have $\mathrm{vd}_{F'}(w) \leq \mathrm{nM}(k - e_1 + 1)$ and $\mathrm{vd}_F(w) = \mathrm{vd}_{F'}(w) + e_1$. Thus $w \in \mathrm{var}_{\mu\mathrm{vd}}(F)$.

Now assume $k \in J$, that is, $k = 2^{m+1} - m - 2$ for $m \geq 1$. For $k = 1$ we have the example $\{1, -1\}$, so assume $k \geq 2$. Then we have $\mathrm{nM}(k) = 2^{m+1} - 2$. Now we obtain an example from A_{m+1} by performing one strict full subsumption resolution: The resolution variable occurs $2^{m+1} - 2$ times, the other $m - 1$ variables occur $2^{m+1} - 1$ times. □

In Lemma 12.11 we formulate the sharpness of the upper bound of Theorem 8.3 for these cases.

9 The min-var-degree upper bound for $\mathcal{LEAN}$

In this section we prove Theorem 9.8, the upper bound $\mathrm{nM}(k)$ on the min-var-degree for lean clause-sets of deficiency k, and the sharpness of this upper bound for any class between $\mathcal{VMU}$ and $\mathcal{LEAN}$ in Theorem 9.12. The proof consists in lifting Theorem 8.3 to the general case in Subsection 9.2, while in Subsection 9.1 we introduce the auxiliary class $\mathcal{SED}$ of clause-sets, where deficiency and surplus coincide; Lemma 9.5 there shows that unsatisfiable elements of $\mathcal{SED}$ are variable-minimally unsatisfiable. Sharpness of the upper bound in considered in Subsection 9.3.

9.1 Clause-sets with extremal surplus

We consider the task of generalising Theorem 8.3 to $F \in \mathcal{LEAN}$. Consider an arbitrary (multi-)clause-set F. Consider a set of variables $\emptyset \neq V \subseteq \mathrm{var}(F)$ realising the surplus of F, i.e., such that $\delta(F[V])$ is minimal (recall Definition 2.19). If $F[V]$ would be satisfiable, then a satisfying assignment would give a non-trivial autarky for F. Assuming that F is lean thus yields that $F[V]$ must be unsatisfiable. So there exists a minimally unsatisfiable $F' \subseteq F[V]$. If now $\mathrm{var}(F') \neq \mathrm{var}(F[V]) = V$ would be the case, then we would loose control over the deficiency of F'. Fortunately this can not happen, as we will show in Lemma 9.5. To understand this result, the following class of clause-sets with maximal surplus (relative to the deficiency) is important.

Definition 9.1 *Let the class $\boldsymbol{\mathcal{SED}} \subset \mathcal{CLS}$ ("surplus equal deficiency") consist of those clause-sets $F \in \mathcal{CLS}$ with $\sigma(F) = \delta(F)$.*

It seems the class $\mathcal{SED}$ crosses the classes considered in this report in interesting extremal cases.

Example 9.2 *Some basic examples:*

1. *We have $\top \in \mathcal{SED}$ and $\{\bot\} \notin \mathcal{SED}$, and more generally, for every $F \in \mathcal{SED}$ we have $\bot \notin F$.*
2. *For a clause $C \in \mathcal{CL}$ we have $\delta(\{C\}) = 1 - |C|$ and $\sigma(\{C\}) = 0$, and thus $\{C\} \in \mathcal{SED} \Leftrightarrow |C| = 1$.*
3. *For $F := \{\{1\}, \{2\}\}$ we have $\sigma(F) = \delta(F) = 0$, and thus $F \in \mathcal{SED}$. However for the multi-clause-set $F' := \{2 * \{1\}, \{2\}\}$ we have $\delta(F') = 1$, while still $\sigma(F') = 0$, and thus $F' \notin \underline{\mathcal{SED}}$.*
4. *Another example for $F \in \mathcal{SED}$ with $\delta(F) = 0$ is $F := \{\{1, 2\}, \{-1, 2\}\}$.*
5. *$A_n \in \mathcal{SED}$ for $n \geq 1$ (Example 2.20).*
6. *$\mathcal{MU}_{\delta=1} \setminus \{\{\bot\}\} \subset \mathcal{SED}$ (since for $F \in \mathcal{MU} \setminus \{\{\bot\}\}$ holds $\sigma(F) \geq 1$).*
7. *$\mathcal{F}_n \in \mathcal{SED}$ for $n \geq 2$ (Example 3.3).*
8. *For $F := \{\{1, 2, 3\}, \{1, 2, -3\}, \{1, -2\}, \{-1, 2\}, \{-1, -2\}\}$ we have $F \in \mathcal{MU}$ with $\delta(F) = 2$, but $\sigma(F) = 1$, and thus $F \notin \mathcal{SED}$.*
9. *In Definition 10.5 we introduce the subclass $\mathcal{MLCR} \subset \mathcal{SED} \cap \mathcal{SAT}$, and Example 10.7 shows elements of this class.*
10. *See also Example 10.11 and Question 10.12.*

Finally we note that $F \in \underline{\mathcal{SED}}$ if $F' \in \underline{\mathcal{SED}}$, where F' is the multi-clause-set obtained from F by forgetting all signs of the literals, i.e., replacing clauses $C \in F$ by $\mathrm{var}(C)$ (since $\delta(F') = \delta(F)$ and $\sigma(F') = \sigma(F)$).

The corresponding class $\underline{\mathcal{SED}}$ of multi-clause-sets is not invariant under multiplicities; consider a multi-clause-set F and the underlying clause-set F':

1. If $F' \in \mathcal{SED}$, then in general we do not have $F \in \underline{\mathcal{SED}}$ (Example 9.2).
2. However in general holds $F \in \underline{\mathcal{SED}} \Rightarrow F' \in \mathcal{SED}$, since if we would have $F' \notin \mathcal{SED}$, then $\sigma(F') < \delta(F')$, and adding a duplicated clause to a multi-clause-set increases δ by $+1$, while σ is at most increased by $+1$ (it may also stay unchanged).

A simple but instructive equivalent formulation of $\underline{\mathcal{SED}}$ is as follows, which also yields a stronger corollary than the above "$F \in \underline{\mathcal{SED}} \Rightarrow F' \in \mathcal{SED}$":

Lemma 9.3 *For a multi-clause-set F we have $F \in \underline{\mathcal{SED}}$ if and only if for all $\emptyset \subseteq V \subset \mathrm{var}(F)$ and for the sub-multi-clause-set $F^V \le F$ consisting of all $C \in F$ with $\mathrm{var}(C) \subseteq V$ (with the same multiplicities) we have $c(F^V) \le |V|$.*

Proof: For arbitrary $F \in \underline{\mathcal{CLS}}$ and every $\emptyset \subset V \subseteq \mathrm{var}(F)$ we have

$$c(F[V]) + c(F^{\mathrm{var}(F)\setminus V}) = c(F),$$

and thus we get, using $V' := \mathrm{var}(F) \setminus V$:

$$\delta(F[V]) \ge \delta(F) \Leftrightarrow c(F[V]) - |V| \ge c(F) - n(F) \Leftrightarrow \\ c(F) - c(F^{\mathrm{var}(F)\setminus V}) - |V| \ge c(F) - n(F) \Leftrightarrow c(F^{V'}) \le |V'|.$$

These V' run through all $\emptyset \subseteq V' \subset \mathrm{var}(F)$. $\square$

We remark that $c(F^{\emptyset}) \le 0 \Leftrightarrow \bot \notin F$. We obtain as an immediate corollary, that decreasing multiplicities in $F \in \underline{\mathcal{SED}}$ does not leave this classes (even if the multiplicity drops to zero):

Corollary 9.4 *For $F \in \underline{\mathcal{SED}}$ and $F' \le F$ we have $F' \in \underline{\mathcal{SED}}$.*

Unsatisfiable elements of $\mathcal{SED}$ have a strong structure:

Lemma 9.5 *$\mathcal{SED} \cap \mathcal{USAT} \subset \mathcal{VMU}$ (and thus $\underline{\mathcal{SED}} \cap \underline{\mathcal{USAT}} \subset \underline{\mathcal{VMU}}$).*

Proof: Consider $F \in \mathcal{SED} \cap \mathcal{USAT}$, and assume there is an unsatisfiable $F' \subseteq F$ with $\mathrm{var}(F') \subset \mathrm{var}(F)$; consider a minimally unsatisfiable sub-clause-set $F'' \subseteq F'$. By definition we have for $V := \mathrm{var}(F) \setminus \mathrm{var}(F'') \neq \emptyset$:

$$\delta(F'') = c(F'') - n(F'') = c(F'') - (n(F) - n(F[V]) \le \\ (c(F) - c(F[V])) - (n(F) - n(F[V]) = \delta(F) - \delta(F[V]) \le \sigma(F) - \sigma(F) = 0$$

contradicting $\delta(F'') \ge 1$ (since $F'' \in \mathcal{MU}$). Thus we have $\mathcal{SED} \cap \mathcal{USAT} \subseteq \mathcal{VMU}$. An example of $F \in \mathcal{MU} \setminus \mathcal{SED}$ is given in (Example 9.2. Finally consider $F \in \underline{\mathcal{SED}} \cap \underline{\mathcal{USAT}}$: thus for the underlying clause-set F' holds $F' \in \mathcal{SED} \cap \mathcal{USAT}$, whence $F' \in \mathcal{VMU}$, and thus $F \in \underline{\mathcal{VMU}}$. $\square$

We conclude this subsection by considering the complexity of SAT decision for $F \in \mathcal{SED}_{\delta=k}$ for parameter $k \in \mathbb{N}$. By Lemma 9.5 we could use Theorem 4.7, however we have $\mathcal{SED}_{\delta=k} \subset \mathcal{MLEAN}$, and thus we can apply the fpt-result discussed in Example 2.18, and thus SAT decision for inputs in $\mathcal{SED}_{\delta=k}$ is fpt in k.

Question 9.6 *Can SAT decision for $\mathcal{SED}$ be done in polynomial time? If so, can we also find a satisfying assignment quickly?*

9.2 The generalised upper bound

Back to the main task, the central lemma utilises Lemma 9.5 to show that from extremal $F[V]$ we obtain variables of low degree for F itself:

Lemma 9.7 *Consider a multi-clause-set F and $\emptyset \subset V \subseteq \mathrm{var}(F)$ such that $F[V]$ is unsatisfiable and $\sigma(F[V]) = \delta(F[V]) \ge 1$ (whence $F[V] \in \underline{\mathcal{SED}} \cap \underline{\mathcal{MLEAN}}$). Then there exists $v \in V$ with $\mathrm{vd}_F(v) \le \mathrm{nM}(\delta(F[V]))$ and $\mathrm{ld}_F(v), \mathrm{ld}_F(\overline{v}) \le \delta(F[V])$.*

Proof: Let $F' := F[V]$ and consider some minimally unsatisfiable $F'' \subseteq F'$. By Lemma 9.5 we have $\mathrm{var}(F'') = \mathrm{var}(F')$. So we get $\delta(F'') = \delta(F') - (c(F') - c(F''))$. By Theorem 8.3 there is $v \in \mathrm{var}(F'')$ with

$$\mathrm{vd}_{F''}(v) \le \mathrm{nM}(\delta(F'')) = \mathrm{nM}(\delta(F') - (c(F') - c(F''))) \le \\ \mathrm{nM}(\delta(F')) - (c(F') - c(F''))$$

and $\mathrm{ld}_{F''}(v), \mathrm{ld}_{F''}(\overline{v}) \le \delta(F'') = \delta(F') - (c(F') - c(F''))$. Finally we have $\mathrm{vd}_F(v) \le \mathrm{vd}_{F''}(v) + (c(F') - c(F''))$ (note that all occurrences of v in F are also in F'), and similarly for the literal degrees. □

We are ready to show the generalisation and strengthening of Theorem 8.3:

Theorem 9.8 *We have $\mu\mathrm{vd}(F) \le \mathrm{nM}(\sigma(F))$ for a lean multi-clause-set F with $n(F) > 0$. More precisely, there exists a variable $v \in \mathrm{var}(F)$ with $\mathrm{vd}_F(v) \le \mathrm{nM}(\sigma(F))$ and $\mathrm{ld}_F(v), \mathrm{ld}_F(\overline{v}) \le \sigma(F)$.*

Proof: Recall that F is a lean multi-clause-set with $n(F) > 0$, and we have to show the existence of a variable v with $\mathrm{vd}_F(v) \le \mathrm{nM}(\sigma(F))$ and $\mathrm{ld}_F(v), \mathrm{ld}_F(\overline{v}) \le \sigma(F)$.

Consider $\emptyset \ne V \subseteq \mathrm{var}(F)$ with $\delta(F[V]) = \sigma(F)$, and let $F' := F[V]$. F' is unsatisfiable, since F is lean. Because of $\delta(F') = \sigma(F)$ we have $\delta(F') = \sigma(F')$. So we can apply Lemma 9.7. □

Since for a variable $v \in \mathrm{var}(F)$ for any $F \in \mathcal{CLS}$ holds

$$\delta(F[\{v\}]) = \mathrm{vd}_F(v) - 1 \ge \mu\mathrm{vd}(F) - 1 \ge \sigma(F),$$

and the surplus is a lower bound for the deficiency, we get:

Corollary 9.9 *For a lean multi-clause-set F, $n(F) > 0$, we have*

$$\sigma(F) + 1 \le \mu\mathrm{vd}(F) \le \mathrm{nM}(\sigma(F)) \le \sigma(F) + 1 + \mathrm{fld}(\sigma(F)) \\ \mu\mathrm{vd}(F) \le \mathrm{nM}(\delta(F)) \le \delta(F) + 1 + \mathrm{fld}(\delta(F)).$$

That the bounds from Corollary 9.9 are sharp in general, is shown by the following examples.

Example 9.10 *First we consider any lean clause-set $F \ne \top$, and perform a non-strict full subsumption extension $F \rightsquigarrow F'$. Obviously F' is lean as well (with $\delta(F') = \delta(F)$). Then we have $\mu\mathrm{vd}(F') = 2$ and $\sigma(F') = 1$, and thus*

$$2 = \sigma(F') + 1 = \mu\mathrm{vd}(F') = \mathrm{nM}(\sigma(F')) = \sigma(F) + 1 + \mathrm{fld}(\sigma(F)),$$

while $\delta(F')$ is unbounded. This construction will be taken up again in Lemma 10.13

Now we turn to the δ-upper bounds. For $n \ge 2$ consider A_n. We have $\sigma(A_n) = \delta(A_n) = 2^n - n$ by Example 2.20. Thus here the inequalities of Corollary 9.9 are

$$2^n - n + 1 = \sigma(A_n) + 1 < \\ 2^n = \mu\mathrm{vd}(A_n) = \mathrm{nM}(\delta(A_n)) = \delta(A_n) + 1 + \mathrm{fld}(\delta(A_n))$$

(using Corollary 7.23).

9.3 Sharpness of the bound for $\mathcal{VMU}$

We now show that for every deficiency k there are variable-minimally unsatisfiable clause-sets where the min-var degree is $\mathrm{nM}(k)$ (strengthening Example 9.10). The examples are obtained as follows:

Lemma 9.11 *For a clause-set $F \in \mathcal{CLS}$, $\bot \notin F$ and $n(F) > 0$, with at least one full clause, consider the following construction of $F' \in \mathcal{CLS}$:*

1. *Let C be a full clause of F.*
2. *Let F'' be a full singular unit-extension of F (recall Definition 5.15).*
3. *Let $F' := F'' \,\dot\cup \{C\}$.*

We have the following properties:

1. *$\sigma(F') = \sigma(F) + 1$, $\delta(F') = \delta(F) + 1$, $\mu\mathrm{vd}(F') = \mu\mathrm{vd}(F) + 1$.*
2. *$F \in \mathcal{SED} \Rightarrow F' \in \mathcal{SED}$.*
3. *$F \in \mathcal{USAT} \Rightarrow F' \in \mathcal{USAT}$.*

Proof: With Lemma 5.17 we get $\sigma(F'') = \sigma(F)$, $\delta(F'') = \delta(F)$, $\mu\mathrm{vd}(F'') = \mu\mathrm{vd}(F)$. Obviously $\delta(F') = \delta(F'') + 1$. Let $\mathrm{var}(F'') \setminus \mathrm{var}(F) = \{v\}$. To see $\mu\mathrm{vd}(F') = \mu\mathrm{vd}(F'') + 1$, we note that for $w \in \mathrm{var}(F)$ we have $\mathrm{vd}_{F'}(w) = \mathrm{vd}_{F''}(w) + 1$, while $\mathrm{vd}_{F'}(v) = c(F') - 1 = c(F'') \geq \mathrm{vd}_{F''}(w) + 1$.

To prove $\sigma(F') = \sigma(F'') + 1$, we consider $\emptyset \subset V \subseteq \mathrm{var}(F') = \mathrm{var}(F'')$. If $v \notin V$, then $\delta(F'[V]) = \delta(F''[V]) + 1$, since C is full for F. If $V = \{v\}$, then $\delta(F'[V]) = \delta(F''[V]) = c(F'') \geq \sigma(F'') + 1$. Finally, if $V \supset \{v\}$, then $\delta(F'[V]) = c(F') - |V| \geq \delta(F') = \delta(F'') + 1 \geq \sigma(F'') + 1$.

The implication $F \in \mathcal{SED} \Rightarrow F' \in \mathcal{SED}$ follows now by definition of $\mathcal{SED}$, and $F \in \mathcal{USAT} \Rightarrow F' \in \mathcal{USAT}$ is trivial. $\square$

With the construction of Lemma 9.11 we now show that the general upper bound on the min-var degree of lean clause-sets is tight for variable-minimally unsatisfiable clause-sets:

Theorem 9.12 *For a class $\mathcal{VMU} \cap \mathcal{SED} \subseteq \mathcal{C} \subseteq \mathcal{LEAN}$ and $k \in \mathbb{N}$ we have $\mu\mathrm{vd}(\mathcal{C}_{\delta=k}) = \mathrm{nM}(k)$.*

Proof: By Theorem 9.8 it remains to show the lower bound $\mu\mathrm{vd}(\mathcal{VMU}_{\delta=k} \cap \mathcal{SED}) \geq \mathrm{nM}(k)$. For deficiencies $k = 2^n - n$, $n \in \mathbb{N}$ we have $\mathrm{nM}(k) = 2^n$, and thus A_n serves as lower bound example (as shown in Example 9.10), while until the next jump position we can use Lemma 9.11 together with Lemma 9.5, where due to Corollary 7.23 in this range also nM increases only by 1 for $k \rightsquigarrow k+1$. $\square$

Using Lemma 4.3, we can now determine the min-var degrees for the classes $\mathcal{LEAN}$ and $\mathcal{VMU}$, separated into layers via deficiency or surplus:

Corollary 9.13 *For $k \in \mathbb{N}$ holds $\mathrm{nM}(k) = \mu\mathrm{vd}(\mathcal{LEAN}_{\delta=k}) = \mu\mathrm{vd}(\mathcal{LEAN}_{\sigma=k}) = \mu\mathrm{vd}(\mathcal{VMU}_{\delta=k}) = \mu\mathrm{vd}(\mathcal{VMU}_{\sigma=k})$.*

10 Algorithmic implications

In Subsections 10.1, 10.2 we consider the algorithmic implications of Theorem 9.8. First in Theorem 10.2 we show that via an autarky-reduction every clause-set $F \in \mathcal{CLS}$ can be reduced to some $F' \subseteq F$, where F' fulfils the min-var-degree upper bound of Theorem 9.8 (although F' might not be lean). For this autarky-reduction we do not know whether we can efficiently compute a certificate, the autarky, and we discuss the Conjecture 10.3, that efficient computation is possible, in Subsection 10.2. We conclude with some remarks on the surplus in Subsection 10.3.

10.1 Autarky reduction

By Theorem 9.8 lean clause-sets fulfil a condition on the minimum variable-degree — if that condition is not fulfilled, then there exists an autarky. In this section we try to pinpoint these autarkies. We consider a vast generalisation of lean clause-sets, namely matching-lean clause-sets (recall Subsection 2.6, especially that a multi-clause-set F with $n(F) > 0$ is matching-lean iff $\sigma(F) \geq 1$). It is also useful to note here the observation that a (multi-)clause-set F has a non-trivial autarky (is not lean) iff there is $\emptyset \subset V \subseteq \mathrm{var}(F)$ such that $F[V]$ is satisfiable, and the corresponding autarky reduction of F removes all clauses containing some variable of V; note that to perform this autarky reduction the autarky itself (the satisfying assignment for $F[V]$) is not needed, only its set V of variables.

We obtain a sufficient criterion for the existence of a non-trivial autarky by considering the converse of Theorem 9.8:

Lemma 10.1 *Consider a matching-lean multi-clause-set F with $n(F) > 0$. If we have $\mu\mathrm{vd}(F) > \mathrm{nM}(\sigma(F))$, then for all $F' := F[V]$ with $\emptyset \subset V \subseteq \mathrm{var}(F)$ and $\delta(F[V]) = \sigma(F)$ we have:*

1. *$\delta(F') = \sigma(F') = \sigma(F)$ (so $F' \in \underline{\mathcal{SED}} \cap \underline{\mathcal{MLEAN}}$).*
2. *$\mu\mathrm{vd}(F') > \mathrm{nM}(\sigma(F'))$.*
3. *$F' \in \underline{\mathcal{SAT}}$ (thus there is $\varphi \in \mathcal{PASS}$ with $\mathrm{var}(\varphi) = V$ and $\varphi * F' = \top$; this φ is a non-trivial autarky for F).*

Proof: Part 1 follows by definitions. For Part 2 note that $\mu\mathrm{vd}(F') \leq \mathrm{nM}(\sigma(F'))$ implies $\mu\mathrm{vd}(F) \leq \mu\mathrm{vd}(F') \leq \mathrm{nM}(\sigma(F))$ contradicting the assumption. And Part 3 follows now by Lemma 9.7. □

To better understand the background, we recall two fundamental facts regarding the surplus $\sigma(F)$ for multi-clause-set F with $n(F) > 0$:

1. $\sigma(F)$ together with some $\emptyset \subset V \subseteq \mathrm{var}(F)$ with $\sigma(F) = \delta(F[V])$ can be computed in polynomial time (see Subsection 11.1 in [57]).
2. If $\sigma(F) \leq 0$, then one can compute a non-trivial matching autarky for F in polynomial time (see Section 7 in [51] or Section 9 in [57]).

We see now that we can reach the conclusion of Theorem 9.8 for arbitrary inputs F in polynomial time, via some autarky reduction (maintaining satisfiability-equivalence):

Theorem 10.2 *Consider a multi-clause-set F. We can find in polynomial time a sub-clause-set $F' \subseteq F$ such that:*

1. *There exists an autarky φ for F with $F' = \varphi * F$.*

2. If $n(F') > 0$, then $\sigma(F') \geq 1$ and $\mu\mathrm{vd}(F') \leq \mathrm{nM}(\sigma(F'))$.

Proof: First F is reduced to the underlying clause-set (and all further computations only handle clause-sets), and if $\bot \in F$, then we reduce F to $\{\bot\}$ and are finished. Otherwise, the reduction process of F yielding the final F' consists now of a loop of two steps: First eliminate matching-autarkies (i.e., compute the matching-lean kernel), such that we reach $\sigma(F) \geq 1$. Then apply the autarky-reduction according to Part 3 of Lemma 10.1 (removing all clauses containing a variable of V) in case of $\mu\mathrm{vd}(F) > \mathrm{nM}(\sigma(F))$. This loop is aborted if $\top$ is reached or the criterions no longer applied. All autarkies are composed together (as shown in [50], also in general the composition of autarkies is again an autarky), yielding the final φ. $\square$

In Theorem 10.2 we can only show the *existence* of an autarky φ for F with $F' = \varphi * F$, however we currently do not know how to compute it efficiently. We conjecture that it can be found in polynomial time:

Conjecture 10.3 *For $F \in \mathcal{CLS}_{\sigma \geq 1}$ there is a poly-time algorithm for computing a non-trivial autarky φ for F in case of $\mu\mathrm{vd}(F) > \mathrm{nM}(\sigma(F))$.*

Note that we ask only to find *some* autarky φ, not necessarily one given by Lemma 10.1 (i.e., with $\mathrm{var}(\varphi) = V$ as in Part 3 of Lemma 10.1). That this is enough follows by the fact that the number of variables is reduced by such a reduction, and this by some autarky:

Lemma 10.4 *If Conjecture 10.3 is true, then for the algorithm from Theorem 10.2, which reduces a multi-clause-set F to some (satisfiability-equivalent) $F' \subseteq F$, we can also compute an autarky φ for F with $F' = \varphi * F$ in polynomial time.*

Proof: In the loop as given in the proof of Theorem 10.2, we can replace the autarky-reduction according to Part 3 of Lemma 10.1 by the reduction $F \leadsto \varphi * F$ according to a (non-trivial) autarky as given by Conjecture 10.3. $\square$

In the subsequent Subsection 10.2 we discuss what we know about Conjecture 10.3.

10.2 On finding the autarky

Consider a matching-lean multi-clause-set F with $n(F) > 0$, where Lemma 10.1 is applicable (recall that we have $\sigma(F) \geq 1$), that is, we have $\mu\mathrm{vd}(F) > \mathrm{nM}(\sigma(F))$. So we know that F has a non-trivial autarky. Conjecture 10.3 states that finding such a non-trivial autarky in this case can be done in polynomial time (recall that finding a non-trivial autarky in general is NP-complete, which was shown in [51]).

The task of actually finding the autarky can be considered as finding a satisfying assignment for the following class $\underline{\mathcal{MLCR}} \subset \underline{\mathcal{SAT}} \cap \underline{\mathcal{MLEAN}}$ of satisfiable(!) multi-clause-sets F, obtained by considering all $F[V]$ for minimal sets of variables V with $\delta(F[V]) = \sigma(F)$ (where "CR" stands for "critical"):

Definition 10.5 *Let $\boldsymbol{\mathcal{MLCR}}$ be the class of clause-sets F fulfilling the following three conditions:*

1. *$F \in \mathcal{MLEAN}$, $\bot \notin F$, $F \neq \top$.*
2. *For all $\emptyset \subset V \subset \mathrm{var}(F)$ holds $\delta(F[V]) > \sigma(F)$.*
3. *$\mu\mathrm{vd}(F) > \mathrm{nM}(\sigma(F))$.*

(The definition of $\underline{\mathcal{MLCR}}$ just uses $F \in \underline{\mathcal{MLEAN}}$ instead.)

The basic properties of this class are collected in the following lemma:

Lemma 10.6 *For $F \in \underline{\mathcal{MLCR}}$ holds:*

1. $\delta(F) = \sigma(F) \geq 1$ *(whence $F \in \underline{\mathcal{SED}}$).*
2. $F \in \underline{\mathcal{SAT}}$.

Proof: Since $F \in \underline{\mathcal{MLEAN}}$ and $n(F) > 0$, we have $\sigma(F) \geq 1$. By $\perp \notin F$ we get $F = F[\mathrm{var}(F)]$, and thus $\sigma(F) = \delta(F[\mathrm{var}(F)]) = \delta(F)$, while $F \in \underline{\mathcal{SAT}}$ follows by Lemma 10.1. $\square$

The examples we know for elements of $\mathcal{MLCR}$ are as follows:

Example 10.7 *A simple example for $F \in \mathcal{MLCR}_{\delta=1} \cap \mathcal{HIT}$ is given by*

$$F := \{\{1, 2\}, \{-1, 2, -3\}, \{-2, 3\}, \{1, -2, -3\}\}.$$

We have $\delta(F) = 4 - 3 = 1$ and $\mu\mathrm{vd}(F) = 3$; for $\sigma(F) = 1$ and Condition 2 of Definition 10.5 notice, that any two variables cover all four clauses, and thus the minimum of $\delta(F[V])$ is only attained for $V = \mathrm{var}(F)$; finally by $\mu\mathrm{vd}(F) = 3 > \mathrm{nM}(1) = 2$ we get $F \in \mathcal{MLCR}$, while $F \in \mathcal{HIT}$ by definition (any two clauses have a clash).

This example also shows that $\underline{\mathcal{MLCR}}$ is not invariant under multiplicities: Obtain F' from F by duplicating the first clause. We still have $F' \in \underline{\mathcal{MLEAN}}$, but $\delta(F') = 2$, while $\delta(F'[\{3\}]) = 2$ as well, and thus $F' \notin \underline{\mathcal{MLCR}}$. So duplicating a clause can lead outside of $\underline{\mathcal{MLCR}}$. In the other direction, removing a duplication, we have the following simple (counter-)example: Let $F := \{2 * \{1\}\}$; trivially we have $F \in \underline{\mathcal{MLCR}}$, but for the underlying clause-set we have $\{\{1\}\} \notin \mathcal{MLEAN}$, thus it is not in $\mathcal{MLCR}$.

A more general class of example is obtained by full clause-sets. Let F be a full clause-set and $n := n(F)$, $m := c(F)$. Then $F \in \mathcal{MLCR}$ iff $n < m < 2^n$:

1. *We have $\delta(F) = m - n$ and thus $\delta(F) \geq 1 \Leftrightarrow m > n$.*
2. *Furthermore $F \in \mathcal{SAT} \Leftrightarrow m < 2^n$.*
3. *For $\emptyset \subset V \subset \mathrm{var}(F)$ we have $\delta(F[V]) = m - |V|$. Thus $\sigma(F) = \delta(F)$, and Condition 2 of Definition 10.5 is fulfilled.*
4. *It remains to show the condition on the min-var degree: We have $\mu\mathrm{vd}(F) = m$, while $\mathrm{nM}(\sigma(F)) = \mathrm{nM}(m - n)$. By Theorem 7.21 we obtain $\mathrm{nM}(m - n) = m - n + \mathrm{fld}(m - n + 1 + \mathrm{fld}(m - n + 1))$. We obtain for $n \geq 1$:*

$$\mu\mathrm{vd}(F) > \mathrm{nM}(\sigma(F)) \Leftrightarrow m > m - n + \mathrm{fld}(m - n + 1 + \mathrm{fld}(m - n + 1)) \Leftrightarrow$$
$$\mathrm{fld}(m - n + 1 + \mathrm{fld}(m - n + 1)) < n \Longleftarrow$$
$$\mathrm{fld}(2^n - 1 - n + 1 + \mathrm{fld}(2^n - 1 - n + 1)) < n \Leftrightarrow$$
$$\mathrm{fld}(2^n - n + \mathrm{fld}(2^n - n)) = \mathrm{fld}(2^n - n + (n - 1)) = \mathrm{fld}(2^n - 1) < n.$$

Finally we note that the class $\underline{\mathcal{MLCR}}$ is invariant against changes of polarities of literal occurrences (if clauses become equal in this way, then their multiplicities have to be added), and thus for example replacing all clauses $C \in F \in \underline{\mathcal{MLCR}}$ by their positive forms, $\mathrm{var}(C)$, we obtain a positive (no complementations occur) multi-clause-set $F' \in \underline{\mathcal{MLCR}}$ (with $c(F') = c(F)$ and $n(F') = n(F)$).

The importance of $\mathcal{MCLS}$ is, that it is sufficient to find a non-trivial autarky for this class of satisfiable clause-sets. In order to show this, we need to strengthen the polytime computation of $\sigma(F)$:

Lemma 10.8 *For a multi-clause-set F with $n(F) > 0$ we can compute in polynomial time a minimal subset $\emptyset \subset V \subseteq \mathrm{var}(F)$ with $\delta(F[V]) = \sigma(F)$.*

Proof: Let $V := \mathrm{var}(F)$. Check whether there is $v \in \mathrm{var}(F)$ with $\sigma(F[V \setminus \{v\}]) = \sigma(F)$ — if yes, then $V := V \setminus \{v\}$ and repeat, if not, then V is the desired result. $\square$

We are ready to show that $\mathcal{MCLS}$ is really the "critical class" for the problem of finding the witness-autarky underlying the reduction $F \rightsquigarrow F'$ of Theorem 10.2:

Theorem 10.9 *Consider $F \in \mathcal{CLS}$ with $\sigma(F) \geq 1$ and $\mu\mathrm{vd}(F) > \mathrm{nM}(\sigma(F))$.*

1. *For every minimal subset $\emptyset \subset V \subseteq \mathrm{var}(F)$ with $\delta(F[V]) = \sigma(F)$ we have $F[V] \in \underline{\mathcal{MLCR}}$.*
2. *Thus we can compute in polytime some $\emptyset \subset V \subseteq \mathrm{var}(F)$ with $F[V] \in \underline{\mathcal{MLCR}}$.*
3. *So Conjecture 10.3 is equivalent to the statement, that finding a non-trivial autarky for clause-sets in $\underline{\mathcal{MLCR}}$ can be achieved in polynomial time.*

Proof: Part 1 follows with Lemma 10.1. Part 2 follows from Part 1 with Lemma 10.8. Part 3 follows with Part 2 (note that every autarky for some $F[V]$ yields an autarky for F). $\square$

Since $\mathcal{MLCR} \subset \mathcal{SED}$, if both questions of Question 9.6 have a yes-answer, then this would prove Conjecture 10.3.

10.3 Final remarks on the surplus

It is instructive to investigate the precise relationship between minimum variable-degree and the surplus for lean clause-sets, which by Corollary 9.9 are indeed very close. Small values behave as follows:

Lemma 10.10 *Consider $F \in \mathcal{LEAN} \setminus \{\top\}$ (so $\sigma(F) \geq 1$ and $\mu\mathrm{vd}(F) \geq 2$).*

1. *$\sigma(F) = 1$ holds if and only if $\mu\mathrm{vd}(F) = 2$ holds.*
2. *$\mu\mathrm{vd}(F) = 3$ implies $\sigma(F) = 2$.*
3. *$\sigma(F) = 2$ implies $\mu\mathrm{vd}(F) \in \{3, 4\}$.*
4. *$\mu\mathrm{vd}(F) = 4$ implies $\sigma(F) \in \{2, 3\}$.*

Proof: First consider Part 1. If $\sigma(F) = 1$ (so $n(F) > 0$), then by Theorem 9.8 we have $\mu\mathrm{vd}(F) \leq \mathrm{nM}(1) = 2$, while in case of $\mu\mathrm{vd}(F) = 1$ there would be a matching autarky for F. If on the other hand $\mu\mathrm{vd}(F) = 2$ holds, then by definition $\sigma(F) \leq 2 - 1 = 1$, while $\sigma(F) \geq 1$ holds since F is matching lean. For Part 2 note that due to $\sigma(F) + 1 \leq \mu\mathrm{vd}(F)$ we have $\sigma(F) \leq 2$, and then the assertion follows by Part 1; Part 4 follows in the same way. Finally Part 3 follows by Part 1 and $\mathrm{nM}(2) = 4$. $\square$

For some examples we use F with $\delta(F) = \sigma(F)$:

Example 10.11 *Examples for cases $\sigma(F) \in \{2, 3\}$ in Lemma 10.10:*

1. *An example for $\mu\mathrm{vd}(F) = 4$ in Part 3 with $F \in \mathcal{UHIT} \cap \mathcal{SED}$ is given by A_2.*

2. *For $\{ \{a,b,c\}, \{\overline{a},b,c\}, \{a,\overline{b},c\}, \{\overline{a},\overline{b},c\}, \{a,\overline{c}\}, \{\overline{a},\overline{c}\} \} \in \mathcal{UHIT} \cap \mathcal{SED}$ we have $\mu\mathrm{vd}(F) = 4$ and $\sigma(F) = 3$ (Part 4).*

Question 10.12 *is there for every $k \in \mathbb{N}$ an $F \in \mathcal{UHIT} \cap \mathcal{SED}$ with $\sigma(F) = k$ and $\mu\mathrm{vd}(F) = k + 1$?*

As we have for $\mathcal{MU}$ the levels $\mathcal{MU}_{\delta=k}$ for $k = 1, 2, \ldots$, we can consider for $\mathcal{LEAN}$ the levels $\mathcal{LEAN}_{\sigma=k}$ for $k = 1, 2, \ldots$. However, while the levels $\mathcal{MU}_{\delta=k}$ as well as $\mathcal{LEAN}_{\delta=k}$ all are decidable in polynomial time, already the first level $\mathcal{LEAN}_{\sigma=1}$ is NP-complete:

Lemma 10.13 *Consider the map $E : \mathcal{CLS} \to \mathcal{CLS}$, which has $E(\top) := \top$, while otherwise for $F \in \mathcal{CLS} \setminus \{\top\}$ it chooses (by some rule — it doesn't matter) a clause $C \in F$ and a variable $v \in \mathcal{VA} \setminus \mathrm{var}(F)$, and replaces C by $C \mathbin{\dot\cup} \{v\}, C \mathbin{\dot\cup} \{\overline{v}\}$; in other words, an non-strict full subsumption extension $F \rightsquigarrow E(F)$ is performed, as in Example 9.10. Then we have for $F \in \mathcal{CLS}$:*

1. *$F \in \mathcal{LEAN}$ iff $E(F) \in \mathcal{LEAN}$.*

2. *$F \in \mathcal{MU}$ iff $E(F) \in \mathcal{MU}$.*

3. *$\sigma(F) \leq 1$.*

Thus $\mathcal{LEAN}_{\sigma=1}$ is coNP-complete, while $\mathcal{MU}_{\sigma=1}$ is D^P-complete.

Proof: The properties of the map E are trivial. The completeness-properties follow with the coNP-completeness of $\mathcal{LEAN}$ ([51]) and the D^P-completeness of $\mathcal{MU}$ ([80]). □

With Lemma 10.13 we also get easy examples for minimally unsatisfiable clause-sets of arbitrary deficiency and surplus 1.

11 Matching lean clause-sets

In this section, which concludes our considerations on generalisations (beyond $\mathcal{MU}$), we consider the question whether Theorem 9.8 can incorporate non-lean clause-sets.

We consider the large class $\mathcal{MLEAN}$ of matching lean clause-sets, which is natural, since a basic property of $F \in \mathcal{MU}$ used in the proof of Theorem 9.8 is $\delta(F) \geq 1$ for $F \neq \top$, and this actually holds for all $F \in \mathcal{MLEAN}$. We will construct for arbitrary deficiency $k \in \mathbb{N}$ and $K \in \mathbb{N}$ clause-sets $F \in \mathcal{MLEAN}$ of deficiency k, where every variable occurs positively at least K times. Thus neither the upper bound $\max(\mathrm{ld}_F(v), \mathrm{ld}_F(\overline{v})) \leq f(\delta(F))$ nor $\mathrm{ld}_F(v) + \mathrm{ld}_F(\overline{v}) = \mathrm{vd}_F(v) \leq f(\delta(F))$ for some chosen variable v and for any function f does hold for $\mathcal{MLEAN}$.

An example for $F \in \mathcal{MLEAN}_{\delta=1}$ with $\mu\mathrm{ld}(F) \geq 2$ (the minimal literal degree; and thus $\mu\mathrm{vd}(F) \geq 4$) is given in Section 5 in [52], displaying a "star-free" (thus satisfiable) clause-set F with deficiency 1. In Subsection 9.3 in [57] it is shown that this clause-set is matching lean. "Star-freeness" in our context means, that there are no singular variables (occurring in one sign only once). Our simpler construction pushes the number of positive occurrences arbitrary high, but there are variables with only one negative occurrence (i.e., there are singular variables).

For a finite set V of variables let $M(V) \subseteq A(V)$ be the full clause-set over V containing all full clauses with at most one complementation; e.g. $M(\{1,2\}) = \{\{1,2\}, \{-1,2\}, \{1,-2\}\}$:

1. Obviously $n(M(V)) = |V|$, $c(M(V)) = |V| + 1$ and $\delta(M(V)) = 1$ holds.

2. We have already seen that $M(V) \in \mathcal{MLEAN}$ (for $\top \neq F' \subset F \subseteq A(V)$ we have $\delta(F') < \delta(F)$, and thus a full clause-set F is matching lean iff $\delta(F) \geq 1$).

3. By definition we have $\mathrm{ld}_{M(V)}(v) = |V|$ and $\mathrm{ld}_{M(V)}(\overline{v}) = 1$ for $v \in V$.

Lemma 11.1 *For $k \in \mathbb{N}$ and $K \in \mathbb{N}$ there are $F \in \mathcal{MLEAN}_{\delta=k}$ with $F \in \mathcal{USAT}$ for $k \geq 2$ such that for all variables $v \in \mathrm{var}(F)$ we have $\mathrm{ld}_F(v) \geq K$.*

Proof: For $k = 1$ we can set $F := M(\{v_1, \dots, v_K\})$; so assume $k \geq 2$. Consider any clause-set $G \in \mathcal{MU}_{\delta=k-1}$ with $n := n(G) \geq K$, and let $V := \mathrm{var}(G)$. Consider a disjoint copy of V, that is a set V' of variables with $V' \cap V = \emptyset$ and $|V'| = |V|$, and consider two enumerations of the clauses $M(V) = \{C_1, \dots, C_{n+1}\}$, $M(V') = \{C'_1, \dots, C'_{n+1}\}$. Now

$$F := G \mathbin{\dot\cup} \{\, C_i \mathbin{\dot\cup} C'_i : i \in \{1, \dots, n+1\} \,\}$$

has no matching autarky: If φ is a matching autarky for F, then $\mathrm{var}(\varphi) \cap V = \emptyset$ since G is matching lean, whence $\mathrm{var}(\varphi) \cap V' = \emptyset$ since $M(V')$ is matching lean, and thus φ must be trivial. Furthermore we have $n(F) = 2n$ and $c(F) = c(G) + n + 1$, and thus $\delta(F) = c(G) + n + 1 - 2n = \delta(G) + 1 = k$. By definition for all variables $v \in \mathrm{var}(F)$ we have $\mathrm{ld}_F(v) \geq n$. $\square$

For $k = 1$ the examples of Lemma 11.1 for $K \geq 3$ are necessarily satisfiable, since $\mathcal{MLEAN}_{\delta=1} \cap \mathcal{USAT} = \mathcal{MU}_{\delta=1}$. It remains the questions whether the singular variables can be eliminated:

Question 11.2 *Are there examples for deficiency $k \in \mathbb{N}$ of $F \in \mathcal{MLEAN}_{\delta=k}$ with $\mu\mathrm{ld}(F) \geq k + 1$?*

1. *The above mentioned star-free clause-sets shows that this is the case for $k = 1$.*

2. *What about the stronger condition $\mu\mathrm{ld}(F) \geq K$ for arbitrary $K \in \mathbb{N}$?*

12 Lower bounds for the min-var-degree of $\mathcal{MU}$

We now return to (boolean) minimally unsatisfiable clause-sets. By Theorem 8.3 we have $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \leq \mathrm{nM}(k)$ for all $k \in \mathbb{N}$. The task of precisely determining $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k})$ for all k seems a deep question, and is the subject of the remainder of this report. First we introduce a notation for the true bound on $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k})$:

Definition 12.1 *For $k \in \mathbb{N}$ let* $\boldsymbol{\mu\mathrm{nM}(k)} := \mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \in \mathbb{N}$ *(the "minimum non-Mersenne number" for (deficiency) k).*

So by Theorem 8.3 we have $\mu\mathrm{nM} \leq \mathrm{nM}$. All our examples yielding lower bounds on $\mu\mathrm{nM}(k)$ are actually (unsatisfiable) hitting clause-sets, and thus we believe

Conjecture 12.2 *For all $k \in \mathbb{N}$ holds $\mu\mathrm{nM}(k) = \mu\mathrm{vd}(\mathcal{UHIT}_{\delta=k})$.*

We will see in Theorem 14.3 that $\mu\mathrm{nM} \neq \mathrm{nM}$. We believe that $\mu\mathrm{nM}$ is a highly complicated function, but the true values deviate only at most by one from nM:

Conjecture 12.3 *For all $k \in \mathbb{N}$ we have $\mu\mathrm{nM}(k) \geq \mathrm{nM}(k) - 1$.*

By Corollary 7.22 we get:

Lemma 12.4 *If Conjecture 12.3 holds, then* $k - 1 + \mathrm{fld}(k+1) \le \mu\mathrm{nM}(k) \le k + 1 + \mathrm{fld}(k)$ *holds for* $k \in \mathbb{N}$.

Later in Lemma 13.1 we will see that $\mu\mathrm{nM} : \mathbb{N} \to \mathbb{N}$ is monotonically increasing. While in Theorem 14.5 we will (implicitly) construct a correction function $\gamma_1 : \mathbb{N} \to \{0,1\}$ such that $\mu\mathrm{nM} \le \mathrm{nM} - \gamma_1$, where we remark in the Conclusion (Section 15) that also $\mu\mathrm{nM} \ne \mathrm{nM} - \gamma_1$ holds. Note that Conjecture 12.3 says that there exists $\gamma : \mathbb{N} \to \{0,1\}$ with $\mu\mathrm{nM} = \mathrm{nM} - \gamma$, while for every $\gamma : \mathbb{N} \to \{0,1\}$ the function $\mathrm{nM} - \gamma$ is still monotonically increasing (by Lemma 7.5), and is thus a possible candidate.

In Subsection 12.1 we provide a general method for obtaining *lower bounds*, via considering full clauses (while in Section 13 we turn to improved upper bounds). Namely we introduce $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k}) \in \mathbb{N}$ for $k \in \mathbb{N}$, the maximal number of full clauses in $F \in \mathcal{MU}_{\delta=k}$. According to our numerical investigations the number of full clauses is very close to $\mu\mathrm{nM}$, and indeed to $\mathrm{nM}(k)$:

Conjecture 12.5 *For all* $k \in \mathbb{N}$ *we have* $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k}) \ge \mathrm{nM}(k) - 1$.

Conjecture 12.5 implies Conjecture 12.3; regarding $\nu\mathrm{fc}(\mathcal{UHIT}_{\delta=k})$, there might be unbounded gaps to $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k})$. The smallest deficiency k with $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k}) = \mathrm{nM}(k) - 1$ (and also $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k}) = \mu\mathrm{nM}(k) - 1$) is $k = 3$, as shown in Lemma 12.16 (together with Theorem 14.3). We show for two infinite classes of deficiencies k that $\nu\mathrm{fc}(\mathcal{UHIT}_{\delta=k}) = \mu\mathrm{nM}(k) = \mathrm{nM}(k)$ holds (Lemmas 12.10, 12.11). Actually, the main point here could be considered as (just) the equalities $\mu\mathrm{nM}(k) = \mathrm{nM}(k)$, for which in these two cases the proofs don't need to consider full clauses, and so the general method for computing lower bounds on $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k})$, with the beginnings developed in Subsection 12.2, is not applied here. However in future work we will employ this method more fully (see Subsection 15.3), and, more important for the report at hand, we need for the proof of Theorem 14.3 (that $\mu\mathrm{nM}(6) = \mathrm{nM}(6) - 1 = 8$) the fact $\nu\mathrm{fc}(\mathcal{MU}_{\delta=3}) \le 4$, shown in Lemma 12.16.

12.1 Some precise values for the min-var-degree of $\mathcal{MU}$

A general lower-bound method for $\mu\mathrm{nM}$ is provided by the number $\mathrm{fc}(F)$ of full clauses in a clause-set F. The supremum $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k})$ of this number over all elements of $\mathcal{MU}_{\delta=k}$ for fixed k is an interesting quantity in its own right, but in this report we only touch on this subject, providing the bare minimum of information needed in our context. See Subsection 15.3 for an outlook on the interesting properties of this quantity.

Definition 12.6 *For a clause-set* $F \in \mathcal{CLS}$ *let* $\mathbf{fc}(\boldsymbol{F}) \in \mathbb{N}_0$ *be the number of full clauses, that is* $\mathrm{fc}(F) := |\{C \in F : \mathrm{var}(C) = \mathrm{var}(F)\}|$. *And for a class* $\mathcal{C} \subseteq \mathcal{CLS}$ *of clause-sets we define* $\mathbf{fc}(\boldsymbol{\mathcal{C}}) := \{\mathrm{fc}(F) : F \in \mathcal{C}\} \subseteq \mathbb{N}_0$ *as the set of all possible numbers of full clauses, while* $\boldsymbol{\nu}\mathbf{fc}(\boldsymbol{\mathcal{C}}) \in \mathbb{N}_0 \cup \{+\infty\}$ *is the supremum of* $\mathrm{fc}(\mathcal{C})$.

Some simple examples:

Example 12.7 $\mathrm{fc}(\top) = 0$, $\mathrm{fc}(\{\bot\}) = 1$, *and* $\mathrm{fc}(\{\{1\},\{-1,2\}\}) = 1$. *While* $\mathrm{fc}(\emptyset) = \emptyset$, *thus* $\nu\mathrm{fc}(\emptyset) = 0$, *and* $\mathrm{fc}(\mathcal{CLS}) = \mathbb{N}_0$, *thus* $\nu\mathrm{fc}(\mathcal{CLS}) = +\infty$.

By definition we have:

Lemma 12.8 $\mathrm{fc}(F) \le \mu\mathrm{vd}(F)$ *holds for every* $F \in \mathcal{CLS}$ *(since every variable in* F *has degree at least* $\mathrm{fc}(F)$*), and thus* $\nu\mathrm{fc}(\mathcal{C}) \le \mu\mathrm{vd}(\mathcal{C})$ *for every* $\mathcal{C} \subseteq \mathcal{CLS}$.

We obtain that for lean clause-sets (especially minimally unsatisfiable clause-sets) of fixed deficiency the number of full clauses is bounded:

Corollary 12.9 *A lean clause-set of deficiency k can have at most* $\mathrm{nM}(k)$ *many full clauses; i.e., for all $k \in \mathbb{N}$ we have* $\nu\mathrm{fc}(\mathcal{LEAN}_{\delta=k}) \le \mathrm{nM}(k)$.

Precise values for $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k}) = \mu\mathrm{nM}(k)$ we show for two infinite classes of deficiencies. The simplest class are the deficiencies directly after the jumps (recall Lemma 7.20), the deficiencies of the A_n:

Lemma 12.10 *For $n \in \mathbb{N}$ and $k := 2^n - n$ holds*

$$\nu\mathrm{fc}(\mathcal{UHIT}_{\delta=k}) = \nu\mathrm{fc}(\mathcal{MU}_{\delta=k}) = \mu\mathrm{vd}(\mathcal{UHIT}_{\delta=k}) = \mu\mathrm{nM}(k) = \mathrm{nM}(k) = 2^n.$$

Proof: We have $\nu\mathrm{fc}(A_n) = 2^n$ (recall Lemma 2.13), and thus $\nu\mathrm{fc}(\mathcal{UHIT}_{\delta=k}) \ge 2^n$, while by Corollary 7.23 we have $\mathrm{nM}(k) = 2^n$. $\square$

Also for the jumps themselves (recall Definition 7.12 and Lemma 7.20) the same conclusions hold, namely by Lemma 8.5, Part 2 (and the proof) we have:

Lemma 12.11 *For all $k \in J$ holds*

$$\nu\mathrm{fc}(\mathcal{UHIT}_{\delta=k}) = \nu\mathrm{fc}(\mathcal{MU}_{\delta=k}) = \mu\mathrm{vd}(\mathcal{UHIT}_{\delta=k}) = \mu\mathrm{nM}(k) = \mathrm{nM}(k).$$

Note that for $k \in J$ there is $n \in \mathbb{N}$, $n \ge 2$, with $k = 2^n - n - 1$ and $\mathrm{nM}(k) = 2^n - 2$. The underlying method of Lemmas 12.10, 12.11 is simple (as we already have explained it in Subsection 1.3): start with A_n and apply strict full subsumption resolution to full clauses. Zero steps have been used in Lemma 12.10, one step in Lemma 12.11, and one example for two steps will be seen in the proof of Theorem 14.1. The further development of this method we have to leave for future work.

12.2 On the number of full clauses

We have a special interest in those $F \in \mathcal{MU}$ where the lower bound $\mathrm{fc}(F)$ meets the upper bound $\mu\mathrm{vd}(F)$. In this case this number must be even, and we obtain another $F' \in \mathcal{MU}$ by resolving on any variable realising the minimum variable degree:

Lemma 12.12 *Consider $F \in \mathcal{MU}$ with* $\mathrm{fc}(F) = \mu\mathrm{vd}(F)$. *Then* $\mathrm{fc}(F)$ *is even.*

Proof: Consider $v \in \mathrm{var}(F)$ with $\mathrm{vd}_F(v) = \mu\mathrm{vd}(F)$. The occurrences of v are now exactly in the full clauses of F. Every full clause C must be resolvable on v with another full clause D, yielding $E := C \diamond D$, and thus the full clauses of F can be partitioned into pairs $\{v\} \mathbin{\dot\cup} E, \{\overline{v}\} \mathbin{\dot\cup} E$ (disjoint unions) for $\frac{\mathrm{fc}(F)}{2}$ many clauses E (of length $n(F) - 1$; note that because of fullness for a given E the clauses C, D are uniquely determined up to order). $\square$

Thus, if lower and upper bound match, they must be even numbers:

Corollary 12.13 *If* $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k}) = \mathrm{nM}(k)$ *or* $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k}) = \mu\mathrm{vd}(\mathcal{MU}_{\delta=k})$ *for some $k \in \mathbb{N}$, then* $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k})$ *is even.*

Another property of $\mathrm{fc}(\mathcal{MU}_{\delta=k})$ related to evenness is that if m is a possible number of full clauses, then $2m$ is a possible number for $\delta = k + m - 1$:

Lemma 12.14 $2m \in \mathrm{fc}(\mathcal{MU}_{\delta=k+m-1})$ *for $k \in \mathbb{N}$ and $m \in \{1, \dots, \nu\mathrm{fc}(\mathcal{MU}_{\delta=k})\}$.*

Proof: Consider $F \in \mathcal{MU}_{\delta=k}$ with $\mathrm{fc}(F) = \nu\mathrm{fc}(\mathcal{MU}_{\delta=k})$. Choose m of the full clauses of F, and choose a new variable $v \notin \mathrm{var}(F)$. Replace each of the chosen full clauses $C \in F$ by two clauses $C \mathbin{\dot\cup} \{v\}, C \mathbin{\dot\cup} \{\overline{v}\}$ (one non-strict and $m-1$ strict full subsumption extensions), obtaining F'. We have $F' \in \mathcal{MU}_{\delta=k+m-1}$ and $\mathrm{fc}(F') = 2m$. □

As a special case we obtain that $2, 4$ are always possible for the number of full clauses (except for $k = 1$):

Corollary 12.15 *For $k \in \mathbb{N}$ holds $2 \in \mathrm{fc}(\mathcal{MU}_{\delta=k})$, and if $k \geq 2$ then $4 \in \mathrm{fc}(\mathcal{MU}_{\delta=k})$.*

Proof: We show the assertion by induction over k, using Lemma 12.14, as follows: We have $\{\{1\}, \{-1\}\} \in \mathcal{MU}_{\delta=1}$, so consider $k \geq 2$. We know $2 \in \mathrm{fc}(\mathcal{MU}_{\delta=k-1})$, thus $4 \in \mathrm{fc}(\mathcal{MU}_{\delta=k-1+2-1=k})$. And once we have any $F \in \mathcal{MU}_{\delta=k}$ with a full clause, we get $F' \in \mathcal{MU}_{\delta=k}$ with $\mathrm{fc}(F') = 2$ by performing a non-strict full subsumption extension on that full clause (i.e., $F' = E(F)$ as in Lemma 10.13, with C a full clause). □

We now turn to the determination of $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k})$ for $k = 1, 2, 3$.

Lemma 12.16 *We have:*

1. $\nu\mathrm{fc}(\mathcal{MU}_{\delta=1}) = 2 = \mathrm{nM}(1)$.
2. $\nu\mathrm{fc}(\mathcal{MU}_{\delta=2}) = 4 = \mathrm{nM}(2)$.
3. $\nu\mathrm{fc}(\mathcal{MU}_{\delta=3}) = 4 = \mathrm{nM}(3) - 1$.

Proof: Part 1: Between two clauses of some $F \in \mathcal{MU}_{\delta=1}$ there is at most one conflict, and thus there are at most two full clauses in F. while by Corollary 12.15 we know $\nu\mathrm{fc}(\mathcal{MU}_{\delta=1}) \geq 2$. Part 2: By Corollary 12.15 we have $\nu\mathrm{fc}(\mathcal{MU}_{\delta=2}) \geq 4$, by Corollary 12.9 we have $\nu\mathrm{fc}(\mathcal{MU}_{\delta=2}) \leq 4$. Part 3: By Corollary 12.15 we have $\nu\mathrm{fc}(\mathcal{MU}_{\delta=3}) \geq 4$, by Corollary 12.13 we have $\nu\mathrm{fc}(\mathcal{MU}_{\delta=3}) \leq 4$. □

13 A method for improving the mvd upper bound for $\mathcal{MU}$

We now present a framework for generalising the argumentation of Theorem 8.3 together with the analysis of the underlying recursion from Section 7. The idea is as follows:

1. We start with upper bounds $\mu\mathrm{nM}(k) \leq a_k$ for $k = 1, \dots, p$, collected in a "valid bounds-function" f.
2. For deficiency $p+1$ and an envisaged min-var-degree m we consider the set $\mathrm{pp}_f(p+1, m)$ of "possible" degree-pairs of variables (the degrees of the positive and negative literals) in an envisaged clause-set $F \in \mathcal{SMU}_{\delta=p+1}$ with $\mu\mathrm{vd}(F) = m$.
3. If $\mathrm{pp}_f(p+1, m) = \emptyset$, then m is "inconsistent", that is, impossible to realise (as shown in Theorem 13.10), whence $\mu\mathrm{nM}(p+1) < m$.
4. While in case of $\mathrm{pp}_f(p+1, m) \neq \emptyset$ there might exist such an F or not (the formal reasoning underlying the definition of $\mathrm{pp}_f(p+1, m)$ is not complete).

So here we generalise the approach of Section 7 for describing the function nM to a general recursion scheme, obtaining a general method for improved *upper bounds*. The applications in this report are as follows:

- In Theorem 13.15 we obtain an alternative description of $\mathrm{nM}(k)$.
- In Section 14 we will first show that the smallest k, where we don't have equality, is $k = 6$, namely $\mu\mathrm{vd}(\mathcal{MU}_{\delta=6}) = 8 = \mathrm{nM}(6) - 1$ (Theorem 14.3).

  By the general recursion scheme then follows from this improvement of the upper bound, that for all $k = 2^m - m + 1$ for $m \geq 3$ we have $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \leq \mathrm{nM}(k) - 1$. This improved upper bound is denoted by by $\mathrm{nM}_1 : \mathbb{N} \to \mathbb{N}$ (Theorem 14.5).

13.1 Analysing splitting-situations

"Valid bounds-functions" shall be monotonically increasing — we know that nM is (strictly) monotonically increasing, and we show that $\mu\mathrm{vd}$ is also monotonically increasing (not strictly, as we will later see in Theorem 14.3):

Lemma 13.1 *The map $\mu\mathrm{nM}$ is monotonically increasing ($\mu\mathrm{nM}(k) \leq \mu\mathrm{nM}(k+1)$ for $k \in \mathbb{N}$).*

Proof: For $F \in \mathcal{MU}_{\delta=k}$, $n(F) \neq 0$, we can construct $F' \in \mathcal{MU}_{\delta=k+1}$ with $\mu\mathrm{vd}(F) \leq \mu\mathrm{vd}(F')$ as follows:

1. If F is full, then obtain a non-full $F'' \in \mathcal{MU}_{\delta=k}$ with $\mu\mathrm{vd}(F) = \mu\mathrm{vd}(F'')$ by a full singular unit-extension (Lemma 5.17), and replace F by F''.
2. If F is not full, then perform a strict full subsumption extension (Lemma 6.5), obtaining the desired F'. □

We define now "valid bounds-functions", which are sensible as upper bounds on $\mu\mathrm{nM}$, and we also define how to obtain such a function from initial upper bounds $\mu\mathrm{nM}(k) \leq a_k$ for $k = 1, \dots, p$:

Definition 13.2 *A **valid bounds-function** is a function $f : \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ fulfilling the following three conditions:*

1. *$f(1) = 2$.*
2. *f is monotonically increasing (i.e., $\forall\, k, k' \in \mathbb{N} : k \leq k' \Rightarrow f(k) \leq f(k')$).*
3. *$f(k)$ is an upper bound for the minimal-variable degree of minimally unsatisfiable clause-sets of deficiency k (i.e., $\forall\, k \in \mathbb{N} : \mu\mathrm{nM}(k) \leq f(k)$).*

The set of all valid bounds-functions is denoted by $\boldsymbol{\mathcal{VB}} \subset \mathbb{N}^{\mathbb{N}}$. And by $\boldsymbol{\mathcal{VB}^*} := \{f \in \mathcal{VB} : f \leq \mathrm{nM}\}$ we denote the set of valid bounds-functions (pointwise) less-or-equal than the non-Mersenne function.

For $a_1, \dots, a_p \in \mathbb{N}$, $p \in \mathbb{N}$. such that $a_1 = 2$, $a_i \leq a_j$ for $i \leq j$, and $a_i \geq \mu\mathrm{nM}(i)$, we define $\boldsymbol{[a_1, \dots, a_p]}$ as that $f \in \mathcal{VB}$ with $f(k) = a_k$ for $k \in \{1, \dots, p\}$, while $f(k) = \infty$ for $k > p$.

By Lemma 13.1, $\mu\mathrm{nM}$ is a valid bounds-function, namely the smallest possible one. By Theorem 8.3 and Corollary 7.6 also nM is a valid bounds-function. In Corollary 13.16 we will see, that the continuation of $[a_1, \dots, a_p]$ by just ∞ is harmless, since nM is automatically taken into account via the improvement of valid bounds-functions through the use of potential degree-pairs defined below.

Lemma 13.3 *$\mathcal{VB}$ as well as $\mathcal{VB}^*$, together with $\leq$, is a complete lattice, where infima resp. suprema are given by pointwise minimum resp. pointwise supremum. The smallest elements of both lattices is* μnM*, while the largest is* [2] *resp.* nM.

The following definition reflects the main method to analyse and improve a given upper bound f for $\mu\mathrm{nM}(k)$, namely it determines the numerical possibilities compatible with f:

Definition 13.4 *Consider $k, m \in \mathbb{N}$ with $k \geq 2$ and $m \geq 4$, together with a valid bounds-function f. The* ***set of potential (variable-)degree-pairs w.r.t.*** *f for (deficiency) k and (minimum variable-degree) m, denoted by $\mathbf{pp}_{\boldsymbol{f}}\boldsymbol{(k, m)}$, is the set of pairs $(e_0, e_1) \in \mathbb{N}^2$ fulfilling the following conditions:*

(i) $e_0, e_1 \geq 2$

(ii) $e_0, e_1 \leq k$

(iii) $e_0 + e_1 = m$

(iv) $e_0 \leq e_1$

(v) $\forall \varepsilon \in \{0, 1\} : f(k - e_\varepsilon + 1) + e_\varepsilon \geq m$.

We set $\mathbf{pp}\boldsymbol{(k, m)} := \mathrm{pp}_{\mu\mathrm{nM}}(k, m)$.

The motivation for Definition 13.4 is to assume $F \in \mathcal{SMU}_{\delta=k}$ with $\mu\mathrm{vd}(F) = m$ and $v \in \mathrm{var}_{\mu\mathrm{vd}}(F)$, and to determine the possible literal-degrees $e_0 = \mathrm{ld}_F(\overline{v})$, $e_1 = \mathrm{ld}_F(v)$, "possible" in a formal sense. "e" stands for "eliminated clauses", namely e_ε is the number of clauses eliminated by $\langle v \to \varepsilon \rangle$. The "high" values of m (for fixed k) are of real interest; compare Lemma 13.7. The basic properties of pp_f are as follows:

1. For every valid f and $k \geq 2$ we have $\mathrm{pp}_f(k, 4) = \{(2, 2)\}$ and $\mathrm{pp}_f(k, m) = \emptyset$ for $m > 2k$.
2. Discussion of the five conditions (i) - (v) in Definition 13.4:
   (i) Only non-singular variables are considered, since only in this way the deficiency strictly decreases.
   (ii) The deficiency of $F_\varepsilon := \langle v \to \varepsilon \rangle * F$ is $k_\varepsilon := k - e_\varepsilon + 1$ (assuming we split on a variable with minimal degree), and we must have $k_\varepsilon \geq 1$.
   (iii) e_0, e_1 are the literal-degrees of $\overline{v}, v$, which sum up to the variable-degree m of v.
   (iv) W.l.o.g. we can restrict attention to such degree-pairs, since F plays a role only up to isomorphism, and thus one can flip the sign of v in F.
   (v) We have $F_\varepsilon \in \mathcal{MU}_{\delta=k-e_\varepsilon+1}$ (assuming F is saturated). And for $w \in \mathrm{var}(F_\varepsilon)$ we have $\mathrm{vd}_F(w) \leq \mathrm{vd}_{F_\varepsilon}(w) + e_\varepsilon$. If for some $\varepsilon \in \{0, 1\}$ we would have $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k-e_\varepsilon+1}) + e_\varepsilon < m$, then for $w \in \mathrm{var}_{\mu\mathrm{vd}}(F_\varepsilon)$ we would have $\mathrm{vd}_F(w) \leq \mathrm{vd}_{F_\varepsilon}(w) + e_\varepsilon \leq \mu\mathrm{vd}(\mathcal{MU}_{\delta=k-e_\varepsilon+1}) + e_\varepsilon < m$, but by assumption on w we have $\mathrm{vd}_F(w) \geq m$.
3. An important special case of $\mathrm{pp}_f(k, m)$ is $\mathrm{pp}_{\mathrm{nM}}(k, m)$; we have $\mathrm{pp}(k, m) \subseteq \mathrm{pp}_{\mathrm{nM}}(k, m)$ (see Lemma 13.8 for a generalisation). The main point in using functions f is that the precise values of $\mu\mathrm{nM}(k)$ might not be known.
4. To compute $\mathrm{pp}_f(k, m)$ according to the definition, only the values $f(k')$ for $k' \in \{1, \ldots, k - 1\}$ are needed.

Example 13.5 *Consider* $f := [\mathrm{nM}(1), \mathrm{nM}(2), \mathrm{nM}(3)] = [2, 4, 5]$*. First we determine* $\mathrm{pp}_f(4, 7)$*:*

1. *By Conditions (i) - (iv) only* $\{(3, 4)\}$ *remains.*
2. *Now* $f(4-3+1)+3 = f(2)+3 = 7$*, but* $f(4-4+1)+4 = f(1)+4 = 6 < 7$*.*
3. *Thus* $\mathrm{pp}_f(4, 7) = \emptyset$*.*

We will see in Theorem 13.10 that from this we can conclude $\mu\mathrm{nM}(4) \leq 6$ *(there is no "formal" possibility to reach the min-var degree of* 7 *for deficiency* 4*). Now we determine* $\mathrm{pp}_f(4, 6)$*:*

1. *By Conditions (i) - (iv),* $\{(2, 4), (3, 3)\}$ *are the possibilities.*
2. *Checking Condition (v) for* $(2, 4)$*:* $f(4-2+1)+2 = f(3)+2 = 7$*,* $f(4-4+1)+4 = f(1)+4 = 6$*.*
3. *Checking Condition (v) for* $(3, 3)$*:* $f(4-3+1)+3 = f(2)+3 = 7$*.*
4. *Thus* $\mathrm{pp}_f(4, 6) = \{(2, 4), (3, 3)\}$*.*

The intuitive meaning of this is, that a min-var-degree of 6 *can not be excluded by this type of formal reasoning, and* 6 *is the first new value according to this reasoning for* $[2, 4, 5]$*.*

We invite the reader to compute the following special case of what we show later (in the proof of Theorem 14.6; it might also be useful to consider Table 2):

Example 13.6 $\mathrm{pp}_{\mathrm{nM}}(13, 17) = \{(8, 9)\}$*, while for any valid bounds-function* f *with* $f(k) = \mathrm{nM}(k)$ *for* $k \in \{1, \dots, 5\}$ *and* $f(6) = \mathrm{nM}(6) - 1 = 8$ *holds* $\mathrm{pp}_f(13, 17) = \emptyset$*.*

If we have a potential degree-pair for m, then also for $m' \leq m$:

Lemma 13.7 *Consider* $k, m, m' \in \mathbb{N}$ *with* $k \geq 2$ *and* $4 \leq m' \leq m$*, and consider a valid bounds-function* f*. If* $\mathrm{pp}_f(k, m) \neq \emptyset$*, then also* $\mathrm{pp}_f(k, m') \neq \emptyset$*.*

Proof: Consider $(e_0, e_1) \in \mathrm{pp}_f(k, m)$. Consider any $2 \leq e'_0 \leq e_0$ and $2 \leq e'_1 \leq e_1$ with $e'_0 \leq e'_1$ and $e'_0 + e'_1 = m'$. Now $f(k - e'_\varepsilon + 1) + e'_\varepsilon \geq f(k - e_\varepsilon + 1) + e'_\varepsilon = f(k - e_\varepsilon + 1) + e_\varepsilon - e_\varepsilon + e'_\varepsilon \geq m - e_\varepsilon + e'_\varepsilon = m - (m - e_{\overline{\varepsilon}}) + (m' - e'_{\overline{\varepsilon}}) = e_{\overline{\varepsilon}} + m' - e'_{\overline{\varepsilon}} \geq m'$ for $\varepsilon \in \{0, 1\}$, and thus $(e'_0, e'_1) \in \mathrm{pp}_f(k, m') \neq \emptyset$. □

Using a smaller bounds-function can not yield more potential degree-pairs, as is obvious from Definition 13.4:

Lemma 13.8 *Consider* $k, m \in \mathbb{N}$ *with* $k \geq 2$*,* $m \geq 4$*, and valid bounds-functions* f_1, f_2 *with* $f_1 \leq f_2$ *(pointwise). Then* $\mathrm{pp}_{f_1}(k, m) \subseteq \mathrm{pp}_{f_2}(k, m)$*. Especially for any valid bounds-function* f *holds* $\mathrm{pp}(k, m) \subseteq \mathrm{pp}_f(k, m)$*.*

Again directly by definition (using monotonicity of valid bounds functions) we get that increasing k while keeping m can not remove potential degree-pairs:

Lemma 13.9 *Consider* $k, m \in \mathbb{N}$ *with* $k \geq 2$*,* $m \geq 4$*, and a valid bounds-function* f*. Then* $\mathrm{pp}_f(k, m) \subseteq \mathrm{pp}_f(k + 1, m)$*.*

The main use of potential degree-pairs is to provide upper bounds on $\mu\mathrm{nM}(k)$:

Theorem 13.10 *Consider* $k, m \in \mathbb{N}$ *with* $k \geq 2$*,* $m \geq 4$*, and a valid bounds-function* f*. If* $\mathrm{pp}_f(k, m) = \emptyset$*, then* $\mu\mathrm{nM}(k) < m$*.*

Proof: Assume $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \geq m$. Then there is $F \in \mathcal{SMU}'_{\delta=k}$ with $\mu\mathrm{vd}(F) \geq m$ (using Corollary 5.5). Consider $v \in \mathrm{var}_{\mu\mathrm{vd}}(F)$; if $\mathrm{ld}_F(\overline{v}) \leq \mathrm{ld}_F(v)$ holds, then let $e := (\mathrm{ld}_F(\overline{v}), \mathrm{ld}_F(v))$, while otherwise flip the components of this pair. Now we have $e \in \mathrm{pp}(k, \mu\mathrm{vd}(F))$ (using Remark 2 to Definition 13.4), and thus $\mathrm{pp}_f(k,m) \neq \emptyset$ by Lemmas 13.7, 13.8, contradicting the assumption. $\square$

13.2 Recursion on potential degree-pairs

Via potential degree-pairs and Theorem 13.10, we obtain a method for improving valid bounds-functions:

Lemma 13.11 *Consider $f \in \mathcal{VB}$. We obtain $f' \in \mathcal{VB}$ recursively as follows:*

1. *$f'(1) := 2$.*
2. *For $k > 1$ consider the largest $4 \leq m \leq 2k$ such that $\mathrm{pp}_{f'}(k,m) \neq \emptyset$, using Remark 4 to Definition 13.4 (that we only need $f'(k')$ for $k' < k$).*
3. *Now $f'(k) := \min(m, f(k))$.*

Proof: $f'(k)$ is well-defined for $k > p$ due to Remark 1 to Definition 13.4. That f' is valid follows by induction as follows. We have to show $f'(k) \leq f'(k+1)$ and $\mu\mathrm{nM}(k) \leq f'(k)$ for all $k \in \mathbb{N}$. For $k = 1$ both properties are true by definition. And the induction step follows for monotonicity by Lemma 13.9, and for the upper-bound-condition by Theorem 13.10. $\square$

The mapping $f \in \mathcal{VB} \mapsto f' \in \mathcal{VB}$ we call the "non-Mersenne operator":

Definition 13.12 *For $f \in \mathcal{VB}$ let the $f' \in \mathcal{VB}$ according to Lemma 13.11 be denoted by* $\mathbf{NM}(\boldsymbol{f}) := f'$ *(defined via "recursion on potential degree-pairs"); we call* $\mathrm{NM} : \mathcal{VB} \to \mathcal{VB}$ *the "non-Mersenne operator".*

The basic properties of the non-Mersenne operator are that of a kernel operator, which are order-theoretic properties as follows:

Lemma 13.13 *The map $\mathrm{NM} : \mathcal{VB} \to \mathcal{VB}$ is a kernel operator of the complete lattice $\mathcal{VB}$, that is, for all $f, g \in \mathcal{VB}$ holds:*

1. $\mathrm{NM}(f) \leq f$ *(intensive)*
2. $\mathrm{NM}(\mathrm{NM}(f)) = \mathrm{NM}(f)$ *(idempotent)*
3. $f \leq g \Rightarrow \mathrm{NM}(f) \leq \mathrm{NM}(g)$ *(monotonically increasing).*

Proof: Intensitivity follows by definition of NM (note that in Lemma 13.11 we have defined $f'(k)$ such that $f'(k) \leq f(k)$ holds). Also idempotence follows directly from the definition in Lemma 13.11, namely that $f'(k)$ for $k > 1$ already uses the improved values $f'(k')$ for $k' < k$. Monotonicity follows by Lemma 13.8. $\square$

By Lemma 13.13 we get that $\mathrm{NM}(f)$ for $f \in \mathcal{VB}$ is the supremum of the set of $f' \leq f$ with $\mathrm{NM}(f') = f'$. By Theorem 13.10 we get $\mathrm{NM}(\mu\mathrm{nM}) = \mu\mathrm{nM}$. In order to show that the non-Mersenne operator at most reproduces nM, that is, for all $f \in \mathcal{VB}$ holds $\mathrm{NM}(f) \leq \mathrm{nM}$, we need to provide potential degree-pairs for nM:

Lemma 13.14 *For $k \geq 2$ we have (recall Definition 7.13):*

1. *$(h(k), \mathrm{i}_{\mathrm{nM}}(k)) \in \mathrm{pp}_{\mathrm{nM}}(k, \mathrm{nM}(k))$.*

2. $\mathrm{pp}_{\mathrm{nM}}(k, \mathrm{nM}(k) + 1) = \emptyset$.

Proof: Let $m := \mathrm{nM}(k)$. For Part 1 let $e_0 := h(k)$, $e_1 := \mathrm{i}_{\mathrm{nM}}(k)$; so we have to show $(e_0, e_1) \in \mathrm{pp}_{\mathrm{nM}}(k, m)$. Consider the conditions (i) - (v) in Definition 13.4. We have $e_0 \geq 2$, since $\mathrm{nM} \geq 2$ in general, and $e_1 \geq 2$ by Definition 7.8. As shown in Corollary 7.11 we have $e_0 \leq e_1$, where $e_1 \leq k$ by definition. Furthermore we have $e_0 + e_1 = m$ by Lemma 7.10, Part 3. Altogether we have now shown conditions (i) - (iv), and it remains to show that $\mathrm{nM}(k - e_\varepsilon + 1) + e_\varepsilon \geq m$ holds for both $\varepsilon \in \{0, 1\}$; for $\varepsilon = 1$ we have equality, as already remarked, and it remains to show $\mathrm{nM}(k - e_0 + 1) + e_0 \geq m$, which is equivalent to

$$\mathrm{nM}(k - e_0 + 1) \geq \mathrm{i}_{\mathrm{nM}}(k).$$

By Definition 7.8 of $\mathrm{i}_{\mathrm{nM}}(k)$ (as the smallest i) this is implied by $\mathrm{nM}(k - e_0 + 1) \geq \mathrm{nM}(k - \mathrm{nM}(k - e_0 + 1) + 1)$. By the monotonicity of nM this is implied by $e_0 \leq \mathrm{nM}(k - e_0 + 1)$, i.e., $\mathrm{nM}(k - \mathrm{i}_{\mathrm{nM}}(k) + 1) \leq \mathrm{nM}(k - e_0 + 1)$. Again by monotonicity, this is implied by $\mathrm{i}_{\mathrm{nM}}(k) \geq e_0$, i.e., $e_1 \geq e_0$, which we have already shown.

For Part 2 we have to show $\mathrm{pp}_{\mathrm{nM}}(k, m+1) = \emptyset$. Assume that we have $(e_0, e_1) \in \mathrm{pp}_{\mathrm{nM}}(k, m + 1)$ according to Definition 13.4. Thus we have $\mathrm{nM}(k - e_1 + 1) + e_1 \geq m+1$, where $2 \leq e_1 \leq k$. Because of $e_0 + e_1 = m+1$ and $e_0 \leq e_1$ we get $e_1 \geq \frac{1}{2}(m+1)$, whence $\min(2 \cdot e_1, \mathrm{nM}(k - e_1 + 1) + e_1) \geq m+1$, and thus $\mathrm{nM}(k) \geq m+1$ by Definition 7.1. □

We obtain an alternative recursion for $\mathrm{nM}(k)$ (recall Definition 7.1):

Theorem 13.15 $\mathrm{NM}([2]) = \mathrm{NM}(\mathrm{nM}) = \mathrm{nM}$.

Proof: By Definition 13.12 and Lemma 13.14 we get $\mathrm{NM}([2]) = \mathrm{nM}$. Since NM is idempotent, we also get $\mathrm{NM}(\mathrm{nM}) = \mathrm{nM}$. □

So the non-Mersenne operator yields nM in the worst-case:

Corollary 13.16 $\mathrm{NM} : \mathcal{VB} \to \mathcal{VB}^*$*, that is, for every* $f \in \mathcal{VB}$ *holds* $\mathrm{NM}(f) \leq \mathrm{nM}$.

14 Strengthening of the mvd upper bound for $\mathcal{MU}$

In this final section many techniques introduced in this report come together, and we give some initial sharpness results (considering small deficiencies), and some non-sharpness results in the form of improved bounds (improving nM for infinitely many deficiencies). In Subsection 14.1 we determine $\mu\mathrm{nM}(k)$ for $1 \leq k \leq 6$ as values $2, 4, 5, 6, 8, 8$, where the main achievement is Theorem 14.3, showing $\mu\mathrm{nM}(6) = 8 = \mathrm{nM}(6) - 1$. Applying the non-Mersenne operator, we obtain the improved upper bound $\mu\mathrm{nM}(k) \leq \mathrm{nM}_1(k)$ in Subsection 14.2, where nM_1 is like nM, but with a duplication after the jump positions, that is, $\Delta\,\mathrm{nM}(k) = \Delta\,\mathrm{nM}_1(k) = 2$ is followed by $\Delta\,\mathrm{nM}_1(k+1) = \Delta\,\mathrm{nM}(k+1) - 1 = 0$.

14.1 Deficiencies $1, \ldots, 6$

We now show that the first deficiency k, for which the bound $\mu\mathrm{nM}(k) \leq \mathrm{nM}(k)$ is not sharp, is $k = 6$. First we show sharpness for the first five values:

Theorem 14.1 *For* $k \in \{1, \ldots, 5\}$ *we have* $\mu\mathrm{nM}(k) = \mu\mathrm{vd}(\mathcal{UHIT}_{\delta=k}) = \mathrm{nM}(k)$.

Proof: We have to give examples showing that the upper bound $\mathrm{nM}(k)$ is attained for examples in $\mathcal{UHIT}_{\delta=k}$). Lemma 12.10 covers deficiencies $k = 1, 2, 5$, namely

1. $A_1 \in \mathcal{UHIT}_{\delta=1}$ has $\mathrm{fc}(A_1) = 2 = \mathrm{nM}(1)$ (recall Example 3.2).

2. $A_2 \in \mathcal{UHIT}_{\delta=2}$ has $\mathrm{fc}(A_2) = 4 = \mathrm{nM}(2)$ (recall Example 3.3).

3. $A_3 \in \mathcal{UHIT}_{\delta=5}$ has $\mathrm{fc}(A_3) = 8 = \mathrm{nM}(5)$.

Deficiency $k = 4$ is a jump position, and thus covered by Lemma 12.11, where the example is as follows:

4. For $F_4 := \{\{1, 2\}, \{-1, 2, 3\}, \{1, -2, 3\}, \{-1, -2, 3\}, \{-1, 2, -3\}, \{1, -2, -3\}, \{-1, -2, -3\}\}$ we have $F_4 \in \mathcal{UHIT}_{\delta=4}$ with $\mathrm{fc}(F_4) = 6 = \mathrm{nM}(4)$.

The remaining case $k = 3$ we obtain via strict full subsumption resolution from F_4:

5. For $F_3 := \{\{1, 2\}, \{-1, 3\}, \{1, -2, 3\}, \{-1, 2, -3\}, \{1, -2, -3\}, \{-1, -2, -3\}\}$ we have $F_3 \in \mathcal{UHIT}_{\delta=3}$ with $\mu\mathrm{vd}(F_3) = 5 = \mathrm{nM}(3)$. □

We note that example F_3 in the proof of Theorem 14.1 shows $\nu\mathrm{fc}(\mathcal{UHIT}_{\delta=3}) = \nu\mathrm{fc}(\mathcal{MU}_{\delta=3}) = 4$ (together with Lemma 12.16, Part 3). In the sequel of this subsection we consider $k = 6$. A computation shows that there is only one potential degree-pair for the min-var-degree as given by $\mathrm{nM}(6) = 9$:

Lemma 14.2 $\mathrm{pp}_{\mathrm{nM}}(6, 9) = \{(4, 5)\}$.

Proof: Conditions (i) - (iv) of Definition 13.4 yield $\mathrm{pp}_{\mathrm{nM}}(6, 9) \subseteq \{(3, 6), (4, 5)\}$. Condition (v) excludes $(3, 6)$, since we have $\mu\mathrm{vd}(\mathcal{MU}_{\delta=6-6+1}) + 6 = 8 \not\geq 9$, while $(4, 5)$ fulfils this condition due to $\mathrm{nM}(6-4+1)+4 = 5+4 \geq 9$ and $\mathrm{nM}(6-5+1)+5 = 4 + 5 \geq 9$. □

However, the potential degree-pair of Lemma 14.2 actually can not be realised, and thus $\mu\mathrm{nM}(6) < \mathrm{nM}(6)$:

Theorem 14.3 $\nu\mathrm{fc}(\mathcal{UHIT}_{\delta=6}) = \nu\mathrm{fc}(\mathcal{MU}_{\delta=6}) = \mu\mathrm{vd}(\mathcal{UHIT}_{\delta=6}) = \mu\mathrm{nM}(6) = 8 = \mathrm{nM}(6) - 1$.

Proof: $\nu\mathrm{fc}(\mathcal{UHIT}_{\delta=6}) \geq 8$ is confirmed by the variable-clause matrix

$$\begin{pmatrix} + & + & + & - & + & - & - & - & + & - \\ + & + & + & + & + & + & + & + & - & - \\ + & + & - & + & - & + & - & - & 0 & 0 \\ + & - & + & + & - & - & + & - & 0 & 0 \end{pmatrix}$$

(where unsatisfiability is given by $8 \cdot 2^{-4} + 2 \cdot 2^{-2} = 1$).

Assume now that there exists $F \in \mathcal{MU}_{\delta=6}$ with $\mu\mathrm{vd}(F) = 9$. By Lemmas 5.3, 5.4 w.l.o.g. we can assume that F is saturated and non-singular. By Theorem 6.13 we know $n(F) \geq 4$. Consider $v \in \mathrm{var}(F)$ with $\mathrm{vd}_F(v) = 9$. W.l.o.g. we assume $\mathrm{ld}_F(v) \geq \mathrm{ld}_F(\overline{v})$. By Lemma 14.2 we have $\mathrm{ld}_F(v) = 5$, $\mathrm{ld}_F(\overline{v}) = 4$, and $\delta(F_0) = 6 - 4 + 1 = 3$, $\delta(F_1) = 6 - 5 + 1 = 2$.

Let the 5 occurrences of v in F be $C_1, \dots, C_5 \in F$, and let $C'_i := C_i \setminus \{v\}$. And let the 4 occurrences of $\overline{v}$ in F be $D_1, \dots, D_4 \in F$, and let $D'_i := D_i \setminus \{\overline{v}\}$. Using $G := \{C \in F : v \notin \mathrm{var}(C)\} = F \setminus \{C_1, \dots, C_5, D_1, \dots, D_4\}$ we get

$$\begin{aligned} F_0 &= \{C'_1, \dots, C'_5\} \mathbin{\dot\cup} G \\ F_1 &= \{D'_1, \dots, D'_4\} \mathbin{\dot\cup} G \end{aligned}$$

where $c(F_0) = 5 + c(G) = c(F) - 4$ and $c(F_1) = 4 + c(G) = c(F) - 5$.

Consider first $F_0 \in \mathcal{MU}_{\delta=3}$. We have $\mu\mathrm{vd}(F_0) \geq 9-4 = 5$, and thus $\mu\mathrm{vd}(F_0) = 5$ (due to $\mu\mathrm{vd}(F_0) \leq \mathrm{nM}(3) = 5$). Every variable $w \in \mathrm{var}(F_0)$ realising the min-var-degree of F_0 has at least 9 occurrences in F, from which at most 4 are eliminated, and thus actually such variables have $\mathrm{vd}_F(w) = 9$, and furthermore $w \in \mathrm{var}(D_i)$ for all $i \in \{1, \dots, 4\}$. By Lemma 8.5, Part 1, there exist two different variables $u_1, u_2 \in \mathrm{var}(F_0)$ with $\mathrm{vd}_{F_0}(u_i) = 5$ for $i \in \{1, 2\}$, and so we have $|D_i| \geq 3$ for all $i \in \{1, \dots, 4\}$.

Now consider $F_1 \in \mathcal{MU}_{\delta=2}$. We have $\mu\mathrm{vd}(F_1) \geq 9 - 5 = 4$, thus $\mu\mathrm{vd}(F_1) = 4$ (due to $\mu\mathrm{vd}(F_1) \leq \mathrm{nM}(2) = 4$), and thus by Lemma 5.13 F_1 is non-singular iff F_1 does not contain unit-clauses. If F_1 would contain a unit-clause, then there would be a binary clause $\{\overline{v}, x\} \in F$, contradicting that all D_i contain at least three literals. So F_1 is non-singular, and thus F_1 is isomorphic to some $\mathcal{F}_m$ for some $m \geq 2$. So F_1 is 4-variable-regular, where all the variables of F_1 have at least 9 occurrences in F, and thus we have $\mathrm{var}(F_1) \subseteq \mathrm{var}(C'_i)$ for all $i \in \{1, \dots, 5\}$, which implies that actually $\mathrm{var}(C'_i) = \mathrm{var}(F_1) = \mathrm{var}(F_0)$ holds.

Coming back to the structure of F_0, we now know that F_0 has five full clauses $C'_1, \dots, C'_5$, which contradicts Lemma 12.16, Part 3. $\square$

14.2 Sharpening the bound

Based on recursion on potential degree-pairs, we can improve the upper bound $\mathrm{nM}(k)$ for $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k})$ for $k \geq 6$ (generalising Example 13.6):

Definition 14.4 *Let* $\mathrm{nM}_1 : \mathbb{N} \to \mathbb{N}$ *be defined as* $\mathrm{nM}_1 := \mathrm{NM}([2, 4, 5, 6, 8, 8])$ *(recall Definition 13.12).*

By Lemma 13.11 together with Theorem 14.3 we get:

Theorem 14.5 *For all* $k \in \mathbb{N}$ *we have* $\mu\mathrm{nM}(k) \leq \mathrm{nM}_1(k)$.

It remains to determine nM_1 numerically:

Theorem 14.6 *In Table 3 we find the values of* $\mathrm{nM}_1(k)$ *for* $k \leq 30$. *We have* $\mathrm{nM}_1(k) = \mathrm{nM}(k)$ *for* $k \notin \{2^m - m + 1 : m \in \mathbb{N}, m \geq 3\}$, *while for* $k = 2^m - m + 1$ *we have* $\mathrm{nM}_1(k) = \mathrm{nM}(k) - 1 = 2^m$.

k	**1**	2	3	**4**	5	6	7	8	9	10	**11**	12	13	14	15
$\mathrm{nM}_1(k)$	2	4	5	6	8	$\underline{8}$	10	11	12	13	14	16	$\underline{16}$	18	19
k	16	17	18	19	20	21	22	23	24	25	**26**	27	28	29	30
$\mathrm{nM}_1(k)$	20	21	22	23	24	25	26	27	28	29	30	32	$\underline{32}$	34	35

Table 3: Values of $\mathrm{nM}_1(k)$ for $k \in \{1, \dots, 30\}$, in bold the jump-values (i.e., $k \in J$), and underlined the changed values compared to $\mathrm{nM}(k)$; we see that directly after the jump we have stagnation, followed by a second jump.

Proof: We show the formula for $\mathrm{nM}_1(k)$ via induction on k. Due to $2^3 - 3 + 1 = 6$ it holds for $k \leq 6$. So assume $k \geq 7$. We show by induction on $k \geq 7$ the following two properties (which imply the assertions of the theorem):

1. For $k = 2^m - m + 1$, $m \geq 4$, we have
   (a) $\mathrm{pp}_{\mathrm{nM}_1}(k, \mathrm{nM}(k)) = \emptyset$ and
   (b) and $\mathrm{pp}_{\mathrm{nM}_1}(k, \mathrm{nM}(k) - 1) \neq \emptyset$.

2. Otherwise $\mathrm{pp}_{\mathrm{nM}_1}(k, \mathrm{nM}(k)) \neq \emptyset$.

By Lemma 13.13 we know $\mathrm{nM}_1 \le \mathrm{nM}$.

Part 1. We consider $k = 2^m - m + 1$, $m \ge 4$. We have $\mathrm{nM}(k) = 2^m + 1$.

Part (a). To show $\mathrm{pp}_{\mathrm{nM}_1}(2^m - m + 1, 2^m + 1) = \emptyset$, we assume $(e_0, e_1) \in \mathrm{pp}_{\mathrm{nM}_1}(2^m - m + 1, 2^m + 1)$. Thus we know $e_0, e_1 \ge 2$, $e_0, e_1 \le 2^m - m + 1$, $e_0 + e_1 = 2^m + 1$, $e_0 \le e_1$, whence $e_0 \le 2^{m-1}$, and

$$\mathrm{nM}_1(2^m - m + 1 - e_\varepsilon + 1) + e_\varepsilon \ge 2^m + 1 \tag{1}$$

for both $\varepsilon \in \{0, 1\}$.

Case (a.1). Assume $e_0 \le 2^{m-1} - 1$, and thus $e_1 \ge 2^{m-1} + 2$.

From (1) we get $\mathrm{nM}(2^m - m + 1 - e_1 + 1) + e_1 \ge 2^m + 1$, where (using Corollary 7.6):

$$\begin{aligned}
&\mathrm{nM}(2^m - m + 1 - e_1 + 1) + e_1 \ge 2^m + 1 \Rightarrow \\
&\qquad \mathrm{nM}(2^m - m + 1 - (2^{m-1} + 2) + 1) + 2^{m-1} + 2 \ge 2^m + 1 \Leftrightarrow \\
&\qquad\qquad \mathrm{nM}(2^{m-1} - m) \ge 2^{m-1} - 1,
\end{aligned}$$

where by Corollary 7.24 we have $\mathrm{nM}(2^{m-1} - m) = \mathrm{nM}(2^{m-1} - (m-1) - 1) = 2^{m-1} - 2$, and we obtained a contradiction, finishing Case (a.1).

Case (a.2). It remains $e_0 = 2^{m-1}$. From (1) we get $\mathrm{nM}_1(2^m - m + 1 - e_0 + 1) + e_0 \ge 2^m + 1$, where $2^m - m + 1 - e_0 + 1 = 2^m - m + 1 - 2^{m-1} + 1 = 2^{m-1} - (m-1) + 1$, and thus by induction hypothesis we get $\mathrm{nM}_1(2^m - m + 1 - e_0 + 1) + e_0 = 2^{m-1} + 2^{m-1} = 2^m$, a contradiction. This concludes Part (a).

Part (b). We show $(2^{m-1}, 2^{m-1}) \in \mathrm{pp}_{\mathrm{nM}_1}(k, \mathrm{nM}(k) - 1)$.[15] For this it remains to show $\mathrm{nM}_1(2^m - m + 1 - 2^{m-1} + 1) + 2^{m-1} \ge 2^m$, and indeed $\mathrm{nM}_1(2^m - m + 1 - 2^{m-1} + 1) = \mathrm{nM}_1(2^{m-1} - (m-1) + 1) = 2^{m-1}$ by induction hypothesis. This concludes Part 1.

Part 2. $k \neq 2^m - m + 1$ for any $m \ge 4$. We have to show $\mathrm{pp}_{\mathrm{nM}_1}(k, \mathrm{nM}(k)) \neq \emptyset$.

Part (a). $k = 2^m - m + 2$; thus $\mathrm{nM}(k) = 2^m + 2$.

We have $(2^{m-1}, 2^{m-1} + 2) \in \mathrm{pp}_{\mathrm{nM}_1}(k, 2^m + 2)$[16], due to $\mathrm{nM}_1(2^m - m + 2 - 2^{m-1} + 1) + 2^{m-1} = \mathrm{nM}_1(2^{m-1} - (m-1) + 2) + 2^{m-1} = 2^{m-1} + 2 + 2^{m-1}$ and $\mathrm{nM}_1(2^m - m + 2 - (2^{m-1} + 2) + 1) + 2^{m-1} + 2 = \mathrm{nM}_1(2^{m-1} - (m-1)) + 2^{m-1} + 2 = 2^{m-1} + 2^{m-1} + 2$.

Part (b). $k = 2^m - m + 3$; thus $\mathrm{nM}(k) = 2^m + 3$.

We have $(2^{m-1}, 2^{m-1} + 3) \in \mathrm{pp}_{\mathrm{nM}_1}(k, 2^m + 3)$[17], due to $\mathrm{nM}_1(2^m - m + 3 - 2^{m-1} + 1) + 2^{m-1} = \mathrm{nM}_1(2^{m-1} - (m-1) + 3) + 2^{m-1} = 2^{m-1} + 3 + 2^{m-1}$ and $\mathrm{nM}_1(2^m - m + 3 - (2^{m-1} + 3) + 1) + 2^{m-1} + 3 = \mathrm{nM}_1(2^{m-1} - (m-1)) + 2^{m-1} + 3 = 2^{m-1} + 2^{m-1} + 3$.

For all remaining cases

$$2^m - m + 4 \le k \le 2^{m+1} - (m+1)$$

we show $(e_0, e_1) := (h(k), \mathrm{i}_{\mathrm{nM}}(k)) \in \mathrm{pp}_{\mathrm{nM}_1}(k, \mathrm{nM}(k))$, as in Lemma 13.14, Part 1. Recall Corollary 7.25 for the computation of $\mathrm{i}_{\mathrm{nM}}(k)$.

For the critical condition "$\mathrm{nM}_1(k - e_1 + 1) + e_1 \ge \mathrm{nM}(k)$" we recall $k - \mathrm{i}_{\mathrm{nM}}(k) + 1 = i'(k)$ (recall Definition 7.13), where $i'(k)$ is monotonically increasing. We want to

[15] We have $(2^{m-1}, 2^{m-1}) = (h(k), \mathrm{i}_{\mathrm{nM}}(k) - 1)$, but we don't need this here.

[16] We have $(2^{m-1}, 2^{m-1} + 2) = (h(k) - 1, \mathrm{i}_{\mathrm{nM}}(k) + 1)$.

[17] We have $(2^{m-1}, 2^{m-1} + 3) = (h(k) - 1, \mathrm{i}_{\mathrm{nM}}(k) + 1)$.

show that $i'(k)$ is never equal to some argument where $\mathrm{nM}_1 \neq \mathrm{nM}$, and thus we need to check the upper and the lower bounds:

For $k = 2^m - m + 4$ holds $\mathrm{i_{nM}}(k) = 2^{m-1} + 2$, thus $i'(k) = 2^m - m + 4 - (2^{m-1} + 2) + 1 = 2^{m-1} - m + 3 > 2^{m-1} - (m-1) + 1$.

For $k = 2^{m+1} - (m+1)$ holds $\mathrm{i_{nM}}(k) = 2^m$, thus $i'(k) = 2^{m+1} - (m+1) - 2^m + 1 = 2^m - m < 2^m - m + 1$.

So all critical condition s for e_1 are fulfilled, and it remains to check e_0.

Part (c). $2^m - m + 4 \le k \le 2^{m+1} - (m+1) - 2$.

Here for the critical conditions "$\mathrm{nM}_1(k - e_0 + 1) + e_0 \ge \mathrm{nM}(k)$" the values $k - e_0 + 1$ are never equal to some argument where $\mathrm{nM}_1 \neq \mathrm{nM}$, since $\Delta(k - h(k) + 1) = 1 - \Delta h(k) \ge 0$ for $k \le 2^{m+1} - (m+1) - 2$ by Theorem 7.15, and thus $k - h(k) + 1$ is monotonically increasing for the k-range we consider.

Checking for the lower bound $k = 2^m - m + 4$: $\mathrm{i_{nM}}(k) = 2^{m-1} + 2$, thus $h(k) = \mathrm{nM}(k) - \mathrm{i_{nM}}(k) = 2^m + 4 - 2^{m-1} - 2 = 2^{m-1} + 2$, which is the same as $\mathrm{i_{nM}}(k)$, and doesn't need to be checked again.

Checking for the upper bound $k = 2^{m+1} - (m+1) - 2$: $\mathrm{i_{nM}}(k) = 2^m - 1$, and thus $h(k) = \mathrm{nM}(k) - \mathrm{i_{nM}}(k) = 2^{m+1} - 3 - 2^m + 1 = 2^m - 2$, and so we check $2^{m+1} - (m+1) - 2 - (2^m - 2) + 1 = 2^m - m < 2^m - m + 1$.

Part (d). $k = 2^{m+1} - (m+1) - 1$; thus $\mathrm{nM}(k) = 2^{m+1} - 2$.

Now $\mathrm{i_{nM}}(k) = 2^m$, and so $h(k) = \mathrm{nM}(k) - \mathrm{i_{nM}}(k) = 2^m - 2$, and so the critical argument is $2^{m+1} - (m+1) - 1 - (2^m - 2) + 1 = 2^m - m + 1$. It remains to check $\mathrm{nM}_1(2^m - m + 1) + 2^m - 2 \ge 2^{m+1} - 2$, and indeed $\mathrm{nM}_1(2^m - m + 1) + 2^m - 2 = 2^m + 2^m - 2$.

Part (e). $k = 2^{m+1} - (m+1)$; thus $\mathrm{nM}(k) = 2^{m+1}$.

Now $h(k) = 2^m = \mathrm{i_{nM}}(k)$, and no check is needed. □

It is instructive to note the new Δ-values explicitly:

Corollary 14.7 *For $k \in \mathbb{N}$ holds $\Delta\,\mathrm{nM}_1(k) \in \{0, 1, 2\}$, with*

1. $\Delta\,\mathrm{nM}_1(k) = 0 \iff k = 2^m - m$ *for some* $m \in \mathbb{N}$, $m \ge 3$.
2. $\Delta\,\mathrm{nM}_1(k) = 2 \iff k = 2^m - m \pm 1$ *for some* $m \in \mathbb{N}$, $m \ge 3$.

15 Conclusion and open problems

The main subject of this report can be seen in the study of $\mu\mathrm{vd}(\mathcal{C}_{\delta=k})$ for classes $\mathcal{UHIT} \subseteq \mathcal{C} \subseteq \mathcal{MLEAN}$ and $k \in \mathbb{N}$, that is, the study of the maximal minimum variable-degree of classes of matching-lean clause-sets containing all minimally unsatisfiable clause-sets, parameterised by the deficiency. If $\mathcal{C} \subseteq \mathcal{LEAN}$, then this quantity is bounded, and indeed we have shown $\mu\mathrm{vd}(\mathcal{LEAN}_{\delta=k}) = \mathrm{nM}(k)$ (more generally this holds for every subclass of $\mathcal{LEAN}$ containing $\mathcal{VMU}$). While for $\mathcal{C} = \mathcal{MLEAN}$ this quantity is unbounded. For $\mathcal{MU}$ we have shown the improved bound $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) \le \mathrm{nM}_1(k)$, where indeed also this bound is not sharp (as will be shown in [68]; see Subsection 15.2) — the question about the determination of $\mu\mathrm{nM}(k) = \mu\mathrm{vd}(\mathcal{MU}_{\delta=k})$ is a major open research question.

For lean clause-sets we have shown the strengthened upper bound $\mu\mathrm{vd}(F) \le \mathrm{nM}(\sigma(F))$, and indeed for every clause-set F we can satisfiability-equivalently remove some clauses in polynomial time such that this upper bound holds.

15.1 Conjectures and questions

We made the following conjectures:

1. Conjecture 10.3: If a clause-set violates the upper bound on the min-var-degree for lean clause-sets, then it must have a non-trivial autarky. As we have seen, we can determine the set of variables involved, but the determination of the autarky itself is open — the conjecture states that there is a poly-time algorithm for computing such an autarky. See Subsection 10.2 for more information on this topic.

2. Conjecture 12.2: the maximum min-var-degree for unsatisfiable hitting clause-sets is the same as for the larger class of minimally unsatisfiable clause-sets. In Conjecture 15.5 we generalise this to non-boolean clause-sets.

3. Conjecture 12.3: nM is not far away from the μnM, more precisely, nM $-1 \leq \mu$nM $\leq$ nM. The stronger Conjecture 12.5: the same holds even for the maximal number of full clauses, that is, $\mathrm{nM}(k) - 1 \leq \nu\mathrm{fc}(\mathcal{MU}_{\delta=k}) \leq \mathrm{nM}(k)$ for all $k \in \mathbb{N}$. In Lemma 15.3 we will state a weaker, but proven (in future work) lower bound.

Five more conjectures will be presented in this final section. We asked also the following questions:

1. Question 3.10 is about some complexity problems around the elimination of literal occurrences in minimally unsatisfiable clause-sets.

2. Question 9.6 is about the complexity of SAT decision for $\mathcal{SED}$. At first sight it might seem easy to translate every $F \in \mathcal{CLS}$ into some sat-equivalent element of $\mathcal{SED}$, and in fact to manipulate deficiency and surplus alone is rather easy, but we do not know how to handle them together.

3. Question 10.12 concerns the existence of unsatisfiable hitting clause-sets of arbitrary surplus equal deficiency and a min-var-degree as low as possible. An underlying question is to understand better the quantity "surplus".

4. Question 11.2 is about strengthening the construction of Lemma 11.1, for finding matching-lean clause-sets of high minimum literal-degree (perhaps completely different constructions are needed).

In the remainder we outline main research areas related to the topics of the report.

15.2 Improved upper bounds for μnM

We know $\mu\mathrm{nM} \leq \mathrm{nM}_1$, and we know $\mu\mathrm{nM}(k)$ precisely for $k \in \{1, \dots, 6\}$ and for $k \in J, J+1$. Also of high relevance here is to determine $\mu\mathrm{vd}(\mathcal{UHIT}_{\delta=k})$, which by Conjecture 12.2 is the same on $\mu\mathrm{nM}(k)$. Another major conjecture is Conjecture 12.3, which says that μnM deviates at most by 1 from nM. Beyond this article, we know the following improvements of the upper bound nM_1:

- Generalising the ideas of Theorem 14.3, which is based on the improved upper bound for deficiency $2^3 - 3 + 1 = 6$, we can show also for deficiency $k = 2^4 - 4 + 2 = 14$ that we have $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) = \mathrm{nM}(k) - 1$. Via the non-Mersenne operator, this yields the improved upper bound nM_2.

- Altogether we obtain a sequence of improved upper bounds nM_{m-2} for $m \in \mathbb{N}$, $m \geq 3$, improving the upper bound at deficiency $k = 2^m - 2$ for nM_{m-3} and applying the non-Mersenne operator.

- The infimum of $\mathrm{nM}_1, \mathrm{nM}_2, \dots$ is nM_ω. This will be developed in [68].

- However, this is not the end of it — also for deficiency $k = 15$ we have $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k}) = \mathrm{nM}(k)-1$, obtaining $\mathrm{nM}_{\omega+1}$. This new improvement depends on new ideas — will there be an infinite chain of ever-increasing complexity of such improvements?

We believe that a closed "nice" formula for $\mu\mathrm{nM}(k)$ is impossible, but that however computation is possible:

Conjecture 15.1 *The function $\mu\mathrm{nM} : \mathbb{N} \to \mathbb{N}$ is "complex", and for no finite tuple $\vec{a}$ holds $\mathrm{NM}(\vec{a}) = \mu\mathrm{nM}$, however $\mu\mathrm{nM}$ is computable in doubly-exponential time.*

See Lemma 15.9 for some conditions which imply the computability-part of Conjecture 15.1.

15.3 Determining $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k})$

While Subsection 15.2 was about improving the upper bound, here now we turn to the lower bound. In Subsection 12.2 we provided only the minimum needed in this report for the measure $\mathrm{fc}(F)$ of full clauses. In the forthcoming [66] we show the following lower bound, using $S_2 : \mathbb{N} \to \mathbb{N}$, the function for the "Smarandache Primitive Numbers" introduced in [92, Unsolved Problem 47], which for $k \in \mathbb{N}$ is defined as the minimal natural number $s \in \mathbb{N}$ such that 2^k divides $s!$.

Lemma 15.2 ([66]) *For all $k \in \mathbb{N}$ holds $\nu\mathrm{fc}(\mathcal{UHIT}_{\delta=k}) \geq S_2(k)$.*

Lemma 15.2 yields the interesting inequality $S_2 \leq \mu\mathrm{nM} \leq \mathrm{nM}$. This is relevant as the upper bound nM on S_2 as well as the lower bound S_2 on $\mu\mathrm{nM}$. From [97] we get that $k + 1 \leq S_2(k)$ and thus by Corollary 7.22 we get

Lemma 15.3 ([66]) $k + 1 \leq S_2(k) \leq \mathrm{nM}(k) \leq k + 1 + \mathrm{fld}(k)$ *for $k \in \mathbb{N}$.*

Recall that in Lemma 12.4 we obtained a much sharper lower bound for $\mu\mathrm{nM}(k)$ from Conjecture 12.3. For sequences $a, b : \mathbb{N} \to \mathbb{R}$ let asymptotic equality be denoted by $a \sim b :\Leftrightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

Corollary 15.4 ([66]) *The six sequences $S_2(k)$, $\nu\mathrm{fc}(\mathcal{UHIT}_{\delta=k})$, $\nu\mathrm{fc}(\mathcal{MU}_{\delta=k})$, $\mu\mathrm{vd}(\mathcal{MU}_{\delta=k})$, $\mu\mathrm{nM}(k)$, $\mathrm{nM}(k)$ are asymptotically equal to $(k)_{k\in\mathbb{N}}$ (these are known facts for $S_2(k)$ and $\mathrm{nM}(k)$).*

In Figure 1 we show the six quantities from Corollary 15.4 and the relations between them (we do not mention the dependencies on the deficiency $k \in \mathbb{N}$ there). An arrow means a (proven) $\leq$-relation. If the arrow is labelled with "-1", then we conjecture the difference is at most -1 (while in all three cases we know cases where the difference is equal to -1), the label "$=$" means that we conjecture equality, and the label "∞" means that we conjecture that the difference is unbounded.

For a more precise asymptotic determination of these six quantities from Corollary 15.4, calling them a_k, we need to consider the six sequences $a_k - k$. Currently we only know $\mathrm{nM}(k) - k \sim \mathrm{ld}(k)$.

15.4 Generalisation to non-boolean clause-sets

It is interesting to generalise Theorem 8.3 for generalised clause-sets; see [57, 58] for a systematic study, while the most general notion of generalised clause-sets, "signed clause-sets" are discussed in [6]. Generalised clause-sets F have literals (v, ε), meaning "$v \neq \varepsilon$", for variables v with non-empty finite domains D_v and values $\varepsilon \in D_v$. The deficiency is generalised by giving every variable a weight

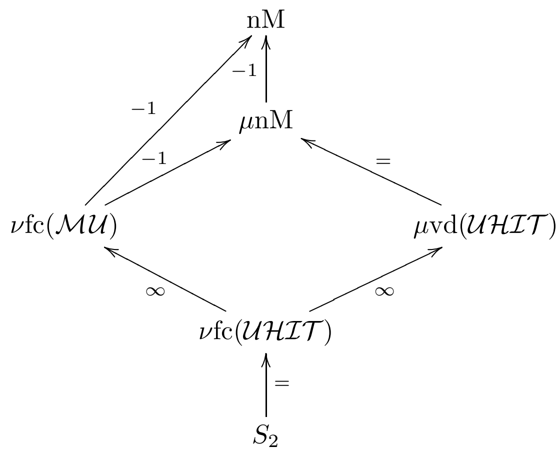

Figure 1: The four main combinatorial quantities, and the two numerical functions

$|D_v|-1 \in \mathbb{N}_0$ (which is 1 in the boolean case), i.e., $\delta(F) = c(F) - \sum_{v\in \mathrm{var}(F)}(|D_v|-1) = c(F) + n(F) - \sum_{v\in\mathrm{var}(F)}|D_v|$; see [57, Subsection 7.2]. A partial assignment is a map φ with some finite set of variables as domain $\mathrm{dom}(\varphi) =: \mathrm{var}(\varphi)$, which maps $v \in \mathrm{var}(\varphi)$ to $\varphi(v) \in D_v$. A partial assignment φ satisfies a clause-set F iff for every $C \in F$ there is $(v, \varepsilon) \in C$ with $v \in \mathrm{var}(\varphi)$ and $\varphi(v) \neq \varepsilon$. Minimally unsatisfiable (generalised) clause-sets are defined as usual (they are unsatisfiable, while every strict subset is satisfiable). In [57, Corollary 9.9] it is shown that also all minimally unsatisfiable generalised clause-sets F fulfil $\delta(F) \geq 1$ (based, like in the boolean case, on matching autarkies).

The degree $\mathrm{vd}_F(v)$ of a variable v in a clause-set F is the sum of the degrees of the literals (v, ε) for $\varepsilon \in D_v$, and thus $\mathrm{vd}_F(v) = |C \in F : C \cap (\{v\} \times D_v) \neq \emptyset|$. For a given deficiency $k \in \mathbb{N}$, the basic question is to determine the supremum of $\mu\mathrm{vd}(F)$ over all minimally unsatisfiable F with $\delta(F) = k$. The base case of deficiency $k = 1$ is handled in [58, Lemma 5.4], showing that for generalised minimally unsatisfiable clause-sets of deficiency 1 we have $\mu\mathrm{vd}(F) \leq \max_{v\in\mathrm{var}(F)}|D_v|$; actually all structural knowledge from [3, 17, 49] has been completely generalised in [58, Subsection 5.2].

But $k \geq 2$ requires more work, since here the basic method of saturation is not available for generalised clause-sets, as discussed in Subsection 5.1 in [58]: saturated generalised clause-sets (i.e., unsatisfiable clause-sets, where no literal occurrence can be added without rendering the clause-set satisfiable) with deficiency at least 2 after splitting do not necessarily generate minimally unsatisfiable (generalised) clause-sets. Thus the proofs for the boolean case seem not to be generalisable for arbitrary minimally unsatisfiable (generalised) clause-sets.

In order to repair this, the "substitution stability parameter regarding irredundancy" $\mathrm{sir}(F) \in \mathbb{Z}_{\geq -1} \uplus \{+\infty\}$ is introduced in [58, Subsection 5.3]), defined as the supremum of $k \in \mathbb{Z}_{\geq -1}$ such that for every partial assignment with $n(\varphi) := |\mathrm{var}(\varphi)| \leq k$ the clause-set $\varphi * F$, obtained as usual by application of φ to F, is minimally unsatisfiable. So $\mathrm{sir}(F) \geq 0$ iff F is minimally unsatisfiable, and as shown in [58, Corollary 4.8], $\mathrm{sir}(F) = +\infty$ iff F is a hitting clause-set (i.e., for all $C, D \in F$, $C \neq D$, there are $x \in C$, $y \in D$ with $x = (v, \varepsilon)$ and $y = (v, \varepsilon')$ for some variable v and $\varepsilon, \varepsilon' \in D_v$ with $\varepsilon \neq \varepsilon'$). And $\mathrm{sir}(F) \geq 1$ iff splitting on any variable yields always a minimally unsatisfiable clause-set. So for a boolean clause-sets F holds $\mathrm{sir}(F) \geq 1$ iff F is saturated, but for generalised clause-sets we only have that $\mathrm{sir}(F) \geq 1$ implies saturatedness ([58, Corollary 5.3]).

In Corollary 5.10 in [58] one finds a generalisation of the basic bound $\mu\mathrm{vd}(F) \leq 2\delta(F)$ for the boolean case. Namely $\mu\mathrm{vd}(F) \leq \max_{v\in\mathrm{var}(F)}|D_v| \cdot \delta(F)$ is shown for F with $\mathrm{sir}(F) \geq 1$. Since for (generalised) saturated F with $\delta(F) = 1$ we have $\mathrm{sir}(F) = \infty$ ([58, Corollary 5.6]), this covers the above mentioned result $\mu\mathrm{vd}(F) \leq \max_{v\in\mathrm{var}(F)}|D_v|$ for (arbitrary) minimally unsatisfiable F with $\delta(F) = 1$ (note that here saturation works as in the boolean case).

In [66] we concentrate on unsatisfiable hitting (generalised) clause-sets, and via generalised non-Mersenne numbers $\mathrm{nM}^d(k)$ we are able to generalise Theorem 8.3 to generalised clause-sets. We believe that in general the minimum variable degree of minimally unsatisfiable clause-sets F with $\mathrm{sir}(F) \geq 1$ for a given deficiency is always obtained by unsatisfiable hitting clause-sets (generalising Conjecture 12.3):

Conjecture 15.5 *Let $\mathcal{UHIT}^d_{\delta=k}$ denote the set of generalised unsatisfiable hitting clause-sets of deficiency $k \in \mathbb{N}$ and with uniform domain-size $d \in \mathbb{N}$, and let $\mathcal{MU}^d_{\delta=k,\mathrm{sir}\geq 1}$ be defined in the same way. Then we have for all $k, d \in \mathbb{N}$ that $\mu\mathrm{vd}(\mathcal{UHIT}^d_{\delta=k}) = \mu\mathrm{vd}(\mathcal{MU}^d_{\delta=k,\mathrm{sir}\geq 1})$.*

Furthermore, the "2" in $S_2(k)$ in Lemma 15.2 is related to the boolean domain, and generalising the results of this report in [66] to the non-boolean domain sheds light on $S_d(k)$ (the minimal $s \in \mathbb{N}$ such that d^k divides $s!$) for arbitrary prime numbers $d \in \mathbb{N}$, as introduced in [92, Unsolved Problem 49] (while for non-prime-numbers d the definition of S_d has to be generalised). See Subsection III.1 in [39] for basic properties of $S_p(k)$.

15.5 Classification of $\mathcal{MU}$

As mentioned in the introduction, a major motivation for us is the project of the classification of minimally unsatisfiable clause-sets in the deficiency (recall Examples 3.2, 3.3), where the main conjecture is:

Conjecture 15.6 *For every deficiency $k \in \mathbb{N}$ there are finitely many "patterns" which determine the nonsingular elements of $\mathcal{MU}_{\delta=k}$, as well as the saturated and hitting cases amongst them. Especially for every k the isomorphism types of $\mathcal{MU}'_{\delta=k}$ can be efficiently enumerated (without repetitions), and for any given $F \in \mathcal{MU}'_{\delta=k}$ its isomorphism type can be determined in polynomial time.*

Conjecture 15.6 has been shown for $k \leq 2$ (recall Examples 3.2, 3.3). As we discussed in Subsection 1.6.1, the translation $e : \mathcal{CLS} \to \mathcal{HYP}$ has the property $F \in \mathcal{MU}_{\delta=k} \Leftrightarrow e(F) \in \mathcal{MNC}_{\delta_{\mathrm{H}}=k-1}$ for $k \in \mathbb{N}$, and so the classification of $\mathcal{MU}_{\delta=k}$ can be seen as a subtask of the classification of $\mathcal{MNC}_{\delta_{\mathrm{H}}=k}$ for $k \in \mathbb{N}_0$. The possibility of a characterisation of $\mathcal{MNC}_{\delta_{\mathrm{H}}=0}$ was already raised in [3] (where concentration on the special case of saturated ("strong" there) minimally non-2-colourable hypergraphs was recommended), but is indeed still outstanding, which is understandable, given that polytime decision of $\mathcal{MU}_{\delta=1}$ is easy when compared with polytime decision of $\mathcal{MNC}_{\delta_{\mathrm{H}}=0}$. In the other direction, the consideration of $\mathcal{MU}_{\delta_{\mathrm{h}}=k} \subset \mathcal{MU}_{\delta\leq k}$ (recall Subsection 1.6.5) could be a stepping stone (recall $\mathcal{MU}_{\delta_{\mathrm{h}}=1} = \mathcal{UHIT}_{\delta=1}$).

A major step towards Conjecture 15.6 should be the classification of unsatisfiable hitting clause-sets in dependency on the deficiency. We remark here that unsatisfiable hitting clause-sets do not seem to have a close correspondence in hypergraph colouring, due to the lack of complementation in hypergraphs. Here the main conjecture (which should follow from Conjecture 15.6, once we found the precise formulation of "finitely many patterns") is:

Conjecture 15.7 *For every deficiency $k \in \mathbb{N}$ there are only finitely many isomorphism types of non-singular unsatisfiable hitting clause-sets, or equivalently, the number of variables of elements of $\mathcal{UHIT}'_{\delta=k}$ is bounded.*

For $k \leq 2$ finiteness has been established (Examples 3.2, 3.3), while recently we were able to prove it for $k = 3$ ([67]). Assuming Conjecture 15.7, the question arises about the computability of the function, which maps $k \in \mathbb{N}$ to the set of isomorphism types. Equivalently one can consider the computability of any function, which maps $k \in \mathbb{N}$ to an upper bound on the number of variables of elements of $\mathcal{UHIT}'_{\delta=k}$. It is conceivable that such functions must grow so quickly that they are not computable, we however believe that actually a very small bound holds, and we conjecture the following strengthened form of Conjecture 15.7:

Conjecture 15.8 *For every $k \in \mathbb{N}$ and every $F \in \mathcal{UHIT}'_{\delta=k}$ holds $n(F) \leq 4k - 5$.*

This conjecture together with the other conjectures implies computability of $\mu\mathrm{nM}$ (using Corollary 5.5):

Lemma 15.9 *Assume that Conjecture 15.8 holds.*

1. *Then the map $k \in \mathbb{N} \mapsto \mu\mathrm{vd}(\mathcal{UHIT}_{\delta=k})$ is computable, by enumerating all possible clause-sets F with at most $4k - 5$ variables, checking whether they are in $\mathcal{UHIT}'_{\delta=k}$, and if so, including $\mu\mathrm{vd}(F)$ into the maximum-computation.*
2. *If also Conjecture 12.2 holds, then the function $\mu\mathrm{nM}$ is also computable.*

Conjecture 15.1 says additionally, that although $\mu\mathrm{nM}$ is computable, it should be "complex".

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